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Meet the editor

Dr. Cheon Seoung Ryoo is a professor of Mathematics at Hannam University, South Korea. He received his PhD in Mathematics from Kyushu University, Fukuoka, Japan. Dr. Ryoo is the author of several research articles on numerical computations with guaranteed accuracy. He has also contributed works to the fields of scientific computing, analytic number theory and p-adic functional analysis. More recently, he has been working with special functions, quantum calculus and differential equations.

Contents

Preface

Polynomials are well known for their evincing properties and wide applicability in interdisciplinary areas of science. The problems arising in physical sciences and engineering are mathematically framed in terms of differential equations. Most of them can only be solved using special polynomials. Special polynomials and orthogonal polynomials provide new means of analysis for solving large classes of differential equations often encountered in physical problems. In particular, sequences of special polynomials play a fundamental role in applied mathematics. Such sequences can be described in various ways, for example, by orthogonality conditions, as solutions to differential equations, by generating functions, by recurrence relations and by operational formulas.

Written by leading researchers and mathematicians, this book provides an overview of the current research in the field of polynomials. Topics include but are not limited to the following:

- The modern umbral calculus (binomial, Appell, and Sheffer polynomial sequences)
- Orthogonal polynomials, matrix orthogonal polynomials, multiple orthogonal polynomials, and orthogonal polynomials of several variables
- Matrix and determinant approach to special polynomial sequences
- Applications of special polynomial sequences in approximation theory and in boundary value problems
- Number theory and special functions
- Asymptotic methods in orthogonal polynomials
- Fractional calculus and special functions
- Symbolic computations and special functions

This timely book will help fill a gap in the literature on the theory of polynomials and related fields. We hope it will promote further research and development in this important area.

We thank the authors for their creative contributions and the referees for their prompt and careful reviews.

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Section 1

Theory of Polynomials

Chapter 1

Cyclotomic and Littlewood Polynomials Associated to Algebras

José-Antonio de la Peña

Abstract

Let A be a finite dimensional algebra over an algebraically closed field k. Assume A is a basic connected and triangular algebra with n pairwise non-isomorphic simple modules. We consider the *Coxeter transformation* $\phi_A(T)$ as the automorphism of the Grothendieck group $K_0(A)$ induced by the Auslander-Reiten translation τ in the derived category $D^b(\text{mod}_A)$ of the module category mod_A of finite dimensional left A-modules. In this paper we study the Mahler measure $\mathbb{M}(\chi_A)$ of the Coxeter polynomial χ_A of certain *algebras* A. We consider in more detail two cases: (a) A is said to be *cyclotomic* if all eigenvalues of χ_A are roots of unity; (b) A is said to be of *Littlewood type* if all coefficients of χ_A are -1, 0 or 1. We find criteria in order that A is of one of those types. In particular, we establish new records according to Mossingshoff's list of *Record Mahler measures* of polynomials q with $1 < \mathbb{M}(q)$ as small as possible, ordered by their number of roots outside the unit circle.

Keywords: finite dimensional algebra, coxeter transformation, derived category, accessible algebra, characteristic polynomial, cyclotomic polynomial, littlewod type

1. Introduction

Assume throughout the paper that K is an algebraically closed field. We assume that A is a triangular finite dimensional basic K-algebra, that is, of the form $A = KQ/I$, where I is an ideal of the path algebra KQ for Q a quiver without oriented cycles. In particular, A has finite global dimension. The Coxeter transformation ϕ_A is the automorphism of the Grothendieck group $K_0(A)$ induced by the Auslander-Reiten translation τ in the derived category $D^b(A)$ see [1]. The characteristic polynomial χ_A of ϕ_A is called the *Coxeter polynomial* χ_A of A, or simply χ_A see [15, 17]. It is a monic self-reciprocal polynomial, therefore it is $\chi_A = a_0 + a_1T + a_2T^2 + \dots + a_{n-2}T^{n-2} + a_{n-1}T^{n-1} + a_nT^n \in \mathbb{Z}[T]$, with $a_i = a_{n-i}$ for $0 \le i \le n$, and $a_0 = 1 = a_n$.

Consider the roots $\lambda_1, \dots, \lambda_n$ of χ_A, the so called *spectrum* of A. There is a number of measures associated to the absolute values $|\lambda|$ for λ in the spectrum $\text{Spec}(\phi_A)$ of A. For instance, the *spectral radius* of A is defined as $\rho_A = \max\{|\lambda| : \lambda \in \text{Spec}(\phi_A)\}$ and the *Mahler measure* of χ_A defined as $\mathbb{M}(\chi_A) = \max\left\{1, \prod_{|\lambda|>1}|\lambda|\right\}$. Recently, some explorations on the relations of the Mahler measure $\mathbb{M}(\chi_A)$ and properties of the algebra A have been initiated.

For a one-point extension $A = B[N]$, we show that $\mathbb{M}(\chi_B) \le \mathbb{M}(\chi_A)$. The strongest statements and examples will be given for the class of accessible algebras. We say that an algebra A is *accessible from B* if there is a sequence $B = B_1, B_2, \ldots, B_s = A$ of algebras such that each B_{i+1} is a one-point extension (resp. coextension) of B_i for some exceptional B_i-module M_i. As a special case, a K-algebra A is called *accessible* if A is accessible from the one vertex algebra K.

We say that A is of *cyclotomic type* if the eigenvalues of ϕ_A lie on the unit circle. Many important finite dimensional algebras are known to be of cyclotomic type: hereditary algebras of finite or tame representation type, canonical algebras, some extended canonical algebras and many others. On the other hand, there are well-known classes of algebras with a mixed behavior with respect to cyclotomicity. For instance, in Section 6 below we consider the class of *Nakayama algebras*. Let $N(n, r)$ be the quotient obtained from the linear quiver with n vertices

$$\bullet \xrightarrow{x} \bullet \xrightarrow{x} \cdots \bullet \xrightarrow{x} \bullet$$

with relations $x^r = 0$. The Nakayama algebras $N(n, 2)$ are easily proven to be of cyclotomic type, while those of the form $N(n, 3)$ are of cyclotomic type as consequence of lengthly considerations in [18]. The case $r = 4$ is more representative: $N(n, 4)$ is of cyclotomic type for all $0 \leq n \leq 100$ except for $n = 10, 22, 30, 42, 50, 62, 70, 82$ and 90. Clearly, if A is of cyclotomic type then $|\mathrm{Tr}\ (\phi_A)^k| \leq n$, for $k \geq 0$. We show the following theorem.

Theorem 1: *Let M be an unimodular n × n-matrix. The following are equivalent:*

a. *M is of cyclotomic type;*

b. for every positive integer $0 \leq k \leq n$, we have $|\mathrm{Tr}\ (M^k)| \leq n$.

We also consider algebras A of *Littlewood type* where χ_A has all its coefficients in the set $\{-1, 0, 1\}$. Among other structure results, we prove.

Proposition. *The closure $\overline{P}$ of the set P of roots of Littlewood polynomials, equals the set R of roots of Littlewood series.*

Our results make use of well established techniques in the *representation theory of algebras*, as well as results from the *theory of polynomials* and *transcendental number theory*, where Mahler measure has its usual habitat. We stress here the natural context of these investigations on the largely unexplored overlapping area of these important subjects. Hence, rather than a comprehensive study we understand our work as a preliminary exploration where examples are most valuable.

2. Measures for polynomials

2.1 Self-reciprocal polynomials

A polynomial $p(z)$ of degree n is said to be self-reciprocal if $p(z) = z^n p(1/z)$. The following table displays the number $a(n)$ of polynomials p of degree n (for small n) with $p(0)$ non-zero, $b(n)$ is the number of such polynomials which are additionally self-reciprocal, and $c(n)$ is the number of those which are self-reciprocal and where $p(-1)$ is the square of an integer.

n	1	2	3	4	5	6	7	8	9	10	11	12	15	20	25
$a(n)$	2	6	10	24	38	78	118	224	330	584	838	1420	4514	30,532	152,170
$b(n)$	1	5	5	19	19	59	59	165	165	419	419	1001	2257	20,399	76,085
$c(n)$	1	3	5	12	19	34	59	99	165	244	419	598	2257	12,526	76,085

Indeed, there is an efficient algorithm to determine such polynomials of given degree n, based on a quadratic bound for $n \leq 4f(n)^2$ in terms of Euler totient function, $f(n)$.

Cyclotomic polynomials Φ_n and their products are a natural source for self-reciprocal polynomials. Clearly, $\Phi_1(z) = z - 1$ is not self-reciprocal, but all remaining Φ_n (with $n \geq 2$) are. Hence, exactly the polynomials $(z-1)^{2k}\prod_{n\geq 2}\Phi_n^{e_n}$ with natural numbers k and e_n are self-reciprocal with spectral radio one and without eigenvalue zero.

It is not a coincidence that in the above tables we have $b(n) = c(n+1)$ for n even and $b(n) = c(n)$ for n odd. Indeed, if p is self-reciprocal of odd degree then $p(-1) = 0$, hence $p(z) = (z+1)q(z)$ where q is also self-reciprocal.

2.2 Mahler measure

Let A be a finite dimensional K-algebra with finite global dimension. The *Grothendieck group* $K_0(A)$ of the category mod_A of finite dimensional (right) A-modules, formed with respect to short exact sequences, is naturally isomorphic to the Grothendieck group of the derived category, formed with respect to exact triangles.

The Coxeter transformation ϕ_A is the automorphism of the Grothendieck group $K_0(A)$ induced by the Auslander-Reiten translation τ. The characteristic polynomial $\chi_A(T)$ of ϕ_A is called the *Coxeter polynomial* $\chi_A(T)$ of A, or simply χ_A. It is a monic self-reciprocal polynomial, therefore it is $\chi_A(T) = a_0 + a_1T + a_2T^2 + \ldots + a_{n-2}T^{n-2} + a_{n-1}T^{n-1} + a_nT^n \in \mathbb{Z}[T]$, with $a_i = a_{n-i}$ for $0 \leq i \leq n$, and $a_0 = 1 = a_n$.

Consider the roots $\lambda_1(A), \ldots, \lambda_n(A)$ of χ_A, the so called *spectrum* of A. In [15], a measure for polynomials was introduced. Namely, the *Mahler measure* of χ_A is $\mathbb{M}(\chi_A) = \max\{1, \prod_{i=1}^{n}|\lambda_i|\}$. By a celebrated result of Kronecker [9], see also [7, Prop. 1.2.1], a monic integral polynomial p, with $p(0) \neq 0$, has $\mathbb{M}(p) = 1$ if and only if p factorizes as product of cyclotomic polynomials. As observed in [18], A is of *cyclotomic type* if and only if $\mathbb{M}(\chi_A) = 1$, that is, $\chi_A(T)$ factorizes as product of cyclotomic polynomials.

2.3 Spectral radius one, periodicity

If the spectrum of A lies in the unit disk, then all roots of χ_A lie on the unit circle, hence A has spectral radius $\rho_A = 1$. Clearly, for fixed degree there are only finitely many monic integral polynomials with this property.

The following finite dimensional algebras are known to produce Coxeter polynomials of spectral radius one:

1. hereditary algebras of finite or tame representation type;

2. all canonical algebras;

3. (some) extended canonical algebras;

4. generalizing (2), (some) algebras which are derived equivalent to categories of coherent sheaves.

We put $v_n = 1 + x + x^2 + \ldots + x^{n-1}$. Note that v_n has degree $n-1$. There are several reasons for this choice: first of all $v_n(1) = n$, second this normalization yields convincing formulas for the Coxeter polynomials of canonical algebras and hereditary stars, third representing a Coxeter polynomial — for spectral radius one — as a rational function in the v_n's relates to a Poincaré series, naturally attached to the setting.

Dynkin type	Star symbol	v-factorization	Cyclotomic factorization	Coxeter number
$\mathbb{A}_n$	$[n]$	v_{n+1}	$\prod_{d\mid n,\, d>1} \Phi_d$	$n+1$
$\mathbb{D}_n$	$[2,2,n-2]$	$\frac{v_2\,(v_2 v_{n-2})}{(v_2 v_{n-2}) v_{n-1}} v_{2(n-1)}$	$\Phi_2 \prod_{\substack{d\mid 2(n-1)\\ d\neq 1,\, d\neq n-1}} \Phi_d$	$2(n-1)$
$\mathbb{E}_6$	$[2,3,3]$	$\frac{v_2 v_3 (v_3)}{(v_3) v_4 v_6} v_{12}$	$\Phi_3\Phi_{12}$	12
$\mathbb{E}_7$	$[2,3,4]$	$\frac{v_2 v_3 (v_4)}{(v_4) v_6 v_9} v_{18}$	$\Phi_2\Phi_{18}$	18
$\mathbb{E}_8$	$[2,3,5]$	$\frac{v_2 v_3 v_5}{v_6 v_{10} v_{15}} v_{30}$	Φ_{30}	30

In the column 'v-factorization', we have added some extra terms in the nominator and denominator which obviously cancel.

Inspection of the table shows the following result:

Proposition. *Let k be an algebraically closed field and A be a connected, hereditary k-algebra which is representation-finite. Then the Coxeter polynomial χ_A determines A up to derived equivalence.* □

2.4 Triangular algebras

Nearly all algebras considered in this survey are triangular. By definition, a finite dimensional algebra is called *triangular* if it has triangular matrix shape

$$\begin{bmatrix} A_1 & M_{12} & \cdots & M_{1n} \\ 0 & A_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

where the diagonal entries A_i are skew-fields and the off-diagonal entries M_{ij}, $j > i$, are A_i, A_j-bimodules. Each triangular algebra has finite global dimension.

Proposition. *Let A be a triangular algebra over an algebraically closed field K. Then $\chi_A(-1)$ is the square of an integer.*

Proof. Let C be the Cartan matrix of A with respect to the basis of indecomposable projectives. Since A is triangular and K is algebraically closed, we get $\det C = 1$, yielding

$$\chi_A = \left|xI + C^{-1}C^t\right| = \left|C^{-1}\right| \cdot \left|xC + C^t\right| = \left|C^t + xC\right|.$$

Hence $\chi_A(-1)$ is the determinant of the skew-symmetric matrix $S = C^t - C$. Using the skew-normal form of S, see [16, Theorem IV.1], we obtain $S' = U^t S U$ for some $U \in \mathrm{GL}_n(\mathbb{Z})$, where S' is a block-diagonal matrix whose first block is the zero matrix of a certain size and where the remaining blocks have the shape $\begin{bmatrix} 0 & m_i \\ -m_i & 0 \end{bmatrix}$ with integers m_i. The claim follows. □

Which self-reciprocal polynomials of spectral radius one are Coxeter polynomials? The answer is not known. If arbitrary base fields are allowed, we conjecture that all self-reciprocal polynomials are realizable as Coxeter polynomials of triangular algebras. Restricting to algebraically closed fields, already the request that $\chi_A(-1)$ is a square discards many self-reciprocal polynomials, for instance the cyclotomic polynomials Φ_4, Φ_6, Φ_8, Φ_{10}. Moreover, the polynomial $f = x^3 + 1$, which is the Coxeter polynomial of the non simply-laced Dynkin diagram $\mathbb{B}_3$, does not appear as the Coxeter polynomial of a triangular algebra over an algebraically closed field, despite of the fact that $f(-1) = 0$ is a square. Indeed, the Cartan matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

yields the Coxeter polynomial $f = x^3 + \alpha x^2 + \alpha x + 1$, where $\alpha = abc - a^2 - b^2 - c^2 + 3$. The equation $a^2 + b^2 + c^2 - abc = 3$ of Hurwitz-Markov type does not have an integral solution. (Use that reduction modulo 3 only yields the trivial solution in $\mathbb{F}_3$.)

2.5 Relationship with graph theory

Given a (non-oriented) graph Δ, its *characteristic polynomial* κ_Δ is defined as the characteristic polynomial of the adjacency matrix M_Δ of Δ. Observe that, since M_Δ is symmetric, all its eigenvalues are real numbers. For general results on graph theory and spectra of graphs see [4].

There are important interactions between the theory of graph spectra and the representation theory of algebras, due to the fact that if C is the Cartan matrix of $A = K\left[\vec{\Delta}\right]$, then M_Δ is determined by the symmetrization $C + C^t$ of C, since $M_\Delta = C + C^t - 2I$. We shall see that information on the spectra of M_Δ provides fundamental insights into the spectral analysis of the Coxeter matrix Φ_A and the structure of the algebra A.

A fundamental fact for a hereditary algebra $A = K\left[\vec{\Delta}\right]$, when $\vec{\Delta}$ is a *bipartite quiver*, that is, every vertex is a sink or source, is that $\mathrm{Spec}(\Phi_A) \subset \mathbb{S}^1 \cup \mathbb{R}^+$. This was shown as a consequence of the following important identity.

Proposition. [2] *Let* $A = K\left[\vec{\Delta}\right]$ *be a hereditary algebra with* $\vec{\Delta}$ *a bipartite quiver without oriented cycles. Then* $\chi_A(x^2) = x^n \kappa_\Delta(x + x^{-1})$, *where n is the number of vertices of* $\vec{\Delta}$ *and* κ_Δ *is the characteristic polynomial of the underlying graph* Δ *of* $\vec{\Delta}$.

Proof. Since $\vec{\Delta}$ is bipartite, we may assume that the first m vertices are sources and the last $n - m$ vertices are sinks. Then the adjacency matrix A of Δ and the Cartan matrix C of A, in the basis of simple modules, take the form: $A = N + N^t$, $C = I_n - N$, where

$$N = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$$

for certain $m \times m$-matrix D. Since $N^2 = 0$, then $C^{-1} = I_n + N$. Therefore

$$\begin{aligned} \det\left(x^2 I_n - \Phi_A\right) &= \det\left(x^2 I_n + (I_n - N)(I_n + N)^t\right)\det(I_n - N^t) \\ &= \det\left(x^2 I_n - x^2 N^t + (I_n - N)\right) \\ &= x^n \det\left((x + x^{-1}) I_n - x N^t - x^{-1} N\right) \\ &= x^n \det\left((x + x^{-1}) I_n - A\right). \end{aligned}$$

□

The above result is important since it makes the spectral analysis of bipartite quivers and their underlying graphs almost equivalent. Note, however, that the representation theoretic context is much richer, given the categorical context behind the spectral analysis of quivers. The representation theory of bipartite quivers may thus be seen as a categorification of the class of graphs, allowing a bipartite structure.

Constructions in graph theory. Several simple constructions in graph theory provide tools to obtain in practice the characteristic polynomial of a graph. We recall two of them (see [4] for related results):

a. Assume that a is a vertex in the graph Δ with a unique neighbor b and Δ'(resp. Δ'') is the full subgraph of Δ with vertices $\Delta_0\backslash\{a\}$ (resp. $\Delta_0\backslash\{a,b\}$), then

$$\kappa_\Delta = x\kappa_{\Delta'} - \kappa_{\Delta''}$$

b. Let Δ_i be the graph obtained by deleting the vertex i in Δ. Then the first derivative of κ_Δ is given by

$$\kappa'_\Delta = \sum_i \kappa_{\Delta_i}$$

The above formulas can be used inductively to calculate the characteristic polynomial of trees and other graphs. They immediately imply the following result that will be used often to calculate Coxeter polynomials of algebras.

Proposition. *Let $A = K\left[\vec{\Delta}\right]$ be a bipartite hereditary algebra. The following holds:*

i. *Let a be a vertex in the graph Δ with a unique neighbor b. Consider the algebras B and C obtained as quotients of A modulo the ideal generated by the vertices a and a, b, respectively. Then*

$$\chi_A = (x+1)\chi_B - x\chi_C$$

ii. *The first derivative of the Coxeter polynomial satisfies:*

$$2x\chi_A{}' = n\chi_A + (x-1)\sum_i \chi_{A^{(i)}}$$

where $A^{(i)} = K\left[\vec{\Delta}\backslash\{i\}\right]$ is an algebra obtained from A by 'killing' a vertex i.

Proof. Use the corresponding results for graphs and A'Campo's formula for the algebras A and its quotients $A^{(i)}$. □

3. Important classes of algebras

In this section we give the definitions and main properties of such classes of finite dimensional algebras where information on their spectral properties is available.

3.1 Hereditary algebras

Let A be a finite dimensional K-algebra. For simplicity we assume $A = K\left[\vec{\Delta}\right]/I$ for a quiver $\vec{\Delta}$ without oriented cycles and I an ideal of the path algebra. The following facts about the Coxeter transformation Φ_A of A are fundamental:

i. Let $S_1, \ldots, S_n$ be a complete system of pairwise non-isomorphic simple A-modules, $P_1, \ldots, P_n$ the corresponding projective covers and $I_1, \ldots, I_n$ the injective envelopes. Then ϕ_A is the automorphism of $K_0(A)$ defined by $\Phi_A[P_i] = -[I_i]$, where $[X]$ denotes the class of a module X in $K_0(A)$.

ii. For a hereditary algebra $A = K\left[\vec{\Delta}\right]$, the spectral radius $\rho_A = \rho_{\Phi_A}$ determines the representation type of A in the following manner:

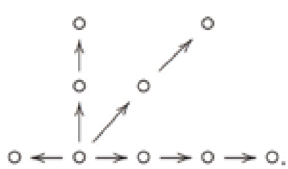
a. A is representation-finite if $1 = \rho_A$ is not a root of the Coxeter polynomial χ_A.

b. A is tame if $1 = \rho_A \in \mathrm{Roots}(\chi_A)$.

c. A is wild if $1 < \rho_A$. Moreover, if A is wild connected, Ringel [20] shows that the spectral radius ρ_A is a simple root of χ_A. Then Perron-Frobenius theory yields a vector $y^+ \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$ with positive coordinates such that $\Phi_A y^+ = \rho_A y^+$. Since χ_A is self reciprocal, there is a vector $y^- \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$ with positive coordinates such that $\Phi_A y^- = \rho_A^{-1} y^-$. The vectors y^+, y^- play an important role in the representation theory of $A = K\left[\vec{\Delta}\right]$, see [5, 17].

Explicit formulas, special values. We are discussing various instances where an explicit formula for the Coxeter polynomial is known.

star quivers. Let A be the path algebra of a hereditary star $[p_1, \ldots, p_t]$ with respect to the standard orientation, see

$[2, 3, 3, 4]:$ 

Since the Coxeter polynomial χ_A does not depend on the orientation of A we will denote it by $\chi_{[p_1, \ldots, p_t]}$. It follows from [11, prop. 9.1] or [2] that

$$\chi_{[p_1, \ldots, p_t]} = \prod_{i=1}^{t} v_{p_i} \left((x+1) - x \sum_{i=1}^{t} \frac{v_{p_i - 1}}{v_{p_i}} \right). \tag{1}$$

In particular, we have an explicit formula for the sum of coefficients of $\chi = \chi_{[p_1, \ldots, p_t]}$ as follows:

$$\chi(1) = \prod_{i=1}^{t} p_i \left(2 - \sum_{i=1}^{t} \left(1 - \frac{1}{p_i} \right) \right). \tag{2}$$

This special value of χ has a specific mathematical meaning: up to the factor $\prod_{i=1}^{t} p_i$ this is just the orbifold-Euler characteristic of a weighted projective line $\mathbb{X}$ of weight type $(p_1, \ldots, p_t)$. Moreover,

1. $\chi(1) > 0$ if and only if the star $[p_1, \ldots, p_t]$ is of Dynkin type, correspondingly the algebra A is representation-finite.

2. $\chi(1) = 0$ if and only if the star $[p_1, ..., p_t]$ is of extended Dynkin type, correspondingly the algebra A is of tame (domestic) type.

3. $\chi(1) < 0$ if and only if $[p_1, ..., p_t]$ is not Dynkin or extended Dynkin, correspondingly the algebra A is of wild representation type.

The above deals with all the Dynkin types and with the extended Dynkin diagrams of type $\tilde{\mathbb{D}}_n$, $n \geq 4$, and $\tilde{\mathbb{E}}_n$, $n = 6, 7, 8$. To complete the picture, we also consider the extended Dynkin quivers of type $\tilde{\mathbb{A}}_n$ $(n \geq 2)$ restricting, of course, to quivers without oriented cycles. Here, the Coxeter polynomial depends on the orientation: If p (resp. q) denotes the number of arrows in clockwise (resp. anticlockwise) orientation $(p, q \geq 1, p + q = n + 1)$, that is, the quiver has type $\mathbb{A}(p, q)$, the Coxeter polynomial χ is given by

$$\chi_{(p,q)} = (x-1)^2 v_p v_q. \tag{3}$$

Hence $\chi(1) = 0$, fitting into the above picture.
The next table displays the v-factorization of extended Dynkin quivers.

Extended Dynkin type	**Star symbol**	**Weight symbol**	**Coxeter polynomial**
$\tilde{\mathbb{A}}_{p,q}$	—	(p, q)	$(x-1)^2 v_p v_q$
$\tilde{\mathbb{D}}_n$, $n \geq 4$	[2,2,n-2]	$(2, 2, n-2)$	$(x-1)^2 v_2^2 v_{n-2}$
$\tilde{\mathbb{E}}_6$	$[3, 3, 3]$	$(2, 3, 3)$	$(x-1)^2 v_2 v_3^2$
$\tilde{\mathbb{E}}_7$	$[2, 4, 4]$	$(2, 3, 4)$	$(x-1)^2 v_2 v_3 v_4$
$\tilde{\mathbb{E}}_8$	$[2, 3, 6]$	$(2, 3, 5)$	$(x-1)^2 v_2 v_3 v_5$

Remark: As is shown by the above table, proposition 2.3 extends to the tame hereditary case. That is, the Coxeter polynomial of a connected, tame hereditary K-algebra A (remember, K is algebraically closed) determines the algebra A up to derived equivalence. This is no longer true for wild hereditary algebras, not even for trees.

3.2 Canonical algebras

Canonical algebras were introduced by Ringel [19]. They form a key class to study important features of representation theory. In the form of tubular canonical algebras they provide the standard examples of tame algebras of linear growth. Up to tilting canonical algebras are characterized as the connected K-algebras with a separating exact subcategory or a separating tubular one-parameter family (see [12]). That is, the module category mod $- \Lambda$ accepts a *separating tubular family* $\mathcal{T} = (T_\lambda)_{\lambda \in P_1 K}$, where T_λ is a homogeneous tube for all λ with the exception of t tubes $T_{\lambda_1}, ..., T_{\lambda_t}$ with T_{λ_i} of rank p_i $(1 \leq i \leq t)$.

Canonical algebras constitute an instance, where the explicit form of the Coxeter polynomial is known, see [11] or [10].

Proposition. *Let Λ be a canonical algebra with weight and parameter data $(\boldsymbol{p}, \boldsymbol{\lambda})$. Then the Coxeter polynomial of Λ is given by*

$$\chi_\Lambda = (x-1)^2 \prod_{i=1}^{t} v_{p_i}. \tag{4}\quad\square$$

The Coxeter polynomial therefore only depends on the weight sequence $\boldsymbol{p}$. Conversely, the Coxeter polynomial determines the weight sequence — up to ordering.

3.3 Incidence algebras of posets

Let X be a finite partially ordered set (poset). The incidence algebra KX is the K-algebra spanned by elements e_{xy} for the pairs $x \leq y$ in X, with multiplication defined by $e_{xy}e_{zw} = \delta_{yz}e_{xw}$. Finite dimensional right modules over KX can be identified with commutative diagrams of finite dimensional K-vector spaces over the Hasse diagram of X, which is the directed graph whose vertices are the points of X, with an arrow from x to y if $x < y$ and there is no $z \in X$ with $x < z < y$.

We recollect the basic facts on the Euler form of posets and refer the reader to [6] for details. The algebra KX is of finite global dimension, hence its Euler form is well-defined and non-degenerate. Denote by C_X, Φ_X the matrices of the bilinear form and the corresponding Coxeter transformation with respect to the basis of the simple KX-modules.

The incidence matrix of X, denoted 1_X, is the $X \times X$ matrix defined by $(1_X)_{xy} = 1$ if $x \leq y$ and otherwise $(1_X)_{xy} = 0$. By extending the partial order on X to a linear order, we can always arrange the elements of X such that the incidence matrix is uni-triangular. In particular, 1_X is invertible over $\mathbb{Z}$. Recall that the Möbius function $\mu_X : X \times X \to \mathbb{Z}$ is defined by $\mu_X(x,y) = (1_X)^{-1}_{xy}$.

Lemma. *a.* $C_X = 1_X^{-1}$.

b. Let $x, y \in X$. *Then* $(\Phi_X)_{xy} = -\sum_{z:z \geq x} \mu_X(y,z)$.

Proposition. *If X and Y are posets, then* $C_{X\times Y} = C_X \otimes C_Y$ *and* $\Phi_{X\times Y} = -\Phi_X \otimes \Phi_Y$.

4. Cyclotomic polynomials and polynomials of Littlewood type

4.1 Cyclotomic polynomials

We recall some facts about *cyclotomic polynomials*.

The n-cyclotomic polynomial $\Phi_n(T)$ is inductively defined by the formula

$$T^n - 1 = \prod_{d|n} \Phi_d(T). \tag{5}$$

The *Möbius function* is defined as follows:

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^r & \text{if } n = p_1, \dots p_r \text{ is a factorization into distinct primes.} \end{cases}$$

A more explicit expression for the cyclotomic polynomials is given by

$$\Phi_n(T) = \prod_{\substack{1 \leq d < n \\ d|n}} v_{n/d}(T)^{\mu(d)} \tag{6}$$

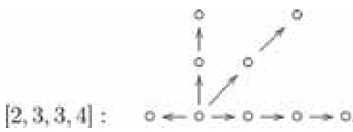
for $n \geq 2$, where $v_n = 1 + T + T^2 + \dots + T^{n-1}$.

4.2 Hereditary stars

A *path algebra* $K\Delta$ is said to be of *Dynkin type* if the underlying graph $|\Delta|$ of Δ is one of the *ADE-series*, that is, of type, $\mathbb{A}_n$, $\mathbb{D}_n$, for some $n \geq 1$ or $\mathbb{E}_k$, for $k = 6, 7, 8$.

There are various instances where an explicit formula for the Coxeter polynomial is known.

Let A be the path algebra of a hereditary star $[p_1, \ldots, p_t]$ with respect to the standard orientation, see [13].



Since the Coxeter polynomial χ_A does not depend on the orientation of A we will denote it by $\chi_{[p_1,\ldots,p_t]}$. It follows that

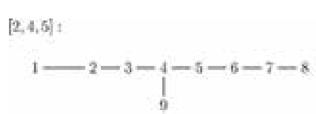
$$\chi_{[p_1,\ldots,p_t]} = \prod_{i=1}^{t} v_{p_i} \left((T+1) - T \sum_{j=1}^{t} \frac{v_{p_j - 1}}{v_{p_j}} \right).$$

In particular, we have an explicit formula for the sum of coefficients of $\chi_{[p_1,\ldots,p_t]}$ as follows:

$$\sum_{i=0}^{n} a_i = \chi_{[p_1,\ldots,p_t]}(1) = \prod_{i=1}^{t} p_i \left(2 - \sum_{i=1}^{t} \left(1 - \frac{1}{p_i} \right) \right).$$

4.3 Wild algebras

Let c be the real root of the polynomial $T^3 - T - 1$, approximately $c = 1.325$. As observed in [21], a wild hereditary algebra A associated to a graph Δ without multiple arrows has spectral radius $\rho_A > c$ unless Δ is one of the following graphs:



In these cases, for $m \geq 8$

$$c > \rho_{[2,4,5]} > \rho_{[2,3,m]} > \rho_{[2,3,7]} = \mu_0$$

where $\mu_0 = 1.176280\ldots$ is the real root of the Coxeter polynomial

$$T^{10} + T^9 - T^7 - T^6 - T^5 - T^4 - T^3 + T + 1$$

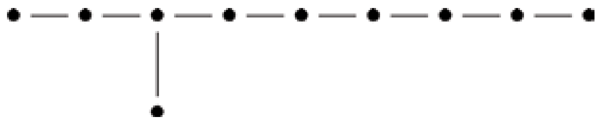
associated to any hereditary algebra whose underlying graph is $[2, 3, 7]$. Observe that in these cases, the Mahler measure of the algebra equals the spectral radius.

4.4 Lehmer polynomial

In 1933, D. H. Lehmer found that the polynomial

$$T^{10} + T^9 - T^7 - T^6 - T^5 - T^4 - T^3 + T + 1$$

has Mahler measure $\mu_0 = 1.176280\ldots$, and he asked if there exist any smaller values exceeding 1. In fact, the polynomial above is the Coxeter polynomial of the hereditary algebra whose underlying graph $[2, 3, 7]$ is depicted below.



We say that a matrix M is of *Mahler type* (resp. *strictly Mahler type*) if either $\mathbb{M}(M) = 1$ or $\mathbb{M}(M) \geq \mu_0$ (resp. $\mathbb{M}(M) > \mu_0$). Earlier this year, Jean-Louis Verger-Gaugry announced a proof of Lehmer's conjecture, see https://arxiv.org/pdf/1709.03771.pdf. The key result (Theorem 5.28, p. 122) is a Dobrowolski type minoration of the Mahler Measure $\mathbb{M}(\beta)$. Experts are still reading the arguments, but there is no conclusive opinion.

4.5 Happel's trace formula

In [8], Happel shows that the trace of the Coxeter matrix can be expressed as follows:

$$-\mathrm{Tr}(\phi_A) = \sum_{k=0}^{\infty} (-1)^k \dim_K H^k(A) \tag{7}$$

where $H^k(A)$ denotes the k-th Hochschild cohomology group. In particular, if the Hochschild cohomology ring $H^*(A)$ is trivial, that is, $H^i(A) = 0$ for $i > 0$ and $H^0(A) = K$, then $\mathrm{Tr}(\phi_A) = -1$.

For an algebra A and a left A-module N we call

$$A[N] = \begin{bmatrix} A & 0 \\ N & K \end{bmatrix}$$

the *one-point extension* of A by N. This construction provides an order of vertices to deal with *triangular algebras*, that is, algebras KQ/I, where I is an ideal of the path algebra KQ for Q a quiver without oriented cycles.

4.6 One-point extensions

Let B be an algebra and M a B-module. Consider the one-point extension $A = B[N]$. In [19] it is shown the Coxeter transformations of A and B are related by

$$\phi_A = \begin{pmatrix} \phi_B & -C_B^T n^T \\ -n\phi_B & nC_B^T n^T - 1 \end{pmatrix} \tag{8}$$

where C_B is the *Cartan matrix* of B which satisfies $\phi_B = -C_B^{-T} C_B$ and n is the class of N in the Grothendieck group $K_0(B)$. In case $A = B[N]$ with N an *exceptional* module, it follows that

$$\mathrm{Tr}(\phi_A) = \mathrm{Tr}(\phi_B)$$

We recall that the *Euler quadratic form* is defined as $q_A(x) = xC_A^t x^t$. Assume that $A = B[M]$ for an algebra B and an indecomposable module M. In many cases, we get that $q_A(m) > 0$, for m the dimension vector of M (for instance, if M is preprojective, or if q_A coincides with the Tits form of A...)

Proposition. *Let A be an accessible algebra, such that $q_A(m) > 0$ for m the dimension vector of M, where $A = B[M]$ for certain algebra B and an indecomposable module M. Then the following happens:*

a. $\mathrm{Tr}(\phi_A) \geq -1$;

b. if $\mathrm{Tr}(\phi_B) = -1$ and $q_B(m) = 1$, then $\mathrm{Tr}(\phi_A) = -1$.

Proof. Assume that $A = B[M]$ for an algebra B and an indecomposable module M such that $q_A(m) > 0$ for m the dimension vector of M. Then B is also accessible. By induction hypothesis, $\mathrm{Tr}(\phi_B) \geq -1$. Then

$$\mathrm{Tr}(\phi_A) = \mathrm{Tr}(\phi_B) + \left(mC_B^T m^T - 1\right) \geq -1 + \left(mC_B^T m^T - 1\right) = -1 + \left(q_B(m) - 1\right) \geq -1$$

This shows (a).

For (b) assume that $\mathrm{Tr}(\phi_B) = -1$ and $q_B(m) = 1$, then

$$\mathrm{Tr}(\phi_A) = \mathrm{Tr}(\phi_B) + \left(mC_B^T m^T - 1\right) = -1 + \left(mC_B^T m^T - 1\right) = -1 + \left(q_B(m) - 1\right) = -1$$

□

4.7 Strongly accessible algebras

Theorem: *A finite dimensional accessible algebra A then it is strongly accessible if and only if* $\mathrm{Tr}(\phi_A) = -1$.

Proof. Assume A is strongly accessible from A_0. Since $q_A(m) \geq 1$, for $A = B[M]$ a one-point extension of the subcategory B of A by the exceptional module M (since then $q_A(m) = \dim_K \mathrm{End}_A(\mathrm{M})$). By the Proposition above

$$\mathrm{Tr}(\phi_A) = \mathrm{Tr}\left(\phi_{A_{n-1}}\right) = ... = \mathrm{Tr}\left(\phi_{A_0}\right) = -1$$

Conversely, assume that $\mathrm{Tr}(\phi_A) = -1$ and write $A = B[M]$ as a one-point extension of the subcategory B of A by the module M. We shall prove that M is exceptional.

$$-1 = \mathrm{Tr}(\phi_A) = \mathrm{Tr}(\phi_B) + \left(mC_B^T m^T - 1\right) \geq -1 + \left(mC_B^T m^T - 1\right) = -1 + \left(q_B(m) - 1\right) \geq -1$$

Equality holds and $q_B(m) = 1$, since M is indecomposable, it follows that the extension ring of M is trivial. □

4.8 Stable matrices

The following statement is Theorem 1 for stable matrices.

Proposition. *Suppose M is a stable unimodular $n \times n$-matrix. Let $\chi_M = c_0 + c_1 T + c_2 T^2 + ... + c_{n-2} T^{n-2} + c_{n-1} T^{n-1} + c_n T^n$ be its characteristic polynomial.*

Suppose that $0 < \mathrm{Tr} M^k \leq m$ for $p \leq k \leq p + n - 1$ and certain integers $1 \leq p$ and m.

Then $0 < \mathrm{Tr} M^k \leq m$ for all integers $p \leq k$.

In particular, M is of cyclotomic type.

Proof. Consider the coefficients $c_0, c_1, ... c_n$ of χ_M. Since M is stable then $c_n = 1, c_{n-1} < 0, c_{n-2} > 0$ and the signs alternate until we meet a j with $c_j c_0 < 0$. Cayley-Hamilton theorem states that $\chi_M(M) = 0$. Then

$$0 = c_0 1_n + c_1 M + c_2 M^2 + \ldots + c_{n-1} M^{n-1} + c_n M^n$$

Then

$$c_0 1_n + c_2 M^2 + \ldots + c_{2m} M^{2m} = c_1 M + c_3 M^3 + \ldots + c_{2m-1} M^{2m-1} + c_{2(m+r)-1} M^{2(m+r)-1}$$

Let $c > 0$ be the common value of the trace of this matrix.
Write $n = 2m + r$ for $r = 0$ or 1. Consider the matrices

$$P = \frac{1}{c}\left(c_0 1_n + c_2 M^2 + \ldots + c_{2m} M^{2m}\right)$$
$$Q = -\frac{1}{c}\left(\left(c_1 M + c_3 M^3 + \ldots + c_{2m-1} M^{2m-1} + c_{2(m+r)-1} M^{2(m+r)-1}\right)\right)$$

so that we get two expressions of P as positive linear combinations of powers of M.
Suppose that $n = 2m + 1$. By hypothesis we have Tr $(P) \leq n$. Moreover, since $c_n = 1$ then

$$\mathrm{Tr}\,(M^n) \leq \mathrm{Tr}\,(Q) = \mathrm{Tr}\,(P) \leq n$$

The claim follows by induction.
Otherwise, $n = 2m$. The claim follows similarly. □

4.9 Theorem 1

Proof of Theorem 1. Observe that $M = \phi_A$ is a real unimodular matrix. One implication of the Theorem was shown before. Suppose that $|\mathrm{Tr}\left(M^k\right)| \leq n$ or equivalently, $-n \leq \mathrm{Tr}\left(M^k\right) \leq n$ for $0 \leq k \leq n$. The Proposition above yields that M is cyclotomic. □

4.10 Polynomials of Littlewood type

An integral self-reciprocal polynomial $p(t) = p_0 + p_1 t + \ldots + p_{n-1} t^{n-1} + p_n t^n$ is of Littlewood type if every coefficient non-zero p_i has modulus 1. A polynomial $p(t)$ of Littlewood type with all $p_i \neq 0$, for $i = 0, 1, \ldots, n$, is said to be Littlewood.

Lemma. *If z is a root of a polynomial of Littlewood type, then*

$$1/2 < |z| < 2$$

Proof. Suppose z is a root of a polynomial of Littlewood type. Then

$$1 = \epsilon_1 z + \epsilon_2 z^2 + \ldots + \epsilon_n z^n$$

for some $\epsilon_i \in \{-1, 0, 1\}$.
If $|z| < 1$ then $1 \leq |z| + |z|^2 + \ldots + |z|^n < |z|/(1 - |z|)$ so $|z| > 1/2$. Since z is the root of a polynomial of Littlewood type if and only if z^{-1} is, then $1/2 < |z| < 2$.
Moreover, if $|z| > 1$, then $1/|z| < 1$ and $1/2 < 1/|z| < 2$. Hence $1/2 < |z| < 2$. □

4.11 Littlewood series

Definition. A *Littlewood series* is a power series all of whose coefficients are 1, 0 or -1.
Let $P = \{z \in \mathbb{C} : z$ is the root of some Littlewood polynomial $\}$.

Remarks:

a. Littlewood series converge for $|z| < 1$.

b. A point $z \in \mathbb{C}$ with $|z| < 1$ lies in P if and only if some Littlewood series vanishes at this point.

c. A Littlewood polynomial is not a Littlewood series. But any Littlewood polynomial, say $p(z) = a_0 + \ldots + a_d z^d$ yields a Littlewood series having the same roots z with $|z| < 1$: indeed, consider the series

$$P(z) = p(z)/\left(1 - z^{d+1}\right) = a_0 + \ldots + a_d z^d + a_0 z^{d+1} + \ldots + a_d z^{2d+1} + a_0 z^{2d+2} + \ldots$$

Thus $P \subset R$, where R is the set of roots of Littlewood series. We shall show the Proposition at the Introduction.

Proof. Let $\mathcal{L}$ be the set of Littlewood series. Then $\mathcal{L} = \{-1, 0, 1\}^{\mathbb{N}}$, so with the product topology it is homeomorphic to the Cantor set. Choose $0 < r < 1$. Let $\mathbb{F}$ be the space of finite multisets of points z with $|z| < r$, modulo the equivalence relation generated by $S \cong S \cup X$ when $|X| = r$.

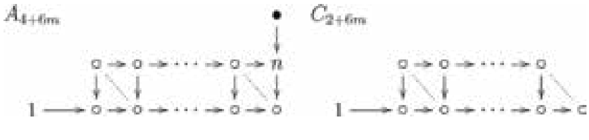
Claim. Any Littlewood series has finitely many roots in the disc $|z| \le r$. The map $f : \mathcal{L} \to \mathbb{F}$ sending a Littlewood series to its multiset of roots in this disc is continuous.

Since $\mathcal{L}$ is compact, the image of f is closed. From this we can show that R, the set of roots of Littlewood series, is closed. Since Littlewood polynomials are densely included in $\mathcal{L}$ and f is continuous, we get that P, the set of roots of Littlewood polynomials, is dense in R. It follows that $\overline{P} = R$, as we wanted to show. $\square$

5. An example

5.1 Construction

For m a natural number and let $n = 3 + 6m$. Let R_n be an algebra formed by n commutative squares. Consider the one-point extension $A_m = R_n[P_n]$ with P_n the unique indecomposable projective R_n-module of K-dimension 2. Observe that A_m (resp. C_{n-1}) is given by the following quiver with $n+1$ vertices and commutative relations (resp. $n-1$ vertices and relations):



We claim:

a. $\chi_{A_m} = T^n + T^{n-1} - T^3 \chi_{A_{m-1}} + T + 1$, for all $n \ge 1$. As consequence, the algebras A_m and C_n are of Littlewood type;

b. the number of eigenvalues of ϕ_{A_m} not lying in the unit disk is at least m;

c. $\mathbb{M}(\chi_{A_m}) \le 8$.

Proof. (a): Consider $m \geq 1$, $n = 3 + 6m$ and the algebra $B_n = R_{3+6m}$ such that $A_m = B_n[P_n]$ and the perpendicular category $P_n^{\perp}$ in $D^b(B_n)$ is derived equivalent to $\mathrm{mod}\,(C_{n-1})$ where C_{n-1} is a proper quotient of an algebra derived equivalent to R_{2+6m}. Therefore

$$\begin{aligned}\chi_{A_{m+1}} &= (T+1)\chi_{R_{n+6}} - T\chi_{C_{n+5}} \\ &= (T+1)\left(T^{n+6} + T^{n+5} + T + 1\right) - T^3(T+1)\chi_{R_n} - T\chi_{C_{n+5}}\end{aligned}$$

We shall calculate $\chi_{C_{2+6m}}$. Observe that C_{2+6m} is tilting equivalent to the one-point extension $R_{1+6m}[P_1]$. Hence

$$\begin{aligned}\chi_{C_{2+6m}} &= (T+1)\chi_{R_{1+6m}} - T\chi_{R_{6m}} = T^{2+6m} + T^{1+6m} - T^3\left\{(T+1)\chi_{R_{1+6(m-1)}} - T\chi_{R_{6(m-1)}}\right\} \\ &\quad + T + 1 = T^{2+6m} + T^{1+6m} - T^3\chi_{C_{2+6(m-1)}} + T + 1\end{aligned}$$

which implies

$$\begin{aligned}\chi_{A_{m+1}} &= (T+1)\left(T^{n+6} + T^{n+5} + T + 1\right) - T^3(T+1)\chi_{R_n} - T\left(T^{n+5} + T^{n+4} + T + 1\right) \\ &\quad - T^3 T\chi_{C_{n-1}} = T^{n+7} + T^{n+6} - T^3\chi_{A_m} + T + 1\end{aligned}$$

as claimed.

As consequence of formula (a) we observe the following:

(a′) $L\left(\chi_{A_m}\right) = 4m + 5$.

(b) By induction, we shall construct polynomials r_m representing χ_{A_m}.

For $m = 0$, we have $\chi_{A_0} = T^4 + T^3 + T^2 + T + 1$, which is represented by the polynomial $r_0 = T^4 - 3T^2 + 1$.

Observe that $\left(T^{n-1} + 1\right) = v_n - Tv_{n-2}$ then $T^n + T^{n-1} + T + 1 = (T+1)\left(T^{n-1} + 1\right)$ is represented by $w_n = T(u_{n-1} - u_{n-3})$.

For $n = 4 + 6m$, we define $r_m = w_n - T^3 r_{m-1}$. We verify by induction on m that r_m represents χ_{A_m}:

$$\begin{aligned}\chi_{A_m}\left(T^2\right) &= \left(T^2 + 1\right)\left(T^{2n-2} + 1\right) - T^6\chi_{A_{m-1}}\left(T^2\right) \\ &= T^n w_n\left(T + T^{-1}\right) - T^6 T^{n-6} r_{m-1}\left(T + T^{-1}\right) = T^n r_m\left(T + T^{-1}\right)\end{aligned}$$

For instance.

$$\begin{aligned}r_1 &= w_{10} - T^3 r_0 = T\left\{\left(T^9 - 8T^7 + 21T^5 - 20T^3 + 5T\right) - \left(T^7 - 6T^5 + 10T^3 - 4T\right)\right\} \\ &\quad - T^3\left\{T^4 - 3T^2 + 1\right\} \\ &= T^{10} - 9T^8 - T^7 + 27T^6 + 3T^5 - 30T^4 - T^3 + 9T^2\end{aligned}$$

which has $\xi(r_1) = 4$ changes of sign in the sequence of coefficients. According to *Descartes rule of signs*, r_1 has at most $\xi(r_1) = 4$ positive real roots. Since r_1 represents χ_{A_1}, then χ_{A_1} has at most $2\xi(r_1) = 8$ roots in the unit circle. That is, χ_{A_1} has at least 2 roots z with $|z| \neq 1$.

We shall prove, by induction, that r_m has at most $\xi(r_m) = 2(m+1)$ positive real roots. Indeed, write

$$r_m = T^n - (n-1)T^{n-2} - T^3 q_m + (n-1)T^2$$

for some polynomial q_m of degree $n-6$ with signs of its coefficients $+--++--\cdots\pm$ so that $\xi(q_m)=2m$. Then

$$r_{m+1}=w_{n+6}-T^3r_m=Tu_{n+5}-Tu_{n+3}-T^3r_m$$

an addition of three polynomials with signs of coefficients given as follows:

$$\begin{array}{ccccccccccc} + & 0 & - & 0 & + & 0 & - & 0 & \cdots & + & 0 & 0 \\ & & - & 0 & + & 0 & - & 0 & \cdots & + & 0 & 0 \\ & & & - & + & + & - & - & \cdots & 0 & 0 & 0 \end{array}$$

Hence $r_{m+1}=T^{n+6}-(n+5)T^{n+4}-T^3q_{m+1}+(n+5)T^2$ where the polynomial q_{m+1} of degree n has signs of its coefficients $+--++--\cdots\pm$ so that $\xi(q_{m+1})=\xi(q_m)+2=2(m+1)$. Hence $\xi(r_m)=2+\xi(q_m)=2(m+1)$.

By the Lemma below, χ_{A_m} has at most $4(m+1)$ roots in the unit circle. Equivalently, χ_{A_m} has at least $4+6m-4(m+1)=2m$ roots outside the unit circle. Hence χ_{A_m} has at least m roots z satisfying $|z|>1$.

Lemma. *Let q be a polynomial representing the polynomial p. Assume q accepts at most s positive real roots, then p has at most $2s$ roots in the unit circle.*

Proof. Let $\mu_1,\ldots,\mu_s$ be the positive real roots of q. Let $z=a+ib$ be a root of p with $a^2+b^2=1$. Consider $w=c+id$ a complex number with $w^2=z$. Then $0=p(z)=w^nq(w+w^{-1})$ where $w+w^{-1}=(c+id)+(c-id)=2c$. Then $2c=\epsilon\lambda_j$ for some $\epsilon\in\{1,-1\}$ and $1\le j\le s$. Hence

$$z=w^2=\left(\frac{1}{2}\lambda_j^2-1\right)+i\left(2\epsilon\lambda_j\sqrt{1-\lambda_j^2}\right)$$

can be selected in two different ways. $\square$

(c) For $n=6m+4$ we have $\chi_{A_m}=T^n+T^{n-1}-T^3\chi_{A_{m-1}}+T+1$. Then

$$\chi_{A_m}=\xi_m+(-1)^{m-1}T^{2m+4}\chi_{10},\text{ where }\xi_m=T^n+T^{n-1}-T^3\xi_{m-1}+T+1$$

for $m\ge 2$ and $\xi_1=0$.

We observe that ξ_m is a product of cyclotomic polynomials. Indeed, since $\xi_m(-1)=0$ we can write

$$\xi_m=(T+1)\sigma_m\text{ and }\sigma_m=T^{n-1}-T^3\sigma_{m-1}+1$$

for $m\ge 2$ and $\sigma_1=0$.

Recall $\Phi_{2^s-1}=T^{s-1}+T^{s-2}+\ldots T+1$ and $\Phi_{2s}(T)=\Phi_s(-T)$. Moreover, $\Phi_{3p}(T)=\Phi_p(T^3)$, if p is a power of 2. Altogether this yields

$$\begin{aligned}\Phi_{6\left(2^{2(m+1)}-1\right)}(T)&=\Phi_{2\left(2^{2(m+1)}-1\right)}\left(T^3\right)=\Phi_{2^{2(m+1)}-1}\left(-T^3\right)\\&=T^{6m+3}-T^{6m}+\ldots-T^3+1=\sigma_m\end{aligned}$$

hence

$$\xi_m=\Phi_2\Phi_{6\left(2^{2(m+1)}-1\right)}$$

confirming the claim.

We estimate the Mahler measure of $\chi_{A_m}=\xi_m+(-1)^{m-1}T^{2m+4}\chi_{A_{10}}$. Write $\chi_{A_m}=f_m+g_m$, where f_m is the cyclotomic summand. Observe that $L(g_m)=L(\chi_{A_{10}})=8$ and apply Lemma (3.4) with $\mathbb{M}(f_m)=1$ to get

$$\mathbb{M}(\chi_{A_m}) \leq \mathbb{M}(f_m) L(g_m) = 8$$

With the help of computer programs we calculate more accurate values of the Mahler measure of some of the above examples:

No. vertices	No. roots outside unit disk	Mahler measure
178	29	1.28368024451292
184	30	1.28327850483340
190	31	1.28386917621114
196	32	1.28395305512596

Comparing with the list of *Record Mahler measures by roots outside the unit circle* in Mossinghoff's web page we see:

i. for the entry 29 the Mahler measure is the same in both tables;

ii. the entries 30 and 31 have a smaller Mahler measure in our table, establishing new records;

iii. the entry 32 of our table seems to be new. Further entries could be calculated.

6. Coefficients of Coxeter polynomials

6.1 Derived tubular algebras

There are interesting invariants associated to the Coxeter polynomial of a triangular algebra $A = k[\Delta]/I$. For instance, the evaluation of the Coxeter polynomial $\chi_A(-1) = m^2$ for some integer m. Clearly, this number is a derived invariant. A simple argument yields that $m = 0$ in case Δ has an odd number of vertices. In [14], it was shown that for a representation-finite accessible algebra A with gl.dim $A \leq 2$ the invariant $\chi_A(-1)$ equals zero or one. The criterion was applied to show that a canonical algebra is derived equivalent to a representation-finite algebra if and only if it has weight type $(2, p, p+k)$, where $p \geq 2$ and $k \geq 0$. In particular, the tubular canonical algebra of type $(3, 3, 3)$ is not derived equivalent to a representation-finite algebra, while the tubular algebras of type $(2, 4, 4)$ or $(2, 3, 6)$ are.

6.2 Strong towers

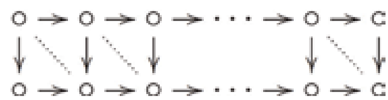
Recall from [14] that a *strong tower* $\mathbb{T} = (A_0 = k, A_1, \ldots, A_n = A)$ of access to A satisfies that $A_{i+1} = A_i[M_i]$ or $[M_i]A_i$ for some exceptional module M_i in such a way that, in case $A_{i+1} = A_i[M_i]$ (resp. $A_{i+1} = [M_i]A_i$), the perpendicular category $\underline{M_i}^{\perp}$ (resp. ${}^{\perp}\underline{M_i}$) of M_i in mod_{A_i} is equivalent to $\mathrm{mod}_{C_{i-1}}$ for some accessible algebra C_{i-1}, $i = 1, \ldots, n-1$. In the extension situation the perpendicular category $M_i^{\perp}$ (resp. ${}^{\perp}M_i$ in the coextension situation) in $\mathrm{D}^b(\mathrm{mod}_{A_i})$ is equivalent to $\mathrm{D}^b(\mathrm{mod}_{C_{i-1}})$ and $B_i$is derived equivalent to a one-point (co-)extension of C_{i-1}. An algebra C_i as above is called an *i-th perpendicular restriction* of the tower $\mathbb{T}$, observe that it is well-defined only up to derived equivalence. We denote by s_i the number of connected components of the algebra C_i; in particular, $s_1 = 1$.

There are many examples of *strongly accessible algebras*, that is, algebras derived equivalent to algebras with a strong tower of access. The following are some instances:

a. A canonical algebra C of weight $(p_1, ..., p_t)$ is strongly accessible if and only if $t = 3$, in that case, C is derived-equivalent to a representation-finite algebra if and only if the weight type does not dominate $(3, 3, 3)$.

b. The following sequence of poset algebras defines strong towers of access:

B_n



6.3 Towering numbers

Consider a strong tower $\mathbb{T} = (A_0 = k, A_1, ..., A_n = A)$ of access to A such that A_{i+1} is an one-point (co)extension of A_i by M_i and C_{i-1} the corresponding i-th perpendicular restriction of $\mathbb{T}$. Let C_{i-1} have s_{i-1} connected components, $i = 2, ..., n-1$. Define the *first towering number of* $\mathbb{T}$ as the sum $s_{\mathbb{T}}(A) = \sum_{i=1}^{n-2} s_i$.

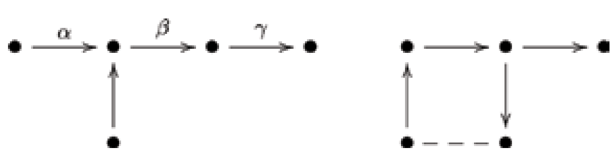
Theorem. *Let A be a strongly accessible algebra with n vertices, then the first towering number $s_{\mathbb{T}}(A) = \sum_{i=1}^{n-2} s_i$ of $\mathbb{T}$ is a derived invariant, that is, depends only on the derived class of A. It is $s_{\mathbb{T}}(A) = n - 1 - a_2$, where a_2 is the coefficient of the quadratic term in the Coxeter polynomial of A.*

Proof. Assume $A = A_n$ and $B = A_{n-1}$ such that $A = B[M]$ for M an exceptional B-module and let $C = C_{n-2}$ be the algebra such that mod_C is derived equivalent to the perpendicular category $M^{\perp}$ formed in $\mathrm{D}^b(\mathrm{mod}_B)$. Then $\chi_A(t) = (1+t)\chi_B(t) - t\chi_C(t)$. Write $\chi_B(t) = 1 + t + \sum_{i=2}^{n-3} b_i t^i + t^{n-2} + t^{n-1}$ and $\chi_C(t) = 1 + \sum_{i=1}^{n-3} c_i t^i + t^{n-2}$. By induction hypothesis we may assume that $s(B) = n - 2 - b_2$. Then $a_2 = b_2 + 1 - c_1$. Moreover, since C is a direct sum accessible algebras, then $c_1 = \sum_{i=0}^{n-2} (-1)^i \dim_k H^i(C) = \dim_k H^0(C) = s_{n-2}$. Hence $a_2 = n - 1 - s(B) - s_{n-2} = n - 1 - s(A)$. □

Corollary. *Let $\mathbb{T} = (A_1 = k, ..., A_n = A)$ be a strong tower of access to A. Let $A = B[M]$ for $B = A_{n-1}$ with M exceptional and C a perpendicular restriction of B via M. Consider the Coxeter polynomials $\chi_A(t) = 1 + t + a_2 t^2 + ... + a_{n-2} t^{n-2} + t^{n-1} + t^n$ and $\chi_B(t) = 1 + t + b_2 t^2 + ... + b_{n-3} t^{n-3} + t^{n-2} + t^{n-1}$, then $a_2 \leq b_2$, with equality if and only if C is connected. In particular, $a_2 \leq 1$.*

Proof. First recall that for a connected accessible algebra the linear term of the Coxeter polynomial has coefficient 1. Let $\chi_C(t) = 1 + c_1 t + c_2 t^2 + ... + c_{n-4} t^{n-4} + c_{n-3} t^{n-3} + t^{n-2}$ be the Coxeter polynomial of C. If C is the direct sum of connected accessible algebras $C_1, ..., C_s$, then $c_1 = s$. Therefore, $a_2 = b_2 + b_1 - c_1 = b_2 - (s-1) \leq b_2$. By induction hypothesis, we get $a_2 \leq 1$. □

Let A be the algebra given by the following quiver with relation $\gamma\beta\alpha = 0$:



which is derived equivalent to the quiver algebra B with the zero relation as depicted in the second diagram. Clearly, $A = A'[M]$, where A' is a quiver algebra of type $\mathbb{A}_4$ and M is an indecomposable module with $M^{\perp}$ the category of modules of the disconnected quiver $\bullet \to \bullet \quad \bullet$, that is $s_3(A) = 2$. Moreover $s_2(A) = s_2(A') = 1$ and $s(A) = 4$. On the other hand $B = [N]B'$ such that B' is not hereditary. A calculation yields $s_3(B) = 1$ and $s_2(B) = s_2(B') = 2$, obviously implying that $s(B) = 4$.

Some properties of the invariant s:

i. Let A and B be accessible algebras and A be accessible from B, then $s(B) \leq s(A)$. Equality holds exactly when $A = B$.

ii. Let A be an accessible schurian algebra (that is for every couple of vertices i, j, $\dim_k A(i,j) \leq 1$), then for every convex subcategory B we have $s(B) \leq s(A)$.

6.4 Totally accessible algebras

An accessible algebra A with $n = 2r + r_0$ vertices, and $r_0 \in \{0, 1\}$, is said to be *totally accessible* if there is a family of (not necessarily connected) algebras $C^{(n)} = A'$, $C^{(n-2)}$, $C^{(n-4)}$, ..., $C^{(r_0)}$ satisfying:

a. A is derived equivalent to A';

b. for each $0 \leq i = n - 2j \leq n$, there is a strong tower $\mathbb{T}^{(j)} = \left(C^{(j,1)} = k, \ldots, C^{(j,i)} = C^{(i)}\right)$ of access to $C^{(i)}$;

c. $C^{(i-2)}$ is an $i - 1$-th perpendicular restriction of $\mathbb{T}^{(j)}$, that is, $C^{(i)}$ is a one-point (co)extension of $C^{(j,i-1)}$ by a module N_{i-1} and $C^{(i-2)}$ is a perpendicular restriction of $C^{(j,i-1)}$ via N_{i-1}.

The tower $\mathbb{T}^{(j)}$ is said to be a *j-th derivative* of the tower $\mathbb{T}^{(0)}$.

Examples that we have encountered of totally accessible algebras are:

i. Hereditary tree algebras: since for any conneceted hereditary tree algebra A with at least 3 vertices, there is an arrow $a \to b$ with a a source (or dually a sink) and $A = B[P_b]$ such that the perpendicular restriction of B via P_b is the algebra hereditary tree algebra C obtained from A by deleting the vertices a, b.

ii. Accessible representation-finite algebras A with gl.dim $A \leq 2$, since then the perpendicular restrictions of any strong tower (which exists by [14]) satisfy the same set of conditions.

iii. Certain canonical algebras: for instance the tame canonical algebra A of weight type $(2, 4, 4)$ is an extension $A = B[M]$ of a hereditary algebra B of extended Dynkin type $[2, 4, 4]$ by a module M in a tube of rank 4, then the perpendicular restriction of B via M is the hereditary algebra C of extended Dynkin type $[3, 3, 3]$, see for example [?](10.1). Since C is totally accessible, so A is. Moreover $s(A) = 8$.

iv. Let A be an accessible algebra of the form $A = B[M]$ for an algebra B and an exceptional module M and let C the perpendicular restriction of B via M. If A is totally accessible, then B and C are totally accessible.

The following results extend some of the features observed in the examples above.

Proposition. *a. Assume that A is a totally accessible algebra, then $\chi_A(-1) \in \{0,1\}$.*

b. Assume that A is an accessible but not totally accessible algebra with gl.dim $A \leq 2$, then one of the following conditions hold:

i. for every exceptional B-module such that $A = B[M]$ and any perpendicular restriction C of B via M, then C is not accessible;

ii. there exists a homological epimorphism $\phi : A \to B$ such that $\chi_B(-1) > 1$.

Proof. (a): Consider the perpendicular restriction C of B via M, such that $\chi_A(t) = (1+t)\chi_B(t) - t\chi_C(t)$. Therefore $\chi_A(-1) = \chi_C(-1)$ and moreover, C is totally accessible. Then by induction hypothesis, $\chi_A(-1) = \chi_{C^{(m)}}(-1)$ for a totally accessible algebra $C^{(m)}$ with number of vertices $m = 1$ or $m = 2$. Clearly, $C^{(m)}$ is either k, $k \oplus k$ or hereditary of type $\mathbb{A}_2$, which yields the desired result.

(b): Assume A is an accessible algebra with gl.dim $A \leq 2$ and such that for every homological epimorphism $\phi : A \to B$ we have $\chi_B(-1) \in \{0,1\}$. Let $A = B[M]$ for an accessible algebra B and an exceptional B-module M such that C is a perpendicular restriction of B via M. Since gl.dim $A \leq 2$ then there is a homological epimorphism $A \to C$ and gl.dim $C \leq 2$. Observe that for every homological epimorphism $\psi : B \to B'$ (resp. $\psi : C \to C'$) there is a homological epimorphism $\phi : A \to B'$ (resp. $\phi : A \to C'$), hence $\chi_{B'}(-1)$ (resp. $\chi_{C'}(-1)$) is 0 or 1. By induction hypothesis, B is totally accessible. Moreover if C is accessible, then the induction hypothesis yields that C is totally accessible and also A is totally accessible, a contradiction. Therefore C is not accessible. □

7. On the quadratic coefficient of the Coxeter polynomial of a totally accessible algebra

7.1 Derived algebras of linear type

Recall that an *extended canonical* algebra of weight type $\langle p_1, \ldots, p_t \rangle$ is a one-point extension of the canonical algebra of weight type $[p_1, \ldots, p_t]$ by an indecomposable projective module. As in (1.3), the extended canonical algebras of type $\langle p_1, p_2, p_3 \rangle$ is strongly accessible. Moreover, the extended canonical algebra A of type $\langle 3, 4, 5 \rangle$ (with 12 points) has Coxeter polynomial $1 + t + t^2 + \ldots + t^{12}$ which is also the Coxeter polynomial of a linear hereditary algebra H with 12 vertices. Clearly A and H are not derived equivalent.

The following generalizes a result of Happel who considers the case of Coxeter polynomials associated to hereditary algebras [8].

Theorem 1. *Let A be a totally accessible algebra with n vertices and let $\chi_A(t) = \sum_{i=0}^{n} a_i t^i$ be the Coxeter polynomial of A. The following are equivalent:*

i. $a_2 = 1$;

ii. *let $\mathbb{T} = (A_1 = k, \ldots, A_{n-1}, A_n = A)$ be a strong tower of access to A and C_i the i-th perpendicular restriction of $\mathbb{T}$, for all $1 \leq i \leq n-2$, then the algebras C_i are connected;*

iii. *A is derived equivalent to a quiver algebra of type $\mathbb{A}_n$.*

Proof. $(i) \Leftrightarrow (ii)$: Let $\mathbb{T} = (A_1 = k, \ldots, A_n = A)$ be a strong tower of access to A. In case each C_i is connected, then $s(A) = n - 2$, that is $a_2 = 1$. If $a_2 = 1$, then $n - 2 = s_{\mathbb{T}}(A) = \sum_{i=1}^{n-2} s_i$ with each $s_i \geq 1$. $(i) \Leftrightarrow (iii)$: We know that an algebra A derived equivalent to a quiver algebra of type $\mathbb{A}_n$ has $\chi_A(t) = \sum_{i=0}^{n} t^i$, in particular, $a_2 = 1$. Assume that an accessible algebra A has the quadratic coefficient of its Coxeter polynomial $a_2 = 1$. Let $A = B[M]$ for an accessible algebra $B = A_{n-1}$ and an exceptional module M. Since B is also totally accessible with a tower $\mathbb{T}' = (A_1 = k, \ldots, A_{n-1} = B)$ satisfying (ii), then the quadratic coefficient of the Coxeter polynomial of B is $b_2 = 1$ and we may assume that B is derived equivalent to a quiver algebra of type $\mathbb{A}_{n-1}$. In particular, B is representation-finite with a preprojective component $\mathcal{P}$ such that the orbit graph $\mathcal{O}(\mathcal{P})^{\tau}$ is of type $\mathbb{A}_{n-1}$ (recall that the orbit graph has vertices the τ-orbits in the quiver $\mathcal{P}$ with Auslander-Reiten translation τ and there is an edge between the orbit of X and the orbit of Y if there is some numbers a, b and an irreducible morphism $\tau^a X \to \tau^b Y$). Observe that for any X in $\mathrm{D}^b(\mathrm{mod}_A)$ not in the orbit of M, there is some translation $\tau^a X$ belonging to $M^{\perp}$, implying that in case M^{τ} has two neighbors in the orbit graph then $M^{\perp}$ is not connected, that is $s_{n-2} > 1$ and $a_2 = n - 1 - s(A) \leq 0$, a contradiction. Therefore, M^{τ} has just one neighbor in $\mathcal{O}(\mathcal{P})^{\tau}$, hence A is derived of type $\mathbb{A}_n$. $\square$

7.2 Theorem 2

Consider a tower $A_1, \ldots, A_n = A$ of accessible algebras where A_{i+1} is a one-point (co)extension of A_i by the indecomposable M_i and C_i is such that $M_i^{\perp}$ is derived equivalent to $\mathrm{D}^b(\mathrm{mod}_{C_i})$. Assume that $C_i^{(j)}$, for $1 \leq j \leq s_i$, are the connected components of the category C_i. Consider the corresponding Coxeter polynomials:

$$\chi_{A_i}(t) = 1 + t + \sum_{j=2}^{i-2} a_j^{(i)} t^j + t^{i-1} + t^i,$$

$$\chi_{C_i}(t) = 1 + s_i t + \sum_{r=2}^{n_i-2} c_{i,r} t^r + s_i t^{n_i-1} + t^{n_i},$$

$$\chi_{C_i^{(j)}}(t) = 1 + t + \sum_{s=2}^{n_{i,j}-2} c_{i,s}^{(j)} t^s + t^{n_{i,j}-1} + t^{n_{i,j}},$$

where clearly, $\sum_{j=1}^{s_i} n_{i,j} = n_i$.

Lemma. *(α) For every $1 \leq j \leq i-2$, we have $a_j^{(i)} \leq 1$.*

($\alpha\alpha$) For every $1 \leq j \leq i-2$, we have $a_j^{(i)} \leq c_{i,j}$ and $a_j^{(i)} \leq a_j^{(i-1)}$.

Proof. We shall check that (α) implies ($\alpha\alpha$), then we show that (a') holds by induction on j.

Indeed, assume that (α) holds and proceed to show ($\alpha\alpha$) by induction on j. If $j = 0, 1$, then $a_j^{(i)} = 1 = a_{i-j}^{(i)}$. Assume that $2 \leq j \leq i-2$ and $a_j^{(i)} \leq c_{i,j}$ and $a_j^{(i)} \leq a_j^{(i-1)}$. Then

$$a_{j+1}^{(i)} = a_{j+1}^{(i-1)} + \left(a_j^{(i-1)} - c_{j,i-1}\right) \leq a_{j+1}^{(i-1)} \leq \ldots \leq a_{j+1}^{(j+1)} = 1.$$

Let $0 \leq j \leq i-2$. If $j = 0, 1$ we have $a_0^{(i)} = 1 = c_{i,0}$ and $a_1^{(i)} \leq s_1(A) = c_{i,1}$. Moreover $a_1^{(i)} = a_1^{(i-1)}$. Assume (α) holds for $j \geq 2$, then.

$$a_{j+1}^{(i)} = a_{j+1}^{(i-1)} + \left(a_j^{(i-1)} - c_{j,i-1}\right) \leq a_{j+1}^{(i-1)},$$
$$a_{j+1}^{(i)} - c_{i,j+1} = a_{j+2}^{(i)} - a_{j+2}^{(i-1)} \leq 0. \qquad \square$$

Theorem 2. *Let A be a totally accessible algebra with Coxeter polynomial $\chi_A(t) = 1 + t + a_2t^2 + \ldots + a_{n-2}t^{n-2} + t^{n-1} + t^n$, then:*

a. $a_j \leq 1$, *for every* $2 \leq j \leq n-2$;

b. *if for some* $2 \leq j \leq n-2$, *we have* $a_j = 1$ *then A is derived equivalent to a hereditary algebra of type* $\mathbb{A}_n$.

Proof. Keep the notation as in (4.1). Then (a) is the case $i = n$ of the Lemma above.

We shall prove (b) by induction on n the number of vertices of A. Let $j = 2$ and assume $a_2 = 1$, then (3.1) implies that A is derived equivalent to $\mathbb{A}_n$. Consider now $2 < j < n-2$ and assume that $a_j = 1$, we get:

$$1 = a_j^{(n)} = a_j^{(n-1)} + \left(a_{j-1}^{(n-1)} - c_{n-1,j-1}\right) \leq a_j^{(n-1)} \leq 1$$

The last inequality due to (a), hence $a_j^{(n-1)} = 1$. Induction hypothesis yields that A_{n-1} is derived equivalent to $\mathbb{A}_{n-1}$ and its Auslander-Reiten quiver consists of a preprojective component $\mathcal{P}$. In particular, $a_2^{(n-1)} = 1$, which implies that $s_{n-3}(A_{n-1}) = 1$, that is, $A = A_{n-1}[M]$ for some exceptional module M such that $M^{\perp}$ is derived equivalent to mod_C for a connected algebra C, that is, $s(A) = n-2$ and by (3.1), $A = B[M]$ is derived equivalent to a hereditary algebra of type $\mathbb{A}_n$. $\square$

7.3 Examples

If A is a representation-finite accessible algebra with gl.dim $A \leq 2$, then A is totally accessible. On the other hand the algebra B with quiver:

$$1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} 4 \ldots \xrightarrow{x} 11 \xrightarrow{x} 12$$

and $x^3 = 0$ is representation-finite and accessible (but not gl.dim $B \leq 2$). The Coxeter polynomial of B is:

$$\chi_B(t) = 1 + t - t^3 - t^4 + t^6 - t^8 - t^9 + t^{11} + t^{12}.$$

Then observe that the 6-th coefficient is 1 but the algebra B is not derived equivalent to Dynkin type $\mathbb{A}_{12}$.

8. On the traces of Coxeter matrices

Let A be an algebra such that not all roots of χ_A are roots of unity. By the result of Kronecker [36], not all of the spectrum of A lies in the unit disk. Equivalently, the spectral radius $\rho_A = \max\{|\lambda| : \lambda \text{ eigenvalue of } \phi_A\} > 1$. Arrange the eigenvalues of ϕ_A so that $\mu_1, \mu_2, \ldots, \mu_n$ have absolute values $\rho_A = r_1 > r_2 > \ldots > r_s$ and multiplicities $m_1, \ldots, m_s$, respectively. Therefore $s \geq 2$ and

$$|\det\phi_A| = r_1^{m_1} r_2^{m_2} \dots r_s^{m_s} = 1.$$

We define the *critical power* $\kappa(A)$ as the minimal k such that

$$|\mathrm{Tr}\left(\phi_A^k\right)| > n$$

Since r_1 is a simple eigenvalue of ϕ_A, then it follows that $\kappa(A)$ is well defined due to the existence of k satisfying the following chain of inequalities:

$$|\mathrm{Tr}\left(\phi_A^k\right)| = |\sum_{j=1}^{n} \mu_j^k| \geq r_1^{km_1} - \sum_{j=2}^{s} r_j^{km_j} \geq r_1^k - (n-1) r_2^k > n.$$

The following is a reformulation of Theorem 2.

Theorem. *Let A be an algebra such that not all roots of χ_A are roots of unity. We have $\kappa(A) \leq n$.*

Proof. Indeed, suppose that A is not of cyclotomic type and $\kappa(A) > n$, that is, $|\mathrm{Tr}\left(\phi_A^k\right)| \leq n$ for all $0 \leq k \leq n$. Observe that $M = \phi_A$ is a unimodular matrix and therefore, Theorem 2 implies that M is of cyclotomic type, which yields a contradiction. □

Remark: We consider explicitly the case $n = 2$ in the above Theorem. Obviously, the Cartan matrix of A is of the form

$$C = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad \phi_A = -C^{-1}C^T = \begin{pmatrix} a^2 - 1 & a \\ -a & -1 \end{pmatrix}$$

for some $a \geq 1$. Then ϕ_A has the indicated shape. If A is not cyclotomic, then $a \geq 3$ and $\mathrm{T}r\left(\phi_A^2\right) = \left(a^2 - 2\right)^2 - 2 > 2$.

9. Stability of a real matrix

9.1 Stability of matrices and the Lyapunov criterion

Let M be a real invertible $n \times n$-matrix with eigenvalues $\lambda_j = r_j e^{i\theta_j}$, for some numbers $\theta_j \in [0, 2\pi)$ and $j = 1, \dots, n$. We will say that M is *stable* (resp. *semi-stable*) if the real part $\mathrm{Re}\left(e^{i\theta_j}\right) = \cos\theta_j$ of the argument of the eigenvalue λ_j is positive (resp. non-negative), for every $j = 1, \dots, n$. The following is well-known, we sketch a proof for the sake of completeness.

Proposition. *Let M be a stable (resp. semi-stable) $n \times n$-matrix. Then the characteristic polynomial $\chi_M = T^n + a_{n-1}T^{n-1} + \dots + a_1 T + a_0$ has coefficients satisfying $(-1)^{n-j} a_j > 0$ (resp. ≥ 0), for $j = 0, 1, \dots, n$;*

Proof. Observe that $(-1)^n p(-T)$ is the product of polynomials $T - \alpha$ with $\alpha \in \mathbb{R}$ and $(T - (\alpha + i\beta))(T - (\alpha - i\beta)) = T^2 - 2\alpha T + \left(\alpha^2 + \beta^2\right)$ with $0 \neq \beta, \alpha \in \mathbb{R}$. Stability (resp. semi-stability) implies that $\alpha < 0$ (resp. $\alpha \leq 0$) above. Therefore, $(-1)^n p(-T)$ is product of polynomials with positive coefficients. □

Remark: In most of the literature the stability concept we use goes by the name of *positive stability*, while the *stability* name is used also as *Hurwitz stability*, or *Lyapunov stability*.

The system of differential equations

$$y'(t) = -My(t)$$

is said to be *stable* if for every vector $d = (d_1, \ldots, d_n)$, the solution $v(t) = e^{-tM}d$ of the above system has the property that $\lim_{t\to\infty} v(t) = 0$.

We recall here the celebrated.

Lyapunov criterion: The system $y'(t) = -My(t)$ is stable if and only if M is a stable matrix, equivalently there is a real positive definite matrix P such that

$$M^T P + PM = I_n.$$

It is not hard to see that given M, the corresponding P is unique. A proof of the criterion and its equivalence to other stability conditions are considered in [13].

9.2 Semi-stable powers

Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of the real matrix M with $\mu_j = \rho_j e^{2\pi i \theta_j}$ in polar form. Observe that μ_j^k, for $j = 1, \ldots, n$, are the eigenvalues of M^k and

$$\mathrm{Tr} M^k = \sum_{j=1}^{n} \rho_j^k \cos\left(k\theta_j\right) \le \sum_{j=1}^{n} |\mu_j^k| |\cos\left(k\theta_j\right)|$$

Lemma. *For a positive integer* $k \ge 1$ *the following assertions are equivalent:*

a. M^k is a semi-stable matrix;

b. $\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^{n} \left|\mu_j\right|^k |\cos\left(k\theta_j\right)|$.

Proof. If M^k is a semi-stable matrix, then $\mu^k = \rho_j^k\left(\cos\left(k\theta_j\right) + i\sin\left(k\theta_j\right)\right)$ has $\cos\left(k\theta_j\right) \ge 0$. Since M is a real matrix then $\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^{n} \rho_j^k \cos\left(k\theta_j\right) \ge 0$. Therefore

$$\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^{n} \rho_j^k |\cos\left(k\theta_j\right)|.$$

Assume that $\mathrm{Tr}\left(M^k\right) = \sum_{j=1}^{n} |\lambda_j|^k |\cos\left(k\theta_j\right)|$. Since $|\lambda_j^k| \ge \rho_j^k \cos\left(k\theta_j\right)$ for $j = 1, \ldots, n$, adding up, we get

$$\mathrm{Tr}\left(M^k\right) \ge \sum_{j=1}^{n} \rho_j^k \cos\left(k\theta_j\right) = \mathrm{Tr}\left(M^k\right)$$

Hence we have equalities $|\lambda_j^k| |\cos\left(k\theta_j\right)| = \rho_j^k \cos\left(k\theta_j\right)$ for $j = 1, \ldots, n$. Then M^k is semi-stable. □

We say that k is a *stable power* (resp. *semi-stable power*) of M if M^k is a stable (resp. semi-stable) matrix.

10. Nakayama algebras

10.1 Cyclotomic Nakayama algebras

As a well-understood example the representation theory of the *Nakayama algebras* stands appart. Let $N(n, r)$ be the quotient obtained from the linear quiver with n vertices with radical rad_A of nilpotency index r.

For instance, for $A = N(6, 3)$ the Cartan matrix C and Coxeter matrix ϕ are:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \text{ and } \phi = \begin{pmatrix} -1 & 1 & 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

whose characteristic polynomial is cyclotomic as we know from [18] or might be verified calculating $\mathrm{Tr}\left(\phi_B^k\right) \leq n$, for $1 \leq k \leq 72$ and applying the criterion of Theorem 1. Indeed, for.

$k =$	$\mathrm{Tr}\chi_A^k =$
11	-1
$1, 2, 5, 7, 9, 10, 13, 14, 17$	$= 1$
$3, 6, 15$	$= 2$
$4, 8, 16$	$= 3$
12	$= 6$

Starting with $k = 17$ the sequence of traces repeats cyclically. Therefore, $\mathrm{Tr}\left(\chi_A^k\right) \leq 6$ for all $0 \leq k$. Then $N(6, 3)$ is of cyclotomic type.

10.2 An example

We recall in some length the argument given in [18] for the cyclotomicity of $N(n, 3)$, for all $n \geq 1$.

Consider the algebra R_{2n} with $2n$ vertices and whose quiver is given as

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & \cdots \arrow[r] & n \arrow[d] \\ 1' \arrow[r] & 2' \arrow[r] & 3' \arrow[r] & \cdots \arrow[r] & n' \end{tikzcd}$$

with all commutative relations. The corresponding Coxeter polynomial

$$\chi_{R_{2n}} = \chi_{\mathbb{A}_n} \otimes \chi_{\mathbb{A}_2} = v_{n+1} \otimes v_3$$

is a product of cyclotomic polynomials, therefore $\chi_{R_{2n}}$ is a cyclotomic polynomial. In fact $R_{2n} = \mathbb{A}_n \otimes \mathbb{A}_2$, where $\mathbb{A}_s$ is the hereditary algebra associated to the linear quiver $1 \to 2 \to \cdots \to s$.

For $2m + 1$ odd, we consider.

$$R_{2m+1} \qquad \begin{tikzcd} & \circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \cdots \arrow[r] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

The following holds for the sequence of algebras R_n and its Coxeter polynomials χ_{R_n}:

a. R_n is derived equivalent to $N(n, 3)$.

b. $\chi_{R_n} = T^n + T^{n-1} - T^3\chi_{R_{n-6}} + T + 1$, for all $n \geq 6$;

c. $\mathbb{M}(\chi_{R_n}) = 1$.

Observe that the sequence of algebras (R_n) forms an *interlaced tower of algebras*, that is, it is a sequence of triangular algebras $R_1, \dots, R_n$, such that R_s is a basic algebra with s simple modules and, among others, the condition

$$\chi_{R_{s+1}} = (T+1)\chi_{R_s} - T\chi_{R_{s-1}}$$

is satisfied for $s = 1, \dots, n-1$. Moreover, A_{s+1} is a one-point extension (or coextension) of an accessible algebra A_s by an exceptional A_s- module M_s such that the perpendicular category $M_s^{\perp}$ formed in the derived category is triangular equivalent to mod (A_{s-1}), for $s = m+1, \dots, n-1$.

The following was shown in [18]: *Consider an interlaced tower of algebras $A_m, \dots, A_n$ with $m \leq n-2$. If* $\mathrm{Spec}\,\phi_{A_n}$ *is contained in the union of the unit circle and the semi-ray of positive real numbers then either all A_i are of cyclotomic type or $M(\chi_{A_m}) < M(\chi_{A_n})$. In the latter case, $M(\chi_{A_n}) < \prod_{s=m}^{n-1} M(\chi_{A_s})$.*

Since we know that $M(\chi_{R_{2n}}) = 1$, for all $n \geq 0$, we conclude that $M(\chi_{R_n}) = 1$, for all $n \geq 0$. That is the Nakayama algebras of the form $N(n,3)$ are of cyclotomic type.

10.3 Non-cyclotomic Nakayama algebras

Calculation of $\mathrm{Tr}\,\phi_A^k$ for $A = N(n,r)$ and k in intervals, for data sets (n,r,k), yield interesting information. Namely,

a. Many Nakayama algebras are of cyclotomic type;

b. Not all Nakayama algebras are of cyclotomic type. The case $r = 4$ illustrates this claim:

   $N(n,4)$ is of cyclotomic type for all $0 \leq n \leq 100$ except for
   $n = 10, 22, 30, 42, 50, 62, 70, 82$ and 90

c. A canonical algebra C of weight $(p_1, \dots, p_t)$ is strongly accessible if and only if $t = 3$, in that case, C is derived-equivalent to a representation-finite algebra if and only if the weight type does not dominate $(3,3,3)$.

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Chapter 2

New Aspects of Descartes' Rule of Signs

Vladimir Petrov Kostov and Boris Shapiro

Abstract

Below, we summarize some new developments in the area of distribution of roots and signs of real univariate polynomials pioneered by R. Descartes in the middle of the seventeenth century.

Keywords: real univariate polynomial, sign pattern, admissible pair, Descartes' rule of signs, Rolle's theorem

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1. Introduction

The classical Descartes' rule of signs claims that the number of positive roots of a real univariate polynomial is bounded by the number of sign changes in the sequence of its coefficients and it coincides with the latter number modulo 2. It was published in French (instead of the usual at that time Latin) as a small portion of *Sur la construction de problèmes solides ou plus que solide* which is the third book of Descartes' fundamental treatise *La Géométrie* which, in its turn, is an appendix to his famous *Discours de la méthode*. It is in the latter chef d'oeuvre that Descartes developed his analytic approach to geometric problems leaving practically all proofs and details to an interested reader. This interested reader turned out to be Frans van Schooten, a professor of mathematics at Leiden who together with his students undertook a tedious work of making Descartes' writings understandable, translating and publishing them in the proper language, that is, Latin. (For the electronic version of this book, see [13].) Mathematical achievements of Descartes form a small fraction of his overall scientific and philosophical legacy, and Descartes' rule of signs is a small but important fraction of his mathematical heritage.

Descartes' rule of signs has been studied and generalized by many authors over the years; one of the earliest can be found in [7], see also [4, 11]. (For some recent contributions, see [1, 2, 6, 10, 12, 14], to mention a few.)

In the present survey, we summarize a relatively new development in this area which, to the best of our knowledge, was initiated only in the 1990s (see [12]).

For simplicity, we consider below only real univariate polynomials with all nonvanishing coefficients. For a polynomial $P := \sum_{j=0}^{d} a_j x^j$ with fixed signs of its coefficients, Descartes' rule of signs tells us what possible values the number of its real positive roots can have. For P as above, we define the sequence of $\pm$ signs of

To René Descartes, a polymath in philosophy and science.

length $d+1$ which we call the *sign pattern* (SP for short) of P, namely, we say that a polynomial P with all nonvanishing coefficients *defines the sign pattern* $\sigma := (s_d, s_{d-1}, \dots, s_0)$ if $s_j = \text{sgn } a_j$. Since the roots of the polynomials P and $-P$ are the same, we can, without loss of generality, assume that the first sign of a SP is always a $+$.

It is true that for a given SP with c sign changes (and hence with $p = d - c$ sign preservations), there always exist polynomials P defining this sign pattern and having exactly pos positive roots, where $pos = 0, 2, \dots, c$ if c is even and $pos = 1, 3, \dots, c$ if c is odd (see, e.g., [1, 3]). (Observe that we do not impose any restriction on the number of negative roots of these polynomials.)

One can apply Descartes' rule of signs to the polynomial $(-1)^d P(-x)$ which has p sign changes and c sign preservations in the sequence of its coefficients and whose leading coefficient is positive. The roots of $(-1)^d P(-x)$ are obtained from the roots of $P(x)$ by changing their sign. Applying the above result of [1] to $(-1)^d P(-x)$, one obtains the existence of polynomials P with exactly neg negative roots, where $neg = 0, 2, \dots, p$ if p is even and $neg = 1, 3, \dots, p$ if p is odd. (Here again we impose no requirement on the number of positive roots.)

A natural question apparently for the first time raised in [12] is whether one can freely combine these two results about the numbers of positive and negative roots. Namely, given a SP σ with c sign changes and $p = d - c$ sign preservations, we define its *admissible pair* (AP for short) as (pos, neg), where $pos \le c$, $neg \le p$, and the differences $c - pos$ and $p - neg$ are even. For the SP σ as above, we call (c, p) the *Descartes' pair* of σ. The main question under consideration in this paper is as follows.

Problem 1. *Given a couple* (*SP, AP*), *does there exist a polynomial of degree d with this SP and having exactly pos positive and exactly neg negative roots* (*and hence exactly* $(d - pos - neg)/2$ *complex conjugate pairs*)?

If such a polynomial exists, then we say that it *realizes* a given couple (SP, AP). The present paper discusses the current status of knowledge in this realization problem.

Example 1. For $d = 4$ and for the sign pattern $\sigma^0 := (+, -, -, -, +)$, the following pairs and only them are admissible: $(2, 2)$, $(2, 0)$, $(0, 2)$, and $(0, 0)$. The first of them is the Descartes' pair of σ^0.

It is clear that if a couple (SP, AP) is realizable, then it can be realized by a polynomial with all simple roots, because the property of having nonvanishing coefficients is preserved under small perturbations of the roots.

In this short survey, we present what is currently known about Problem 1. After the pioneering observations of Grabiner [12] which started this line of research, important contributions to Problem 1 have been made by Albouy and Fu [1] who, in particular, described all non-realizable combinations of the numbers of positive and negative roots and respective sign patterns up to degree 6. Our results on this topic which we summarize below can be found in [5, 8, 9] and [15–19]. On the other hand, we find it surprising that such a natural classical question has not deserved any attention in the past, and we hope that this survey will help to change the situation. The current status of Problem 1 is not very satisfactory in spite of the complete results in degrees up to 8 as well as several series of non-realizable cases in all degrees. There is still no general conjecture describing all non-realizable cases. It might happen that the answer to Problem 1 in sufficiently high degrees is very complicated.

On the other hand, besides Problem 1 as it is stated, there is a significant number of related basic questions which can be posed in connection to the latter Problem and are still waiting for their researchers. (Very few of them are listed in Section 5.)

One should also add that there is a number of completely different directions in which mathematicians are trying to extend Descartes' rule of signs. They include, for example, rule of signs for other univariate analytic functions including exponential functions, trigonometric functions and orthogonal polynomials, multivariate Descartes' rule of signs, tropical rule of signs, rule of signs in the complex domain, etc. (see, e.g., [6, 10, 14]) and references therein. But we think that Problem 1 is the closest one to the original investigations by Descartes himself.

The structure of this chapter is as follows. In Section 2, we provide the information about the solution of Problem 1 in degrees up to 11. In Section 3, we present several infinite series of non-realizable couples (SP, AP). Finally, in Section 4 we discuss two generalizations of Problem 1 and their partial solutions.

2. Solution of the realization problem 1 in small degrees

2.1 Natural $\mathbb{Z}_2 \times \mathbb{Z}_2$-action and degrees $d = 1$, 2, and 3

Let us start with the following useful observation.

To shorten the list of cases (SP, AP) under consideration, we can use the following $\mathbb{Z}_2 \times \mathbb{Z}_2$-action whose first generator acts by.

$$P(x) \mapsto (-1)^d P(-x), \tag{1}$$

and the second one acts by

$$P(x) \mapsto P^R(x) \ := \ x^d P(1/x)/P(0). \tag{2}$$

Obviously, the first generator exchanges the components of the AP. Concerning the second generator, to obtain the SP defined by the polynomial P^R, one has to read the SP defined by $P(x)$ backward. The roots of P^R are the reciprocals of these of P which implies that both polynomials have the same numbers of positive and negative roots. Therefore, the SPs which they define have the same AP.

Remark 1. A priori the length of an orbit of any $\mathbb{Z}_2 \times \mathbb{Z}_2$-action could be 1, 2, or 4, but for the above action, orbits of length 1 do not exist since the second components of the SPs defined by the polynomials $P(x)$ and $(-1)^d P(-x)$ are always different. When an orbit of length 2 occurs and d is even, then both SPs are symmetric w.r.t. their middle points (hence their last component equal $+$). Similarly, when d is odd, then one of the two SPs is symmetric w.r.t. its middle (with the last component equal to $+$), and the other one is antisymmetric. Thus, its last components equal $-$.

It is obvious that all pairs or quadruples (SP, AP) constituting a given orbit are simultaneously (non-)realizable.

As a warm-up exercise, let us consider degrees $d = 1, 2$ and 3. In these cases, the answer to Problem 1 is positive. We give the list of SPs, with the respective values c and p of their APs and examples of polynomials realizing the couples (SP, AP). In order to shorten the list, we consider only SPs beginning with two $+$ signs; the cases when these signs are $(+, -)$ are realized by the respective polynomials $(-1)^d P(-x)$. All quadratic factors in the table below have no real roots.

d	SP	c	p	AP	P
1	$(+,+)$	0	1	$(0,1)$	$x+1$
2	$(+,+,+)$	0	2	$(0,2)$	$x^2+3x+2=(x+1)(x+2)$
				$(0,0)$	x^2+x+1
	$(+,+,-)$	1	1	$(1,1)$	$x^2+x-2=(x-1)(x+2)$
3	$(+,+,+,+)$	0	3	$(0,3)$	$x^3+6x^2+11x+6=(x+1)(x+2)(x+3)$
				$(0,1)$	$x^3+3x^2+4x+2=(x+1)(x^2+2x+2)$
	$(+,+,+,-)$	1	2	$(1,2)$	$x^3+2x^2+x-6=(x-1)(x+2)(x+3)$
				$(1,0)$	$x^3+5x^2+4x-10=(x-1)(x^2+6x+10)$
	$(+,+,-,+)$	2	1	$(2,1)$	$x^3+x^2-24x+36=(x+6)(x-2)(x-3)$
				$(0,1)$	$x^3+2x^2-19x+30=(x+6)(x^2-4x+5)$
	$(+,+,-,-)$	1	2	$(1,2)$	$x^3+x^2-4x-4=(x-2)(x+1)(x+2)$
				$(1,0)$	$x^3+2x^2-3x-10=(x-2)(x^2+4x+5)$

Example 2. For $d=4$, an example of an orbit of length 2 is given by the couples

$$((+,-,-,-,+),(2,2)) \quad \text{and} \quad ((+,+,-,+,+),(2,2)).$$

Here, both SPs are symmetric w.r.t. its middle.
For $d=5$, such an example is given by the couples

$$((+,-,-,-,-,+),(2,3)) \quad \text{and} \quad ((+,+,-,+,-,-),(3,2)).$$

The first of the SPs is symmetric, and the second one is antisymmetric w.r.t. their middles.
Finally, for $d=3$, the following four couples (SP, AP)

$$((+,+,+,-),(1,2)); \quad ((+,-,+,+),(2,1));$$
$$((+,-,-,-),(1,2)); \quad ((+,+,-,+),(2,1)).$$

constitute one orbit for $d=3$. In this example all admissible pairs are Descartes' pairs.

2.2 Degrees $d \geq 4$

It turns out that for $d \geq 4$, it is no longer true that all couples (SP, AP) are realizable by polynomials of degree d. Namely, the following result can be found in [12]:

Theorem 1. *The only couples (SP, AP) which are non-realizable by univariate polynomials of degree* 4 *are*

$$((+,-,-,-,+),(0,2)) \quad \text{and} \quad ((+,+,-,+,+),(2,0)).$$

It is clear that these two cases constitute one orbit of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action of length 2 (the SPs are the same when read the usual way and backward).

Proof. The argument showing non-realizability in Theorem 1 is easy. Namely, if a polynomial

$$P \ := \ x^4+a_3x^3+a_2x^2+a_1x+a_0$$

realizes the second of these couples and has two positive roots $\alpha < \beta$ and no negative roots, then for any $u \in (\alpha, \beta)$, the values of the monomials x^4, a_2x^2, and a_0 are the same at u and $-u$, while the monomials a_3x^3 and a_1x are positive at u and negative at $-u$. Hence, $P(-u) < P(u) < 0$. As $P(0) > 0$ and $\lim_{x\to-\infty}P(x) = +\infty$, the polynomial P has two negative roots as well—a contradiction.

For $d = 4$, realizability of all other couples (SP, AP) can be proven by producing explicit examples.

Remark 2. In [19] a geometric illustration of the non-realizability of the two cases mentioned in Theorem 1 is proposed. Namely, one considers the family of polynomials $Q := x^4 + x^3 + ax^2 + bx + c$ and the *discriminant set*

$$\Delta := \{ (a, b, c) \in \mathbb{R}^3 \,|\, \mathrm{Res}(Q, Q') = 0 \},$$

where Res (Q, Q') is the resultant of the polynomials Q and Q'. The hypersurface $\Delta = 0$ partitions $\mathbb{R}^3$ into three open domains, in which the polynomial Q has 0, 1, or 2 complex conjugate pairs of roots, respectively. These domains intersect the 8 open orthants of $\mathbb{R}^3$ defined by the coordinate system (a, b, c), and in each of these intersections, the polynomial Q has one and the same number of positive, negative, and complex roots, as well as the same signs of its coefficients. The non-realizability of the couple $((+,+,-,+,+),(2,0))$ can be interpreted as the fact that the corresponding intersection is empty. Pictures of discriminant sets allow to construct easily the numerical examples mentioned in the proof of Theorem 1.

It remains to be noticed that for $\alpha > 0$ and $\beta > 0$, the polynomials $P(x)$ and $\beta P(\alpha x)$ have one and the same numbers of positive, negative, and complex roots. Therefore, it suffices to consider the family of polynomials Q in order to cover all SPs beginning with $(+,+)$. The ones beginning with $(+,-)$ will be covered by the family $Q(-x)$.

For degrees $d = 5$ and 6, the following result can be found in [1].

Theorem 2. (1) *The only two couples* (*SP, AP*) *which are non-realizable by univariate polynomials of degree* 5 *are*:

$$((+,-,-,-,-,+),(0,3)) \quad \text{and} \quad ((+,+,-,+,-,-),(3,0)).$$

(2) *For degree $d = 6$, up to the above $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, the only non-realizable couples (SP, AP) are*:

$$((+,-,-,-,-,-,+),(0,2)); \quad ((+,-,-,-,-,-,+),(0,4));$$
$$((+,-,+,-,-,-,+),(0,2)); \quad ((+,+,-,-,-,-,+),(0,4)).$$

The two cases of Part (1) of Theorem 2 also form an orbit of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action of length 2. Each of the first two cases of Part (2) defines an orbit of length 2, while each of the last two cases defines an orbit of length 4.

For $d = 7$, the following theorem is contained in [8].

Theorem 3. *For univariate polynomials of degree* 7, *among their 1472 possible couples* (*SP*, *AP*) (*up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action*), *exactly the following* 6 *are non-realizable*:

$$((+,+,-,-,-,-,-,+),(0,5)); \quad ((+,+,-,-,-,-,+,+),(0,5));$$
$$((+,-,-,-,-,+,-,+),(0,3)); \quad ((+,+,+,-,-,-,-,+),(0,5));$$
$$((+,-,-,-,-,-,-,+),(0,3)); \quad ((+,-,-,-,-,-,-,+),(0,5)).$$

The lengths of the respective orbits in these 6 cases are 4, 2, 4, 4, 2, and 2.

The case $d = 8$ has been partially solved in [8] and completely in [16]:

Theorem 4. *For degree* $d = 8$, *among the 3648 possible couples* (SP, AP) *(up to the* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-action), exactly the following* 19 *are non-realizable*:

$$\begin{array}{ll}
((+,+,-,-,-,-,-,+,+),(0,6)); & ((+,+,-,-,-,-,-,-,+),(0,6)); \\
((+,+,+,-,-,-,-,-,+),(0,6)); & ((+,+,+,+,-,-,-,-,+),(0,6)); \\
((+,-,+,-,-,-,+,-,+),(0,2)); & ((+,-,+,-,+,-,-,-,+),(0,2)); \\
((+,-,+,-,-,-,-,-,+),(0,2)); & ((+,-,+,-,-,-,-,-,+),(0,4)); \\
((+,-,-,-,+,-,-,-,+),(0,2); & ((+,-,-,-,+,-,-,-,+),(0,4)); \\
((+,-,-,-,-,-,-,-,+),(0,2)); & ((+,-,-,-,-,-,-,-,+),(0,4); \\
((+,-,-,-,-,-,-,-,+),(0,6)); & ((+,+,+,-,-,-,-,+,+),(0,6)); \\
((+,-,-,-,-,+,-,-,+),(0,4)); & ((+,-,-,-,-,-,-,+,+),(0,4)); \\
((+,-,+,+,-,-,-,-,+),(0,4)); & ((+,-,+,-,-,-,-,+,+),(0,4)); \\
((+,-,-,-,-,+,-,+,+),(0,4)). &
\end{array}$$

The lengths of the respective orbits are 2, 4, 4, 4, 2, 4, 4, 4, 2, 2, 2, 2, 2, 4, 4, 4, 4, 4, and 4.

Remark 3. As we see above, for $d = 4, 5, 6, 7$, and 8, up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, the numbers of non-realizable cases are 1, 1, 4, 6, and 19, respectively. The fact that these numbers increase more when $d = 5$ and $d = 7$ than when $d = 4$ and $d = 6$ could be related to the fact that the maximal possible number of complex conjugate pairs of roots of a real univariate degree d polynomial is $[d/2]$. This number increases w.r.t. $[(d-1)/2]$ when d is even and does not increase when d is odd.

Observe that for $d \leq 8$, all examples of couples (SP, AP) which are non-realizable are with APs of the form $(\nu, 0)$ or $(0, \nu)$ and $\nu \in \mathbb{N}$. Initially, we thought that this is always the case. However, recently it was proven that, for higher degrees, this fact is no longer true (see [17]):

Theorem 5. *For* $d = 11$, *the following couple* (SP, AP)

$$((+,-,-,-,-,-,+,+,+,+,+,-),(1,8))$$

is non-realizable. *The Descartes' pair in this case equals* $(3, 8)$.

There is a strong evidence that for $d = 9$, the similar couple (SP, AP)

$$((+,-,-,-,-,+,+,+,+,-),(1,6))$$

is also non-realizable. (Its Descartes' pair equals $(3, 6)$.) If this were true, then 9 would be the smallest degree with an example of a non-realizable couple (SP, AP) for which both components of the AP are nonzero. When studying the cases $d = 8$ and $d = 11$ (see [16] and [17]), discriminant sets have been considered (see Remark 2).

Summarizing the above, we have to admit that the information in low degrees available at the moment does not allow us to formulate a consistent conjecture describing all non-realizable couples in an arbitrary degree which we could consider as sufficiently well motivated.

3. Series of examples of (non-)realizable couples (SP, AP)

In this section we present a series of couples (non-)realizable for infinitely many degrees. We decided to include those proofs of the statements formulated below which are short and instructive.

3.1 Some examples of realizability and a concatenation lemma

Our first examples of realizability deal with polynomials with the minimal possible number of real roots:

Proposition 1. *For d even, any SP whose last component is a $+$ (resp. is a $-$) is realizable with the AP* $(0,0)$ *(resp.* $(1,1)$*). For d odd, any SP whose last component is a $+$ (resp. is a $-$) is realizable with the AP* $(0,1)$ *(resp.* $(1,0)$*).*

Proof. Indeed, for any given SP, it suffices to choose any polynomial defining this SP and to increase (resp. decrease) its constant term sufficiently much if the latter is positive (resp. negative). The resulting polynomial will have the required number of real roots. □

Our next example deals with *hyperbolic* polynomials, that is, real polynomials with all real roots. Several topics concerning hyperbolic polynomials are developed in [18].

Proposition 2. *Any SP is realizable with its Descartes' pair.*

Proposition 2 will follow from the following *concatenation lemma* whose proof can be found in [8].

Lemma 1. *Suppose that monic polynomials P_1 and P_2, of degrees d_1 and d_2 resp., realize the SPs $(+,\hat{\sigma}_1)$ and $(+,\hat{\sigma}_2)$, where $\hat{\sigma}_j$ are the SPs defined by P_j in which the first $+$ is deleted. Then*:

1. *If the last position of $\hat{\sigma}_1$ is a $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the SP $(+,\hat{\sigma}_1,\hat{\sigma}_2)$ and the AP $\left(pos_1+pos_2, neg_1+neg_2\right)$.*

2. *If the last position of $\hat{\sigma}_1$ is a $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the SP $(+,\hat{\sigma}_1,-\hat{\sigma}_2)$ and the AP $\left(pos_1+pos_2, neg_1+neg_2\right)$. (Here $-\hat{\sigma}_2$ is the SP obtained from $\hat{\sigma}_2$ by changing each $+$ by a $-$ and vice versa.)*

The concatenation lemma allows to deduce the realizability of couples (SP, AP) with higher values of d from that of couples with smaller d in which cases explicit constructions are usually easier to obtain. On the other hand, non-realizability of special cases cannot be concluded using this lemma.

Example 3. Denote by τ the last entry of the SP $\hat{\sigma}_1$. We consider the cases

$$\begin{array}{ccccccc} P_2(x) & = & x-1, & x+1, & x^2+2x+2, & x^2-2x+2 & \text{with} \\ \left(pos_2, neg_2\right) & = & (1,0), & (0,1), & (0,0), & (0,0) & \text{resp.} \end{array}$$

When $\tau=+,$ then one has, respectively,

$$\hat{\sigma}_2 = (-), (+), (+,+), (-,+),$$

and the SP of $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ equals

$$(+,\hat{\sigma}_1,-),\quad (+,\hat{\sigma}_1,+),\quad (+,\hat{\sigma}_1,+,+),\quad (+,\hat{\sigma}_1,-,+).$$

When $\tau=-,$ then one has, respectively,

$$\hat{\sigma}_2 = \quad (+),\quad (-),\quad (-,-),\quad (+,-),$$

and the SP of $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ equals

$$(+,\hat{\sigma}_1,+),\quad (+,\hat{\sigma}_1,-),\quad (+,\hat{\sigma}_1,-,-),\quad (+,\hat{\sigma}_1,+,-).$$

Proof of Proposition 2. We will use induction on the degree d of the polynomial. For $d = 1$, the SP $(+, -)$ (resp. $(+, +)$) is realizable with the AP $(1, 0)$ (resp. $(0, 1)$) by the polynomial $x - 1$ (resp. $x + 1$).

For $d = 2$, we apply Lemma 1. Set $P_1 := x + 1$ and $P_2 := x - 1$. Then, for $\varepsilon > 0$ small enough, the polynomials

$$\begin{aligned} \varepsilon P_1(x)P_2(x/\varepsilon) &= (x+1)(x-\varepsilon) &= x^2 + (1-\varepsilon)x - \varepsilon \quad \text{and} \\ \varepsilon P_2(x)P_1(x/\varepsilon) &= (x-1)(x+\varepsilon) &= x^2 + (-1+\varepsilon)x - \varepsilon \end{aligned}$$

define the SPs $(+, +, -)$ and $(+, -, -)$, respectively, and realize them with the AP $(1, 1)$. In the same way, one can concatenate P_1 (resp. P_2) with itself to realize the SP $(+, +, +)$ with the AP $(0, 2)$ (resp. the SP $(+, -, +)$ with the AP $(2, 0)$). These are all possible cases of monic hyperbolic degree 2 polynomials with nonvanishing coefficients.

For $d \geq 2$, in order to realize a SP σ with its Descartes' pair (c, p), we represent σ in the form $(\sigma^\dagger, u, v)$, where u and v are the last two components of σ and $\sigma^\dagger$ is the SP obtained from σ by deleting u and v. Then, we choose P_1 to be a monic polynomial realizing the SP $(\sigma^\dagger, u)$:

i. With the AP $(c - 1, p)$, and we set $P_2 := x - 1$, if $u = -v$.

ii. With the AP $(c, p - 1)$, and we set $P_2 := x + 1$, if $u = v$. □

Our next result discusses (non-)realizability for polynomials with only two sign changes (see [8, 9]).

Proposition 3. *Consider a sign pattern $\overline{\sigma}$ with 2 sign changes, consisting of m consecutive pluses followed by n consecutive minuses and then by q consecutive pluses, where $m + n + q = d + 1$. Then*:

i. *For the pair $(0, d - 2)$, this sign pattern is not realizable if*

$$\kappa := \frac{d - m - 1}{m} \cdot \frac{d - q - 1}{q} \geq 4; \tag{3}$$

ii. *The sign pattern $\overline{\sigma}$ is realizable with any pair of the form $(2, v)$, except in the case when d and m are even, $n = 1$ (hence q is even), and $v = 0$.*

Certain results about realizability are formulated in terms of the ratios between the quantities *pos*, *neg*, and d. The following proposition is proven in [8].

Proposition 4. *For a given couple (SP, AP), if* $\min(pos, neg) > [(d - 4)/3]$, *then this couple is realizable.*

3.2 The even and the odd series

Suppose that the degree d is even. Then, the following result holds (see Proposition 4 in [8]):

Proposition 5. *Consider the SPs satisfying the following three conditions*:

i. *Their last entry (i.e., the sign of the constant term) is a $+$.*

ii. *The signs of all odd monomials are $+$.*

iii. *Among the remaining signs of even monomials, there are exactly $\ell \geq 1$ signs $-$ (at arbitrary positions).*

Then, for any such SP, the APs $(2, 0)$, $(4, 0)$, …, $(2\ell, 0)$, *and only they, are non-realizable*.

Suppose now that the degree $d \geq 5$ is odd. For $1 \leq k \leq (d-3)/2$, denote by σ_k the SP beginning with two pluses followed by k pairs $(-, +)$ and then by $d - 2k - 1$ minuses. Its Descartes' pair of σ_k equals $(2k+1, d-2k-1)$. The following proposition is proven in [19].

Theorem 6. (1) *The SP* σ_k *is not realizable with any of the pairs* $(3, 0)$, $(5, 0)$, …, $(2k+1, 0)$; (2) *The SP* σ_k *is realizable with the pair* $(1, 0)$; (3) *The SP* σ_k *is realizable with any of the APs* $(2\ell + 1, 2r)$, $\ell = 0, 1, \ldots, k$, and $r = 1, 2, \ldots, (d-2k-1)/2$.

One can observe that Cases (1), (2), and (3) exhaust all possible APs (pos, neg).

4. Similar realization problems

In this section, we consider realization problems similar or motivated by Problem 1. A priori it is hard to tell which of these or similar problems might have a reasonable answer.

4.1 $\mathcal{D}$-Sequences

Consider a real polynomial P of degree d and its derivative. By Rolle's theorem, if P has exactly r real roots (counted with multiplicity), then the derivative P' has $r - 1 + 2\ell$ real roots (counted with multiplicity), where $\ell \in \mathbb{N} \cup 0$. It is possible that P' has more real roots than P. For example, for $d = 2$ and $P = x^2 + 1$, one gets $P' = 2x$ which has a real root at 0, while P has no real roots at all. For $d = 3$, the polynomial $P = x^3 + 3x^2 - 8x + 10 = (x+5)((x-1)^2 + 1)$ has one negative root and one complex conjugate pair, while its derivative $P' = 3x^2 + 6x - 8$ has one positive and one negative root.

Now, for $j = 0, \ldots,$ and $d - 1$, denote by r_j and c_j the numbers of real roots and complex conjugate pairs of roots of the polynomial $P^{(j)}$ (both counted with multiplicity). These numbers satisfy the conditions

$$r_j \leq r_{j+1} + 1, \quad r_j + 2c_j = d - j. \tag{4}$$

Definition 1. A sequence $((r_0, 2c_0), (r_1, 2c_1), \ldots, (r_{d-1}, 2c_{d-1}))$ satisfying conditions (4) will be called a *$\mathcal{D}$-sequence of length d*. We say that a given $\mathcal{D}$-sequence of length d is *realizable* if there exists a real polynomial P of degree d with this $\mathcal{D}$-sequence, where for $j = 0, \ldots, d-1$, all roots of $P^{(j)}$ are distinct.

Example 4. One has $r_{d-1} = 1$ and $c_{d-1} = 0$. Clearly, one has either $r_{d-2} = 2$, $c_{d-2} = 0$ or $r_{d-2} = 0$, $c_{d-2} = 1$. For small values of d, one has the following $\mathcal{D}$-sequences and respective polynomials realizing them:

$$\begin{array}{lll} d = 1 & (1, 0) & x \\ d = 2 & ((2, 0), (1, 0)) & x^2 - 1 \\ & ((0, 2), (1, 0)) & x^2 + 1 \\ d = 3 & ((3, 0), (2, 0), (1, 0)) & x^3 - x \\ & ((1, 2), (0, 2), (1, 0)) & x^3 + x \\ & ((1, 2), (2, 0), (1, 0)) & x^3 + 10x^2 + 26x. \end{array}$$

The following question where a positive answer to which can be found in [15] seems very natural.

Problem 2. *Is it true that for any* $d \in \mathbb{N}$, *any $\mathcal{D}$-sequence is realizable*?

4.2 Sequences of admissible pairs

Now, we are going to formulate a problem which is a refinement of both Problems 1 and 2.

Recall that for a real polynomial P of degree d, the signs of its coefficients a_j define the sign patterns $\sigma_0, \sigma_1, \ldots, \sigma_{d-1}$ corresponding to P and to all its derivatives of order $\leq d-1$ since the SP σ_j is obtained from σ_{j-1} by deleting the last component. We denote by (c_k, p_k) and (pos_k, neg_k) the Descartes' and admissible pairs for the SPs σ_k, $k = 0, \ldots, d-1$. The following restrictions follow from Rolle's theorem:

$$\begin{aligned} & pos_{k+1} \geq pos_k - 1 \ , \ \ neg_{k+1} \geq neg_k - 1 \\ \text{and} \qquad & pos_{k+1} + neg_{k+1} \geq pos_k + neg_k - 1. \end{aligned} \tag{5}$$

It is always true that

$$pos_{k+1} + neg_{k+1} + 3 - pos_k - neg_k \in 2\mathbb{N}. \tag{6}$$

Definition 2. Given a sign pattern σ_0 of length $d+1$, suppose that for $k = 0, \ldots, d-1$, the pair (pos_k, neg_k) satisfies the conditions

$$\begin{aligned} & pos_k \leq c_k, \quad && c_k - pos_k \in 2\mathbb{Z}, \\ & neg_k \leq p_k, \quad && p_k - neg_k \in 2\mathbb{Z}, \\ & \text{and} && \mathrm{sgn}\, a_k = (-1)^{pos_k}. \end{aligned} \tag{7}$$

as well as the inequalities (5)–(6). Then, we say that

$$((pos_0, neg_0), \ldots, (pos_{d-1}, neg_{d-1})) \tag{8}$$

is a *sequence of admissible pairs* (*SAPs*). In other words, it is a sequence of pairs admissible for the sign pattern σ_0 in the sense of these conditions. We say that a given couple (SP, SAP) is *realizable* if there exists a polynomial P whose coefficients have signs given by the SP σ_0, and such that for $k = 0, \ldots, d-1$, the polynomial $P^{(k)}$ has exactly pos_k positive and neg_k negative roots, all of them being simple. Complex roots are also supposed to be distinct.

Remark 4. If one only knows the SAP (8), the SP σ_0 can be restituted by the formula

$$\sigma_0 = (+, \ \ (-1)^{pos_{d-1}}, \ \ (-1)^{pos_{d-2}}, \ldots, (-1)^{pos_0} \ ).$$

Nevertheless, in order to make comparisons with Problem 1 more easily, we consider couples (SP, SAP) instead of just SAPs. But for a given SP, there are, in general, several possible SAPs which is illustrated by the following example.

Example 5. Consider the SP of length $d+1$ with all pluses. For $d = 2$ and 3, there are, respectively, two and three possible SAPs:

$((0,2),(0,1))$, $((0,0),(0,1))$, for $d = 2$

and

$((0,3),(0,2),(0,1))$, $((0,1),(0,2),(0,1))$, $((0,1),(0,0),(0,1))$ for $d = 3$.

For $d = 4, 5, 6, 7, 8, 9, 10$, the numbers $A(d)$ of SAPs compatible with the SP of length $d + 1$ having all pluses are

$$7, \quad 12, \quad 30, \quad 55, \quad 143, \quad 273, \quad \text{and} \quad 728,$$

respectively. One can show that $A(d) \geq 2A(d-1)$, if $d \geq 2$ is even, and $A(d) \geq 3A(d-1)/2$, if $d \geq 3$ is odd (see [5]).

Example 6. There are two couples (SP, SAP) corresponding to the couple (SP, AP) $C := ((+,+,-,+,+), (0,2))$; we also say that the couple C *can be extended* into these couples (SP, SAP). These are

$$\begin{array}{ccccccccccccl} (& (+,+,-,+,+) & , & (0,2) & , & (2,1) & , & (1,1) & , & (0,1) &) & \text{and} \\ (& (+,+,-,+,+) & , & (0,2) & , & (0,1) & , & (1,1) & , & (0,1) &) & . \end{array}$$

Indeed, by Rolle's theorem, the derivative of a polynomial realizing the couple C has at least one negative root. By conditions (7), this derivative (whose degree equals 3) has an even number of positive roots. This yields just two possibilities for (pos_1, neg_1), namely, $(2,1)$ and $(0,1)$. The second derivative is a quadratic polynomial with positive leading coefficient and negative constant term. Hence, it has a positive and a negative root. The realizability of the above two couples (SP, SAP) is proven in [5].

Our final realization problem is as follows:

Problem 3. *For a given degree d, which couples (SP, SAP) are realizable?*

Remarks 1. (1) This problem is a refinement of Problem 1, because one considers the APs of the derivatives of all orders and not just the one of the polynomial itself (see Remark 4). Therefore, if a given couple (SP, AP) is non-realizable, then all couples (SP, SAP) corresponding to it in the sense of Example 6 are automatically non-realizable.

(2) Obviously, Problem 3 is a refinement of Problem 2—in the latter case, one does not take into account the signs of the real roots of the polynomial and its derivatives.

(3) When we deal with couples (SP, SAP), we can use the $\mathbb{Z}_2$-action defined by (1). Therefore, it suffices to consider the cases of SPs beginning with $(+,+)$. The generator (2.2) of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action cannot be used, because when the derivatives of a polynomial are involved, the polynomial loses its last coefficients. Due to this circumstance, the two ends of the SP cannot be treated equally.

The following proposition is proven in [5]:

Proposition 6. *For any given SP of length $d + 1$ and $d \geq 1$, there exists a unique SAP such that $pos_0 + neg_0 = d$. This SAP is realizable. For the given SP, this pair (pos_0, neg_0) is its Descartes' pair.*

Example 7. For even d, consider the SP with all pluses. Any hyperbolic polynomial with all negative and distinct roots realizes this SP with SAP

$$((0,d), (0,d-1), \ldots, (0,1)).$$

One can choose such a polynomial P with all $d - 1$ distinct critical values. Hence, in the family of polynomials $P + t$ and $t > 0$, one encounters polynomials realizing this SP with any of the SAPs

$$((0,d-2\ell), (0,d-1), (0,d-2), \ldots, (0,1)), \quad \ell = 0, 1, \quad \ldots \quad d/2.$$

In the same way, for odd d, the SP $(+,+,\ldots,+,-)$ is realizable with the SAP

$$((1,d-1),\ \ (0,d-1),\ \ (0,d-2),\ \ \ldots,\ \ (0,1))$$

by some hyperbolic polynomial R with all distinct roots and critical values. In the family of polynomials $R-s$ and $s>0$, one encounters polynomials realizing this SP with any of the SAPs

$$((1,d-1-2\ell),\ \ (0,d-1),\ \ (0,d-2),\ \ \ldots,\ \ (0,1)),\quad \ell=0,\ \ 1,\ \ \ldots(d-1)/2.$$

For $d\leq 5$, the following exhaustive answer to Problem 3 is given in [5]:

A. For $d=1$, 2, and 3, all couples (SP, SAP) are realizable.

B. For $d=4$, the couple (SP, SAP)

$$(\ (+,+,-,+,+),\ \ \ (2,0),\ \ (2,1),\ \ (1,1),\ \ \ (0,1)\ ),$$

and only it (up to the $\mathbb{Z}_2$-action), is non-realizable. Its non-realizability follows from one of the couples (SP, AP) $C^\dagger := ((+,+,-,+,+),(2,0))$ (see Theorem 1).

One can observe that the couple $C^\dagger$ can be uniquely extended into a couple (SP, SAP). Indeed, the first derivative has a positive constant term hence an even number of positive roots. This number is positive by Rolle's theorem. Hence, the AP of the first derivative is $(2,1)$. In the same way, one obtains the APs $(1,1)$ and $(0,1)$ for the second and third derivatives, respectively.

C. For $d=5$, the following couples (SP, SAP), and only they, are non-realizable:

$$\begin{array}{lllllll}
(& (+,+,-,+,+,+), & (2,1), & (2,0), & (2,1), & (1,1), & (0,1)\ ), \\
(& (+,+,-,+,+,+), & (0,1), & (2,0), & (2,1), & (1,1), & (0,1)\ ), \\
(& (+,+,-,+,+,-), & (3,0), & (2,0), & (2,1), & (1,1), & (0,1)\ ), \\
(& (+,+,-,+,+,-), & (1,0), & (2,0), & (2,1), & (1,1), & (0,1)\ ), \\
(& (+,+,-,+,-,-), & (3,0), & (3,1), & (2,1), & (1,1), & (0,1)\ ).
\end{array}$$

The non-realizability of the first four of them follows from that of the couple $C^\dagger$. The last one is implied by part (1) of Theorem 2; it is true that the couple (SP, AP) $((+,+,-,+,-,-),(3,0))$ extends in a unique way into a couple (SP, SAP), and this is the fifth of the five such couples cited above.

One of the methods used in the study of couples (SP, AP) or (SP, SAP) is the explicit construction of polynomials with multiple roots which define a given SP. Such constructions are not difficult to carry out because one has to use families of polynomials with fewer parameters. Once a polynomial with multiple roots is constructed, one has to justify the possibility to deform it continuously into a nearby polynomial with all distinct roots. Multiple roots can give rise to complex conjugate pairs of roots. An example of such a construction is the following lemma from [5].

Lemma 2. *Consider the polynomials $S := (x+1)^3(x-a)^2$ and $T := (x+a)^2(x-1)^3$ and $a>0$. Their coefficients of x^4 are positive if and only if, respectively, $a<3/2$ and $a>3/2$. The coefficients of the polynomial S define the SP*

$$\begin{array}{lll}
(+,+,+,+,-,+) & \text{for} & a\in(\,0,(3-\sqrt{6})/3\,)\quad , \\
(+,+,+,-,-,+) & \text{for} & a\in(\,(3-\sqrt{6})/3,3-\sqrt{6}\,)\quad , \\
(+,+,-,-,-,+) & \text{for} & a\in(\,3-\sqrt{6},2/3\,)\quad \text{and} \\
(+,+,-,-,+,+) & \text{for} & a\in(\,2/3,3/2\,)\quad .
\end{array}$$

The coefficients of T define the SP

$$\begin{array}{lll} (+,+,-,+,+,-) & \text{for} \quad a \in \left(3/2, (3+\sqrt{6})/3\right) & , \\ (+,+,-,-,+,-) & \text{for} \quad a \in \left((3+\sqrt{6})/3, 3+\sqrt{6}\right) & \text{and} \\ (+,+,+,-,+,-) & \text{for} \quad a > 3+\sqrt{6} & . \end{array}$$

5. Outlook

1. Our first open question deals with the limit of the ratio between the quantities $R(d)$ of all realizable and $A(d)$ of all possible cases of couples (SP, AP) as $d \to \infty$. In principle, one does not have to take into account the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action in order not to face the problem of the two different possible lengths of orbits (2 and 4).

   A priori, for $d \geq 4$, one has $R(d)/A(d) \in (0, 1)$. It would be interesting to find out whether this ratio has a limit as $d \to \infty$ and, if "yes," whether this limit is 0 and 1 or belongs to $(0, 1)$. In the latter case, it would be interesting to find the exact value.

   A less ambitious open problem is to find an interval $[\alpha, \beta] \subset (0, 1)$ to which this ratio belongs for any $d \in \mathbb{N}$, $d \geq 4$, or at least for d sufficiently large.

2. A related problem would be to find sufficient conditions for realizability based on the ratios between the quantities pos, neg, and d. On the one hand, when the ratios pos/d and neg/d are both large enough, one has realizability (see Proposition 4). On the other hand, in all examples of non-realizability known up to now, one of the quantities pos and neg is either 0 or is very small compared to the other one. Thus, it would be interesting to understand the role of these ratios for the (non)-realizability of the couples (SP, AP).

3. Our third open question is about the realizability of couples (SP, SAP). For $d \leq 5$, the non-realizability of all non-realizable couples (SP, SAP) results from the non-realizability of the corresponding couples (SP, AP). In principle, one could imagine a situation in which there exists a couple (SP, AP) extending into several couples (SP, SAP) some of which are realizable and the remaining are not. Whether, for $d \geq 6$, such couples (SP, AP) exist or not is unknown at present.

4. Our final natural and important question deals with the topology of intersections of the set of real univariant polynomials with a given number of real roots with orthants in the coefficient space (which means fixing the signs of the coefficients). It is well known that the set of monic univariate polynomials of a given degree and with a given number of real roots is contractible. When we cut this set with the union of coordinate hyperplanes (coordinates being the coefficients of polynomials), then it splits into a number of connected components. In each such connected component, the number of positive and negative roots is fixed. But, in principle, it can happen that different connected components correspond to the same pair (pos, neg). Could this really happen? Are all such connected components contractible, or they can have some nontrivial topology?

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Chapter 3

Obtaining Explicit Formulas and Identities for Polynomials Defined by Generating Functions of the Form $F(t)^x \cdot G(t)^\alpha$

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Abstract

In this chapter, we study properties of polynomials defined by generating functions of the form $A(t,x,\alpha) = F(t)^x \cdot G(t)^\alpha$. Based on the Lagrange inversion theorem and the theorem of logarithmic derivative for generating functions, we obtain new properties related to the compositional inverse generating functions of those polynomials. Also we study the composition of generating functions $R(tA(t))$, where $A(t)$ is the generating function of the form $F(t)^x \cdot G(t)^\alpha$. We apply those results for obtaining explicit formulas and identities for such polynomials as the generalized Bernoulli, generalized Euler, Frobenius-Euler, generalized Sylvester, generalized Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials.

Keywords: polynomial, identity, generating function, composita, composition, compositional inverse

1. Introduction

Generating functions are a powerful tool for solving problems in number theory, combinatorics, algebra, probability theory, and other fields of mathematics. One of the advantages of generating functions is that an infinite number sequence can be represented in a form of a single expression. Many authors have studied generating functions and their properties and found applications for them (for instance, Comtet [1], Flajolet and Sedgewick [2], Graham et al. [3], Robert [4], Stanley [5], and Wilf [6]).

Generating functions have an important role in the study of polynomials. Vast investigations related to the generating functions for many polynomials can be found in many books and articles (e.g., see [7–17]).

A special place in this area is occupied by research in the field of obtaining new identities for polynomials and special numbers with using their generating functions. Interesting results in the field of obtaining new identities for polynomials can be found in some recent works by Simsek [18–20], Kim et al. [21, 22], and Ryoo [23–25].

Another trend in study of polynomials is getting new representation and explicit formulas for those polynomials. For instance, Qi has recently established explicit formulas for the generalized Motzkin numbers in [26] and the central Delannoy numbers in [27]. One can find interesting results in papers of Srivastava [28, 29], Cenkci [30], and Boyadzhiev [31].

In this chapter, we obtain some interesting properties of polynomials defined by generating functions of the form $F(t)^x \cdot G(t)^\alpha$. As an application, we give some new identities for the Bernoulli, Euler, Frobenius-Euler, Sylvester, Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials.

According to Stanley [32], ordinary generating functions are defined as follows:

Definition 1. *An ordinary generating function of the sequence* $(a_n)_{n\geq 0}$ *is the formal power series*

$$A(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{n\geq 0} a_n x^n. \tag{1}$$

Kruchinin et al. [33–35] introduced the mathematical notion of the *composita* of a given generating function, which can be used for calculating the coefficients of a composition of generating functions.

Definition 2. The composita of the generating function $F(x) = \sum_{n>0} f_n x^n$ is the function with two variables

$$F^\Delta(n,k) = \sum_{\pi_k\in C_n} f_{\lambda_1} f_{\lambda_2} \cdots f_{\lambda_k}, \tag{2}$$

where C_n is the set of all compositions of an integer n and π_k is the composition n into k parts such that $\sum_{i=1}^k \lambda_i = n$.

Using the expression of the composita of a given generating function $F^\Delta(n,k)$, we can get powers of the generating function $F(x)$:

$$(F(x))^k = \sum_{n\geq k} F^\Delta(n,k) x^n. \tag{3}$$

Compositae also can be used for calculating the coefficients of generating functions obtained by addition, multiplication, composition, reciprocation, and compositional inversion of generating functions (for details see [33–35]).

By the reciprocal generating function we mean the following [6]:

Definition 1. A reciprocal generating function $A(x)$ of a generating function $B(x) = \sum_{n\geq 0} b_n x^n$ is a power series such that satisfies the following condition:

$$A(x)B(x) = 1. \tag{4}$$

By the compositional inverse generating function we mean the following:

Definition 2. A compositional inverse $\overline{F(x)}$ of generating function $F(x) = \sum_{n>0} f_n x^n$ with $f(1) \neq 0$ is a power series such that satisfies the following condition:

$$F\left(\overline{F(x)}\right) = x. \tag{5}$$

Also the compositional inverse can be written as $F^{[-1]}(x)$ or $\overline{F(x)} = RevF$.

For example, we will use the following formulas:

If we consider the composition $A(x) = R(F(x)) = \sum_{n\geq 0} a_n x^n$ of generating functions $R(x) = \sum_{n\geq 0} r_n x^n$ and $F(x) = \sum_{n>0} f_n x^n$, then we can get the values of the coefficients a_n by using the following formula ([35], Eq. (17)):

$$a_n = \begin{cases} r_0, & \text{for} n = 0; \\ \sum\limits_{k=1}^{n} F^{\Delta}(n,k) r_k, & \text{otherwise}. \end{cases} \tag{6}$$

If we consider the composition $A(x) = R(F(x)) = \sum_{n>0} a_n x^n$ of generating functions $R(x) = \sum_{n>0} r_n x^n$ and $F(x) = \sum_{n>0} f_n x^n$, then we can get the values of the composita $A^{\Delta}(n,k)$ by using the following formula ([35]):

$$A^{\Delta}(n,k) = \sum_{m=k}^{n} F^{\Delta}(n,m) R^{\Delta}(m,k). \tag{7}$$

2. Main results

Let us consider a special case of generating functions that can be presented as the product of the powers of generating functions $F(t)^x \cdot G(t)^{\alpha}$. For such generating functions, we obtain several properties, which are given in the following theorem:

Theorem 1. *If $A(t)$ is a generating function of the following form:*

$$A(t) = F(t)^x \cdot G(t)^{\alpha} = \sum_{n \geq 0} A_n(x,\alpha) t^n, \tag{8}$$

then:

1. *For the composition of generating functions $D(t) = C(B(t)) = \sum_{n \geq 0} B_n t^n$, where $B(t) = tA(t)$ and $C(t) = \sum_{n \geq 0} C_n t^n$, we have*

$$D_n = D_n(x,\alpha) = \sum_{k=1}^{n} A_{n-k}(kx, k\alpha) C_k, \quad D_0 = C_0; \tag{9}$$

2. *For the compositional inverse generating function $\overline{B}(t)$ of $B(t) = tA(t)$, we have*

$$\overline{B}(t) = \sum_{n>0} \frac{1}{n} A_{n-1}(-nx, -n\alpha) t^n; \tag{10}$$

3. *We have the following identities*

$$\sum_{m=k}^{n} A_{n-m}(mx, m\alpha) \frac{k}{m} A_{m-k}(-mx, -m\alpha) = \delta_{n,k} \tag{11}$$

and

$$\sum_{m=k}^{n} \frac{m}{n} A_{n-m}(-nx, -n\alpha) A_{m-k}(kx, k\alpha) = \delta_{n,k}, \tag{12}$$

where $\delta_{n,k}$ is the Kronecker delta.

Proof. First we get the k-th power of the generating function $B(t) = tA(t)$

$$\begin{aligned} (B(t))^k = (tA(t))^k = t^k (F(t))^{xk} (G(t))^{\alpha k} = \\ = t^k \sum_{n \geq 0} A_n(kx, k\alpha) t^n = \sum_{n \geq k} A_{n-k}(kx, k\alpha) t^n. \end{aligned}$$

Hence, the composita of $B(t) = tA(t)$ is

$$B^{\Delta}(n,k) = A_{n-k}(kx, k\alpha). \tag{13}$$

Using Eqs. (6) and (13), we get Eq. (9).

According to [36], the composita of the compositional inverse generating function $\overline{A}(t)$ of $A(t) = \sum_{n>0} a_n t^n$ is

$$\overline{A}^{\Delta}(n,k) = \frac{k}{n} R^{\Delta}(2n-k, n), \tag{14}$$

where $R^{\Delta}(n,k)$ is the composita of the generating function $R(t) = \frac{t^2}{A(t)}$.

For getting the composita of the compositional inverse generating function $\overline{B}(t)$ of $B(t) = tA(t)$, we need to know the composita of the generating function

$$R(t) = \frac{t^2}{B(t)} = \frac{t^2}{tA(t)} = \frac{t}{A(t)}. \tag{15}$$

Then we get the k-th power of the generating function $R(t) = \frac{t}{A(t)}$

$$\begin{aligned} (R(t))^k = \left(\tfrac{t}{A(t)}\right)^k = t^k (F(t))^{-xk} (G(t))^{-\alpha k} = \\ = t^k \sum_{n\geq 0} A_n(-kx, -k\alpha) t^n = \sum_{n\geq k} A_{n-k}(-kx, -k\alpha) t^n. \end{aligned} \tag{16}$$

Hence, the composita of Eq. (15) is

$$R^{\Delta}(n,k) = A_{n-k}(-kx, -k\alpha). \tag{17}$$

Using Eqs. (14) and (17), we get

$$\overline{B}^{\Delta}(n,k) = \frac{k}{n} R^{\Delta}(2n-k, n) = \frac{k}{n} A_{2n-k-n}(-nx, -n\alpha) = \frac{k}{n} A_{n-k}(-nx, -n\alpha). \tag{18}$$

For $k = 1$, we get Eq. (10).

Applying Eq. (7) for the composition $C(t) = B\left(\overline{B}(t)\right) = t$, we get

$$\begin{aligned} C^{\Delta}(n,k) = \sum_{m=k}^{n} \overline{B}^{\Delta}(n,m) B^{\Delta}(m,k) = \\ = \sum_{m=k}^{n} \frac{m}{n} A_{n-m}(-nx, -n\alpha) A_{m-k}(kx, k\alpha) = \delta_{n,k}. \end{aligned} \tag{19}$$

Applying Eq. (7) for the composition $D(t) = \overline{B}(B(t)) = x$, we get

$$\begin{aligned} D^{\Delta}(n,k) = \sum_{m=k}^{n} B^{\Delta}(n,m) \overline{B}^{\Delta}(m,k) = \\ = \sum_{m=k}^{n} A_{n-m}(mx, m\alpha) \frac{k}{m} A_{m-k}(-mx, -m\alpha) = \delta_{n,k}. \end{aligned} \tag{20}$$

□

As an application of Theorem 1, we present several examples of its usage for such polynomials as the Bernoulli, Euler, Frobenius-Euler, Sylvester, Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner.

2.1 Generalized Bernoulli polynomials

The generalized Bernoulli polynomials are defined by the following generating function [37, 38]:

$$B(t,x,\alpha) = e^{xt}\left(\frac{t}{e^t-1}\right)^{\alpha} = (e^t)^x\left(\frac{t}{e^t-1}\right)^{\alpha} = \sum_{n\geq 0} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{21}$$

where

$$B_n^{(\alpha)}(x) = \sum_{i=0}^{n}\frac{n!}{(n+i)!}\binom{n+\alpha}{n-i}\binom{i+\alpha-1}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(x+j)^{n+i}. \tag{22}$$

According to Eq. (13), the composita for the generating function $D(t) = tB(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{B_{n-k}^{(k\alpha)}(kx)}{(n-k)!}. \tag{23}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{2x-\alpha}{2} & 1 & & \\
\frac{12x^2-12\alpha x+3\alpha^2-\alpha}{24} & 2x-\alpha & 1 & \\
\frac{8x^3-12\alpha x^2+(6\alpha^2-2\alpha)x-\alpha^3+\alpha^2}{48} & \frac{24x^2-24\alpha x+6a^2-\alpha}{12} & \frac{6x-3\alpha}{2} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tB(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{B_{n-k}^{(-n\alpha)}(-nx)}{(n-k)!}. \tag{24}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{-2x+\alpha}{2} & 1 & & \\
\frac{36x^2-36\alpha x+9a^2+\alpha}{24} & -2x+\alpha & 1 & \\
\frac{-32x^3+48\alpha x^2-(24\alpha^2+2\alpha)x+4\alpha^3+\alpha^2}{12} & \frac{48x^2-48\alpha x+12\alpha^2+\alpha}{12} & \frac{-6x+3\alpha}{2} & 1
\end{array}$$

Also we can get the following new identities for the generalized Bernoulli polynomials:

$$\sum_{m=k}^{n}\frac{m}{n}\frac{B_{n-m}^{(-n\alpha)}(-nx)}{(n-m)!}\frac{B_{m-k}^{(k\alpha)}(kx)}{(m-k)!} = \delta_{n,k} \tag{25}$$

and

$$\sum_{m=k}^{n} \frac{B_{n-m}^{(m\alpha)}(mx)}{(n-m)!} \frac{k}{m} \frac{B_{m-k}^{(-m\alpha)}(-mx)}{(m-k)!} = \delta_{n,k}. \tag{26}$$

2.2 Generalized Euler polynomials

The generalized Euler polynomials are defined by the following generating function [37]:

$$E(t,x,\alpha) = e^{xt}\left(\frac{2}{e^t+1}\right)^{\alpha} = (e^t)^x\left(\frac{2}{e^t+1}\right)^{\alpha} = \sum_{n\geq 0} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \tag{27}$$

where

$$E_n^{(\alpha)}(x) = \sum_{i=0}^{n} \frac{1}{2^i}\binom{i+\alpha-1}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j}(x+j)^n. \tag{28}$$

According to Eq. (13), the composita for the generating function $D(t) = tE(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{E_{n-k}^{(k\alpha)}(kx)}{(n-k)!}. \tag{29}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{2x-\alpha}{2} & 1 & & \\
\frac{4x^2-4\alpha x+\alpha^2-\alpha}{8} & 2x-\alpha & 1 & \\
\frac{8x^3-12\alpha x^2+(6\alpha^2-6\alpha)x-\alpha^3+3\alpha^2}{48} & \frac{8x^2-8\alpha x+2a^2-\alpha}{4} & \frac{6x-3\alpha}{2} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tE(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{E_{n-k}^{(-n\alpha)}(-nx)}{(n-k)!}. \tag{30}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{-2x+\alpha}{2} & 1 & & \\
\frac{12x^2-12\alpha x+3a^2+\alpha}{8} & -2x+\alpha & 1 & \\
\frac{-32x^3+48\alpha x^2-(24\alpha^2+6\alpha)x+4\alpha^3+3\alpha^2}{12} & \frac{16x^2-16\alpha x+4\alpha^2+\alpha}{4} & \frac{-6x+3\alpha}{2} & 1
\end{array}$$

Also we can get the following new identities for the generalized Euler polynomials:

$$\sum_{m=k}^{n} \frac{m}{n}\frac{E_{n-m}^{(-n\alpha)}(-nx)}{(n-m)!}\frac{E_{m-k}^{(k\alpha)}(kx)}{(m-k)!} = \delta_{n,k} \tag{31}$$

and

$$\sum_{m=k}^{n} \frac{E_{n-m}^{(m\alpha)}(mx)}{(n-m)!} \frac{k}{m} \frac{E_{m-k}^{(-m\alpha)}(-mx)}{(m-k)!} = \delta_{n,k}. \tag{32}$$

2.3 Frobenius-Euler polynomials

The Frobenius-Euler polynomials are defined by the following generating function [39]:

$$H(t,x,\alpha,\lambda) = e^{xt}\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} = (e^t)^x\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} = \sum_{n\geq 0} H_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!}, \tag{33}$$

where

$$H_n^{(\alpha)}(x,\lambda) = \sum_{i=0}^{n} \frac{1}{(1-\lambda)^i}\binom{i+\alpha-1}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(x+j)^n. \tag{34}$$

According to Eq. (13), the composita for the generating function $D(t) = tH(t,x,\alpha,\lambda)$ is

$$D^{\Delta}(n,k) = \frac{H_{n-k}^{(k\alpha)}(kx,\lambda)}{(n-k)!}. \tag{35}$$

The triangular form of this composita is

$$\begin{array}{ccc}
1 & & \\
\dfrac{(\lambda-1)x+\alpha}{\lambda-1} & 1 & \\
\dfrac{(\lambda^2-2\lambda+1)x^2+(2\lambda-2)\alpha x+\alpha^2+\lambda\alpha}{2\lambda^2-4\lambda+2} & \dfrac{(2\lambda-2)x+2\alpha}{\lambda-1} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tH(t,x,\alpha,\lambda)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{H_{n-k}^{(-n\alpha)}(-nx,\lambda)}{(n-k)!}. \tag{36}$$

The triangular form of this composita is

$$\begin{array}{ccc}
1 & & \\
-\dfrac{(\lambda-1)x+\alpha}{\lambda-1} & 1 & \\
\dfrac{(3\lambda^2-6\lambda+3)x^2+(6\lambda-6)\alpha x+3\alpha^2-\lambda\alpha}{2\lambda^2-4\lambda+2} & -\dfrac{(2\lambda-2)x+2\alpha}{\lambda-1} & 1
\end{array}$$

Also we can get the following new identities for the Frobenius-Euler polynomials:

$$\sum_{m=k}^{n} \frac{m}{n}\frac{H_{n-m}^{(-n\alpha)}(-nx,\lambda)}{(n-m)!}\frac{H_{m-k}^{(k\alpha)}(kx,\lambda)}{(m-k)!} = \delta_{n,k} \tag{37}$$

and

$$\sum_{m=k}^{n} \frac{H_{n-m}^{(m\alpha)}(mx,\lambda)}{(n-m)!} \frac{k}{m} \frac{H_{m-k}^{(-m\alpha)}(-mx,\lambda)}{(m-k)!} = \delta_{n,k}. \tag{38}$$

2.4 Generalized Sylvester polynomials

The generalized Sylvester polynomials are defined by the following generating function [40]:

$$F(t,x,\alpha) = (1-t)^{-x} e^{\alpha x t} = \left(\frac{e^{\alpha t}}{1-t}\right)^{x} = \sum_{n \geq 0} F_n(x,\alpha) t^n, \tag{39}$$

where

$$F_n(x,\alpha) = \sum_{i=0}^{n} \frac{(\alpha x)^{n-i}}{(n-i)!} \binom{i+x-1}{i}. \tag{40}$$

According to Eq. (13), the composita for the generating function $D(t) = tF(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = F_{n-k}(kx,\alpha). \tag{41}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
(\alpha+1)x & 1 & & \\
\dfrac{(\alpha^2+2\alpha+1)x^2+x}{2} & (2\alpha+2)x & 1 & \\
\dfrac{(\alpha^3+3\alpha^2+3\alpha+1)x^3+(3\alpha+3)x^2+2x}{6} & (2\alpha^2+4\alpha+2)x^2+x & (3\alpha+3)x & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tF(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n} F_{n-k}(-nx,\alpha). \tag{42}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
-(\alpha+1)x & 1 & & \\
\dfrac{(3\alpha^2+6\alpha+3)x^2-x}{2} & -(2\alpha+2)x & 1 & \\
-\dfrac{(8\alpha^3+24\alpha^2+24\alpha+8)x^3-(6\alpha+6)x^2+x}{3} & (4\alpha^2+8\alpha+4)x^2-x & -(3\alpha+3)x & 1
\end{array}$$

Also we can get the following new identities for the generalized Sylvester polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} F_{n-m}(-nx,\alpha) F_{m-k}(kx,\alpha) = \delta_{n,k} \tag{43}$$

and

$$\sum_{m=k}^{n} F_{n-m}(mx,\alpha)\frac{k}{m}F_{m-k}(-mx,\alpha)=\delta_{n,k}. \tag{44}$$

2.5 Generalized Laguerre polynomials

The generalized Laguerre polynomials are defined by the following generating function [8]:

$$L(t,x,\alpha)=(1-t)^{-\alpha-1}e^{\frac{xt}{t-1}}=\left(e^{\frac{t}{t-1}}\right)^{x}\left(\frac{1}{1-t}\right)^{\alpha+1}=\sum_{n\geq 0}L_n^{(\alpha)}(x)t^n, \tag{45}$$

where

$$L_n^{(\alpha)}(x)=\sum_{i=0}^{n}\frac{(-x)^i}{i!}\binom{n+\alpha}{n-i}. \tag{46}$$

According to Eq. (13), the composita for the generating function $D(t)=tL(t,x,\alpha)$ is

$$D^{\Delta}(n,k)=L_{n-k}^{(k\alpha+k-1)}(kx). \tag{47}$$

The triangular form of this composita is

$$\begin{array}{ccc}
1 & & \\
-x+\alpha+1 & 1 & \\
\dfrac{x^2-(2\alpha+4)x+\alpha^2+3\alpha+2}{2} & -2x+2\alpha+2 & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t)=tL(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k)=\frac{k}{n}L_{n-k}^{(-n\alpha-n-1)}(-nx). \tag{48}$$

The triangular form of this composita is

$$\begin{array}{ccc}
1 & & \\
x-\alpha-1 & 1 & \\
\dfrac{3x^2-(6\alpha+4)x+3\alpha^2+5\alpha+2}{2} & 2x-2\alpha-2 & 1
\end{array}$$

Also we can get the following new identities for the generalized Laguerre polynomials:

$$\sum_{m=k}^{n}\frac{m}{n}L_{n-m}^{(-n\alpha-n-1)}(-nx)L_{m-k}^{(k\alpha+k-1)}(kx)=\delta_{n,k} \tag{49}$$

and

$$\sum_{m=k}^{n}L_{n-m}^{(m\alpha+m-1)}(mx)\frac{k}{m}L_{m-k}^{(-m\alpha-m-1)}(-mx)=\delta_{n,k}. \tag{50}$$

2.6 Abel polynomials

The Abel polynomials are defined by the following generating function [8, 41]:

$$A(t,x,\alpha) = e^{\frac{W(\alpha t)x}{\alpha}} = \left(e^{\frac{W(\alpha t)}{\alpha}}\right)^x = \sum_{n\geq 0} A_n(x,\alpha)\frac{t^n}{n!}, \tag{51}$$

where $W(t)$ is the Lambert W function and

$$A_n(x,\alpha) = x(x-\alpha n)^{n-1}. \tag{52}$$

According to Eq. (13), the composita for the generating function $D(t) = tA(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{A_{n-k}(kx,\alpha)}{(n-k)!}. \tag{53}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
x & 1 & & & \\
\frac{x^2-2\alpha x}{2} & 2x & 1 & & \\
\frac{x^3-6\alpha x^2+9\alpha^2 x}{6} & 2x^2-2\alpha x & 3x & 1 & \\
\frac{x^4-12\alpha x^3+48\alpha^2x^2-64\alpha^3x}{24} & \frac{4x^3-12\alpha x^2+9\alpha^2x}{3} & \frac{9x^2-6\alpha x}{2} & 4x & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tA(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{A_{n-k}(-nx,\alpha)}{(n-k)!}. \tag{54}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
-x & 1 & & & \\
\frac{3x^2+2\alpha x}{2} & -2x & 1 & & \\
-\frac{16x^3+24\alpha x^2+9\alpha^2 x}{6} & 4x^2+2\alpha x & -3x & 1 & \\
\frac{125x^4+300\alpha x^3+240\alpha^2x^2+64\alpha^3x}{24} & -\frac{25x^3+30\alpha x^2+9\alpha^2x}{3} & \frac{15x^2+6\alpha x}{2} & -4x & 1
\end{array}$$

Also we can get the following new identities for the Abel polynomials:

$$\sum_{m=k}^{n} \frac{m}{n}\frac{A_{n-m}(-nx,\alpha)}{(n-m)!}\frac{A_{m-k}(kx,\alpha)}{(m-k)!} = \delta_{n,k} \tag{55}$$

and

$$\sum_{m=k}^{n} \frac{A_{n-m}(mx,\alpha)}{(n-m)!} \frac{k}{m} \frac{A_{m-k}(-mx,\alpha)}{(m-k)!} = \delta_{n,k}. \tag{56}$$

2.7 Bessel polynomials

The Bessel polynomials are defined by the following generating function [8]:

$$B(t,x) = e^{x\left(1-\sqrt{1-2t}\right)} = \left(e^{1-\sqrt{1-2t}}\right)^x = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}, \tag{57}$$

where

$$B_n(x) = \begin{cases} 1, & n=0; \\ \sum_{k=1}^{n} \frac{(2n-k-1)!}{(n-k)!(k-1)!}\frac{x^k}{2^{n-k}}, & n>0. \end{cases} \tag{58}$$

According to Eq. (13), the composita for the generating function $D(t) = tB(t,x)$ is

$$D^{\Delta}(n,k) = \frac{B_{n-k}(kx)}{(n-k)!}. \tag{59}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
2 & 1 & & & \\
\frac{x^2+x}{2} & 2x & 1 & & \\
\frac{x^3+3x^2+3x}{6} & 2x^2+x & 3x & 1 & \\
\frac{x^4+6x^3+15x^2+15x}{24} & \frac{4x^3+6x^2+3x}{3} & \frac{9x^2+3x}{2} & 4x & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tB(t,x)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{B_{n-k}(-nx)}{(n-k)!}. \tag{60}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
-x & 1 & & & \\
\frac{3x^2-x}{2} & -2x & 1 & & \\
-\frac{16x^3-12x^2+3x}{6} & 4x^2-x & -3x & 1 & \\
\frac{125x^4-150x^3+75x^2-15x}{24} & -\frac{25x^3-15x^2+3x}{3} & \frac{15x^2-3x}{2} & -4x & 1
\end{array}$$

Also we can get the following new identities for the Bessel polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} \frac{B_{n-m}(-nx)}{(n-m)!} \frac{B_{m-k}(kx)}{(m-k)!} = \delta_{n,k} \tag{61}$$

and

$$\sum_{m=k}^{n} \frac{B_{n-m}(mx)}{(n-m)!} \frac{k}{m} \frac{B_{m-k}(-mx)}{(m-k)!} = \delta_{n,k}. \tag{62}$$

2.8 Stirling polynomials

The Stirling polynomials are defined by the following generating function [8, 42]:

$$S(t,x) = \left(\frac{t}{1-e^{-t}}\right)^{x} = \sum_{n\geq 0} S_n(x) \frac{t^n}{n!}, \tag{63}$$

where

$$S_n(x) = \sum_{i=0}^{n} \binom{x+i}{i} \sum_{j=0}^{i} \frac{j!}{(n+j)!} (-1)^{n+j} \binom{i}{j} \left\{ {n+j \atop j} \right\}. \tag{64}$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t,x)$ is

$$D^{\Delta}(n,k) = S_{n-k}(kx+k-1). \tag{65}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
\frac{x+1}{2} & 1 & & & \\
\frac{3x^2+5x+2}{24} & x+1 & 1 & & \\
\frac{x^3+2x^2+x}{48} & \frac{6x^2+11x+5}{12} & \frac{3x+3}{2} & 1 & \\
\frac{15x^4+30x^3+5x^2-18x-8}{5760} & \frac{2x^3+5x^2+4x+1}{12} & \frac{9x^2+17x+8}{8} & 2x+2 & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t,x)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n} S_{n-k}(-nx-n-1). \tag{66}$$

The triangular form of this composita is

$$\begin{array}{ccccc}
1 & & & & \\
-\frac{x+2}{2} & 1 & & & \\
\frac{9x^2+19x+10}{24} & -x-1 & 1 & & \\
-\frac{4x^3+13x^2+14x+5}{12} & \frac{12x^2+25x+13}{12} & -\frac{3x+3}{2} & 1 & \\
\frac{1875x^4+8250x^3+13525x^2+9798x+2648}{5760} & -\frac{25x^3+80x^2+85x+30}{24} & \frac{15x^2+31x+16}{8} & -2x-2 & 1
\end{array}$$

Also we can get the following new identities for the Stirling polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} S_{n-m}(-nx-n-1) S_{m-k}(kx+k-1) = \delta_{n,k} \tag{67}$$

and

$$\sum_{m=k}^{n} S_{n-m}(mx+m-1) \frac{k}{m} S_{m-k}(-mx-m-1) = \delta_{n,k}. \tag{68}$$

2.9 Narumi polynomials

The Narumi polynomials are defined by the following generating function [8]:

$$S(t,x,\alpha) = \left(\frac{t}{\ln(1+t)}\right)^{\alpha} (1+t)^x = (1+t)^x \left(\frac{t}{\ln(1+t)}\right)^{\alpha} = \sum_{n \geq 0} S_n(x,\alpha) \frac{t^n}{n!}, \tag{69}$$

where

$$S_n(x,\alpha) = n! \sum_{i=0}^{n} \binom{x}{n-i} \sum_{j=0}^{i} \binom{j+\alpha-1}{j} \sum_{l=0}^{j} (-1)^l \binom{j}{l} \frac{l!}{(l+i)!} \begin{bmatrix} l+i \\ l \end{bmatrix}. \tag{70}$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = \frac{S_{n-k}(kx,k\alpha)}{(n-k)!}. \tag{71}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
\frac{2x+\alpha}{2} & 1 & & \\
\frac{12x^2+(12\alpha-12)x+3\alpha^2-5\alpha}{24} & 2x+\alpha & 1 & \\
\frac{8x^3+(12\alpha-24)x^2+(6\alpha^2-22\alpha+16)x+\alpha^3-5\alpha^2+6\alpha}{48} & \frac{24x^2+(24\alpha-12)x+6\alpha^2-5\alpha}{12} & \frac{6x+3\alpha}{2} & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n} \frac{S_{n-k}(-nx,-n\alpha)}{(n-k)!}. \tag{72}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
-\frac{2x+\alpha}{2} & 1 & & \\
\frac{36x^2+(36\alpha+12)x+9\alpha^2-5\alpha}{24} & -2x-\alpha & 1 & \\
-\frac{64x^3(96\alpha+48)x^2+(48\alpha^2+44\alpha+8)x+8\alpha^3+10\alpha^2+3\alpha}{24} & \frac{48x^2+(48\alpha+12)x+12\alpha^2+5\alpha}{12} & -\frac{6x+3\alpha}{2} & 1
\end{array}$$

Also we can get the following new identities for the Narumi polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} \frac{S_{n-m}(-nx,-n\alpha)}{(n-m)!} \frac{S_{m-k}(kx,k\alpha)}{(m-k)!} = \delta_{n,k} \tag{73}$$

and

$$\sum_{m=k}^{n} \frac{S_{n-m}(mx, m\alpha)}{(n-m)!} \frac{k}{m} \frac{S_{m-k}(-mx, -m\alpha)}{(m-k)!} = \delta_{n,k}. \tag{74}$$

2.10 Peters polynomials

The Peters polynomials are defined by the following generating function [8]:

$$S(t, x, \mu, \lambda) = \left(1 + (1+t)^{\lambda}\right)^{-\mu} (1+t)^{x} = (1+t)^{x} \left(\frac{1}{1+(1+t)^{\lambda}}\right)^{\mu} = \sum_{n \geq 0} S_n(x, \mu, \lambda) \frac{t^n}{n!}, \tag{75}$$

where

$$S_n(x, \mu, \lambda) = n! \sum_{i=0}^{n} \binom{x}{n-i} \sum_{j=0}^{i} \frac{1}{2^{j+\mu}} \binom{j+\mu-1}{j} \sum_{l=0}^{j} (-1)^{l} \binom{j}{l} \binom{l\lambda}{i}. \tag{76}$$

According to Eq. (13), the composita for the generating function $D(t) = tS(t, x, \mu, \lambda)$ is

$$D^{\Delta}(n, k) = \frac{S_{n-k}(kx, k\mu, \lambda)}{(n-k)!}. \tag{77}$$

The triangular form of this composita is

$$\begin{array}{ccc} 2^{-\mu} & & \\ 2^{-\mu-1}(2x-\lambda\mu) & 2^{-2\mu} & \\ 2^{-\mu-3}\left(4x^2-(4\lambda\mu+4)x+\lambda^2\mu^2-\lambda^2\mu+2\lambda\mu\right) & 2^{-2\mu}(2x-\lambda\mu) & 2^{-3\mu} \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tS(t, x, \mu, \lambda)$ is

$$\overline{D}^{\Delta}(n, k) = \frac{k}{n} \frac{S_{n-k}(-nx, -n\mu, \lambda)}{(n-k)!}. \tag{78}$$

The triangular form of this composita is

$$\begin{array}{ccc} 2^{\mu} & & \\ 2^{2\mu-1}(-2x+\lambda\mu) & 2^{2\mu} & \\ 2^{3\mu-3}\left(12x^2+(4-12\lambda\mu)x+3\lambda^2\mu^2+\lambda^2\mu-2\lambda\mu\right) & 2^{3\mu}(\lambda\mu-2x) & 2^{3\mu} \end{array}$$

Also we can get the following new identities for the Peters polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} \frac{S_{n-m}(-nx, -n\mu, \lambda)}{(n-m)!} \frac{S_{m-k}(kx, k\mu, \lambda)}{(m-k)!} = \delta_{n,k} \tag{79}$$

and

$$\sum_{m=k}^{n} \frac{S_{n-m}(mx, m\mu, \lambda)}{(n-m)!} \frac{k}{m} \frac{S_{m-k}(-mx, -m\mu, \lambda)}{(m-k)!} = \delta_{n,k}. \tag{80}$$

2.11 Gegenbauer polynomials

The Gegenbauer polynomials are defined by the following generating function [43]:

$$C(t,x,\alpha) = \left(1 - 2xt + t^2\right)^{-\alpha} = \left(\frac{1}{1 - 2xt + t^2}\right)^{\alpha} = \sum_{n \geq 0} C_n^{(\alpha)}(x)t^n, \tag{81}$$

where

$$C_n^{(\alpha)}(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{i}{n-i} \binom{i+\alpha-1}{i} (2x)^{2i-n}. \tag{82}$$

According to Eq. (13), the composita for the generating function $D(t) = tC(t,x,\alpha)$ is

$$D^{\Delta}(n,k) = C_{n-k}^{(k\alpha)}(x). \tag{83}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
2\alpha x & 1 & & \\
(2\alpha^2 + 2\alpha)x^2 - \alpha & 4\alpha x & 1 & \\
\dfrac{(4\alpha^3 + 12\alpha^2 + 8\alpha)x^3 - (6\alpha^2 + 6\alpha)x}{3} & (8\alpha^2 + 4\alpha)x^2 - 2\alpha & 6\alpha x & 1
\end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tC(t,x,\alpha)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n} C_{n-k}^{(-n\alpha)}(x). \tag{84}$$

The triangular form of this composita is

$$\begin{array}{cccc}
1 & & & \\
-2\alpha x & 1 & & \\
(6\alpha^2 - 2\alpha)x^2 + \alpha & -4\alpha x & 1 & \\
\dfrac{(64\alpha^3 - 48\alpha^2 + 8\alpha)x^3 + (24\alpha^2 - 6\alpha)x}{3} & (16\alpha^2 - 4\alpha)x^2 + 2\alpha & -6\alpha x & 1
\end{array}$$

Also we can get the following new identities for the Gegenbauer polynomials:

$$\sum_{m=k}^{n} \frac{m}{n} C_{n-m}^{(-n\alpha)}(x) C_{m-k}^{(k\alpha)}(x) = \delta_{n,k} \tag{85}$$

and

$$\sum_{m=k}^{n} C_{n-m}^{(m\alpha)}(x) \frac{k}{m} C_{m-k}^{(-m\alpha)}(x) = \delta_{n,k}. \tag{86}$$

2.12 Meixner polynomials of the first kind

The Meixner polynomials of the first kind are defined by the following generating function [8, 44]:

$$M(t,x,\beta,c) = \left(1-\frac{t}{c}\right)^{x}(1-t)^{-x-\beta} = \left(\frac{c-t}{c(1-t)}\right)^{x}\left(\frac{1}{1-t}\right)^{\beta} = \sum_{n\geq 0} M_n(x,\beta,c)\frac{t^n}{n!}, \tag{87}$$

where

$$M_n(x,\beta,c) = (-1)^n n! \sum_{i=0}^{n} \binom{x}{i}\binom{-x-\beta}{n-i}c^{-i}. \tag{88}$$

According to Eq. (13), the composita for the generating function $D(t) = tM(t,x,\beta,c)$ is

$$D^{\Delta}(n,k) = \frac{M_{n-k}(kx,k\beta,c)}{(n-k)!}. \tag{89}$$

The triangular form of this composita is

$$\begin{array}{ccc} 1 & & \\ \dfrac{(c-1)x+\beta c}{c} & 1 & \\ \dfrac{(c^2-2c+1)x^2+((2\beta+1)c^2-2\beta c-1)x+\left(\beta^2+\beta\right)c^2}{2c^2} & \dfrac{(2c-2)x+2\beta c}{c} & 1 \end{array}$$

Using Eq. (17), the composita for the compositional inverse generating function $\overline{D}(t)$ of $D(t) = tM(t,x,\beta,c)$ is

$$\overline{D}^{\Delta}(n,k) = \frac{k}{n}\frac{M_{n-k}(-nx,-n\beta,c)}{(n-k)!}. \tag{90}$$

The triangular form of this composita is

$$\begin{array}{ccc} 1 & & \\ \dfrac{(1-c)x-\beta c}{c} & 1 & \\ \dfrac{(3c^2-6c+3)x^2+((6\beta-1)c^2-6\beta c+1)x+\left(3\beta^2-\beta\right)c^2}{2c^2} & \dfrac{(2-2c)x-2\beta c}{c} & 1 \end{array}$$

Also we can get the following new identities for the Meixner polynomials of the first kind:

$$\sum_{m=k}^{n} \frac{m}{n}\frac{M_{n-m}(-nx,-n\beta,c)}{(n-m)!}\frac{M_{m-k}(kx,k\beta,c)}{(m-k)!} = \delta_{n,k} \tag{91}$$

and

$$\sum_{m=k}^{n} \frac{M_{n-m}(mx,m\beta,c)}{(n-m)!}\frac{k}{m}\frac{M_{m-k}(-mx,-m\beta,c)}{(m-k)!} = \delta_{n,k}. \tag{92}$$

3. Conclusions and future developments

In this chapter, we find new explicit formulas and identities for such polynomials as the generalized Bernoulli, generalized Euler, Frobenius-Euler, generalized

Sylvester, generalized Laguerre, Abel, Bessel, Stirling, Narumi, Peters, Gegenbauer, and Meixner polynomials that are defined by generating functions of the form $A(t, x, \alpha) = F(t)^x \cdot G(t)^\alpha$.

A lot of studies have recently showed that polynomials are a solution for practical problems related to modeling, quantum mechanics, and other areas. So a study of obtaining explicit formulas and representations of polynomials will be important and influential. Also the further research can be conducted to find practical means of obtained properties.

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Chapter 4

Polynomials with Symmetric Zeros

Ricardo Vieira

Abstract

Polynomials whose zeros are symmetric either to the real line or to the unit circle are very important in mathematics and physics. We can classify them into three main classes: the self-conjugate polynomials, whose zeros are symmetric to the real line; the self-inversive polynomials, whose zeros are symmetric to the unit circle; and the self-reciprocal polynomials, whose zeros are symmetric by an inversion with respect to the unit circle followed by a reflection in the real line. Real self-reciprocal polynomials are simultaneously self-conjugate and self-inversive so that their zeros are symmetric to both the real line and the unit circle. In this survey, we present a short review of these polynomials, focusing on the distribution of their zeros.

Keywords: self-inversive polynomials, self-reciprocal polynomials, Pisot and Salem polynomials, Möbius transformations, knot theory, Bethe equations

1. Introduction

In this work, we consider the theory of self-conjugate (SC), self-reciprocal (SR), and self-inversive (SI) polynomials. These are polynomials whose zeros are symmetric either to the real line $\mathbb{R}$ or to the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. The basic properties of these polynomials can be found in the books of Marden [1], Milovanović et al. [2], and Sheil-Small [3]. Although these polynomials are very important in both mathematics and physics, it seems that there is no specific review about them; in this work, we present a bird's eye view to this theory, focusing on the zeros of such polynomials. Other properties of these polynomials (e.g., irreducibility, norms, analytical properties, etc.) are not covered here due to short space, nonetheless, the interested reader can check many of the references presented in the bibliography to this end.

2. Self-conjugate, self-reciprocal, and self-inversive polynomials

We begin with some definitions:

Definition 1. Let $p(z) = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1} + p_n z^n$ be a polynomial of degree n with complex coefficients. We shall introduce three polynomials, namely the *conjugate polynomial* $\overline{p}(z)$, the *reciprocal polynomial* $p^*(z)$, and the *inversive polynomial* $p^\dagger(z)$, which are, respectively, defined in terms of $p(z)$ as follows:

$$\begin{aligned} \overline{p}(z) &= \overline{p_0} + \overline{p_1} z + \cdots + \overline{p_{n-1}} z^{n-1} + \overline{p_n} z^n, \\ p^*(z) &= p_n + p_{n-1} z + \cdots + p_1 z^{n-1} + p_0 z^n, \\ p^\dagger(z) &= \overline{p_n} + \overline{p_{n-1}} z + \cdots + \overline{p_1} z^{n-1} + \overline{p_0} z^n, \end{aligned} \tag{1}$$

where the bar means complex conjugation. Notice that the conjugate, reciprocal, and inversive polynomials can also be defined without making reference to the coefficients of $p(z)$:

$$\overline{p}(z) = \overline{p(\overline{z})}, \qquad p^*(z) = z^n p(1/z), \qquad p^\dagger(z) = z^n \overline{p(1/\overline{z})}. \tag{2}$$

From these relations, we plainly see that if $\zeta_1, \ldots, \zeta_n$ are the zeros of a complex polynomial $p(z)$ of degree n, then, the zeros of $\overline{p}(z)$ are $\overline{\zeta_1}, \ldots, \overline{\zeta_n}$, the zeros of $p^*(z)$ are $1/\zeta_1, \ldots, 1/\zeta_n$, and finally, the zeros of $p^\dagger(z)$ are $1/\overline{\zeta_1}, \ldots, 1/\overline{\zeta_n}$. Thus, if $p(z)$ has k zeros on $\mathbb{R}$, l zeros on the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, and m zeros in the lower half-plane $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$ so that $k + l + m = n$, then $\overline{p}(z)$ will have the same number k of zeros on $\mathbb{R}$, l zeros in $\mathbb{C}^-$ and m zeros in $\mathbb{C}^+$. Similarly, if $p(z)$ has k zeros on $\mathbb{S}$, l zeros inside $\mathbb{S}$ and m zeros outside $\mathbb{S}$, so that $k + l + m = n$, then both $p^*(z)$ as $p^\dagger(z)$ will have the same number k of zeros on $\mathbb{S}$, l zeros outside $\mathbb{S}$ and m zeros inside $\mathbb{S}$.

These properties encourage us to introduce the following classes of polynomials:

Definition 2. A complex polynomial $p(z)$ is called[1] *self-conjugate* (SC), *self-reciprocal* (SR), or *self-inversive* (SI) if, for any zero ζ of $p(z)$, the complex-conjugate $\overline{\zeta}$, the reciprocal $1/\zeta$, or the reciprocal of the complex-conjugate $1/\overline{\zeta}$ is also a zero of $p(z)$, respectively.

Thus, the zeros of any SC polynomial are all symmetric to the real line $\mathbb{R}$, while the zeros of the any SI polynomial are symmetric to the unit circle $\mathbb{S}$. The zeros of any SR polynomial are obtained by an inversion with respect to the unit circle followed by a reflection in the real line. From this, we can establish the following:

Theorem 1. *If $p(z)$ is an SC polynomial of odd degree, then it necessarily has at least one zero on $\mathbb{R}$. Similarly, if $p(z)$ is an SR or SI polynomial of odd degree, then it necessarily has at least one zero on $\mathbb{S}$.*

Proof. From Definition 2 it follows that the number of non-real zeros of an SC polynomial $p(z)$ can only occur in (conjugate) pairs; thus, if $p(z)$ has odd degree, then at least one zero of it must be real. Similarly, the zeros of $p^\dagger(z)$ or $p^*(z)$ that have modulus different from 1 can only occur in (inversive or reciprocal) pairs as well; thus, if $p(z)$ has odd degree then at least one zero of it must lie on $\mathbb{S}$. $\square$

Theorem 2. *The necessary and sufficient condition for a complex polynomial $p(z)$ to be SC, SR, or SI is that there exists a complex number ω of modulus 1 so that one of the following relations, respectively, holds:*

$$p(z) = \omega\overline{p}(z), \qquad p(z) = \omega p^*(z), \qquad p(z) = \omega p^\dagger(z). \tag{3}$$

Proof. It is clear in view of (1) and (2) that these conditions are sufficient. We need to show, therefore, that these conditions are also necessary. Let us suppose first that $p(z)$ is SC. Then, for any zero ζ of $p(z)$ the complex-conjugate number $\overline{\zeta}$ is also a zero of it. Thus, we can write

[1] The reader should be aware that there is no standard in naming these polynomials. For instance, what we call here self-inversive polynomials are sometimes called self-reciprocal polynomials. What we mean positive self-reciprocal polynomials are usually just called self-reciprocal or yet palindrome polynomials (because their coefficients are the same whether they are read from forwards or backwards), as well as, negative self-reciprocal polynomials are usually called skew-reciprocal, anti-reciprocal, or yet anti-palindrome polynomials.

$$p(z) = p_n \prod_{k=1}^{n} \left(z - \overline{\zeta_k}\right) = p_n \prod_{k=1}^{n} \overline{\left(\overline{z} - \zeta_k\right)} = \left(p_n/\overline{p_n}\right)\overline{p(\overline{z})} = \omega \overline{p}(z), \tag{4}$$

with $\omega = p_n/\overline{p_n}$ so that $|\omega| = \left|p_n/\overline{p_n}\right| = 1$. Now, let us suppose that $p(z)$ is SR. Then, for any zero ζ of $p(z)$, the reciprocal number $1/\zeta$ is also a zero of it; thus,

$$p(z) = p_n \prod_{k=1}^{n} \left(z - \frac{1}{\zeta_k}\right) = \frac{(-1)^n z^n p_n}{\zeta_1 \cdots \zeta_n} \prod_{k=1}^{n} \left(\frac{1}{z} - \zeta_k\right) = \frac{(-1)^n z^n}{\zeta_1 \cdots \zeta_n} p\left(\frac{1}{z}\right) = \omega p^*(z), \tag{5}$$

with $\omega = (-1)^n/(\zeta_1 \ldots \zeta_n) = p_n/p_0$; now, for any zero ζ of $p(z)$ (which is necessarily different from zero if $p(z)$ is SR), there will be another zero whose value is $1/\zeta$ so that $|\zeta_1 \ldots \zeta_n| = 1$, which implies $|\omega| = 1$. The proof for SI polynomials is analogous and will be concealed; it follows that $\omega = p_n/\overline{p_0}$ in this case. □

Now from (1), (2) and (3), we can conclude that the coefficients of an SC, an SR, and an SI polynomial of degree n satisfy, respectively, the following relations:

$$p_k = \omega \overline{p_k}, \qquad p_k = \omega p_{n-k}, \qquad p_k = \omega \overline{p_{n-k}}, \qquad |\omega| = 1, \qquad 0 \leqslant k \leqslant n. \tag{6}$$

We highlight that any real polynomial is SC—in fact, many theorems which are valid for real polynomials are also valid for, or can be easily extended to, SC polynomials.

There also exist polynomials whose zeros are symmetric with respect to *both* the real line $\mathbb{R}$ and the unit circle $\mathbb{S}$. A polynomial $p(z)$ with this double symmetry is, at the same time, SC and SI (and, hence, SR as well). This is only possible if all the coefficients of $p(z)$ are real, which implies that $\omega = \pm 1$. This suggests the following additional definitions:

Definition 3. A real self-reciprocal polynomial $p(z)$ that satisfies the relation $p(z) = \omega z^n p(1/z)$ will be called a *positive self-reciprocal* (PSR) polynomial if $\omega = 1$ and a *negative self-reciprocal* (NSR) polynomial if $\omega = -1$.

Thus, the coefficients of any PSR polynomial $p(z) = p_0 + \cdots + p_n z^n$ of degree n satisfy the relations $p_k = p_{n-k}$ for $0 \leqslant k \leqslant n$, while the coefficients of any NSR polynomial $p(z)$ of degree n satisfy the relations $p_k = -p_{n-k}$ for $0 \leqslant k \leqslant n$; this last condition implies that the middle coefficient of an NSR polynomial of even degree is always zero.

Some elementary properties of PSR and NSR polynomials are the following: first, notice that, if ζ is a zero of any PSR or NSR polynomial $p(z)$ of degree $n \geqslant 4$, then the three complex numbers $1/\zeta$, $\overline{\zeta}$ and $1/\overline{\zeta}$ are also zeros of $p(z)$. In particular, the number of zeros of such polynomials which are neither in $\mathbb{S}$ or in $\mathbb{R}$ is always a multiple of 4. Besides, any NSR polynomial has $z = 1$ as a zero and $p(z)/(z-1)$ is PSR; further, if $p(z)$ has even degree then $z = -1$ is also a zero of it and $p(z)/(z^2 - 1)$ is a PSR polynomial of even degree. In a similar way, any PSR polynomial $p(z)$ of odd degree has $z = -1$ as a zero and $p(z)/(z+1)$ is also PSR. The product of two PSR, or two NSR, polynomials is PSR, while the product of a PSR polynomial with an NSR polynomial is NSR. These statements follow directly from the definitions of such polynomials.

We also mention that any PSR polynomial of even degree (say, $n = 2m$) can be written in the following form:

$$p(z) = z^m \left[p_0\left(z^m + \frac{1}{z^m}\right) + p_1\left(z^{m-1} + \frac{1}{z^{m-1}}\right) + \cdots + p_{m-1}\left(z + \frac{1}{z}\right)\right] + p_m, \tag{7}$$

an expression that is obtained by using the relations $p_k = p_{2m-k}$, $0 \leqslant k \leqslant 2m$, and gathering the terms of $p(z)$ with the same coefficients. Furthermore, the expression $Z_s(z) = (z^s + z^{-s})$ for any integer s can be written as a polynomial of degree s in the new variable $x = z + 1/z$ (the proof follows easily by induction over s); thus, we can write $p(z) = z^m q(x)$, where $q(x) = q_0 + \cdots + q_m x^m$ is such that the coefficients $q_0, \ldots, q_m$ are certain functions of $p_0, \ldots, p_m$. From this we can state the following:

Theorem 3. *Let $p(z)$ be a PSR polynomial of even degree $n = 2m$. For each pair ζ and $1/\zeta$ of self-reciprocal zeros of $p(z)$ that lie on $\mathbb{S}$, there is a corresponding zero ξ of the polynomial $q(x)$, as defined above, in the interval $[-2, 2]$ of the real line.*

Proof. For each zero ζ of $p(x)$ that lie on S, write $\zeta = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Thereby, as $q(x) = q(z + 1/z) = p(z)/z^m$, it follows that $\xi = \zeta + 1/\zeta = 2\cos\theta$ will be a zero of $q(x)$. This shows us that ξ is limited to the interval $[-2, 2]$ of the real line. Finally, notice that the reciprocal zero $1/\zeta$ of $p(z)$ is mapped to the same zero ξ of $q(x)$. □

Finally, remembering that the Chebyshev polynomials of first kind, $T_n(z)$, are defined by the formula $T_n\left[\frac{1}{2}(z + z^{-1})\right] = \frac{1}{2}(z^n + z^{-n})$ for $z \in \mathbb{C}$, it follows as well that $q(x)$, and hence any PSR polynomial, can be written as a linear combination of Chebyshev polynomials:

$$q(x) = 2\left[p_0 T_m(x) + p_1 T_{m-1}(x) + \cdots + p_{m-1} T_1(x) + \frac{1}{2} p_m T_0(x)\right]. \tag{8}$$

3. How these polynomials are related to each other?

In this section, we shall analyze how SC, SR, and SI polynomials are related to each other. Let us begin with the relationship between the SR and SI polynomials, which is actually very simple: indeed, from (1), (2), and (3) we can see that each one is nothing but the conjugate polynomial of the other, that is

$$p^{\dagger}(z) = \overline{p^{*}}(z) = \overline{p^{*}(\overline{z})}, \qquad \text{and} \qquad p^{*}(z) = \overline{p^{\dagger}}(z) = \overline{p^{\dagger}(\overline{z})}. \tag{9}$$

Thus, if $p(z)$ is an SR (SI) polynomial, then $\overline{p}(z)$ will be SI (SR) polynomial. Because of this simple relationship, several theorems which are valid for SI polynomials are also valid for SR polynomials and vice versa.

The relationship between SC and SI polynomials is not so easy to perceive. A way of revealing their connection is to make use of a suitable pair of *Möbius transformations*, that maps the unit circle onto the real line and vice versa, which is often called *Cayley transformations*, defined through the formulas:

$$M(z) = (z - i)/(z + i), \qquad \text{and} \qquad W(z) = -i(z + 1)/(z - 1). \tag{10}$$

This approach was developed in [4], where some algorithms for counting the number of zeros that a complex polynomial has on the unit circle were also formulated.

It is an easy matter to verify that $M(z)$ maps $\mathbb{R}$ onto $\mathbb{S}$ while $W(z)$ maps $\mathbb{S}$ onto $\mathbb{R}$. Besides, $M(z)$ maps the upper (lower) half-plane to the interior (exterior) of $\mathbb{S}$, while $W(z)$ maps the interior (exterior) of $\mathbb{S}$ onto the upper (lower) half-plane. Notice that $W(z)$ can be thought as the inverse of $M(z)$ in the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, if we further assume that $M(-i) = \infty$, $M(\infty) = 1$, $W(1) = \infty$, and $W(\infty) = -i$.

Given a polynomial $p(z)$ of degree n, we define two *Möbius-transformed polynomials*, namely

$$Q(z) = (z+i)^n p(M(z)), \qquad \text{and} \qquad T(z) = (z-1)^n p(W(z)). \tag{11}$$

The following theorem shows us how the zeros of $Q(z)$ and $T(z)$ are related with the zeros of $p(z)$:

Theorem 4. *Let $\zeta_1, \ldots, \zeta_n$ denote the zeros of $p(z)$ and $\eta_1, \ldots, \eta_n$ the respective zeros of $Q(z)$. Provided $p(1) \neq 0$, we have that $\eta_1 = W(\zeta_1), \ldots, \eta_n = W(\zeta_n)$. Similarly, if $\tau_1, \ldots \tau_n$ are the zeros of $T(z)$, then we have $\tau_1 = M(\zeta_1), \ldots, \tau_n = M(\zeta_n)$, provided that $p(-i) \neq 0$.*

Proof. In fact, inverting the expression for $Q(z)$ and evaluating it in any zero ζ_k of $p(z)$ we get that $p(\zeta_k) = (-i/2)^n (\zeta_k - 1)^n Q(W(\zeta_k)) = 0$ for $0 \leqslant k \leqslant n$. Provided that $z = 1$ is not a zero of $p(z)$ we get that $\eta_k = W(\zeta_k)$ is a zero of $Q(z)$. The proof for the zeros of $T(z)$ is analogous. □

This result also shows that $Q(z)$ and $T(z)$ have the same degree as $p(z)$ whenever $p(1) \neq 0$ or $p(-i) \neq 0$, respectively. In fact, if $p(z)$ has a zero at $z = 1$ of multiplicity m then $Q(z)$ will be a polynomial of degree $n - m$, the same being true for $T(z)$ if $p(z)$ has a zero of multiplicity m at $z = -i$. This can be explained by the fact that the points $z = 1$ and $z = -i$ are mapped to infinity by $W(z)$ and $M(z)$, respectively.

The following theorem shows that the set of SI polynomials are isomorphic to the set of SC polynomials:

Theorem 5. *Let $p(z)$ be an SI polynomial. Then, the transformed polynomial $Q(z) = (z+i)^n p(M(z))$ is an SC polynomial. Similarly, if $p(z)$ is an SC polynomial, then $T(z) = (z-1)^n p(W(z))$ will be an SI polynomial.*

Proof. Let ζ and $1/\overline{\zeta}$ be two inversive zeros an SI polynomial $p(z)$. Then, according to Theorem 4, the corresponding zeros of $Q(z)$ will be:

$$W(\zeta) = -i\frac{\zeta+1}{\zeta-1} = \eta \qquad \text{and} \qquad W\left(\frac{1}{\overline{\zeta}}\right) = -i\frac{1/\overline{\zeta}+1}{1/\overline{\zeta}-1} = i\frac{\overline{\zeta}+1}{\overline{\zeta}-1} = \overline{W(\zeta)} = \overline{\eta}. \tag{12}$$

Thus, any pair of zeros of $p(z)$ that are symmetric to the unit circle are mapped in zeros of $Q(z)$ that are symmetric to the real line; because $p(z)$ is SI, it follows that $Q(z)$ is SC. Conversely, let ζ and $\overline{\zeta}$ be two zeros of an SC polynomial $p(z)$; then the corresponding zeros of $T(z)$ will be:

$$M(\zeta) = \frac{\zeta - i}{\zeta + i} = \tau \qquad \text{and} \qquad M(\overline{\zeta}) = \frac{\overline{\zeta} - i}{\overline{\zeta} + i} = -\frac{1/\overline{\zeta} + i}{1/\overline{\zeta} - i} = 1\overline{M(\zeta)} = \frac{1}{\overline{\tau}}. \tag{13}$$

Thus, any pair of zeros of $p(z)$ that are symmetric to the real line are mapped in zeros of $T(z)$ that are symmetric to the unit circle. Because $p(z)$ is SC, it follows that $T(z)$ is SI. □

We can also verify that any SI polynomial with $\omega = 1$ is mapped to a real polynomial through $M(z)$ and any real polynomial is mapped to an SI polynomial with $\omega = 1$ through $W(z)$. Thus, the set of SI polynomials with $\omega = 1$ is isomorphic to the set of real polynomials. Besides, an SI polynomial with $\omega \neq 1$ can be transformed into another one with $\omega = 1$ by performing a suitable uniform rotation of its zeros. It can also be shown that the action of the Möbius transformation over a PSR polynomial leads to a real polynomial that has only even powers. See [4] for more.

4. Zeros location theorems

In this section, we shall discuss some theorems regarding the distribution of the zeros of SC, SR, and SI polynomials on the complex plane. Some general theorems relying on the number of zeros that an arbitrary complex polynomial has inside, on, or outside $\mathbb{S}$ are also discussed. To save space, we shall not present the proofs of these theorems, which can be found in the original works. Other related theorems can be found in Marden's book [1].

4.1 Polynomials that do not necessarily have symmetric zeros

The following theorems are classics (see [1] for the proofs):

Theorem 6. (Rouché). *Let $q(z)$ and $r(z)$ be polynomials such that $|q(z)| < |r(z)|$ along all points of $\mathbb{S}$. Then, the polynomial $p(z) = q(z) + r(z)$ has the same number of zeros inside $\mathbb{S}$ as the polynomial $r(z)$, counted with multiplicity.*

Thus, if a complex polynomial $p(z) = p_0 + \cdots + p_k z^k + \cdots + p_n z^n$ of degree n is such that $|p_k| > |p_0 + \cdots + p_{k-1} + p_{k+1} + \cdots + p_n|$, then $p(z)$ will have exactly k zeros inside $\mathbb{S}$, counted with multiplicity.

Theorem 7. (Gauss and Lucas) *The zeros of the derivative $p'(z)$ of a polynomial $p(z)$ lie all within the convex hull of the zeros of the $p(z)$.*

Thereby, if a polynomial $p(z)$ has all its zeros on $\mathbb{S}$, then all the zeros of $p'(z)$ will lie in or on $\mathbb{S}$. In particular, the zeros of $p'(z)$ will lie on $\mathbb{S}$ if, and only if, they are multiple zeros of $p(z)$.

Theorem 8. (Cohn) *A necessary and sufficient condition for all the zeros of a complex polynomial $p(z)$ to lie on $\mathbb{S}$ is that $p(z)$ is SI and that its derivative $p'(z)$ does not have any zero outside $\mathbb{S}$.*

Cohn introduced his theorem in [5]. Bonsall and Marden presented a simpler proof of Conh's theorem in [6] (see also [7]) and applied it to SI polynomials—in fact, this was probably the first paper to use the expression "self-inversive." Other important result of Cohn is the following: all the zeros of a complex polynomial $p(z) = p_n z^n + \cdots + p_0$ will lie on $\mathbb{S}$ if, and only if, $|p_n| = |p_0|$ and all the zeros of $p(z)$ do not lie outside $\mathbb{S}$.

Restricting ourselves to polynomials with real coefficients, Eneström and Kakeya [8–10] independently presented the following theorem:

Theorem 9. (Eneström and Kakeya) *Let $p(z)$ be a polynomial of degree n with real coefficients. If its coefficients are such that $0 < p_0 \leqslant p_1 \leqslant \cdots \leqslant p_{n-1} \leqslant p_n$, then all the zeros of $p(z)$ lie in or on $\mathbb{S}$. Likewise, if the coefficients of $p(z)$ are such that $0 < p_n \leqslant p_{n-1} \leqslant \cdots \leqslant p_1 \leqslant p_0$, then all the zeros of $p(z)$ lie on or outside $\mathbb{S}$.*

The following theorems are relatively more recent. The distribution of the zeros of a complex polynomial regarding the unit circle $\mathbb{S}$ was presented by Marden in [1] and slightly enhanced by Jury in [11]:

Theorem 10. (Marden and Jury) *Let $p(z)$ be a complex polynomial of degree n and $p^*(z)$ its reciprocal. Construct the sequence of polynomials $P_j(z) = \sum_{k=0}^{n-j} P_{j,k} z^k$ such that $P_0(z) = p(z)$ and $P_{j+1}(z) = \overline{p_{j,0}} P_j(z) - \overline{p_{j,n-j}} P_j^*(z)$ for $0 \leqslant j \leqslant n-1$ so that we have the relations $p_{j+1,k} = \overline{p_{j,0}} p_{j,k} - p_{j,n-j} \overline{p_{j,n-j-k}}$. Let δ_j denote the constant terms of the polynomials $P_j(z)$, i.e., $\delta_j = p_{j,0}$ and $\Delta_k = \delta_1 \cdots \delta_k$. Thus, if N of the products Δ_k are negative and $n - N$ of the products Δ_k are positive so that none of them are zero, then $p(z)$ has N zeros inside $\mathbb{S}$, $n - N$ zeros outside $\mathbb{S}$ and no zero on $\mathbb{S}$. On the other hand, if $\Delta_k \neq 0$ for some $k < n$ but $P_{k+1}(z) = 0$, then $p(z)$ has either $n - k$ zeros on $\mathbb{S}$ or $n - k$ zeros symmetric to $\mathbb{S}$. It has additionally N zeros inside $\mathbb{S}$ and $k - N$ zeros outside $\mathbb{S}$.*

A simple necessary and sufficient condition for all the zeros of a complex polynomial to lie on $\mathbb{S}$ was presented by Chen in [12]:

Theorem 11. (Chen) *A necessary and sufficient condition for all the zeros of a complex polynomial $p(z)$ of degree n to lie on $\mathbb{S}$ is that there exists a polynomial $q(z)$ of degree $n-m$ whose zeros are all in or on $\mathbb{S}$ and such that $p(z)=z^m q(z)+\omega q^{\dagger}(z)$ for some complex number ω of modulus 1.*

We close this section by mentioning that there exist many other well-known theorems regarding the distribution of the zeros of complex polynomials. We can cite, for example, the famous *rule of Descartes* (the number of positive zeros of a real polynomial is limited from above by the number of sign variations in the ordered sequence of its coefficients), the *Sturm Theorem* (the exact number of zeros that a real polynomial has in a given interval $(a,b]$ of the real line is determined by the formula $N=\mathrm{var}[S(b)]-\mathrm{var}[S(a)]$, where $\mathrm{var}[S(\xi)]$ means the number of sign variations of the Sturm sequence $S(x)$ evaluated at $x=\xi$) and *Kronecker Theorem* (if all the zeros of a monic polynomial with integer coefficients lie on the unit circle, then all these zeros are indeed roots of unity), see [1] for more. There are still other important theorems relying on matrix methods and quadratic forms that were developed by several authors as Cohn, Schur, Hermite, Sylvester, Hurwitz, Krein, among others, see [13].

4.2 Real self-reciprocal polynomials

Let us now consider real SR polynomials. The theorems below are usually applied to PSR polynomials, but some of them can be extended to NSR polynomials as well.

An analog of Eneström-Kakeya theorem for PSR polynomials was found by Chen in [12] and then, in a slightly stronger version, by Chinen in [14]:

Theorem 12. (Chen and Chinen) *Let $p(z)$ be a PSR polynomial of degree n that is written in the form $p(z)=p_0+p_1z+\cdots+p_kz^k+p_kz^{n-k}+p_{k-1}z^{n-k+1}+\cdots+p_0z^n$ and such that $0<p_k<p_{k-1}<\cdots<p_1<p_0$. Then all the zeros of $p(z)$ are on $\mathbb{S}$.*

Going in the same direction, Choo found in [15] the following condition:

Theorem 13. (Choo) *Let $p(z)$ be a PSR polynomial of degree n and such that its coefficients satisfy the following conditions: $np_n\geqslant(n-1)p_{n-1}\geqslant\cdots\geqslant(k+1)p_{k+1}>0$ and $(k+1)p_{k+1}\ \geqslant\ \sum_{j=0}^{k}\left|(j+1)p_{j+1}-jp_j\right|$ for $0\leqslant k\leqslant n-1$. Then, all the zeros of $p(z)$ are on $\mathbb{S}$.*

Lakatos discussed the separation of the zeros on the unit circle of PSR polynomials in [16]; she also found several sufficient conditions for their zeros to be all on $\mathbb{S}$. One of the main theorems is the following:

Theorem 14. (Lakatos) *Let $p(z)$ be a PSR polynomial of degree $n>2$. If $|p_n|\geqslant\sum_{k=1}^{n-1}|p_n-p_k|$, then all the zeros of $p(z)$ lie on $\mathbb{S}$. Moreover, the zeros of $p(z)$ are all simple, except when the equality takes place.*

For PSR polynomials of odd degree, Lakatos and Losonczi [17] found a stronger version of this result:

Theorem 15. (Lakatos and Losonczi) *Let $p(z)$ be a PSR polynomial of odd degree, say $n=2m+1$. If $|p_{2m+1}|\geqslant\cos^2(\phi_m)\sum_{k=1}^{2m}|p_{2m+1}-p_k|$, where $\phi_m=\pi/[4(m+1)]$, then all the zeros of $p(z)$ lie on $\mathbb{S}$. The zeros are simple except when the equality is strict.*

Theorem 14 was generalized further by Lakatos and Losonczi in [18]:

Theorem 16. (Lakatos and Losonczi) *All zeros of a PSR polynomial $p(z)$ of degree $n>2$ lie on $\mathbb{S}$ if the following conditions hold: $|p_n+r|\geqslant\sum_{k=1}^{n-1}|p_k-p_n+r|$, $p_nr\geqslant0$, and $|p_n|\geqslant|r|$, for $r\in\mathbb{R}$.*

Other conditions for all the zeros of a PSR polynomial to lie on $\mathbb{S}$ were presented by Kwon in [19]. In its simplest form, Kown's theorem can be enunciated as follows:

Theorem 17. (Kwon) *Let $p(z)$ be a PSR polynomial of even degree $n \geqslant 2$ whose leading coefficient p_n is positive and $p_0 \leqslant p_1 \leqslant \cdots \leqslant p_n$. In this case, all the zeros of $p(z)$ will lie on $\mathbb{S}$ if, either $p_{n/2} \geqslant \sum_{k=0}^{n} \left| p_k - p_{n/2} \right|$, or $p(1) \geqslant 0$ and $p_n \geqslant \frac{1}{2} \sum_{k=1}^{n-1} \left| p_k - p_{n/2} \right|$.*

Modified forms of this theorem hold for PSR polynomials of odd degree and for the case where the coefficients of $p(z)$ do not have the ordination above—see [19] for these cases. Kwon also found conditions for all but two zeros of $p(z)$ to lie on $\mathbb{S}$ in [20], which is relevant to the theory of Salem polynomials—see Section 5.

Other interesting results are the following: Konvalina and Matache [21] found conditions under which a PSR polynomial has at least one non-real zero on $\mathbb{S}$. Kim and Park [22] and then Kim and Lee [23] presented conditions for which all the zeros of certain PSR polynomials lie on $\mathbb{S}$ (some open cases were also addressed by Botta et al. in [24]). Suzuki [25] presented necessary and sufficient conditions, relying on matrix algebra and differential equations, for all the zeros of PSR polynomials to lie on $\mathbb{S}$. In [26] Botta et al. studied the distribution of the zeros of PSR polynomials with a small perturbation in their coefficients. Real SR polynomials of height 1—namely, special cases of *Littlewood*, *Newman*, and *Borwein polynomials*—were studied by several authors, see [27–35] and references therein.[2] Zeros of the so-called *Ramanujan Polynomials* and generalizations were analyzed in [37–39]. Finally, the Galois theory of PSR polynomials was studied in [40] by Lindstrøm, who showed that any PSR polynomial of degree less than 10 can be solved by radicals.

4.3 Complex self-reciprocal and self-inversive polynomials

Let us consider now the case of complex SR polynomials and SI polynomials. Here, we remark that many of the theorems that hold for SI polynomials either also hold for SR polynomials or can be easily adapted to this case (the opposite is also true).

Theorem 18. (Cohn) *An SI polynomial $p(z)$ has as many zeros outside $\mathbb{S}$ as does its derivative $p'(z)$.*

This follows directly from Cohn's Theorem 8 for the case where $p(z)$ is SI. Besides, we can also conclude from this that the derivative of $p(z)$ has no zeros on $\mathbb{S}$ except at the multiple zeros of $p(z)$. Furthermore, if an SI polynomial $p(z)$ of degree n has exactly k zeros on $\mathbb{S}$, while its derivative has exactly l zeros in or on $\mathbb{S}$, both counted with multiplicity, then $n = 2(l+1) - k$.

O'Hara and Rodriguez [41] showed that the following conditions are always satisfied by SI polynomials whose zeros are all on $\mathbb{S}$:

Theorem 19. (O'Hara and Rodriguez) *Let $p(z)$ be an SI polynomial of degree n whose zeros are all on $\mathbb{S}$. Then, the following inequality holds: $\sum_{j=0}^{n} \left| p_j \right|^2 \leqslant \|p(z)\|^2$, where $\|p(z)\|$ denotes the maximum modulus of $p(z)$ on the unit circle; besides, if this inequality is strict then the zeros of $p(z)$ are rotations of nth roots of unity. Moreover, the following inequalities are also satisfied: $|a_k| \leqslant \frac{1}{2}\|p(z)\|$* if $k \neq n/2$ and $|a_k| \leqslant \frac{\sqrt{2}}{2}\|p(z)\|$ for $k = n/2$.

Schinzel in [42], generalized Lakatos Theorem 14 for SI polynomials:

[2] The zeros of such polynomials present a fractal behavior, as was first discovered by Odlyzko and Poonen in [36].

Theorem 20. (Schinzel) *Let $p(z)$ be an SI polynomial of degree n. If the inequality $|p_n| \geqslant \inf_{a,b \in \mathbb{C}:|b|=1} \sum_{k=0}^{n} \left| ap_k - b^{n-k} p_n \right|$, then all the zeros of $p(z)$ lie on $\mathbb{S}$. These zero are simple whenever the equality is strict.*

In a similar way, Losonczi and Schinzel [43] generalized theorem 15 for the SI case:

Theorem 21. (Losonczi and Schinzel) *Let $p(z)$ be an SI polynomial of odd degree, i.e., $n = 2m + 1$. If $|p_{2m+1}| \geqslant \cos^2(\phi_m) \inf_{a,b \in \mathbb{C}:|b|=1} \sum_{k=1}^{2m+1} \left| ap_k - b^{2m+1-k} p_{2m+1} \right|$, where $\phi_m = \pi/[4(m+1)]$, then all the zeros of $p(z)$ lie on $\mathbb{S}$. The zeros are simple except when the equality is strict.*

Another sufficient condition for all the zeros of an SI polynomial to lie on $\mathbb{S}$ was presented by Lakatos and Losonczi in [44]:

Theorem 22. (Lakatos and Losonczi) *Let $p(z)$ be an SI polynomial of degree n and suppose that the inequality $|p_n| \geqslant \frac{1}{2} \sum_{k=1}^{n-1} |p_k|$ holds. Then, all the zeros of $p(z)$ lie on $\mathbb{S}$. Moreover, the zeros are all simple except when an equality takes place.*

In [45], Lakatos and Losonczi also formulated a theorem that contains as special cases many of the previous results:

Theorem 23. (Lakatos and Losonczi) *Let $p(z) = p_0 + \dots + p_n z^n$ be an SI polynomial of degree $n \geqslant 2$ and a, b, and c be complex numbers such that $a \neq 0$, $|b| = 1$, and $c/p_n \in \mathbb{R}$, $0 \leqslant c/p_n \leqslant 1$. If $|p_n + c| \geqslant |ap_0 - b^n p_n| + \sum_{k=1}^{n-1} \left| ap_k - b^{n-k}(c - p_n) \right| + |ap_n - p_n|$, then, all the zeros of $p(z)$ lie on $\mathbb{S}$. Moreover, these zeros are simple if the inequality is strict.*

In [46], Losonczi presented the following necessary and sufficient conditions for all the zeros of a (complex) SR polynomial of even degree to lie on $\mathbb{S}$:

Theorem 24. (Losonczi) *Let $p(z)$ be a monic complex SR polynomial of even degree, say $n = 2m$. Then, all the zeros of $p(z)$ will lie on $\mathbb{S}$ if, and only if, there exist real numbers $\alpha_1, \dots, \alpha_{2m}$, all with moduli less than or equal to 2, that satisfy the inequalities: $p_k = (-1)^k \sum_{l=0}^{[k/2]} \binom{m-k+2l}{l} \sigma_{k-2l}^{2m}(\alpha_1, \dots, \alpha_{2m})$, $0 \leqslant k \leqslant m$, where $\sigma_k^{2m}(\alpha_1, \dots, \alpha_{2m})$ denotes the kth elementary symmetric function in the 2m variables $\alpha_1, \dots, \alpha_{2m}$.*

Losonczi, in [46], also showed that if all the zeros of a complex monic reciprocal polynomial are on $\mathbb{S}$, then its coefficients are all real and satisfy the inequality $|p_n| \leqslant \binom{n}{k}$ for $0 \leqslant k \leqslant n$.

The theorems above give conditions for *all* the zeros of SI or SR polynomials to lie on $\mathbb{S}$. In many cases, however, we need to verify if a polynomial has a *given number* of zeros (or *none*) on the unit circle. Considering this problem, Vieira in [47] found sufficient conditions for an SI polynomial of degree n to have a determined number of zeros on the unit circle. In terms of the length, $L[p(z)] = |p_0| + \dots + |p_n|$ of a polynomial $p(z)$ of degree n, this theorem can be stated as follows:

Theorem 25. (Vieira) *Let $p(z)$ be an SI polynomial of degree n. If the inequality $|p_{n-m}| \geqslant \frac{1}{4}\left(\frac{n}{n-m}\right) L[p(z)]$, $m < n/2$, holds true, then $p(z)$ will have exactly $n - 2m$ zeros on $\mathbb{S}$; besides, all these zeros are simple when the inequality is strict. Moreover, $p(z)$ will have no zero on $\mathbb{S}$ if, for n even and $m = n/2$, the inequality $|p_m| > \frac{1}{2} L[p(z)]$ is satisfied.*

The case $m = 0$ corresponds to Lakatos and Losonczi Theorem 14 for all the zeros of $p(z)$ to lie on $\mathbb{S}$. The necessary counterpart of this theorem was considered by Stankov in [48], with an application to the theory of Salem numbers—see Section 5.1.

Other results on the distribution of zeros of SI polynomials include the following: Sinclair and Vaaler [49] showed that a monic SI polynomial $p(z)$ of degree n

satisfying the inequalities $L^r[p(z)] \leqslant 2 + 2^r(n-1)^{1-r}$ or $L^r[p(z)] \leqslant 2 + 2^r(l-2)^{1-r}$, where $r \geqslant 1$, $L^r[p(z)] = |p_0|^r + \cdots + |p_n|^r$, and l is the number of non-null terms of $p(z)$, has all their zeros on $\mathbb{S}$; the authors also studied the geometry of SI polynomials whose zeros are all on $\mathbb{S}$. Choo and Kim applied Theorem 11 to SI polynomials in [50]. Hypergeometric polynomials with all their zeros on $\mathbb{S}$ were considered in [51, 52]. Kim [53] also obtained SI polynomials which are related to Jacobi polynomials. Ito and Wimmer [54] studied SI polynomial operators in Hilbert space whose spectrum is on $\mathbb{S}$.

5. Where these polynomials are found?

In this section, we shall briefly discuss some important or recent applications of the theory of polynomials with symmetric zeros. We remark, however, that our selection is by no means exhaustive: for example, SR and SI polynomials also find applications in many fields of mathematics (e.g., information and coding theory [55], algebraic curves over a finite field and cryptography [56], elliptic functions [57], number theory [58], etc.) and physics (e.g., Lee-Yang theorem in statistical physics [59], Poincaré Polynomials defined on Calabi-Yau manifolds of superstring theory [60], etc.).

5.1 Polynomials with small Mahler measure

Given a monic polynomial $p(z)$ of degree n, with integer coefficients, the Mahler measure of $p(z)$, denoted by $M[p(z)]$, is defined as the product of the modulus of all those zeros of $p(z)$ that lie in the exterior of $\mathbb{S}$ [61]. That is

$$M[p(z)] = \prod_{i=1}^{n} \max\{1, |\zeta_i|\}, \tag{14}$$

where $\zeta_1, \ldots, \zeta_n$ are the zeros[3] of $p(z)$. Thus, if a monic integer polynomial $p(z)$ has all its zeros in or on the unit circle, we have $M[p(z)] = 1$; in particular, all cyclotomic polynomials (which are PSR polynomials whose zeros are the primitive roots of unity, see [1]) have Mahler measure equal to 1. In a sense, the Mahler measure of a polynomial $p(z)$ measures how close it is to the cyclotomic polynomials. Therefore, it is natural to raise the following:

Problem 1. (Mahler) *Find the monic, integer, non-cyclotomic polynomial with the smallest Mahler measure.*

This is an 80-year-old open problem of mathematics. Of course, we can expect that the polynomials with the smallest Mahler measure be among those with only a few number of zeros outside $\mathbb{S}$, in particular among those with only one zero outside $\mathbb{S}$. A monic integer polynomial that has exactly one zero outside $\mathbb{S}$ is called a *Pisot polynomial* and its unique zero of modulus greater than 1 is called its *Pisot number* [62]. A breakthrough towards the solution of Mahler's problem was given by Smyth in [63]:

Theorem 26. (Smyth) *The Pisot polynomial $S(z) = z^3 - z - 1$ is the polynomial with smallest Mahler measure among the set of all monic, integer, and non-SR polynomials. Its Mahler measure is given by the value of its Pisot number, which is,*

[3] The Mahler measure of a monic integer polynomial $p(z)$ can also be defined without making reference to its zeros through the formula $M[p(z)] = \exp\left\{\int_0^1 \log\left[p\left(e^{2\pi i t}\right)\right]dt\right\}$—see [61].

$$\sigma = \sqrt[3]{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{23}{27}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{23}{27}}} \approx 1.32471795724. \tag{15}$$

The Mahler problem is, however, still open for SR polynomials. A monic integer SR polynomial with exactly two (real and positive) zeros (say, ζ and $1/\zeta$) not lying on $\mathbb{S}$ is called a *Salem polynomial* [62, 64]. It can be shown that a Pisot polynomial with at least one zero on $\mathbb{S}$ is also a Salem polynomial. The unique positive zero greater than one of a Salem polynomial is called its *Salem number*, which also equals the value of its Mahler measure. A Salem number s is said to be small if $s < \sigma$; up to date, only 47 small Salem numbers are known [65, 66] and the smallest known one was found about 80 years ago by Lehmer [67]. This gave place to the following:

Conjecture 1. (Lehmer) *The monic integer polynomial with the smallest Mahler measure is the Lehmer polynomial* $\mathcal{L}(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$, *a Salem polynomial whose Mahler measure is* $\Lambda \approx 1.17628081826$, *known as Lehmer's constant.*

The proof of this conjecture is also an open problem. To be fair, we do not even know if there exists a smallest Salem number at all. This is the content of another problem raised by Lehmer:

Problem 2. (Lehmer) *Answer whether there exists or not a positive number* ϵ *such that the Mahler measure of any monic, integer, and non-cyclotomic polynomial* $p(z)$ *satisfies the inequality* $M[p(z)] > 1 + \epsilon$.

Lehmer's polynomial also appears in connection with several fields of mathematics. Many examples are discussed in Hironaka's paper [68]; here we shall only present an amazing identity found by Bailey and Broadhurst in [69] in their works on polylogarithm ladders: if λ is any zero of the aforementioned Lehmer's polynomial $\mathcal{L}(z)$, then,

$$\frac{\left(\lambda^{315}-1\right)\left(\lambda^{210}-1\right)\left(\lambda^{126}-1\right)^2\left(\lambda^{90}-1\right)\left(\lambda^{3}-1\right)^3\left(\lambda^{2}-1\right)^5(\lambda-1)^3}{\left(\lambda^{630}-1\right)\left(\lambda^{35}-1\right)\left(\lambda^{15}-1\right)^2\left(\lambda^{14}-1\right)^2\left(\lambda^{5}-1\right)^6\lambda^{68}} = 1. \tag{16}$$

5.2 Knot theory

A *knot* is a closed, non-intersecting, one-dimensional curve embedded on $\mathbb{R}^3$ [70]. Knot theory studies topological properties of knots as, for example, criteria under which a knot can be unknot, conditions for the equivalency between knots, the classification of prime knots, etc.; see [70] for the corresponding definitions. In **Figure 1**, we plotted all prime knots up to six crossings.

One of the most important questions in knot theory is to determine whether or not two knots are equivalent. This, however, is not an easy task. A way of attacking this question is to look for abstract objects—mainly the so-called *knot invariants*—rather than to the knots themselves. A knot invariant is a (topologic, combinatorial, algebraic, etc.) quantity that can be computed for any knot and that is always the same for equivalents knots.[4] An important class of knot invariants is constituted by the so-called *Knot Polynomials*. Knot polynomials were introduced in 1928 by Alexander [71]. They consist in polynomials with integer coefficients that can be written down for every knot. For about 60 years since its creation, Alexander polynomials were the only known kind of knot polynomial. It was only in 1985 that Jones [72]

[4] We remark, however, that different knots can have the same knot invariant. Up to date, we do not know whether there exists a knot invariant that distinguishes all non-equivalent knots from each other (although there do exist some invariants that distinguish every knot from the trivial knot). Thus, until now the concept of knot invariants only partially solves the problem of knot classification.

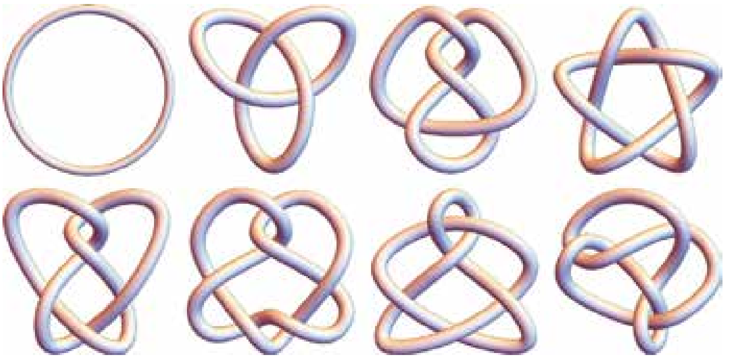

Figure 1.
A table of prime knots up to six crossings. In the Alexander-Briggs notation these knots are, in order, 0_1, 3_1, 4_1, 5_1, 5_2, 6_1, 6_2, *and* 6_3.

came up with a new kind of knot polynomials—today known as *Jones polynomials*—and since then other kinds were discovered as well, see [70].

What is interesting for us here is that the Alexander polynomials are PSR polynomials of even degree (say, $n = 2m$) and with integer coefficients.[5] Thus, they have the following general form:

$$\Delta(t) = \delta_0 + \delta_1 z + \cdots + \delta_{m-1} t^{m-1} + \delta_m t^m + \delta_{m-1} t^{m+1} + \cdots + \delta_1 t^{2m-1} + \delta_0 t^{2m}, \quad (17)$$

where $\delta_i \in \mathbb{N}$, $0 \leqslant i \leqslant m$. In **Table 1**, we present the δ_{m-1}Alexander polynomials for the prime knots up to six crossings.

Knots theory finds applications in many fields of mathematics in physics—see [70]. In mathematics, we can cite a very interesting connection between Alexander polynomials and the theory of Salem numbers: more precisely, the Alexander polynomial associated with the so-called *Pretzel Knot* $\mathcal{P}(-2, 3, 7)$ is nothing but the Lehmer polynomial $\mathcal{L}(z)$ introduced in Section 5.1; it is indeed the Alexander polynomial with the smallest Mahler measure [73]. In physics, knot theory is connected with quantum groups and it also can be used to one construct solutions of the Yang-Baxter equation [74] through a method called baxterization of braid groups.

5.3 Bethe equations

Bethe equations were introduced in 1931 by Hans Bethe [75], together with his powerful method—the so-called *Bethe Ansatz Method*—for solving spectral problems associated with exactly integrable models of statistical mechanics. They consist in a system of coupled and non-linear equations that ensure the consistency of the

Knot	Alexander polynomial $\Delta(t)$	Knot	Alexander polynomial $\Delta(t)$
0_1	1	5_2	$2 - 3t + 2t^2$
3_1	$1 - t + t^2$	6_1	$2 - 5t + 2t^2$
4_1	$1 - 3t + t^2$	6_2	$1 - 3t + 3t^2 - 3t^3 + t^4$
5_1	$1 - t + t^2 - t^3 + t^4$	6_3	$1 - 3t + 5t^2 - 3t^3 + t^4$

Table 1.
Alexander polynomials for prime knots up to six crossings.

[5] Alexander polynomials can also be defined as Laurent polynomials, see [70].

Bethe Ansatz. In fact, for the XXZ Heisenberg spin chain, the Bethe Equations consist in a coupled system of trigonometric equations; however, after a change of variables is performed, we can write them in the following rational form:

$$x_i^L = (-1)^{N-1} \prod_{k=1,\, k\neq i}^{N} \frac{x_i x_k - 2\Delta x_i + 1}{x_i x_k - 2\Delta x_k + 1}, \qquad 1 \leqslant i \leqslant N, \tag{18}$$

where $L \in \mathbb{N}$ is the length of the chain, $N \in \mathbb{N}$ is the excitation number and $\Delta \in \mathbb{R}$ is the so-called spectral parameter. A solution of (18) consists in a (non-ordered) set $X = \{x_1, \dots, x_N\}$ of the unknowns $x_1, \dots, x_N$ so that (18) is satisfied. Notice that the Bethe equations satisfy the important relation $x_1^L x_2^L \cdots x_N^L = 1$, which suggests an inversive symmetry of their zeros.

In [76], Vieira and Lima-Santos showed that the solutions of (18), for $N = 2$ and arbitrary L, are given in terms of the zeros of certain SI polynomials. In fact, (18) becomes a system of two coupled algebraic equations for $N = 2$, namely,

$$x_1^L = -\frac{x_1 x_2 - 2\Delta x_1 + 1}{x_1 x_2 - 2\Delta x_2 + 1}, \qquad \text{and} \qquad x_2^L = -\frac{x_1 x_2 - 2\Delta x_2 + 1}{x_1 x_2 - 2\Delta x_1 + 1}. \tag{19}$$

Now, from the relation $x_1^L x_2^L = 1$ we can eliminate one of the unknowns in (19) —for instance, by setting $x_2 = \omega_a / x_1$, where $\omega_a = \exp(2\pi i a/L)$, $1 \leqslant a \leqslant L$, are the roots of unity of degree L. Replacing these values for x_2 into (19), we obtain the following polynomial equations fixing x_1:

$$p_a(z) = (1 + \omega_a) z^L - 2\Delta \omega_a z^{L-1} - 2\Delta z + (1 + \omega_a) = 0, \qquad 1 \leqslant a \leqslant L. \tag{20}$$

We can easily verify that the polynomial $p_a(z)$ is SI for each value of a. They also satisfy the relations $p_a(z) = z^L p(\omega_a / z)$, $1 \leqslant a \leqslant L$, which means that the solutions of (19) have the general form $X = \{\zeta, \omega_a/\zeta\}$ for ζ any zero of $p_a(z)$. In [76], the distribution of the zeros of the polynomials $p_a(z)$ was analyzed through an application of Vieira's Theorem 25. It was shown that the exact behavior of the zeros of the polynomials $p_a(z)$, for each a, depends on two critical values of Δ, namely

$$\Delta_a^{(1)} = \frac{1}{2} |\omega_a + 1|, \qquad \text{and} \qquad \Delta_a^{(2)} = \frac{1}{2} \left(\frac{L}{L-2} \right) |\omega_a + 1|, \tag{21}$$

as follows: if $|\Delta| \leqslant \Delta_a^{(1)}$, then all the zeros of $p_a(z)$ are on $\mathbb{S}$; if $|\Delta| \geqslant \Delta_a^{(2)}$, then all the zeros of $p_a(z)$ but two are on $\mathbb{S}$; (see [76] for the case $\Delta_a^{(1)} < |\Delta| < \Delta_a^{(2)}$ and more details).

Finally, we highlight that the polynomial $p_a(z)$ becomes a Salem polynomial for $a = L$ and integer values of Δ. This was one of the first appearances of Salem polynomials in physics.

5.4 Orthogonal polynomials

An infinite sequence $\mathcal{P} = \{P_n(z)\}_{n \in \mathbb{N}}$ of polynomials $P_n(z)$ of degree n is said to be an *orthogonal polynomial sequence* on the interval (l, r) of the real line if there exists a function $w(x)$, positive in $(l, r) \in \mathbb{R}$, such that

$$\int_l^r P_m(z) P_n(z) w(z) dz = \begin{cases} K_n, & m = n, \\ 0, & m \neq n, \end{cases} \qquad m, n \in \mathbb{N}, \tag{22}$$

where K_0, K_1, etc. are positive numbers. Orthogonal polynomial sequences on the real line have many interesting and important properties—see [77].

Hermite polynomials	Möbius-transformed Hermite polynomials
$H_0(z) = 1$	$\mathcal{H}_0(z) = 1$
$H_1(z) = 2z$	$\mathcal{H}_1(z) = -2i - 2iz$
$H_2(z) = -2 + 4z^2$	$\mathcal{H}_2(z) = -6 - 4z - 6z^2$
$H_3(z) = -12z + 8z^3$	$\mathcal{H}_3(z) = -20i + 12iz + 12iz^2 + 20iz^3$
$H_4(z) = 12 - 48z^2 + 16z^4$	$\mathcal{H}_4(z) = 76 + 16z + 72z^2 + 16z^3 + 76z^4$

Table 2.
Hermite and Möbius-transformed Hermite polynomials, up to 4th degree.

Very recently, Vieira and Botta [78, 79] studied the action of Möbius transformations over orthogonal polynomial sequences on the real line. In particular, they showed that the infinite sequence $\mathcal{T} = \{T_n(z)\}_{n\in\mathbb{N}}$ of the Möbius-transformed polynomials $T_n(z) = (z-1)^n P_n(W(z))$, where $W(z) = -i(z+1)/(z-1)$, is an SI polynomial sequence with all their zeros on the unit circle $\mathbb{S}$—see **Table 2** for an example. We highlight that the polynomials $T_n(z) \in \mathcal{T}$ also have properties similar to the original polynomials $P_n(z) \in \mathcal{P}$ as, for instance, they satisfy an orthogonality condition on the unit circle and a three-term recurrence relation, their zeros lie all on $\mathbb{S}$ and are simple, for $n \geqslant 1$ the zeros of $T_n(z)$ interlaces with those of $T_{n+1}(z)$ and so on—see [78, 79] for more details.

6. Conclusions

In this work, we reviewed the theory of self-conjugate, self-reciprocal, and self-inversive polynomials. We discussed their main properties, how they are related to each other, the main theorems regarding the distribution of their zeros and some applications of these polynomials both in physics and mathematics. We hope that this short review suits for a compact introduction of the subject, paving the way for further developments in this interesting field of research.

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Chapter 5

A Numerical Investigation on the Structure of the Zeros of the Q-Tangent Polynomials

Jung Yoog Kang and Cheon Seoung Ryoo

Abstract

We introduce q-tangent polynomials and their basic properties including q-derivative and q-integral. By using Mathematica, we find approximate roots of q-tangent polynomials. We also investigate relations of zeros between q-tangent polynomials and classical tangent polynomials.

Keywords: q-tangent polynomials, q-derivative, q-integral, Newton dynamical system, fixed point

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1. Introduction

For a long time, studies on q-difference equations appeared in intensive works especially by F. H. Jackson [1, 2], R. D. Carmichael [3], T. E. Mason [4], and other authors [5–26]. An intensive and somewhat surprising interest in q-numbers appeared in many areas of mathematics and applications including q-difference equations, special functions, q-combinatorics, q-integrable systems, variational q-calculus, q-series, and so on. In this paper, we introduce some basic definitions and theorems (see [1–26]).

For any $n \in \mathbb{C}$, the q-number is defined by

$$[n]_q = \frac{1-q^n}{1-q}, \qquad |q| < 1. \tag{1}$$

Definition 1.1. [1, 2, 9, 13] The q-derivative operator of any function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad x \neq 0, \tag{2}$$

and $D_q f(0) = f'(0)$. We can prove that f is differentiable at 0, and it is clear that $D_q x^n = [n]_q x^{n-1}$.

Definition 1.2. [1, 2, 9, 13, 17] We define the q-integral as

$$\int_0^b f(x) d_q x = (1-q)b \sum_{j=0}^{\infty} q^j f\left(q^j b\right). \tag{3}$$

If this function, $f(x)$, is differentiable on the point x, the q-derivative in Definition 1.1 goes to the ordinary derivative in the classical analysis when $q \to 1$.

Definition 1.3. [5, 17, 18, 21] The Gaussian binomial coefficients are defined by

$$\binom{m}{r}_q = \begin{bmatrix} m \\ r \end{bmatrix}_q = \begin{cases} 0 & \text{if } r > m \\ \dfrac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)} & \text{if } r \leq m \end{cases}, \tag{4}$$

where m and r are non-negative integers. For $r = 0$ the value is 1 since the numerator and the denominator are both empty products. Like the classical binomial coefficients, the Gaussian binomial coefficients are center-symmetric. There are analogues of the binomial formula, and this definition has a number of properties.

Theorem 1.4. Let n, k be non-negative integers. Then we get.

i. $$\prod_{k=0}^{n-1} \left(1 + q^k t\right) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k, \tag{5}$$

ii. $$\prod_{k=0}^{n-1} \frac{1}{(1-q^k t)} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k.$$

Definition 1.5. [5, 26] Let z be any complex number with $|z| < 1$. Two forms of q-exponential functions are defined by

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad e_{q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}. \tag{6}$$

Bernoulli, Euler, and Genocchi polynomials have been studied extensively by many mathematicians(see [22–25]). In 2013, C. S. Ryoo introduced tangent polynomials and he developed several properties of these polynomials (see [22, 23]). The tangent numbers are closely related to Euler numbers.

Definition 1.6. [22–25] Tangent numbers T_n and tangent polynomials $T_n(x)$ are defined by means of the generating functions

$$\begin{aligned} \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} &= \frac{2}{e^{2t}+1} = 2\sum_{m=0}^{\infty} (-1)^m e^{2mt}, \\ \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} &= \frac{2}{e^{2t}+1} e^{tx} = 2\sum_{m=0}^{\infty} (-1)^m e^{(2m+x)t}. \end{aligned} \tag{7}$$

Theorem 1.7. For any positive integer n, we have

$$T_n(x) = (-1)^n T_n(2-x). \tag{8}$$

Theorem 1.8. For any positive integer $m(= odd)$, we have

$$T_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i T_n\left(\frac{2i+x}{m}\right), \quad n \in \mathbb{Z}_+. \tag{9}$$

Theorem 1.9. For $n \in \mathbb{Z}_+$, we have

$$T_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) y^{n-k}. \tag{10}$$

The main aim of this paper is to extend tangent numbers and polynomials, and study some of their properties. Our paper is organized as follows: In Section 2, we define q-tangent polynomials and find some properties of these polynomials. We consider q-tangent polynomials in two parameters and establish some relations between q-tangent polynomials and q-Euler or Bernoulli polynomials. In Section 3, we observe approximate roots distributions of q-tangent polynomials and demonstrate interesting phenomenon.

2. Some properties of the q-tangent polynomials

In this section we define the q-tangent numbers and polynomials and establish some of their basic properties. we shall also study the q-tangent polynomials involving two parameters. We shall find some important relations between these polynomials and q-other polynomials.

Definition 2.1. For $x, q \in \mathbb{C}$, we define q-tangent polynomials as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}e_q(tx), \qquad |t| < \frac{\pi}{2}. \tag{11}$$

From Definition 2.1, it follows that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(0)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}, \tag{12}$$

where $\mathcal{T}_{n,q}$ is q-tangent number. If $q \to 1$, then it reduces to the classical tangent polynomial(see [22–25]).

Theorem 2.2. Let $x, q \in \mathbb{C}$. Then, the following hold.

$$\text{i. } \mathcal{T}_{n,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k}\mathcal{T}_{k,q} = \begin{cases} [2]_q & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}, \tag{13}$$

$$\text{ii. } \mathcal{T}_{n,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k}\mathcal{T}_{k,q}(x) = [2]_q x^n.$$

Proof. From the Definition 2.1, we have

$$\begin{aligned}[2]_q &= \left(1+e_q(2t)\right)\sum_{n=0}^{\infty}\mathcal{T}_{n,q}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\mathcal{T}_{n,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k}\mathcal{T}_{k,q}\right)\frac{t^n}{n!}.\end{aligned} \tag{14}$$

Now comparing the coefficients of t^n we find (i). For (ii) we use the relation

$$\begin{aligned}[2]_q e_q(tx) &= \left(1+e_q(2t)\right)\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\left(\mathcal{T}_{n,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q 2^{n-k}\mathcal{T}_{k,q}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{15}$$

and again compare the coefficients of t^n. □

Theorem 2.3. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k,q} x^k. \tag{16}$$

Proof. From the definition of the q-exponential function, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1} e_q(tx) = \sum_{n=0}^{\infty} \mathcal{T}_{n,q} \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k,q}(x) x^k \right) \frac{t^n}{[n]_q!}.\end{aligned} \tag{17}$$

The required relation now follows on comparing the coefficients of t^n on both sides. □

Theorem 2.4. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,q}(x) x^{n-k}. \tag{18}$$

Proof. From the property of q-exponential function, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{n,q} \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1} e_q(tx) e_{q^1}(-tx) \\ &= \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-1)^n x^n \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,q}(x) x^{n-k} \right) \frac{t^n}{[n]_q!}.\end{aligned} \tag{19}$$

The required relation now follows immediately. □

In what follows, we consider q-derivative of $e_q(tx)$. Using the Mathematical Induction, we find.

$$\text{i. } k=1: \quad D_q^{(1)} e_q(tx) = \sum_{n=1}^{\infty} x^{n-1} \frac{t^n}{[n-1]_q!}. \tag{20}$$

$$\text{ii. } k=i: \quad D_q^{(i)} e_q(tx) = \sum_{n=i}^{\infty} x^{n-i} \frac{t^n}{[n-i]_q!}.$$

If (ii) is true, then it follows that.

$$\begin{aligned}\text{iii. } k=i+1: \quad D_q^{(i+1)} e_q(tx) &= D_{q;x}^{(1)} \left(\sum_{n=i}^{\infty} x^{n-i} \frac{t^n}{[n-i]_q!} \right) \\ &= \sum_{n=i+1}^{\infty} x^{n-(i+1)} \frac{t^n}{[n-(i+1)]_q!} \\ &= t^{i+1} e_q(tx).\end{aligned} \tag{21}$$

We are now in the position to prove the following theorem.

Theorem 2.5. For $k \in \mathbb{N}$, the following holds

$$D_q^{(k)} \mathcal{T}_{n,q}(x) = \frac{[n]_q!}{[n-k]_q!} \mathcal{T}_{n-k,q}(x). \tag{22}$$

Proof. Considering q-derivative of $e_q(tx)$, we find

$$\begin{aligned} D_q^{(i+1)} \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} D_q^{(i+1)} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} \\ &= \frac{[2]_q}{e_q(2t)+1} D_q^{(i+1)} e_q(tx) \\ &= t^{i+1} \frac{[2]_q}{e_q(2t)+1} e_q(tx) \\ &= \sum_{n=0}^{\infty} [n+(i+1)]_q \cdots [n+2]_q [n+1]_q \\ &\times \mathcal{T}_{n,q}(x) \frac{t^{n+i+1}}{[n+(i+1)]_q!} \\ &= \sum_{n=0}^{\infty} \frac{[n]_q}{[n+(i+1)]_q!} \mathcal{T}_{n-(i+1),q}(x) \frac{t^n}{[n]_q!}, \end{aligned} \tag{23}$$

which immediately gives the required result. □

Theorem 2.6. Let a, b be any real numbers. Then, we have

$$\int_a^b \mathcal{T}_{n,q}(x) d_q x = \sum_{k=0}^{n+1} \frac{1}{[n+1]_q} \left(\mathcal{T}_{n+1,q}(b) - \mathcal{T}_{n+1,q}(a) \right). \tag{24}$$

Proof. From Theorem 2.3, we find

$$\begin{aligned} \int_a^b \mathcal{T}_{n,q}(x) d_q x &= \int_a^b \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{k,q} x^{n-k} d_q x \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{k,q} \frac{1}{[n-k+1]_q} x^{n-k+1} \Bigg|_a^b \\ &= \sum_{k=0}^{n+1} \frac{\mathcal{T}_{n+1,q}(b) - \mathcal{T}_{n+1,q}(a)}{[n+1]_q}. \end{aligned} \tag{25}$$

□

Definition 2.7. For $x, y \in \mathbb{C}$, we define q-tangent polynomial with two parameters as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,y) \frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1} e_q(tx) e_q(ty), \qquad |t| < \frac{\pi}{2}. \tag{26}$$

From the Definition 2.7, it is clear that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,0)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}e_q(tx),$$
$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(0,0)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}, \tag{27}$$

where $\mathcal{T}_{n,q}$ is q-tangent number. We also note that the original tangent number, $\mathcal{T}_n$,

$$\lim_{q\to 1}\sum_{n=0}^{\infty} \mathcal{T}_{n,q}\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} T_n\frac{t^n}{n!} = \frac{2}{e^{2t}+1}, \tag{28}$$

where $q \to 1$.

Theorem 2.8. Let x, y be any complex numbers. Then, the following hold.

$$\text{i. } \mathcal{T}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-k,q}(x)y^k, \tag{29}$$

$$\text{ii. } \mathcal{T}_{n,q}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{T}_{n-l,q}\sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_q x^{l-k}y^k.$$

Proof. From the Definition 2.7, we have

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty)$$
$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} y^n\frac{t^n}{[n]_q!}. \tag{30}$$

Using Cauchy's product and the method of coefficient comparison in the above relation, we find (i). Next, we transform q-tangent polynomials in two parameters as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty)$$
$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,q}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} x^n\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty} y^n\frac{t^n}{[n]_q!}. \tag{31}$$

Now following same procedure as in (i), we obtain (ii). □

Theorem 2.9. Setting $y = 2$ in q-tangent polynomials with two parameters, the following relation holds

$$[2]_q x^n = \mathcal{T}_{n,q}(x,2) + \mathcal{T}_{n,q}(x). \tag{32}$$

Proof. Using q-tangent polynomials and its polynomials with two parameters, we have

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,2)\frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{[2]_q e_q(2t)}{e_q(2t)+1}e_q(tx) + \frac{[2]_q}{e_q(2t)+1}e_q(tx)$$
$$= [2]_q e_q(tx) \tag{33}$$

Now from the definition of q-exponential function, the required relation follows. □

Theorem 2.9 is interesting as it leads to the relation

$$x^n = \frac{\mathcal{T}_{n,q}(x,2) + \mathcal{T}_{n,q}(x)}{[2]_q}. \tag{34}$$

Theorem 2.10. Let $|q| < 1$. Then, the following holds

$$\mathcal{T}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \mathcal{T}_{k,\frac{1}{q}}(2) x^{n-k}. \tag{35}$$

Proof. To prove the relation, we note that

$$e_{\frac{1}{q}}(-2t) = \mathcal{E}_q(-2t), \tag{36}$$

where $\mathcal{E}_q(t) = e_{q^{-1}}(t)$. Using the above equation we can represent the q-tangent polynomials as

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(2t)+1} e_q(tx) \\ &= \frac{[2]_q}{1+\mathcal{E}_q(-2t)} \mathcal{E}_q(-2t) e_q(tx) \\ &= \frac{[2]_q}{e_{\frac{1}{q}}(-2t)+1} e_{\frac{1}{q}}(-2t) e_q(tx) \\ &= \sum_{n=0}^{\infty} \mathcal{T}_{n,\frac{1}{q}}(2) \frac{(-t)^n}{[n]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k T_{k,\frac{1}{q}}(2) x^{n-k} \right\} \frac{t^n}{[n]_q!}, \end{aligned} \tag{37}$$

which leads to the required relation immediately. □

Now we shall find relations between q-tangent polynomials and others polynomials. For this, first we introduce well known polynomials by using q-numbers.

Definition 2.11. We define q-Euler polynomials, $E_{n,q}(x)$, and q-Bernoulli polynomials, $B_{n,q}(x)$, as

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{[2]_q}{e_q(t)+1} e_q(tx), \qquad |t| < \pi, \\ \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} &= \frac{t}{e_q(t)-1} e_q(tx), \qquad |t| < 2\pi. \end{aligned} \tag{38}$$

Theorem 2.12. For $x, y \in \mathbb{C}$, the following relation holds

$$\mathcal{T}_{n,q}(x,y) = \frac{1}{[2]_q} \sum_{l=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{\mathcal{T}_{n-l,q}(x)}{m^l} + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\mathcal{T}_{k,q}(x)}{m^{n-k}} \right) E_{l,q}(my). \tag{39}$$

Proof. Transforming q-tangent polynomials containing two parameters, we find

$$\frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty) = \left(\frac{[2]_q}{e_q\left(\frac{t}{m}\right)+1}e_q(ty)\right)\left(\frac{e_q\left(\frac{t}{m}\right)+1}{[2]_q}\right)\left(\frac{[2]_q}{e_q(2t)+1}e_q(tx)\right). \quad (40)$$

Thus, for the relation between q-tangent polynomials of two parameters and q-Euler polynomials, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}E_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{1}{[2]_q}\frac{t^n}{m^n[n]_q!}+\frac{1}{[2]_q}\right)\\
&=\frac{1}{[2]_q}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q E_{l,q}(my)\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}\frac{t^n}{[n]_q!}\\
&+\frac{1}{[2]_q}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q E_{l,q}(my)\frac{\mathcal{T}_{n-l,q}(x)}{m^l}\frac{t^n}{[n]_q!},
\end{aligned} \quad (41)$$

which on comparing the coefficients immediately gives the required relation. □

Corollary 2.13. From Theorem 2.12, the following hold.

i. $$\mathcal{T}_{n,q}(x,y) = \frac{1}{[2]_q}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\left(\frac{\mathcal{T}_{n-l,q}(x)}{m^l}+\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}\right)E_{l,q}(my). \quad (42)$$

ii. $$\mathcal{T}_n(x,y) = \frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\mathcal{T}_{n-l}(x)}{m^l}+\sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\mathcal{T}_k(x)}{m^{n-k}}\right)E_l(my).$$

Theorem 2.14. For $x, y \in \mathbb{C}$, the following relation holds

$$\mathcal{T}_{n-1,q}(x,y) = \frac{1}{[n]_q}\sum_{l=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}-\frac{\mathcal{T}_{n-l,q}(x)}{m^l}\right)B_{l,q}(my). \quad (43)$$

Proof. We note that

$$\frac{[2]_q}{e_q(2t)+1}e_q(tx)e_q(ty) = \left(\frac{t}{e_q\left(\frac{t}{m}\right)-1}e_q(ty)\right)\left(\frac{e_q\left(\frac{t}{m}\right)-1}{t}\right)\left(\frac{[2]_q}{e_q(2t)+1}e_q(tx)\right). \quad (44)$$

Thus as in Theorem 2.12, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\
&=\left(\sum_{n=0}^{\infty}\frac{t^{n-1}}{m^n[n]_q!}-\frac{1}{t}\right)\sum_{n=0}^{\infty}B_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q\frac{\mathcal{T}_{k,q}(x)}{m^{n-k}}B_{l,q}(my)\right)\frac{t^{n-1}}{[n]_q!}\\
&-\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q\frac{\mathcal{T}_{n-l,q}(x)}{m^l}B_{l,q}(my)\right)\frac{t^{n-1}}{[n]_q!}.
\end{aligned} \quad (45)$$

The required relation now follows on comparing the coefficients. □

Corollary 2.15. From the Theorem 2.14, the following relations hold.

i. $$\mathcal{T}_{n-1,q}(x,y) = \frac{1}{[n]_q}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_q \frac{\mathcal{T}_{k,q}(x)}{m^{n-k}} - \frac{\mathcal{T}_{n-l,q}(x)}{m^l}\right) B_{l,q}(my). \quad (46)$$

ii. $$\mathcal{T}_{n-1}(x,y) = \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-l}\binom{n-l}{k}\frac{\mathcal{T}_k(x)}{m^{n-k}} - \frac{\mathcal{T}_{n-l}(x)}{m^l}\right) B_l(my).$$

3. The observation of scattering zeros of the q-tangent polynomials

In this section, we will find the approximate structure and shape of the roots according to the changes in n and q. We will extend this to identify the fixed points and try to understand the structure of the composite function using the Newton method.

The first five q-tangent polynomials are:

$$\begin{aligned}
\mathcal{T}_{0,q}(x) &= \frac{1+q}{2}, \\
\mathcal{T}_{1,q}(x) &= \frac{1}{2}(1+q)(-1+x), \\
\mathcal{T}_{2,q}(x) &= \frac{1}{2}(1+q)\left(1+q(-1+x)+x-x^2\right), \\
\mathcal{T}_{3,q}(x) &= \frac{1}{2}(1+q)\left(-1+q(2-(-2+q)q)-x+q^3x-\left(1+q+q^2\right)x^2+x^3\right), \\
\mathcal{T}_{4,q}(x) &= \frac{1}{2}(1+q)((-1+q)(1+q)(1+(-4+q)q)\left(1+q+q^2\right) \\
&\quad -(1+q)^2(1+(-3+q)q)(1+q^2)x \\
&\quad +(-1+q)(1+q^2)(1+q+q^2)x^2-(1+q)(1+q^2)x^3+x^4).
\end{aligned} \quad (47)$$

Using Mathematica, we will examine the approximate movement of the roots. In **Figure 1**, the x-axis means the numbers of real zeros and the y-axis means the numbers of complex zeros in the q-tangent polynomials. When it moves from left to right, it changes to n = 30, 40, 50, and when it is fixed at q = 0.1, the approximate shape of the root appears to be almost circular. The center is identified as the origin, and it has 2.0 as an approximate root, which is unusual.

Figure 2 shows the shape of the approximate roots when n is changed to the above conditions and fixed at q = 0.5.

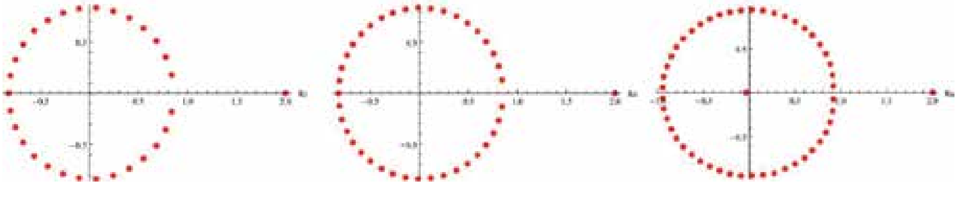

Figure 1.
Zeros of $\mathcal{T}_{n,0.1}(x)$ for n = *30, 40, 50.*

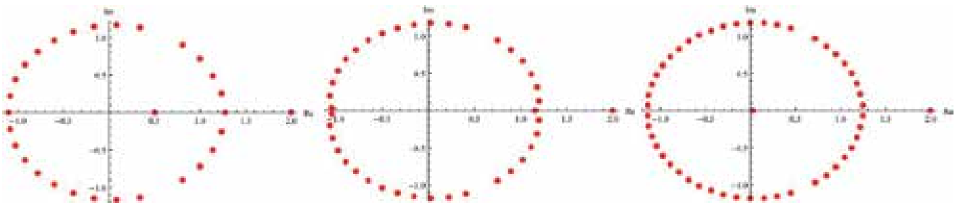

Figure 2.
Zeros of $T_{n,0.5}(x)$ *for* n = *30, 40, 50.*

In **Figure 2**, the shape of the root changes to an ellipse, unlike the $q = 0.1$ condition, and the widening phenomenon appears when the real number is 0.5. In addition, like the previous **Figure 1**, we can see that it has a common approximate root at 2.0. In the following **Figure 3**, n of the far-left figure is 30, and it increases by 10 while moving to the right, and the far-right figure shows the shape of the root when $n = 50$ and is fixed at $q = 0.9$.

In **Figure 3**, the roots have a general tangent polynomial shape with similar properties (see [22–25]). If each approximate root obtained in the previous step is piled up according to the value of n, it will appear as shown in **Figure 4**. The left **Figure 4** is $q = 0.1$ with n from 1 to 50. The middle **Figure 4** is $q = 0.5$ with n from 1 to 50. The right **Figure 4** is $q = 0.9$ with n from 1 to 50.

Let $f : D \to D$ be a complex function, with D as a subset of $\mathbb{C}$. We define the iterated maps of the complex function as the following:

$$f_r : z_0 \mapsto \underbrace{f(f(\cdots(f}_{r}(z_0)\cdots))) \tag{48}$$

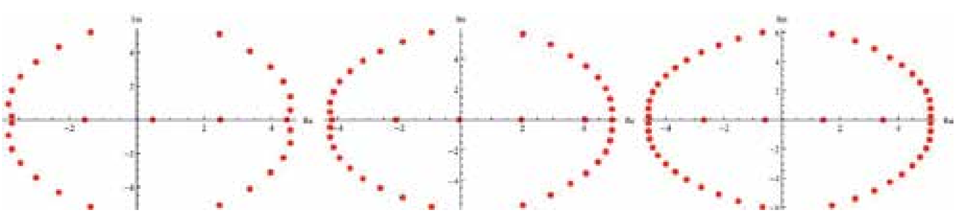

Figure 3.
Zeros of $T_{n,0.9}(x)$ *for* n = *30, 40, 50.*

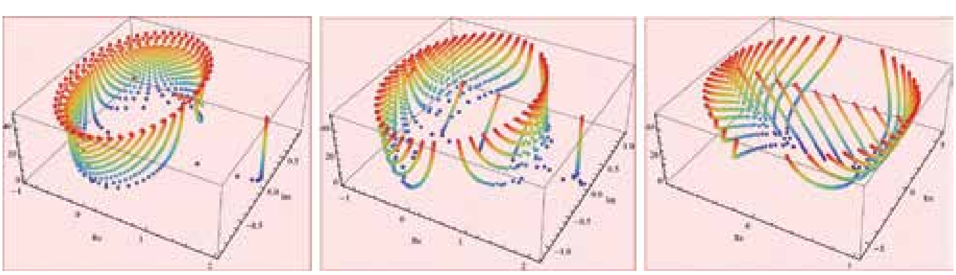

Figure 4.
Zeros of $T_{n,q}(x)$ *for* q = *0.1, 0.5, 0.9,* $1 \le n \le 50$*.*

The iterates of f are the functions $f, f \circ f, f \circ f \circ f, \ldots$, which are denoted $f^1, f^2, f^3, \ldots$ If $z \in \mathbb{C}$, and then the orbit of z_0 under f is the sequence $< z_0, f(z_0), f(f(z_0)), \cdots >$.

We consider the Newton's dynamical system as follows [12, 15, 20]:

$$\left\{ \mathbb{C}_\infty : R(x) = x - \frac{\mathcal{T}(x)}{\mathcal{T}'(x)} \right\}. \tag{49}$$

R is called the Newton iteration function of $\mathcal{T}$. It can be considered that the fixed points of R are the zeros of $\mathcal{T}$ and all the fixed points of R are attracting. R may also have one or more attracting cycles.

For $x \in \mathbb{C}$, we consider $\mathcal{T}_{4,q}(x)$, and then this polynomial has four distinct complex numbers, $a_i (i = 1, 2, 3, 4)$ such that $\mathcal{T}_{4,q}(a_i) = 0$. Using a computer, we obtain the approximate zeros (**Table 1**) as follows:

In Newton's method, the generalized expectation is that a typical orbit $\{R(x)\}$ will converge to one of the roots of $\mathcal{T}_{4,q}(x)$ for $x_0 \in \mathbb{C}$. If we choose x_0, which is sufficiently close to a_i, then this proves that

$$\lim_{r \to \infty} R(x_0) = a_i, \text{ for } i = 1, 2, 3, 4. \tag{50}$$

When it is given a point x_0 in the complex plane, we want to determine whether the orbit of x_0 under the action of $R(x)$ converges to one of the roots of the equation. The orbit of x_0 under the action of R also appears by calculating until 30 iterations or the absolute difference value of the last two iterations is within 10^{-6}.

The output in **Figure 5** is the last calculated orbit value. We construct a function, which assigns one of four colors for each point according to the outcome of R in the plane. If an orbit of x_0 for q = 0.1 converges to -0.672809, $-0.0821877 - 0.710388i$, $-0.0821877 + 0.710388i$ and 1.94818, then we denote the red, blue, yellow, and sky-blue, respectively(the left figure). For example, the yellow region for the left figure represents the part of the basin of attraction of $a_3 = -0.0821877 + 0.710388i$.

i	*q* = 0.1	*q* = 0.5	*q* = 0.9
1	−0.672809	−0.581881 − 0.412941i	−1.10249
2	−0.0821877 − 0.710388*i*	−0.581881 + 0.412941i	−0.158841
3	−0.0821877 + 0.710388*i*	0.907024	1.84004
4	1.94818	2.13174	2.86029

Table 1.
Approximate zeros of $\mathcal{T}_{4,q}(x)$.

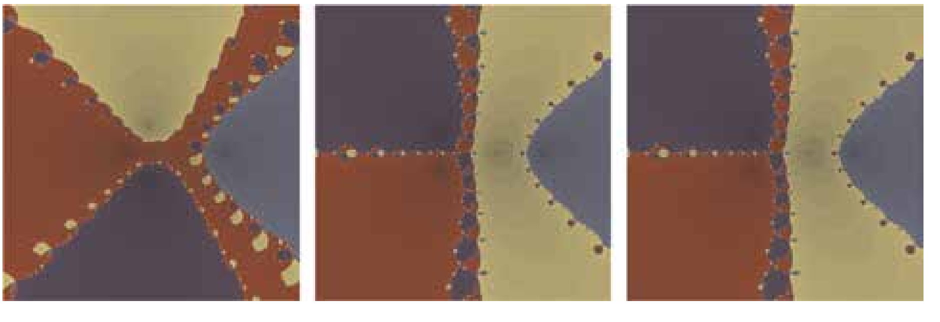

Figure 5.
Orbit of x_0 *under the action of* R *for* $\mathcal{T}_{4,q}(x)$ *for* q = 0.1, 0.5, 0.9.

If we use $\mathcal{T}_{3,0.1}(x)$ to draw a figure using the Newton method, we can obtain **Figure 6**. The picture on the left shows three roots, and the colors are blue, red, and ivory in the counterclockwise direction. When we examine the area closely, we can see that it converges to an approximate value in each color area. The convergence value in the blue area is $-0.379202 + 0.523651i$, that in the red area is $-0.379202 - 0.523651i$, and that in the ivory area is 1.8684. We can also see that it shows self-similarity at the boundary point as divided into three areas. The figure on the right is obtained by 2-times iterated q-tangent polynomials, $\mathcal{T}^2_{3,0.1}(x)$, and the area is divided into nine colors "gray ($x = 2.31831$), scarlet ($x = 1.76736 + 0.216319i$), light brown ($x = 0.137247 + 0.59473i$), sky blue ($x = -0.604153 + 1.19884i$), blue ($x = -0.794606 + 0.378411i$), red ($x = -0.794606 - 0.378411i$), ivory ($x = -0.604153 - 1.19884i$), green ($x = 0.137247 - 0.59473i$), and navy blue ($x = 1.76736 - 0.216319i$) in the counterclockwise direction. This also shows self-similarity at the boundary.

In **Figure 7**, we express the coloring for $\mathcal{T}^2_{3,0.1}(x)$.

Conjecture 3.1. The q-tangent polynomials always have self-similarity at the boundary.

We know that the fixed point is divided as follows. Suppose that the complex function f is analytic in a region D of $\mathbb{C}$, and f has a fixed point at $z_0 \in D$. Then z_0 is said to be (see [6, 16, 20]):

an attracting fixed point if $|f'(z_0)| < 1$;
a repelling fixed point if $|f'(z_0)| > 1$;
a neutral fixed point if $|f'(z_0)| = 1$.

For example, $\mathcal{T}_{3,0.1}(x)$ has three points satisfying $\mathcal{T}_{3,0.1}(x) = x$.
That is, $x_0 = -0.967484, \ -0.33466, 2.41214$. Since

$$\left|\frac{d}{dt}\mathcal{T}_{3,0.1}(-0.967484)\right| = 0 < 1, \quad \left|\frac{d}{dt}\mathcal{T}_{3,0.1}(-0.33466)\right| = 0 < 1 \tag{51}$$

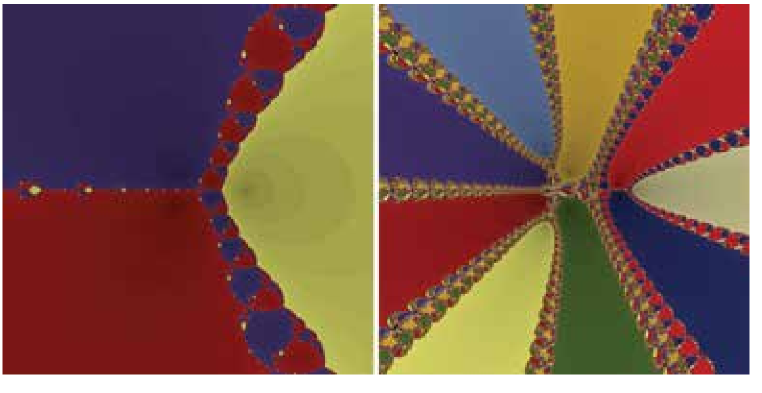

Figure 6.
Orbit of x_o *under the action of* R *for* $\mathcal{T}_{3,0.1}(x)$, $\mathcal{T}^2_{3,0.1}(x)$.

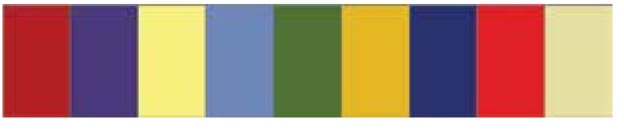

Figure 7.
Palette for escaping points.

Degree n	Attractor	Repellor	Neutral
1	0	1	0
2	1	1	0
3	2	1	0
4	1	3	0
5	1	4	0

Table 2.
Numbers of fixed points of $\mathcal{T}_{n,0.1}(x)$.

r	$\mathbf{R}_{\mathcal{T}^r_{3,0.1}(x)}$	$\mathbf{RF}_{\mathcal{T}^r_{3,0.1}(x)}$
1	3	2
2	3	2
3	3	2
4	23	2
5	2	2
6	1	1

Table 3.
The numbers of $\boldsymbol{R}_{\mathcal{T}^r_{3,0.1}(x)}$ and $\boldsymbol{RF}_{\mathcal{T}^r_{3,0.1}(x)}$ for $1 \le r \le 6$.

Theorem 3.2. $\mathcal{T}_{3,0.1}(x)$ for $q = 0.1$ has two attracting fixed points.

Using Mathematica, we can separate the numerical results for fixed points of $\mathcal{T}_{n,0.1}(x)$. From **Table 2**, we know that $\mathcal{T}_{n,0.1}(x)$ have no neutral fixed point for $1 \le n \le 4$. We can also reach Conjecture 3.3.

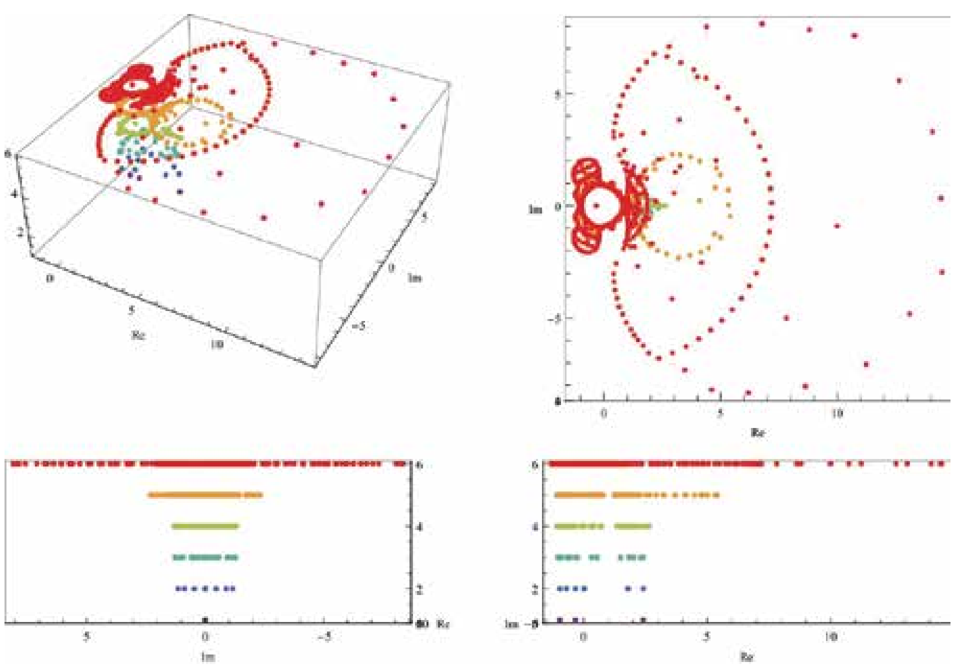

Figure 8.
Stacks of fixed point of $\mathcal{T}^r_{3,0.1}(x)$ for $1 \le r \le 6$.

Conjecture 3.3. The q-tangent polynomials for $n \geq 2$ have at least one attracting fixed point except for infinity.

In **Table 3**, we denote $\mathbf{R}_{\mathcal{T}^r_{n,q}(x)}$ as the numbers of real zeros for rth iteration and $\mathbf{RF}_{\mathcal{T}^r_{n,q}(x)}$ as the numbers of attracting fixed point on real number. From this table, we can know that number of real fixed points of $\mathcal{T}^r_{3,q}(x)$ are less than two. Here, we can suggest Conjecture 3.4.

Conjecture 3.4. The q-tangent polynomials that are iterated, $\mathcal{T}^r_{3,0.1}(x)$, have real fixed point, $\alpha = -0.33466$.

In the top-left of **Figure 8**, we can see the forms of 3D structure related to stacks of fixed points of $\mathcal{T}^r_{3,0.1}(x)$ for $1 \leq r \leq 6$. When we look at the top-left of **Figure 8** in the below position, we can draw the top-right figure. The bottom-left of **Figure 8** shows that image and n-axes exist but not real axis in three dimensions. In three dimensions, the bottom-right of **Figure 8** is the right orthographic viewpoint for the top-left figure,-that is, there exist real and n-axes but there is no image axis (**Figure 8**).

4. Conclusion

We can see that when q comes closer to 0, the approximate shape of the roots become increasingly more circular. Also in this situation, we can observe scattering of zeros in q-tangent polynomials around 2 in three-dimension. When q comes closer to 1, it has properties that are more symmetrical. We can also assume that the property that appears when iterating $\mathcal{T}_{n,q}(x)$ has self-similarity. By iterating, we can conjecture some properties about fixed points. This property warrants further study so that we can create a new property.

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Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Section 2

Applications of Polynomials

Chapter 6

Investigation and Synthesis of Robust Polynomials in Uncertainty on the Basis of the Root Locus Theory

Nesenchuk Alla

Abstract

The root locus method is proposed in the chapter for searching intervals of uncertainty for coefficients of the given (source) polynomial with constant or interval coefficients under perturbations, which ensures its robust stability regardless of whether the given polynomial is Hurwitz or not. The method is based on introduction and application of the "extended root locus" notion. Polynomial adjustment is performed by setting up each one of its coefficients separately and sequentially and determining permissible values of coefficient variation intervals (intervals of uncertainty). The effect of each coefficient variation upon the polynomial root dynamics (behavior) is considered and analyzed separately, and this influence could be observed in the root locus portraits. Root locus method is thus generalized to the cases when the number of polynomial variable coefficients is arbitrary. The root locus parameter distribution diagram along the asymptotic stability bound is introduced and applied for observing the roots behavior regularities. On this basis, the stability conditions are derived, and analytical and graphic-analytical methods are worked out for calculating intervals of variation for the 4th order polynomial family parameters ensuring its robust stability. It also allows to extract Hurwitz subfamilies from the non-Hurwitz families of interval polynomials and to determine whether there exists at least one stable polynomial in the unstable polynomial family.

Keywords: polynomial, dynamic system, uncertainty, stability, robustness, root locus portrait, extended root locus, root locus parameter function

1. Introduction

As it is emphasized in [1, 2], the tasks of analysis and synthesis of control processes occurring in dynamic systems of different physical nature, operating in conditions of substantial plant parametric uncertainty, including the engineering ones, are currently the most urgent and challenging within the framework of the control theory. Among these tasks, one could mention the problem of flux control for the electric motor vector control systems operating in uncertainty because the flux control quality strongly affects the electromagnetic torque and speed control quality, and thus the drive power efficiency. For this reason, of great importance are the tasks of stability investigation and parametric synthesis of robust control systems (their characteristic polynomials) for the plants which parameters vary within the given or unknown intervals of values.

In the area of investigation and synthesis of dynamic system characteristic polynomials, there exists a lot of approaches and methods. For the first time, the necessary and sufficient conditions for systems up to the 3-rd order were formulated by James Maxwell in 1868. Later appeared the stability criteria of Routh–Hurwitz, Mikhajlov, Nyquist, and Bode, which made it possible to check stability of the systems of order n. The frequency Nyquist criterion was the first one that could be used for synthesis by estimation of the system degree of stability. Among the modern methods of synthesis [1, 2] together with the frequency ones, the root locus and state-space methods could be listed. In his book [1], Jurgen Ackermann gives, in particular, the algebraic approach to uncertainty considering different, including the nonlinear, types of the coefficient functions and generating stability regions in the parameter space of real physical parameters of the system (polynomial). The main results in the area of the frequency approach to analysis and synthesis of robust dynamic (control) systems are given in [3], where the stability of uncertain polynomials, including interval ones, is also considered.

The methods for analysis and synthesis of polynomial families represent a separate group. One of the most effective solutions for the task of interval polynomial family investigation within the algebraic approach has been proposed by Kharitonov [4], where in the general case, the task of polynomial stability analysis is reduced to consideration of only four specific polynomials of the whole family with constant coefficients. In [3, 5], the frequency criteria of Hurwitz robust stability are considered, which allow to define the coefficient perturbation sweep for the nominally stable polynomial and various types of uncertainties. Hurwitz robust stability is also investigated in [6–10]. In [6], the maximal deviation intervals of perturbed Hurwitz polynomial coefficients assuring strict Hurwitz property are determined on the basis of the algebraic method worked out using Kharitonov's polynomials [4]. The similar task is solved in [7] but using the Hermite-Biler theorem, which allows to reduce twice the power of investigated polynomial. The way for calculation of perturbed polynomial coefficients' maximal limit values that guarantee sector stability is given in [8]. The linear dependence of coefficient perturbation is considered by Bartlett, et al. for a class of polynomial families generated by convex polytopes in the coefficient space [9]. Here the so-called edge theorem was proved assuring derivation of the stability analysis task to investigation of root location for the finite number of the parametric families. The edge theorem allows to analyze both stability and quality characteristics of the family. A combination of the stochastic and worst-case approaches to the problem of uncertainty is proposed in [10]. It certainly widens the scope of types of treatable uncertainties and reduces conservatism. However, it works properly only in the cases permitting an arbitrarily small probability of specification violation. Thus, to the specific extent, it still bears the drawbacks of the stochastic approach to control, which guarantees only the "average" performance.

An analog of Kharitonov theorem [11] was formulated for the unstable interval polynomials' homogeneous classes of equivalence. Criteria of existence of such classes of equivalence were obtained. Based on the new interval polynomial stability criterion and Lyapunov theorem, a robust optimal proportional-integral-derivative (PID) controller is proposed in paper [12] to carry out design for different plants that contain perturbations of multiple parameters. A new stability criterion of the interval polynomial is presented to determine whether the interval polynomial belongs to Hurwitz polynomial or not. Time-delay systems involving multiple imaginary roots (MIRs) and their stability analysis, which becomes much more complicated than that in the case with only simple imaginary roots, are treated in [13]. For a class of time-delay systems, it was proved that the invariance between the multiple imaginary roots and the simple imaginary roots holds for any multiplicity as well as for the degenerate cases. In paper [14], monic complex polynomials are identified with the sets of their roots instead of being identified with the vectors of their coefficients. A proof is given that the space of Hurwitz polynomials of degree n with positive (resp. negative) coefficients is contractible and also that the space of monic (Schur or Hurwitz) aperiodic polynomials is contractible. A computational method to verify the stability of a convex combination of polynomials is considered in [15] and aimed at the robust stability analysis of a linear system. A simple algebraic test (a matrix inequality) for the stability of the segment of polynomials determined by the given two Hurwitz stable polynomials is proposed. Kučera gives a survey [16] where he navigates the area of the polynomial approach in the control system design technique. Such areas as parameterization of stabilizing controllers, called Youla–Kučera parameterization, are explained; the results on reference tracking, disturbance elimination, pole placement, deadbeat control, robust stabilization, and some others are described.

Of great interest are the problems of ensuring system stability and quality being solved in the modern statements of the problem [2] as tasks of guaranteeing system robustness, which could be solved by application of the root locus approach. The basic benefit of this approach is that its application itself, by its nature, implies parametric variations (i.e., uncertainty). The root locus approach is a powerful method used for the system synthesis [2] and is notable for its descriptiveness ensuring both calculation of the system robust parameters' values and possibility of detailed overview of the dynamic properties variation changes, the system response to uncertainties that is particularly important when investigating systems with uncertain and in particular interval parameters.

Root locus approach to the problem is considered in [17–23]. Paper [17] gives a solution for a compensator synthesis on the basis of the root locus method application. The task of a stable characteristic polynomial synthesis for the interval dynamic system (IDS) by setting up coefficients of the given (initial) unstable one for the case of location of its root locus initial point (where the variable parameter is equal to zero) family within the left half-plane is solved in [21], where the stability is attained via simple setting up the interval of the free term variation.

The above analyzed literature covers various approaches to the uncertainty treatment. However, most of the theoretical works are focused on the tasks of robust stability analysis. The methods for synthesis are not that widely represented, often suffer from complexity and in most cases are enough narrow, which means that they certainly provide instruments for system synthesis, but they are mostly "closed on themselves," which means that they do not provide the complete picture in the sense of showing up what is happening "under cover," which is especially important for the qualitative robust system (polynomials) synthesis. The root locus approach is rarely applied even though it represents the dynamic picture of the system response to uncertainties in the most comprehensive way and thus seems to be the most suitable one to deal with uncertainties.

As for polynomial families, the root locus approach gives us the transparent picture of root dynamics making it possible to see as if from the inside, for example, what subfamilies constitute the whole family of uncertain polynomials in terms of their configuration and stability or some other dynamic indicators bearing significant information about the system behavior and thus leading the way for its investigation and synthesis.

In this work, the root locus methods are described for calculating intervals of uncertainty for coefficients of the given (initial) stable or unstable polynomial with coefficients subject to perturbations, which ensure its robust stability. The proposed methods are based on introduction and application of the notions "extended root locus," "diagram of the root locus parameter function values distribution along the stability bound" and can be used for both synthesis of interval stable polynomials by setting up (adjusting) the unstable ones and analysis of the polynomial behavior under coefficient perturbations. The influence of every coefficient upon the polynomial behavior could be observed.

The work further develops results represented in the papers of Anderson [22] and Kharitonov [4] where they consider the issues of analysis and synthesis of robust interval polynomial families.

2. The problem formulation

Define a polynomial like

$$g_n(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n, \tag{1}$$

where a_j are given (initial) values of real polynomial coefficients, $j = 1, 2, \dots, n$.

In the event of coefficient perturbations, a vector of coefficients of (1), $a = (a_1, \dots, a_{n-1}, a_n)$, belongs to some connected set $A \subset R^n$, $a \in A$; n is a degree of the polynomial (integer value); s is a complex variable, $s = \sigma + i\omega$.

Suppose that coefficients of (1) vary within the following intervals:

$$\underline{a}_j \le a_j \le \overline{a}_j, \quad j = \overline{1, \ n}. \tag{2}$$

where $\underline{a}_j$ and $\overline{a}_j$ are the minimal and maximal limit values of closed interval (2) of coefficients a_j variation correspondingly. Polynomial (1) can be both, non-Hurwitz or Hurwitz one.

After substituting $s = \sigma + i\omega$, write the root locus and parameter equations [18] correspondingly:

$$v(\sigma, \omega) = 0, \text{ and} \tag{3}$$

$$a_n = u(\sigma, \omega), \tag{4}$$

where $u(\sigma,\omega)$ and $v(\sigma,\omega)$ are the real functions of two independent variables σ and ω.

The root locus method represents a powerful and effective tool for stable and qualitative polynomial synthesis and analysis. However, as it is known, this method allows to consider polynomials with only a single variable coefficient (parameter) and cannot be applied in the cases when all coefficients are uncertain. Therefore, the task is to generalize the root locus method for the cases when the number of variable coefficients is arbitrary and thus to solve the problem of investigation of the uncertain polynomial dynamics and working out methods for synthesis of the robustly stable uncertain (interval) polynomial by setting up the given polynomial (non-Hurwitz or Hurwitz) with constant/variable coefficients and determining intervals of all its coefficients (stability intervals) assuring its robust stability.

3. Root locus portraits of uncertain polynomials

Definition 1. The algebraic equation coefficient or the parameter of the dynamic system, described by this algebraic equation, which is being varied in a definite way for generating the root locus, when it is assumed that all the rest coefficients (parameters) are constant, is called the *root locus parameter* or *free parameter*.

If the root locus parameter is a_j, it is named the *root locus relative to parameter (coefficient)* a_j.

Definition 2. The root locus relative to the algebraic equation free term is called the *free root locus*.

Definition 3. Points, where the root locus branches begin and where the root locus parameter is equal to zero are called the *root locus initial points*.

Definition 4. The family P of root loci of interval polynomial (1) with coefficients varying within (2) name as the *interval polynomial root locus portrait (interval polynomial root locus)* or *interval dynamic system root locus portrait (interval dynamic system root locus)*.

Let us along with the parameter a_n vary also parameter a_{n-1} of (1). Thus, we generate a (*free*) *root locus field* F_k (k = 1. 2, ...) in the plane s of system roots, which could also be named a *two-parameter root locus field* or a (*interval*) *root locus subfamily*. Parameter a_{n-1} used for the field generation is named a *root locus field parameter*.

It is evident that the root locus Eq. (3) represents also the equation of level lines of the *free root locus field* F_k. Root locus portrait P is then represented by the family of root locus fields,

$$P = \{F_k \mid k = 1, 2, \ldots\} \tag{5}$$

that represents the infinite set of root locus fields and therefore possesses their properties, and from the mathematical point of view, all root locus fields of P feature the same qualities. Therefore, the portrait P can be investigated as a single root locus field F_k.

Hereinafter the term "root locus" is used in the sense of "Teodorchik – Ewans free root locus" [18].

4. Polynomial analysis and synthesis based on the extended root locus

4.1 Extended root locus

Introduce the following system of polynomials:

$$E_n = \begin{cases} s + a_1 = g_1(s) & (6.1) \\ s^2 + a_1 s + a_2 = g_2(s) & (6.2) \\ \ldots & \\ s^i + a_1 s^{i-1} + \ldots + a_{i-1}s + a_i = g_i(s) & (6.i) \\ \ldots & \\ s^{n-1} + \ldots + a_{n-2}s + a_{n-1} = g_{n-1}(s) & (6.(n-1)) \\ s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = g_n(s) & (6.n) \end{cases}, \tag{6}$$

where

$$g_i(s) = s^i + a_1 s^{i-1} + \ldots + a_{i1}s + a_i, \tag{7}$$

$$g_{i-1}(s) = \left(g_i(s) - a_i\right)/s; \tag{8}$$

i–sequential number of the polynomial in (6), which is equal to its degree, $i = \overline{1, n}$; a_j—coefficients, $j = \overline{1, i}$.

Every polynomial (8) of (i−1) degree is generated from the i-degree polynomial supposed that $a_i = 0$. Polynomials of (6) have common coefficients, but not common roots.

Definition 5. System of polynomials (6) name as the *extension of polynomial* (1) or *extended polynomial*.

Definition 6. Complete set of extension (6) root loci name as the *extended root loci* of (1).

Extension E_n of polynomial $g_n(s)$ could be represented by the finite set of polynomials,

$$E_n = \{g_i(s)\}. \tag{9}$$

Statement 1. In case of variation of any coefficient $a_j, j = \overline{1, (i-1)}$, of polynomial $g_i(s)$ (7) within the specific interval, $\underline{a}_j \leq a_j \leq \overline{a}_j$, every initial point of its free root locus (excluding the point located at the origin) moves along its unique trajectory, representing itself one of the branches of polynomial $g_{i-1}(s)$ (8) root locus, generated relative to this coefficient, and its current position is determined at a point corresponding to the current value of a_j.

Proof. As at initial points of polynomial (6) free root locus the free term a_j is equal to zero, it is evident that (8) represents the equation of initial points of the free root locus of (6), that is, when varying $a_j \left(j = \overline{1, (i-1)}\right)$, the root locus of (8) relative to a_j represents the geometric place of initial points of the root locus of (7). Therefore, every initial point of the free root locus of (7) at fixed a_j coincides in the complex plane s with one of the polynomial (8) roots at the given value of a_j. It is evident, that while varying a_j, this root (and hence, this initial point) moves in the complex plane s, generating one of the $(i-1)$ branches (trajectories) of the root loci of (8) relative to a_j. Thus, the statement has been proved.

Definition 7. Name $g_{i-1}(s)$ (8) as the *originative polynomial* relative to $g_i(s)$ (7) and the root locus of (8)—*the originative root locus* of polynomial (7) free root loci.

Every (i−1)-th polynomial of (6) is the originative one relative to i-th polynomial (6).

Consequence 1. In case of continuous variation of the polynomial $g_i(s)$ coefficient $a_j, j = \overline{1, (n-1)}$, every branch of this polynomial root locus, initiated at the specific initial point, migrates continuously along the corresponding branch of the originative root locus relative to a_{j-1}, being the trajectory of this initial point, correspondingly in direction of increase or decrease of the originative root loci parameter a_j.

Consequence 2. If polynomial $g_{i-1}(s)$ being the originative one for the polynomial $g_i(s)$ is asymptotically stable, all initial points of polynomial $g_i(s)$ free root locus, excluding zero one, are located in the left half-plane s:

$$\forall s_\mu^{i-1}\left[\operatorname{Re} s_\mu^{i-1} < 0 \rightarrow \operatorname{Re} p_\mu^i < 0\right], \tag{10}$$

$$s_\mu^{i-1} = p_\mu^i, \tag{11}$$

where s_μ^i—roots of $g_i(s)$; p_μ^i—initial points of polynomial $g_i(s)$ free root locus; μ—root (initial point) sequential number, $\mu = \overline{1, i-1}$.

Further in the text, polynomial $g_{i\text{-}1}(s)$ free root locus is referred to as the *originative* one relative to that of $g_i(s)$ and $g_i(s)$ free root locus—as the *originated* one relative to that of $g_{i\text{-}1}(s)$.

Statement 1 is illustrated by **Figures 1** and 2. Initial points here are designated by signs "x" (crosses) and letters "p" with the lower indexes, designating the point sequential numbers, and upper indexes, designating the sequential numbers of the corresponding root locus. The root locus sequential number is indicated by a digit next to its corresponding branch.

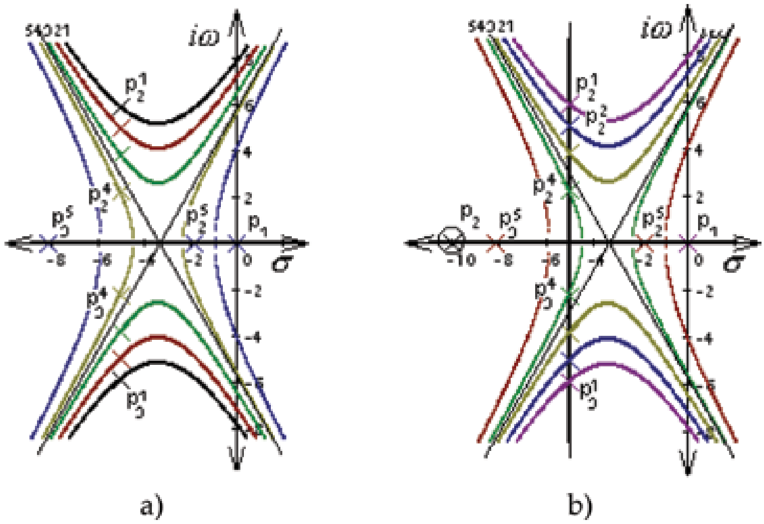

Figure 1.
Polynomial (1) root locus portrait (field) at n = 3, $5 \le a_2 \le 45$: *(a) originated portrait and (b) originated portrait combined with its originative root locus (*n = 2*).*

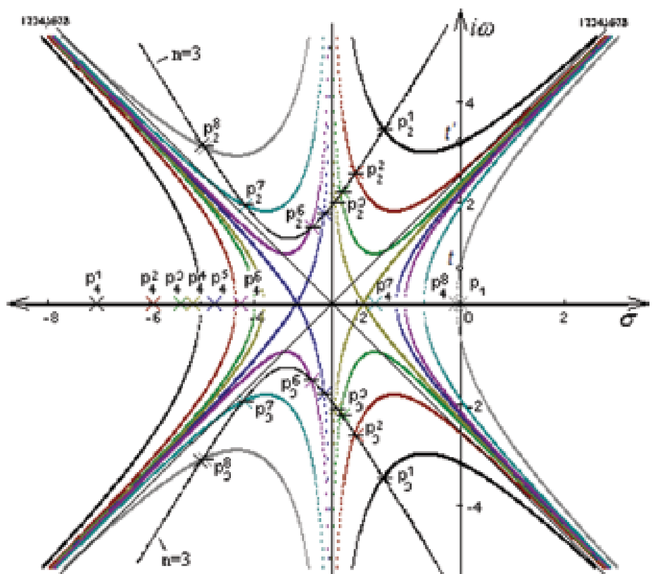

Figure 2.
Free root locus portrait (field) for polynomial $g_4(s) = s^4 + 10\,s^3 + 35\,s^2 + a_3 s + a_4$, $100 \le a_3 \le 5$ *combined with its originative root locus (*n = 3*).*

4.2 Synthesis of stable interval polynomials based on the extended root locus

Consider Eq. (3) in the sense of four following possible cases: n is uneven, $(n-1)/2$ is even/uneven, n is even, and $n/2$ is even/uneven. The root locus parameter equations (as it is in the general form see (4)) are composed in the same way.

Specify the set A_i^+ of a_i values at the cross points of polynomial (7) root locus positive branches with axis ω:

$$A_i^+ = \left\{ a_i^+ l, l = \overline{1, n_i^+} \right\}, \tag{12}$$

where n_i^+ is a number of cross points.

Statement 2. If all initial points of polynomial (7) root locus, excluding a single one at the origin, are located in the left half-plane s, and this polynomial is asymptotically stabile, when the following condition holds:

$$0 < a_i < \inf A_i^+. \tag{13}$$

Proof. Based on the root locus properties [2, 18] and expressions (10) and (11), it can be stated, that provided all initial points of polynomial (7) root locus are located in the left half-plane s (excluding the initial point at the origin), the specific number n_i of root locus branches ($n_i = i - 2$ when i is even and $n_i = i - 1$ when i is uneven), initiating at these points, cross the stability bound $i\omega$ striving along the asymptotes directed to the right half-plane. As the rest of the root locus branches does not cross the stability bound, they are completely stable. For positive branches, crossing the stability bound, specify the set

$$S_i = \{S_{il}\} = \left\{ \left(0, a_i^+ l\right) \right\} \tag{14}$$

of intervals S_{il} of values a_i within the segments from the initial point p_{il} (where $a_i = 0$) of every branch up to its cross point with axis $i\omega$. Thus, the maximal possible interval of a_i values, ensuring stability of (6), is equal to

$$S_{i\max} = \bigcap_{S_{il} \subset S_i} S_{il} = \inf S_i = \left(0, \inf A_i^+\right), \tag{15}$$

that proofs the statement being considered.

For the 4-th degree polynomial represented in **Figure 2**, the interval $S_{i\max} = (0, a_4(t))$.

Theorem 1. For ensuring asymptotic stability of regular or interval polynomial (1), it is enough to.

a. find among polynomials of extension (6), the stable polynomial of degree $i = k$ being the closest one to n;

b. set up sequentially every coefficient a_j of (1), beginning with $a_j = a_k + 1$, within interval $(k+1) < j \leq n$ by setting up the free term a_i of the corresponding i-th polynomial of extension (5) as per condition (13) assuming $i = j$.

Proof. If polynomial $g_i(s) = g_k(s)$ is stable, then on the basis of Consequence 2 of Statement 1 (expressions (10) and (11)), the stability of $g_{i+1}(s)$ can be ensured by simple application of condition (13). Thus, stability of all polynomials $g_i(s)$ is sequentially ensured beginning with the polynomial of degree $i = k + 1$ up to the polynomial of degree $i = n$ inclusive, that is, for $i = \overline{(k+1), n}$. Thus, Theorem 1 has been proved.

An algorithm for the robustly stable regular or interval polynomial synthesis is given below.

Step 1. Composing the extension E_n (6) of the given initial nominal polynomial $g_n(s)$ (1).

Step 2. Sequential check for stability of the extension polynomials, beginning with the polynomial of degree n, until finding the stable polynomial of degree $i = k$.

In case of synthesis of the whole interval polynomial, begin the procedure with the 1-st degree polynomial, $i = k = 1$, specifying interval of a_1 according to the appropriate requirements or arbitrarily.

Step 3. Transfer to the polynomial of the next higher degree, $i = k + 1$.

Step 4. Calculating coordinates $\omega_{i_l}^+$ of cross points of the polynomial $g_i(s)$ free root locus positive branches with the axis $i\omega$ by solving its appropriate root locus Eq. (3).

Cross points $\omega_{i_l}^+$ generate on the axis $i\omega$ a so-called "crossing domain" W_i^+:

$$\omega_{i_l}^+ \in W_i^+ \tag{16}$$

Properties of this domain and behavior of the interval root locus portrait at the stability bound $i\omega$ have been investigated in [18]. On the basis of the fact, that every function of (3) represents continuous differentiable function (steadily increasing/decreasing function), it has been found in [18] that for ensuring stability of the whole interval family, it is required to calculate the parameter $a_i = a_{i_l}^+$ (13) values at only two extreme "dominating points":

$$\omega_{i\min}^+ \in \inf W_i^+, \omega_{i\min}^+ \in \sup W_i^+, \tag{17}$$

by solving the corresponding Eq. (3) after substituting preliminarily into this equation, the appropriate combination [18] of the limit values of each coefficient, from a_1 to a_{i-1}, which have been calculated already in this algorithm when generating the originative polynomial $g_{i-1}(s)$. For finding two coordinates (17), two different combinations of coefficients should be substituted into the root locus equation and thus two different equations should be solved.

Step 5. Determining the value of $\inf A_i^+$ (12) for polynomial $g_i(s)$ by calculating minimal values $a' = a''_{i\min}(\omega_i^+ \min)$ and $a'' = a_i^+ \min(\omega_i^+ \max)$ of coefficient a_i correspondingly at points $\omega_i^+ \min$ and $\omega_i^+ \max$ solving twice Eq. (4) for polynomial $g_i(s)$ at the stability bound:

$$a_i = u(\omega), \tag{18}$$

after substituting previously into (18) the corresponding combinations of coefficients (from a_1 to a_{i-1}) [18]. Thus,

$$\inf A_i^+ = \min(a'a'') = \overline{a}_i, \tag{19}$$

where $\overline{a}_i$ is the upper limit of a_i variation interval. The required interval (13) is: $0 < a_i < \overline{a}_i$.

Step 6. If the last polynomial of extension (6), that is, that of degree n, has been already processed ($i = n$), the calculation is considered finished. Otherwise proceed to step 3.

4.3 Example

Synthesis of the interval polynomial of the 3-rd degree.

Consider polynomial family

$$g^0{}_3\,(s) = s^3 + a_1 s^2 + a_2 s + a_3, \tag{20}$$

where $a_j \in [\underline{a}_j, \overline{a}_j], j \in \{1,2,3\}$; $a_1 \in [10, 15]$, $a_2 \in [25, 35]$, $a_3 \in [350, 450]$.

Step 1. Compose the extended polynomial (6) for (20):

$$\begin{cases} s + a_1 = 0 & (21.1) \\ s^2 + a_1 s + a_2 = 0 & (21.2) \\ s^3 + a_1 s^2 + a_2 s + a_3 = 0 & (21.3) \end{cases} . \tag{21}$$

Step 2. As coefficients of polynomials (21.1) and (21.2) are positive, then both families of these polynomials are asymptotically stable ($i = k = 2$), and therefore, on the basis of Consequence 2 of Statement 1, the root loci family of (21.3) initial points is located in the left half-pane. Thus, for making stable, the polynomial (21.3) uses Statement 2 and Theorem 1.

Step 3. Transfer to the polynomial of the next higher degree, $i = 2 + 1 = 3$.

Step 4. Calculating coordinates (16) of the "dominating points" for polynomial $g_3(s)$. For this purpose, consider the appropriate root locus (3) and parameter (18) equations:

$$\omega^3 - a_2\omega = 0, \tag{22}$$

and parameter function (18) at the stability bound:

$$a_1\omega^2 = a_3 = f_p(\omega). \tag{23}$$

Find the 1-st order derivative of (23) and equate it zero:

$$f_p{}'(\omega) = 2a_1\omega = 0. \tag{24}$$

On the basis of (23) and (24), it can be stated that the character of parameter (23) distribution along the axis σ is steadily increasing and the single extreme point is located at the origin. Thus, there exists the only one extreme point:

$$\omega_3^+ \min = \pm\sqrt{\underline{a}_2} = \pm 5, \tag{25}$$

where function (23) gets the minimal value of the set A_3^+ (see Eq. (12)).

Step 5. Determine $\inf A_3^+$ (12) for $g_3(s)$ using (23), (25):

$\inf A_3^+ = a_3^+ \min\left(\omega_3^+ \min\right) = \underline{a}_1 \cdot \left(\omega_3^+ \min\right)^2 = 10 \cdot 5^2 = 250$. Thus,$0 < a_3 < 250$.

Step 6. As $i = 3 = n$, the algorithm is considered finished.

Thus, coefficient intervals for the resulting robustly stable polynomial $\hat{g}_3(s)$ are as follows:

$a_1 \in [10, 15]$, $a_2 \in [25, 35]$, $a_3 \in (0, 250)$.

5. Investigation of behavior at the stability bound and synthesis of interval polynomial families: root locus parameter function distribution diagram

Consider a dynamic system described by the family of interval characteristic polynomials [4, 18, 20, 22] like.

$$g_4(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4. \tag{26}$$

Coefficients of Eq. (26) to be real, positive, and variable within the intervals

$$\underline{a}_j \le a_j \le \overline{a}_j, \quad j = 1, \ \dots, 4, \quad a_0 = 1. \tag{27}$$

Substitute $s = \sigma + i\omega = i\omega$ ($\sigma = 0$) into (26) and rewrite:

$$\omega^4 - a_1\omega^3 i - a_2\omega^2 + a_3\omega\, i + a_4 = 0 \tag{28}$$

and on the base of (28), write the root locus equation [18, 20] at the stability boundary:

$$-a_1\omega^3 + a_3\omega = 0 \tag{29}$$

and the parameter equation (parameter function) [18, 20] at the stability boundary:

$$f(\omega) = -\omega^4 + a_2\omega^2 = a_4. \tag{30}$$

5.1 Crossing region of the polynomial root locus portrait

Functions (29) and (30) imply properties of analyticity and continuity and, thus, the points where axis $i\omega$ is crossed by the branches of the root locus family P (5), given the condition.

$$0 < a_j < +\infty, \tag{31}$$

constitute on the stability boundary, axis $i\omega$, a specific *crossing region*, D_ω^P.

Definition 8. The region at the asymptotic stability boundary $i\omega$ of the interval system root locus portrait P, described by characteristic polynomial (26), where the given portrait parameter function (30) values family is located, name the c*rossing region* D_ω^P of the root locus portrait P.

The region D_ω^P is a continuous one and, thus, each root locus field F_k (5) and each branch $b_{ki}, \ \ i = 1, 2, \dots$ of the field root loci generate specific subregions, correspondingly subregion $D_\omega^F k$ and continuous subregion $D_\omega^b i$, within the above specified region D_ω^P.

Over the symmetry of the portrait hereinafter, the only upper half-plane s is considered.

5.2 Majorant and minorant of the extremum region

Obtain the extremum parameter function values within $D_\omega^F k \subset D_\omega^P$. To do so, it is necessary to carry out investigation of this function for extremum. It is evident that the majorant parameter function (majorant) can be obtained through rewriting Eq. (30):

$$a_{4\max} = -\omega^4 + \overline{a}_2\omega^2. \tag{32}$$

Take the first-order derivative of (32) and set it to zero:

$$-4\omega^3 + 2\overline{a}_2\omega = 0. \tag{33}$$

After solving Eq. (33), obtain three points of extremum for the majorant parameter function for the field when $a_2 = \overline{a}_2$:

$$\omega_{e_{\max}} = 0, \qquad a_{4e_{\max}} = 0; \ \ \omega_{e_{\max}} = \pm\sqrt{\frac{\overline{a}_2}{2}}, \quad a_{4e_{\max}} = -\omega^4_{e_{\max}} + \overline{a}_2 \cdot \omega^2_{e_{\max}}. \quad (34)$$

Rewrite (30) for determination of a minorant parameter function (or a minorant):

$$a_{4\min} = -\omega^4 + \underline{a}_2\omega^2. \quad (35)$$

In the same way obtain three points of extremum for the minorant, when $a_2 = \underline{a}_2$:

$$\omega_{e_{\min}} = 0, \qquad a_{4e_{\min}} = 0; \ \ \omega_{e_{\min}} = \pm\sqrt{\frac{\underline{a}_2}{2}}, \quad a_{4e_{\min}} = -\omega^4_{e_{\min}} + \underline{a}_2 \cdot \omega^2_{e_{\min}}. \quad (36)$$

Evidently, for n = 4, Eqs. (32) and (35) are the majorant and the minorant for the whole portrait.

Definition 9. *Extremum region* D^e_ω of the interval system root locus portrait described by the characteristic polynomial (26) is a region $[0, \omega_{e\max}]$ at the system asymptotic stability boundary *i*ω where the given portrait parameter function (30) extremum values, $a_{4e\max}$ (34) and $a_{4e\min}$ (36), family is located provided all coefficients a_j vary within limits (31).

5.3 Diagram of the parameter function distribution along the stability boundary

Figure 3 represents the character (diagram) of the parameter function (30) distribution along the boundary of stability by its majorant (32) and minorant (35). For better understanding and descriptiveness, the diagram in **Figure 3** is shown by strait lines, although it constitutes curves. Region D^P_ω constitutes three subregions (see **Figure 3**):

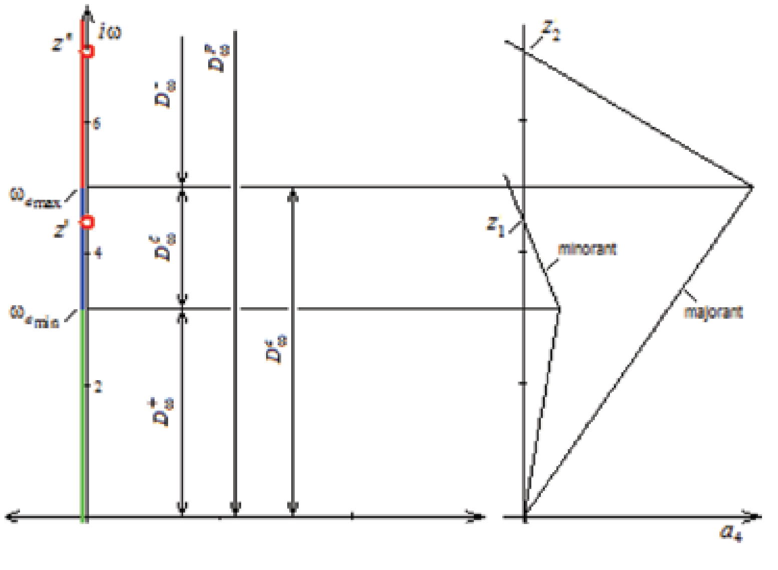

Figure 3.
A diagram for distribution of the interval system root locus portrait parameter function along the asymptotic stability boundary.

- $D_\omega{}^+$ where the parameter function is getting increased (*increase region*);
- $D\omega^-$ where the parameter function is getting decreased (*decrease region*);
- $D\omega^c$ where increase and decrease regions combine (*mixed region*).

Analyze the region Z_ω with the interval $\left[z', z''\right] \subseteq Z_\omega$ where the initial points of the root locus portrait migrate through the stability boundary to the right half-plane. In the diagram, zero points z', z'' are mapped by points z_1, z_2.

Within interval $[0, z']$, covering completely region $D_\omega{}^+$ and partly region $D_\omega{}^c$ ($D_\omega{}^+ \subset \left[0, z'\right]$, $\left[0, z'\right] \cap D_\omega{}^c$), only the positive branches cross the stability boundary, and here the whole family Z of initial points is located in the left half-plane L,

$$Z \subset L. \tag{37}$$

But specific pieces of the positive branches are situated within the right half-plane. For this reason, in some cases, the unstable polynomials could have been found within the whole family (26). However, there certainly could always be found the intervals (27) of stability where the whole family is stable. Name the interval $[0, z']$ the *system stability region*.

The interval $[z', z'']$ covers some piece of the region $D_\omega{}^c$ and some of the region $D_\omega{}^-$, $\left[z', z''\right] \cap D_\omega{}^c$, $\left[z', z''\right] \cap D_\omega{}^-$. In this case, axis $i\omega$ is crossed by combination of both positive and negative branches, and the root locus portrait certainly includes a series of initial points, and thus the whole branches, that have migrated over the boundary to the right half-plane. Therefore, this case always gives us the family (26) that includes combination of stable and unstable polynomials. Name the interval $[z', z'']$ the *system instability region*.

If the interval $[z'', \infty]$ completely belongs to the region $D_\omega{}^-$,

$$\left[z'' \infty\right] \subset D_\omega{}^-, \tag{38}$$

only the negative branches cross the stability boundary $i\omega$, and the family Z together with the corresponding positive branches are located in the right half-pane,

$$Z \subset R. \tag{39}$$

No stable polynomial could be found in (26). This region name the *system complete instability region*.

5.4 Real crossing region of the portrait

Specify the region D_ω^R where the branches of the given real root locus portrait cross the stability boundary. To find its limits, consider Eq. (29) and determine the values of its roots. When $\omega > 0$.

$$\omega_{\max} = \sqrt{\frac{\overline{a}_3}{\underline{a}_1}}, \omega_{\min} = \sqrt{\frac{\underline{a}_3}{\overline{a}_1}}, \tag{40}$$

where $\omega_{\max}$, $\omega_{\min}$ represent the *real crossing region*.

Definition 10. The region $[\omega_{\min}, \omega_{\max}]$ at the stability boundary $i\omega$, where the polynomial (26) root locus portrait branches migrate through to the right half-plane, name the *real crossing region* D_ω^R of the system root locus portrait:

$$[\omega_{\min}, \omega_{\max}] \subseteq D_\omega^R. \tag{41}$$

5.5 Graphic-analytical stability conditions for interval polynomials

Define below three possible ways of the real crossing region location and the corresponding stability conditions.

5.5.1 Real crossing region belongs to the increase region $D_\omega{}^+$

$$D_\omega^R \subset D_\omega{}^+. \tag{42}$$

In this case $\omega_{\max} < \omega_{e_{\min}}$.

Statement 3. When the dynamic system root locus portrait, described by polynomial (26), satisfies relationship (42), the whole family Z of the portrait initial points is located in the left half-plane L,

$$Z \subset L. \tag{43}$$

Then, define the set S of the root locus portrait P branches' intervals s_i:

$$S = \{s_i = [0, a_4(\omega_i)],\ \ i = 1, 2, ...\}. \tag{44}$$

$a_4(\omega_i)$ represents the parameter function (30) at points with the coordinates ω_i; $S \subset P$ and $S \subset L$ (40). Thus, from (42) and (43) obtain:

$$\bigcap_{i=1}^{\infty} s_i = \inf S = \left[0, \underline{a}_4(\omega_{\min})\right], \tag{45}$$

where $\underline{a}_4(\omega_{\min})$—function (30) minimal value at point $\omega_{\min}$ (40). Hence,

$$\forall\ a_4 \in \left[\underline{a}_4, \overline{a}_4\right] \quad \left[a_4 \in \left[0, \underline{a}_4(\omega_{\min})\right] \ \ \rightarrow \ \ a_4 \in S \& P \subset L\right], \tag{46}$$

$$\forall\ a_4 \in \left[\underline{a}_4, \overline{a}_4\right] \quad \left[a_4 \notin \left[0, \underline{a}_4(\omega_{\min})\right] \ \ \rightarrow \ \ a_4 \notin S \& P \not\subset L\right]. \tag{47}$$

The following statement can be formulated on the basis of expressions (42) and (47).

Statement 4. The dynamic system, described by the interval characteristic polynomial family (26) and satisfying expression (42), is asymptotically stable if

$$\overline{a}_4 < \underline{a}_4(\omega_{\min}). \tag{48}$$

Definition 11. One or more stable polynomials with constant coefficients within the family (26) that guarantee stability of the whole family name the d*ominating polynomials*.

From Statement 4 and the previous conclusions, the following stability condition goes.

Stability condition 1. The asymptotic stability of the interval system family, described by the root locus portrait P (5) satisfying expression (42), is guaranteed if polynomial

$$s^4 + \overline{a}_1 s^3 + \underline{a}_2 s^2 + \underline{a}_3 s + \overline{a}_4 = 0 \tag{49}$$

of the family is stable. Polynomial (49) represents the dominating one.

Stability is verified using the Stability condition 1. The polynomial parameters are calculated with application of the Statement 4.

5.5.2 Real crossing region belongs to the decrease region $D_\omega{}^-$

$$D_\omega^R \subset D_\omega{}^-. \tag{50}$$

It happens in case if $\omega_{\min} \ge \omega_{e_{\max}}$.

The above made conclusions allow to formulate the following statement.

Statement 5. If the interval system root locus portrait P satisfies condition (50), the whole family Z of its initial points satisfies Eq. (39), and the system is asymptotically unstable.

5.5.3 Real crossing region completely or partially belongs to the mixed region $D_\omega{}^c$

$$D_\omega^R \subset D_\omega{}^c \ \vee\ D_\omega^R \cap\ D_\omega{}^c. \tag{51}$$

We have this when the following conditions are not satisfied: $\omega_{\max} < \omega_{e_{\min}}$, $\omega_{\min} \ge \omega_{e\max}$.

For this case

$$P = P^+ + P^-, \tag{52}$$

We have already discussed the increase part of (52), when $P^- = \varnothing$. Hence, this section considers the decrease part, P^-. Consider first the family Z of the root locus portrait P^-.

Statement 6. If condition (51) holds, family Z of initial points of the dynamic system root locus portrait, described by characteristic polynomial (26), can be located in both left half-plane L and right half-plane R, that is, the following options of Z location may take place:

$$Z \subset L, \tag{53}$$

$$Z \subset (L + R), \tag{54}$$

$$Z \subset R. \tag{55}$$

Evidently, options (54) and (55) take place when

$$D_\omega^R \subset D_\omega{}^- \tag{56}$$

$$\text{or } D_\omega^R \cap D_\omega{}^-. \tag{57}$$

As options (54)–(57) deliberately indicate instability of the system in whole, consider below only option (53) of the system poles location,

$$\omega_{\max} < \omega(z'), \tag{58}$$

where $\omega(z')$ is coordinate ω at point z' (**Figure 3**).

In this case proceed just as in (44)–(47) but only substituting $\omega_{\max}$ instead of $\omega_{\min}$.

Statement 7. The asymptotic stability of the dynamic system, described by polynomial family (26) and satisfying expression (51), is ensured when the following condition holds:

$$\overline{a}_4 < \min\left\{\underline{a}_4(\omega_{\min}), \underline{a}_4(\omega_{\max})\right\}. \tag{59}$$

From condition (59) follows that the system asymptotic stability for part P^- of portrait (52), provided that condition (53) holds, is defined by the value of

$\underline{a}_4(\omega_{\max})$. Therefore, for checking stability of P^- (52), it is enough to check the only one following dominating polynomial of (26):

$$s^4 + \underline{a}_1 s^3 + \underline{a}_2 s^2 + \overline{a}_3 s + \overline{a}_4 = 0. \tag{60}$$

Because in this case, the portrait represents the compound one (52), check the stability by checking both polynomials, (49) and (60).

Stability condition 2. If the interval dynamic system root locus portrait P (52), describing the family of characteristic polynomials (26), satisfies expression (51), the system asymptotic stability is ensured when the following dominating polynomials

$$s^4 + \overline{a}_1 s^3 + \underline{a}_2 s^2 + \underline{a}_3 s + \overline{a}_4 = 0, \tag{61}$$

$$s^4 + \underline{a}_1 s^3 + \underline{a}_2 s^2 + \overline{a}_3 s + \overline{a}_4 = 0 \tag{62}$$

of family (26) are both stable.

From the results obtained above also goes that in case (51) the system asymptotic stability can be verified by only a single polynomial of (26) having constant coefficients. The equation to choose depends of condition (49) verification results. If the verification shows that $\min\{\underline{a}_4(\omega_{\min}), \underline{a}_4(\omega_{\max})\} = \underline{a}_4(\omega_{\min})$, then Eq. (61) is applied for the stability check. If it shows that $\min\{\underline{a}_4(\omega_{\min}), \underline{a}_4(\omega_{\max})\} = \underline{a}_4(\omega_{\max})$, then the stability is verified by (62).

To determine the coefficients of (26), ensuring satisfaction of expressions (53) and (58), Eqs. (30) and (31) are applied. Thus, coefficients a_1 and a_3 must satisfy the inequality:

$$\sqrt{\frac{\overline{a}_3}{\underline{a}_1}} < \omega\left(z'\right), \overline{a}_3 < \underline{a}_1 \omega^2\left(z'\right). \tag{63}$$

To verify the system stability, the stability conditions 1 and 2 are used. For calculation of the system (polynomial) parameters, expressions (48), (49) and (63) are used.

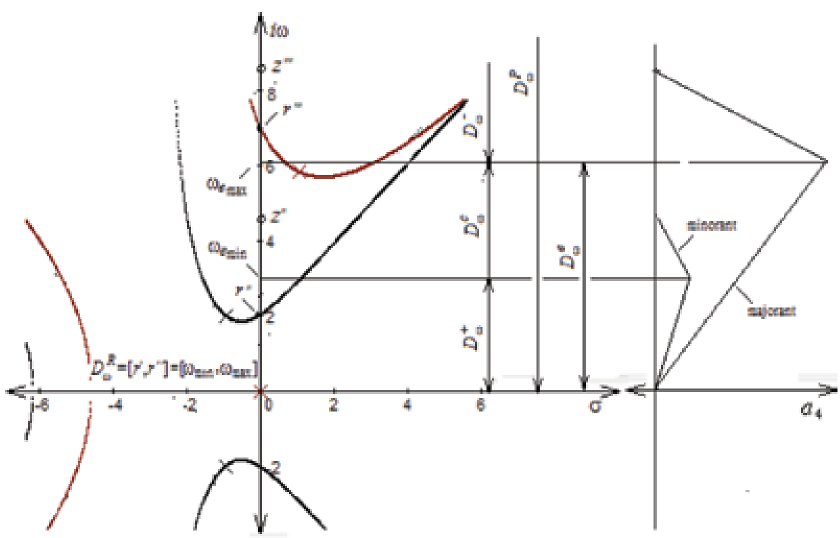

Figure 4.
Dynamics of the interval system root locus portrait at the asymptotic stability boundary.

Polynomial stability could be estimated graphically directly from the plots (see **Figures 3** and **4**).

5.6 Example

Coefficients of the given polynomial (26): $a_1 \in [5, 10]$, $a_2 \in [5, 10]$, $a_3 \in [5, 10]$, $a_4 \in [5, 10]$.

Extremum region: D^e_ω: $\omega_{e\ \min}$ = 3,16; $\omega_{e\ \max}$ = 5,92; $a_{4e\ \min}$ = 100,04; $a_{4e\ \max}$ = 1223,96.

Real region: D^R_ω: $\omega_{\min}$ = 2; $\omega_{\max}$ = 7,1; $\underline{a}_4(\omega_{\min})$= 64; $\underline{a}_4(\omega_{\max})$= − 1532,97.

[*z'*, *z"*]: ω(z') = 4,47; ω(z") = 8,37.

In **Figure 4**, the above indicated regions are shown. The points, corresponding to the dominating polynomials (61), (62), are designated by *r'* and *r"*. The real crossing region in this case completely covers the extremum region, $D^e_\omega \subset D^R_\omega$, $[r', r''] \subseteq D^R_\omega$.

It is evident that the given polynomial family in whole is unstable. Within region Z_ω = [*z'*, *z"*], there exist poles that have migrated to the right half-plane (see (54)), which is confirmed by the negative value of the parameter $\underline{a}_4(\omega_{\max})$I.

Dominating polynomials of the family are the following:

$$s^4 + 10s^3 + 20s^2 + 40s + 30 = 0. \tag{64}$$

$$s^4 + 5s^3 + 20s^2 + 250s + 30 = 0. \tag{65}$$

Polynomials stability check shows that polynomial (6), which root loci crosses the stability boundary at point $\underline{a}_4(\omega_{\min})$, is stable, and polynomial (66), which root loci crosses the stability boundary at point $\underline{a}_4(\omega_{\max})$, has two roots with positive real parts.

Extraction of the stable polynomial subfamily of the given unstable family:

The stable root locus family, satisfying conditions (58) and (59), should cross the stability boundary within the region bounded by interval [*r'*, *z'*] as in this case all initial points of the root locus family are located in the left half-plane (53) (see Section 5.5).

To calculate the maximal value of a_3 that defines the stable subfamily within the given root locus portrait, apply formula (63):

$$\overline{a}_3 < \underline{a}_1 \cdot 4,47^2, \quad \overline{a}_3 < 99,9. \tag{66}$$

Based on (66), accept $\overline{a}_3 = 80$.

Based on (59), accept: $\overline{a}_4 < \underline{a}_4(\omega_{\min})$, $\overline{a}_4 = 60$ and write the dominating polynomials:

$$s^4 + 10s^3 + 20s^2 + 40s + 60 = 0,$$

$$s^4 + 5s^3 + 20s^2 + 80s + 60 = 0.$$

As per stability condition 2, the root locus portrait subfamily having new modified values of a_3 and a_4 ($\overline{a}_3 = 80, \overline{a}_4 = 60$) is asymptotically stable.

6. Conclusions and future developments

A method has been worked out for synthesis of asymptotically stable regular or interval polynomial from the given Hurwitz or non-Hurwitz source polynomial with constant/interval coefficients by setting up coefficients of the given one. The root locus approach is used. The task is solved by introduction of notions of the “extended polynomial” (“generalized polynomial”) and the polynomial “extended root locus,” which allows to obtain a descriptive picture of the polynomial root dynamics under coefficient variations and to disclose on this basis the cause of instability. The intervals of uncertainty for each coefficient being set up are specified along the root locus branches.

The above described method based on the “extended root locus” notion is new and allows to extend the application sphere of the root locus method, which is traditionally considered to be the method of system synthesis by only a single parameter (coefficient) variation and with only one variable parameter (coefficient), in both directions: system synthesis by many parameter variations and system synthesis with many parameter variations.

Investigation of the fourth power dynamic system behavior in conditions of the interval parameter variations has also been carried out on the basis of root locus portraits and introduction of the notion of the “diagram of the root locus parameter function values distribution along the stability bound.” Behavior regularities for interval system root locus portraits at the stability boundary have been formulated. On this basis, the stability conditions have been derived, and graphic-analytical method has been worked out for calculating intervals of parameter variation ensuring the system robust stability.

In continuation of the results of Anderson [22] and Kharitonov [4] in this work, it is proved that for the 4th power interval system family asymptotic stability analysis, it is enough to use the only one polynomial of this kind. It is also shown, how to find and extract the stable families from the unstable ones.

The above discussed topic is certainly worth further investigation in the light of continuous progress of both theory and technology. When speaking of the practical implementations, it could be noted that most of the control system synthesis tasks, especially those in the area of robust control, are currently still being solved in a somewhat “local domestic” way, when a designer each time tries to invent a solution to be suitable for the specific application experiencing the lack of more generalized methods. Besides this, a great deal of existing robust control methods share and suffer complexity. In this connection, further in-depth investigation of the uncertain polynomials’ root locus portraits seems helpful, especially the analysis of its composition in terms of configurations variety, constituting subfamilies, placement of various root domains within the prescribed regions in the complex plane and, of course, dynamics. They also could be distinguished for their undoubted descriptiveness.

Polynomial equation approach in the design technique [16], and root locus technique in particular, is descriptive, clear, and easy to use and computerize and thus could be helpful in many application areas including the areas of industry, biology, medicine, etc. It can be used for proper parameterization of robust drive controllers, for example, in the area of railway traffic control, in particular for the cases of tackling the problems of breaking and skidding.

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Chapter 7

Pricing Basket Options by Polynomial Approximations

Pablo Olivares

Abstract

In this paper, we use polynomial approximations in terms of Taylor, Chebyshev, and cubic splines to compute the price of basket options. The paper extends the use of a similar pricing technique applied under a multivariate Black-Scholes model to a framework where the dynamic of the underlying assets is described by dependent exponential Levy processes generated by a combination of Brownian motions and compound Poisson processes. This model captures some empirical features of the asset dynamics such as common and idiosyncratic random jumps. The approach is implemented in the context of spread options and a multivariate Merton model, i.e., a jump diffusion with Gaussian jumps. Our findings show that, within the range of parameters analyzed, polynomial approximations are comparable in accuracy to a standard Monte Carlo approach with a considerable reduction in computational effort. Among the three expansions, cubic splines show the best performance.

Keywords: Taylor approximations, Chebyshev polynomials, cubic splines, basket options, spread options, jump-diffusion model

1. Introduction

We study the pricing of basket contracts under a multivariate jump-diffusion process. The paper extends the use of a similar pricing technique applied under a multivariate Black-Scholes model, see [1], to a framework where the dynamic of the underlying assets is described by dependent exponential Levy processes generated by a combination of Brownian motions and compound Poisson processes. This model captures some empirical features of the asset dynamics such as common and idiosyncratic random jumps. The dependence between assets is reflected in both the covariance structure of the Brownian motion and the joint probability law of the common jump sizes.

For such class of models, no pricing closed-form formula is available. In single-asset contracts, well-established numerical methods have proven to be effective, but their extensions to several dimensions reveal important instabilities and a costly computational effort. Our paper introduces a novel approach based on polynomial approximations of the conditional price. It is, in the framework considered, less time demanding than a standard Monte Carlo approach to achieve similar results. Moreover, the use of Chebyshev polynomials and cubic splines improves the convergence over previous attempts based on Taylor expansions.

We consider a pricing methodology consisting in a two-step procedure. First, conditioning on $d-1$ out of the total number of d assets, we find the price of a payoff based on a single asset with a more complex conditional distribution.

Secondly, we consider some expansions of the conditional price, given either in terms of Taylor, Chebyshev, or cubic spline polynomials, allowing to write the corresponding price as a linear combination of mixed exponential-power moments.

This approach is implemented in the context of spread options and a multivariate Merton model, that is, a jump diffusion with Gaussian jumps. Our findings show that, within the range of parameters analyzed, polynomial approximations are comparable in accuracy to a standard Monte Carlo approach with a considerable reduction in computational effort. Among the three expansions, cubic splines show the best performance.

The use of a Taylor expansion to pricing has been considered in the pioneering work of [2] for a vanilla European option and in [3, 4] for spread contracts under a bivariate Black-Scholes model. See also [5]. A Chebyshev expansion has been recently considered in [6]. Applications under a multivariate jump-diffusion model have been less explored. Our paper intends to fill this gap.

Although a comparison with alternative approaches is beyond the scope of this paper, it is worth noticing the existence of pricing methods based on Fourier or Hilbert transforms. For example, for spread contracts under a different class of Levy processes, a Fast Fourier transform method can be found in [7]. See also [8] for expansions in terms of Fourier series and [9] for Hilbert transforms.

The organization of the paper is as follows: in Section 2, we introduce the model and obtain the pricing expressions for basket contracts under the approximations. In Section 3, we specialize the three expansions in the case of spreads contracts. In Section 4, we discuss the implementation of the methods and present our numerical findings. Finally in Section 5, we present conclusions. Proofs are deferred to the appendix.

2. Pricing under jump-diffusion models

Let $\left(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a filtered probability space. We define the filtration $\mathcal{F}^{X_t} := \sigma(X_s, 0 \leq s \leq t)$ as the σ-algebra generated by the random variables $\{X_s, 0 \leq s \leq t\}$ completed in the usual way. Denote by Q an equivalent martingale measure (EMM), respectively, by E_Q, φ_X, and M_X the expectation, characteristic, and moment-generating functions of a random variable X under Q. The function f_X is its probability density function.

By r we denote the (constant) interest rate, $A \circ B$ is the componentwise product between matrices A and B, and A' represents the transpose of matrix $A = \left(a_{ij}\right)_{1\leq i,j\leq d}$, while $\text{diag}(A)$ is a vector with components $(a_{ii})_{1\leq i\leq d}$. The symbol δ_{ij} is the usual Kronecker's number. The vector $\tilde{Y}$ is created from the vector Y after eliminating the first component. For a function f with domain in $\mathbb{R}^d$ and a vector $L = (l_1, l_2, ..., l_d)$ with $l_k \in \mathbb{N}$, the symbol $D^L f$ represents the mixed partial derivative of the function f differentiated l_k times w.r.t. the k-th variable.

For vectors $v = (v_1, v_2, ..., v_d)$ and $n = (n_1, n_2, ..., n_d)$, we set $v! = \prod_{k=1}^d v_k$ and $v^n = \prod_{k=1}^d v_k^{n_k}$.

We introduce the following convenient notations. For a $1 \times (n+1)$ vector Va, $b \in \mathbb{R}$, and $n \in \mathbb{N}$

$$bin(n, Va, b) \quad = \quad \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} Va_m b^{n-m}$$

$$PVa \quad = \quad \left(1, Va_1, \dots, Va_n^n\right)$$

Also, for a differentiable function f, we set the vector $DVf = \left(f, D^1 f, \dots, D^n f\right)$.

The d-dimensional process of spot prices is denoted by $(S_t)_{t\geq 0}$, while $(Y_t)_{t\geq 0}$ is the corresponding log-price process. They are related by

$$S_t^{(j)} = S_0^{(j)} \exp\left(Y_t^{(j)}\right), \ \ j = 1, 2, \dots, d \tag{1}$$

We analyze European basket options whose payoff at maturity T, for a strike price K, are given by

$$h(S_T) = \left(\sum_{j=1}^{d} w_j S_T^{(j)} - K\right)_+$$

where $(w_j)_{1\leq j\leq d}$ are some deterministic weights and $x_+ = max(x, 0)$.

Furthermore, for the log-prices, we assume a multidimensional jump-diffusion dynamics under $\mathcal{Q}$ given by

$$dY_t = \mu dt + \Sigma^{\frac{1}{2}} dB_t + dZ_t \tag{2}$$

where $(B_t)_{t\geq 0}$ is a multivariate Brownian motion with independent components and $\mu = r - \frac{1}{2}diag(\Sigma) - m$. The matrix $\Sigma = \left(\sigma_{jl}\right)_{j,l}$ is symmetric, positive definite, while $\Sigma^{\frac{1}{2}}$ is such that $\Sigma^{\frac{1}{2}}\left(\Sigma^{\frac{1}{2}}\right)' = \Sigma$. The value m is the compensator of a compound Poisson process $m = \log \varphi_{Z_1}(-i)$.

We define two sequences of independent and identically distributed $1 \times d$-dimensional random vectors $(X_k)_{k\in\mathbb{N}}$ and $(X_{0,k})_{k\in\mathbb{N}}$. The components of the random vectors in the first sequence are independent.

The process $(Z_t)_{t\geq 0}$ is a d-variate compound Poisson process, independent of $(B_t)_{t\geq 0}$ such that

$$Z_t^{(j)} = \sum_{k=1}^{N_t^{(j)}} X_k^{(j)} + \sum_{k=1}^{N_t^{(0)}} X_{0,k}^{(j)}, \ \ j = 1, \dots, d$$

where $(N_t)_{t\geq 0} = \left(N_t^{(0)}, N_t^{(1)}, \dots, N_t^{(d)}\right)_{t\geq 0}$ is a vector of independent Poisson processes with respective intensities λ_j.

The processes $\left(N_t^{(j)}\right)_{t\geq 0}$ and $\left(N_t^{(0)}\right)_{t\geq 0}$ correspond, respectively, to idiosyncratic and common jumps of the j-th underlying asset on the interval $[0, t]$. Their jump sizes are $X_k^{(j)}$ and $X_{0,k}^{(j)}$.

For the sake of concreteness, we assume Gaussian jumps, i.e., we assume for any $k\in\mathbb{N}$ that $X_k \sim N\left(\mu_J, D_J\right)$, where D_J is a diagonal matrix with components $D_J(j, l) = \delta_{jl}\left(\sigma_J^{(j)}\right)^2$ and $X_{0,k} \sim N\left(\mu_{0,J}, \Sigma_{0,J}\right)$, with $\Sigma_{0,J}$ a matrix of components $\Sigma_{0,J}(j, l) = \sigma_0^{j,l}$. The compensator across each dimension takes the form

$$m_j = \lambda_j \left(\exp\left(\mu_J^{(j)} + \frac{1}{2}\left(\sigma_J^{(j)}\right)^2 \right) - 1 \right) + \lambda_0 \left(\exp\left(\mu_{0,J}^{(j)} + \frac{1}{2}\left(\sigma_0^{jj}\right)^2 \right) - 1 \right), \; j = 1, 2, \ldots d$$

Let C_{JD} denote the price of a European basket option with payoff $h(S_T)$ under the model given by Eqs. (1) and (2).

First, we write the price of the basket contract in terms of its conditional price when the number of jumps and $d-1$ underlying assets are fixed. Results are given in Theorem 1 below.

Notice that, for any $k \in \mathbb{N}^{d+1}$

$$p_k = P(N_T = k) = \frac{\exp\left(-\sum_{j=0}^{d} \lambda_j T\right) \prod_{j=0}^{d} \lambda_j^{k_j} T^{\sum_{j=0}^{d} k_j}}{k!} \tag{3}$$

We also introduce the vector $\overline{\mu}(k)$ with components

$$\overline{\mu}_j(k) = \mu_j T + k_j \mu_J^{(j)} + k_0 \mu_{0,J}^{(j)} \quad j = 1, 2, \ldots, d.$$

Theorem 1. *Let C_{JD} be the price of a European basket contract with maturity T, strike price K, and payoff $h(Y_T)$, under a model given by Eqs.* (1) *and* (2)*. See proof in Appendix A.2.*

In addition assume $X_k \sim N(\mu_J, D_J)$ and $X_{0,k} \sim N(\mu_{0,J}, \Sigma_{0,J})$ for any $k \in \mathbb{N}$, where D_J is a $d \times d$ diagonal matrix with components $D_J(j,l) = \delta_{jl}\left(\sigma_J^{(j)}\right)^2$ and $\Sigma_J^{(0)}$ is also a $d \times d$ matrix with components $\Sigma_{0,J}(j,l) = \sigma_0^{j,l}$.

Then, we have

$$C_{JD} = \sum_{k \in \mathbb{N}^{d+1}} C(k) p_k \tag{4}$$

where for any $k \in \mathbb{N}^{d+1}$

$$C(k) := w_1 \exp\left(\frac{1}{2}\sigma^2(k)T\right) E_Q\left[\exp\left(\mu(\tilde{Y}_T, N_T)\right) C(\tilde{Y}_T, N_T) / N_T = k\right] \tag{5}$$

$$\begin{aligned} C(y,k) = e^{-rT} E_Q\Bigg[\Bigg(S_0^{(1)} \exp\left(\left(r - \frac{1}{2}\sigma^2(N_T)\right)T\right. \\ \left. + \; \sigma(N_T)\sqrt{T}Z)\right) - K(\tilde{Y}_T, N_T)\Bigg)_+ / N_T = k, \tilde{Y}_T = y\Bigg] \end{aligned} \tag{6}$$

with Z a standard normal random variable independent of N_T and $\tilde{Y}_T$.

Also

$$K(y,k) = \exp\left(\left(r - \frac{1}{2}\sigma^2(k)\right)T - \mu(y,k)\right)\left[\frac{K}{w_1} - \sum_{j=2}^{d} \frac{w_j}{w_1} S_0^{(j)} \exp\left(y^{(j)}\right)\right], \;\; \text{for } y \in \mathbb{R}^{d-1}$$

$$\mu(y,k) = \overline{\mu}_1(k) + \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)(y - \tilde{\mu}(k))'$$

$$\sigma(k) = \frac{1}{T}\left(\sigma_{11}(k) - \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)\Sigma'_{1\tilde{Y}}(k)\right)$$

Here $\sigma_{jl}(k)$ is the (j,l) component of the matrix:

$$\Sigma_Y(k) = \Sigma T + D_J \circ D_N + k_0 \Sigma_{0,J} \text{ and}$$

$$D_N(j,l) = \delta_{jl} N_T^{(j)}$$

$$\Sigma_{1\tilde{Y}}(k) = (\sigma_{12}(k), \sigma_{13}(k), \ldots, \sigma_{1,d-1}(k))'$$

Remark 2. *Notice that when $K(y,k)$ is nonnegative, $C(y,k)$ is the well-known Black-Scholes price of a call option with maturity at $T > 0$, volatility $\sqrt{\sigma(k)}$, spot price $S_0^{(1)}$, and strike price $K(y,k)$. A sufficient condition for $K(y,k)$ to be positive is $w_1 \geq 0$ while $w_j \leq 0, 2 \leq j \leq d$. It is the case of spreads and crack spreads. When $K(y,k)$ is negative, it does not have the meaning of a strike price anymore.*

Remark 3. *The values $\mu(y,k)$ and $\sigma(k)$ are, respectively, the mean and variance of the first asset after conditioning on a value y of the remaining assets and the certain number of jumps k.*

For any fixed $k \in \mathbb{N}^{d+1}$, we approximate the conditional price $C(y,k)$ on the variable y by a suitable polynomial. In particular we consider Taylor, Chebyshev polynomials and cubic splines.

Approximations based on the three expansions are discussed below.

(i) An order n Taylor approximation of $C(y,k)$ around $y^* \in \mathbb{R}^{d-1}$ is described by

$$C^T(y,y^*,k,n) = \sum_{l=0}^{n} \sum_{L \in R_l} \frac{D^L C(y^*,k)}{L!} (y - y*)^L \tag{7}$$

with $L = (l_1, l_2, \dots, l_{d-1})$, where the second sum is taken on the set

$$R_l = \left\{L \in \mathbb{N}^{d-1} / l_1 + l_2 + \dots + l_{d-1} = l, \quad 0 \leq l_j \leq l\right\}.$$

Notice the existence of the derivatives of any order in the functions $K(y)$ and $C(y,k)$.

(ii) An approximation based on Chebyshev polynomials is given as follows:

In a region $D \subset \mathbb{R}^{d-1}$, we consider an expansion of order $n = (n_1, n_2, \dots, n_{d-1})$ of the function $C(y,k)$ as

$$\begin{aligned} C^{Ch}(y,k,n) &= \frac{1}{2}\hat{c}_0(k) 1_D(y) + \sum_{l \in B_n} \hat{c}_l(k) T_l^D(y) 1_D(y) \\ &= \frac{1}{2}\hat{c}_0(k) 1_D(y) + \sum_{l \in B_n} \sum_{m \in C_l} \hat{c}_l(k) b_{m,l} y^{l-2m} 1_D(y) \end{aligned} \tag{8}$$

where the sums are taken over the sets

$$\begin{aligned} B_n &= \left\{l \in \mathbb{N}^{d-1} / 0 \leq l \leq n_j ; j = 1,2,\dots,d-1.\right\} \\ C_l &= \left\{m \in \mathbb{N}^{d-1} / 0 \leq m_j \leq \left[\frac{l_j}{2}\right], j = 1,2,\dots,d-1\right\} \end{aligned}$$

Here $\left(T_l^D\right)_{l \in B_n}$ is a family of $d-1$-dimensional Chebyshev polynomials with degrees $l \in B_n$ defined in the region D, while the quantities $\hat{c}_l(k)$ are suitable approximations of the corresponding Chebyshev coefficients $c_l(k)$, computed using the trapezoidal rule.

Notice that, by the orthogonality of the polynomials, the coefficients in the expansion are $c_l(k) = \langle C, T_l^D \rangle_W$, where $\langle f, g \rangle_W$ is the scalar product of functions f and g, conveniently weighted by a function W. See, for example, [10] for a general account on Chebyshev polynomials.

For convenience, we write the Chebyshev polynomials in terms of powers of their variables, where $b_{m,l}$ are the coefficients of this expansion.

In particular, for a rectangular region $D = [a,b]^{d-1}$ and valued vectors $a = (a_1, a_2, \dots, a_{d-1})$ and $b = (b_1, b_2, \dots, b_{d-1})$, we write

$$T^D(y) := T_l^{a,b}(y) = T_l^{-1,1}\left(-1 + 2\frac{y-a}{b-a}\right)$$

Hence, for $d = 2$

$$C^{Ch}(y,k) = \frac{1}{2}\hat{c}_0(k)1_D(y) + \sum_{l=1}^{n}\sum_{m=0}^{\left[\frac{l}{2}\right]}(b-a)^{2m-l}\hat{c}_l(k)b_{m,l}(2y-(a+b))^{l-2m}1_D(y) \tag{9}$$

See, for example, [11] for specific expressions of $b_{m,l}$ in one dimension.

(iii) Approximation by cubic splines.

On a rectangular region $D = [a,b]^{d-1}$, we consider an approximation based on cubic splines given by

$$C^{spl}(y,k) = \sum_{j=1}^{N}\sum_{l\in B_3}\alpha_{j,l}(k)\left(y-b_{j-1}\right)^l 1_{D_j}(y) \tag{10}$$

where b_j is some point on a (d − 1)-dimensional grid $\{b_0, b_1..., b_N\}$ with $N+1$ points in D.

The local coefficients $\alpha_{j,l}(k)$ are determined by imposing the conditions $C\left(y_j,k\right) = z_{jk}, j, k = 1, ..., N+1$. The family of sets $\{D_j,\ j = 0, 1, ..., N\}$ is a partition of D. Notice that the coefficients $\alpha_{j,l}(k)$ depend on the particular rectangle in the grid. See [12] for a general account on multivariate splines.

In the case of $d = 2$, splines used to approximate the conditional price become one-dimensional polynomials. Additional conditions on the derivatives to smoothen these curves are imposed, namely, $D^l_- C\left(y_j,k\right) = D^l_+ C\left(y_j,k\right)$, $j = 1, 2, ..., N, l = 1, 2,$ where $D^l_- C\left(y_j,k\right)$ and $D^l_+ C\left(y_j,k\right)$ are, respectively, the derivatives from the left and the right of the function $C(y,k)$ at point $y = y_j$. Moreover, for end points in the grid, $D^2(y_0,k) = D^2(y_N,k) = 0$.

In order to approximate the prices, we replace the function $C(y,k)$ by its respective expansions. The conditional prices on the event $[N_T = k]$ are estimated by approximating the corresponding conditional expected values. Substituting the approximations of conditional prices into Eq. (4), we obtain, after truncation, estimates of the price of the basket contract, under the jump-diffusion model described by Eqs. (1) and (2). They are denoted, respectively, by $C^T_{JD}(y^*)$, C^{Ch}_{JD}, and C^{spl}_{JD}.

Notice that these estimates depend on the mixing exponential-power moments of the log-prices. The latter can be computed from its conditional moment-generating function under the selected EMM. Hence, for a vector X and a Borel set D, we define

$$M_X(u,k) = E_Q[\exp(uX)/N_T = k]$$
$$M_X(u,k,D) = E_Q[\exp(uX)1_D(X)/N_T = k]$$

In particular when $D = [a,b]^{d-1}$, we write $M_X(u,k,D) = M_X(u,k,a,b)$.

Concrete expressions of these approximations under a two-dimensional Gaussian model are shown in Theorem 4.

As it is well known, the conditional mixed exponential-power moments of a random vector X are related to the partial derivatives of the corresponding moment-generating function Indeed, for $\nu \in \mathbb{N}^{d-1}$, we have

$$D^{\nu}M_X(u,k,D) = E_{\mathcal{Q}}[\exp(uX)X^{\nu}1_D(X)/N_T = k], \quad u \in \mathbb{R}^{d-1}$$

In order to simplify notations, we introduce the following quantities:

$$\begin{aligned} A_1(k) &= \frac{1}{2}\sigma^2(k)T + \overline{\mu}_1(k) - \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)\tilde{\mu}(k)' \\ A_2(k) &= A_1(k) + \Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)y^* \\ A_3(k) &= A_1(k) + \frac{1}{2}(a+b)\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k) \end{aligned}$$

and the set

$$\mathbb{N}_M^{d+1} = \{k = (k_0, k_1, \ldots, k_d)/k_j = 0, 1, \ldots, M_j, j = 0, 1, \ldots, d\}$$

Theorem 4. *Let C_{JD} be the price of a European basket contract with maturity T, strike price K, and payoff $h(Y_T)$ under a model given by Eqs. (1) and (2). In addition assume $X_k \sim N(\mu_J, D_J)$ and $X_{0,k} \sim N(\mu_{0,J}, \Sigma_{0,J})$ for any $k \in \mathbb{N}$, where D_J is a $d \times d$ diagonal matrix with components $D_J(j,l) = \delta_{jl}\left(\sigma_J^{(j)}\right)^2$. Let $\Sigma_J^{(0)}$ be a $d \times d$ matrix with components $\Sigma_{0,J}(j,l) = \sigma_0^{j,l}$.*

Then, its n-th-order approximation around $y^* \in \mathbb{R}^{d-1}$ in terms of Taylor polynomials is given by

$$C_{JD}^T(y^*) = w_1 \sum_{k \in \mathbb{N}_M^{d+1}} \sum_{l=0}^{n} \sum_{L \in R_l} \exp(A_2(k)) \frac{D^L C(y^*,k)}{L!} D^L M_{\tilde{Y}_T - y^*}\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k\right)p_k \tag{11}$$

for some truncation vector $M \in \mathbb{N}^{d+1}$.

The n-th-order Chebyshev approximation on a region $D = [a,b]^{d-1}$ is

$$\begin{aligned} C_{JD}^{Ch} = \frac{w_1}{2} &\sum_{k \in \mathbb{N}_M^{d+1}} [\hat{c}_0(k)K_1(a,b,k) \\ &+ w_1 \sum_{k \in \mathbb{N}_M^{d+1}} \sum_{l \in B_n} \sum_{m \in C_l} \exp(A_3(k))\hat{c}_l(k)b_{m,l}(b-a)^{2m-l} \\ &D^{l-2m}M_{\tilde{V}_T}(\frac{1}{2}\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k, -(b-a), b-a)]p_k \end{aligned} \tag{12}$$

where $\tilde{V}_T = 2\tilde{Y}_T - (a+b)$ and $K_1(a,b,n) = \exp(A_1(k))M_{\tilde{Y}_T}\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k, a, b\right)$.

The n-th-order approximation by cubic splines on the region $D = [a,b]^{d-1}$ is given by

$$\begin{aligned} C_{JD}^{spl} = w_1 &\sum_{k \in \mathbb{N}_M^{d+1}} \left[\exp\left(\frac{1}{2}\sigma^2(k)T\right)\right. \\ &\sum_{j=1}^{N} \sum_{l \in B_3} \exp\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)b_{j-1}\right)\alpha_{j,l}(k) \\ &\left. D^m M_{\tilde{Y}-b_{j-1}}\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k), k, D_j\right)\right]p_k \end{aligned} \tag{13}$$

Remark 5. *The point y^* around which the Taylor expansion is taken, in general, depends on k.*

3. Approximating the price of spread contracts

Spread contracts are the most common basket derivatives. In this case the payoff is written as $h(S_T) = \left(S_T^{(1)} - S_T^{(2)} - K\right)_+$.

Hence for $d = 2$, conditionally on $\left[Y_T^{(2)} = y\right] \cap [N_T = k]$, the log-prices of the first asset are normally distributed, i.e., $Y_T^{(1)} \sim N(\mu(y,k), \sigma^2(k))$, with

$$\mu(y,k) = \overline{\mu}_1(k) + \sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)(y - \overline{\mu}_2(k))$$
$$\sigma^2(k) = \frac{1}{T}\left(\sigma_{11}(k) - \frac{\sigma_{12}^2(k)}{\sigma_{22}(k)}\right) = \frac{1}{T}\left[1 - (\overline{\rho}(k))^2\right]\sigma_{11}(k)$$

where

$$\overline{\rho}(k) = \frac{\sigma_{12}(k)}{\sqrt{\sigma_{11}(k)}\sqrt{\sigma_{22}(k)}}$$

is the conditional correlation coefficient between the two assets.

A result about the derivatives of the moment-generating function of a constrained standard normal random variable Z on the interval $(-\infty, b)$ is needed. To this end we have

$$\begin{aligned} D^m M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k), k, -\infty, b\right) = \ & \exp\left(\frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right) \\ & bin\left(m, \mu V\left(-\infty, b - \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right), \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right) \end{aligned} \tag{14}$$

where $\mu(m,a,b) = \mu(m,-\infty,b) - \mu(m,-\infty,a) = \int_a^b z^m f_Z(z)\ dz$ is the m-th moment of a standard normal random variable constrained to the interval (a,b) and $\mu V(a,b)$ is a vector with components $\mu(j,a,b), j = 0, 1, \ldots, m$.

By integration by parts, the later can be calculated recursively as

$$\begin{aligned} \mu(0,a,b) &= N(b) - N(a) \\ \mu(1,a,b) &= f_Z(a) - f_Z(b) \\ \mu(m,a,b) &= (m-1)\mu(m-2,a,b) + a^{m-1}f_Z(a) - b^{m-1}f_Z(b)), \quad m \geq 2 \end{aligned}$$

For a Taylor expansion, derivatives of the moment-generating function and constrained moment-generating function for the second component of the log-prices are computed as follows:

$$\begin{aligned} D^l M_{Y_T^{(2)} - y*}\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k), k\right) = \ & \exp\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)(\overline{\mu}_2(k) - y^*)\right) \\ & bin\left(l, PV\left(\sqrt{\sigma_{22}(k)}\right)1 \circ DV\left(M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right)\right), \overline{\mu}_2(k) - y^*\right) \end{aligned}$$

Now, combining the expressions above with Eq. (11), we have

$$\begin{aligned} C_{JD}^T(y^*) = w_1 \sum_{k \in \mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T + \overline{\mu}_1(k)\right) \sum_{l=0}^{n} \binom{l}{m} \left(\frac{D^l C(y^*, k)}{l!}\right)(\overline{\mu}_2(k) - y^*)^{l-m} \\ bin\left(l, PV\left(\sqrt{\sigma_{22}(k)}\right)1 \circ DV\left(M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right)\right), \overline{\mu}_2(k) - y^*\right)p_k \end{aligned} \tag{15}$$

Next, we obtain the Taylor approximations up to third order. By elementary calculation we can compute the derivatives of the function $C(y,k)$ with respect to y.

First, notice that, from the Black-Scholes pricing formula:

$$C(y,k) = S_0^{(1)} N\big(d_1(K(y,k)) - K(y,k)e^{-rT} N(d_2(K(y,k))\big)$$

where

$$d_1(K(y,k)) = \frac{\log\left(\dfrac{S_0^{(1)}}{K(y,k)}\right) + \left(r + \dfrac{\sigma(k)}{2}\right)T}{\sigma(k)\sqrt{T}}$$
$$d_2(K(y,k)) = d_1(K(y,k)) - \sigma(k)\sqrt{T}$$

and $N(.)$ is the cumulative distribution function of a standard normal distribution. Hence

$$D^1C(y,k) = -\frac{1}{\sqrt{\sigma(k)T}} T_1(y,k)A(y,k)$$

where

$$T_1(y,k) = \frac{D^1K(y,k)}{K(y,k)}$$
$$A(y,k) = S_0^{(1)} f_Z(d_1(K(y,k))) + \sigma(k)\sqrt{T}e^{-rT}K(y,k)N(d_2(K(y,k)))$$
$$-e^{-rT}K(y,k)f_Z(d_2(K(y,k)))$$

Higher derivatives can be calculated recursively.

$$D^nC(y,k) = -\frac{1}{\sqrt{\sigma(k)T}} \sum_{l=0}^{n-1} \binom{n-1}{l} D^l T_1(y,k) D^{n-l-1} A(y,k)$$

Concrete expressions for second- and third-order derivatives are shown in the appendix.

Regarding the approximation based on Chebyshev polynomials, we first compute the moment-generating function of the random variables $Y_T^{(2)}$ and $V_T^{(2)}$ constrained to the interval (a,b). To this end we denote

$$\tilde{b} = \frac{b - \overline{\mu}_2(k)}{\sqrt{\sigma_{22}(k)}}, \quad \tilde{a} = \frac{a - \overline{\mu}_2(k)}{\sqrt{\sigma_{22}(k)}} \tag{16}$$

Notice that, taking into account Eq. (14),

$$\begin{aligned} &D^m M_Z\left(\sqrt{\sigma_{11}}\overline{\rho}(k), \tilde{a}, \tilde{b}\right) = \exp\left(\frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right) \\ &bin\left(m, \mu\left(m_1, \tilde{a} - \sqrt{\sigma_{11}(k)}\overline{\rho}(k), \tilde{b} - \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right), \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right) \end{aligned} \tag{17}$$

Moreover

$$M_{Y_T^{(2)}}\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k), k, -\infty, b\right) = \exp\left(\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)\overline{\mu}_2(k) + \frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right)$$
$$N\left(\tilde{b} - \sqrt{\sigma_{11}(k)}\overline{\rho}(k)\right)$$

Hence

$$D^{\nu}M_{V_T^{(2)}}\left(\frac{1}{2}\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k),k,-(b-a),b-a\right)$$
$$= \exp\left(\frac{1}{2}\sqrt{\frac{\sigma_{11}(k)}{\sigma_{22}(k)}}\overline{\rho}(k)(2\overline{\mu}_2(k)-a-b)\right)G(\nu,k)$$

where

$$G(\nu,k)=bin\left(\nu,M_Z\left(\sqrt{\sigma_{11}(k)}\overline{\rho}(k),k,\tilde{a},\tilde{b}\right)\circ PV\left(2\sigma_{22}^{\frac{1}{2}}(k)1\right),2\overline{\mu}_2(k)-a-b\right)$$

Then, combining Eq. (12) with the results above, we get

$$C^{Ch}(k,n)=\frac{w_1}{2}\hat{c}_0(k)K_1(a,b,k)$$
$$+w_1\exp\left(\frac{1}{2}\sigma(k)T+\overline{\mu}_1(k)\right)\sum_{l=1}^{n}\sum_{m=0}^{\left[\frac{l}{2}\right]}\hat{c}_l(k)b_{m,l}K(a,b,l,m)G(l-2m,k)$$

Finally, the n-th-order Chebyshev approximation is given by

$$C_{JD}^{Ch}=\sum_{k\in\mathbb{N}_M^3}C^{Ch}(k,n)p_k$$

Similarly for a cubic spline approximation, we specialize Eq. (13) with $D=(a,b), D_j=\left(b_{j-1},b_j\right), b_0=a, b_{N+1}=b$. Therefore, we have

$$C_{JD}^{spl}=w_1\sum_{k\in\mathbb{N}_M^{d+1}}\left[\exp\left(\frac{1}{2}\sigma(k)T+\overline{\mu}_1(k)\right)\right.$$
$$\left.\sum_{j=1}^{N}\sum_{l=0}^{3}\alpha_{j,l}(k)(\sigma_{22}(k))^{\frac{l}{2}}bin(l,DVM_Z(\sqrt{\sigma_{11}(k)}\overline{\rho}(k),\tilde{b}_{j-1},\tilde{b}_j),-\tilde{b}_{j-1})\right]p_k \tag{18}$$

where $\tilde{b}_j$ is defined as $\tilde{b}$ in Eq. (16) but replacing b by b_j.

4. Numerical results

We implement the results from the previous section to price spread contracts and show that the approximations considered above produce accurate price values when compared with a standard Monte Carlo approach, at a lesser computational effort.

To this end we consider the following benchmark set of parameters:

The contract specifications consist a strike price of $K=\$1$, maturity $T=1$ year, spot prices $S_0^{(1)}=\$100$, $S_0^{(2)}=\$96$, and a fix interest rate of 3%.

Volatilities corresponding to the diffusion part of both assets are $\sigma_1=10\%$ and $\sigma_2=30\%$, while the correlation coefficient between the two Brownian noises is $\rho=0.3$. Regarding the jump part, we consider an average intensity of the common jumps equal to $\lambda_0=3$ jumps per year and idiosyncratic intensities $\lambda_1=\lambda_2=2$ jumps per year for the respective assets, while jump sizes have means equal to zero; volatilities of common jump sizes are $\sigma_{0,1}=1\%$, $\sigma_{0,2}=5\%$, with a linear correlation $\rho_J=0.5$. Volatilities of the idiosyncratic jumps are taken as $\sigma_{J,1}=10\%$ and $\sigma_{J,2}=20\%$.

	MC	Taylor (f.o)	Taylor (s.o.)	Spl.
Price	14.7784	10.2980	14.29068	14.8842
Interval	(14.7683, 14.7885)	—	—	—
Run time	624.312	1.68806	1.68806	54.1720

In row three the average computer time (in seconds) for different pricing methods is shown.

Table 1.
Prices obtained using the benchmark parameter set and Monte Carlo, first- and second-order Taylor, and cubic spline approximations.

Although these values are somehow arbitrary, they have been selected to produce *reasonable* asset prices in connection with contracts based on crude oil prices. It is worth noting that there is not a general agreement about the range of the parameters in a jump-diffusion model. Indeed they may depend on the market into consideration.

In **Table 1** prices of spread contracts under different methods are shown. Prices are obtained using Taylor and cubic splines approximations and contrasted with a Monte Carlo approach. For the latter we carry 10^7 repetitions to achieve stable results, with a relative average error of 0.1%. In addition, 95% Monte Carlo confidence intervals and running times are provided. Implementation is done on a Surface Pro 4 i7 computer, using MATLAB language.

The efficiency of the Monte Carlo method can be improved by considering only the simulation of a single asset with the corresponding conditional probability and then computing the discounted average of the conditional Black-Scholes price. It reduces the computational time by half, still considerably higher than those based on polynomial expansions. Chebyshev polynomial approximation is discussed in [1].

The expansions also require repetitive evaluations of conditional prices, which turn out to be given by simple Black-Scholes closed formulas.

For a Taylor approach of order n, evaluations in the order of nM^3 are needed, where M is the maximum truncation level in the number of jumps. In a Chebyshev approach of the same order about n^2NM^3, evaluations of the conditional price should be performed, when a grid of N points is used in a trapezoidal approximation of the corresponding integrals. In a cubic splines approximation $3NM^3$. Here N is also the number of points in the grid where the polynomial coefficients are adjusted.

For a theoretical analysis of the error using Taylor and Chebyshev expansions, although in different contexts, see [13] for Taylor and [6] for Chebyshev cases.

In **Figure 1a**, a graph of conditional prices in function of log-price values of the first asset (blue line) with average number of jumps equal to $k_0 = 3$ and $k_1 = k_2 = 2$

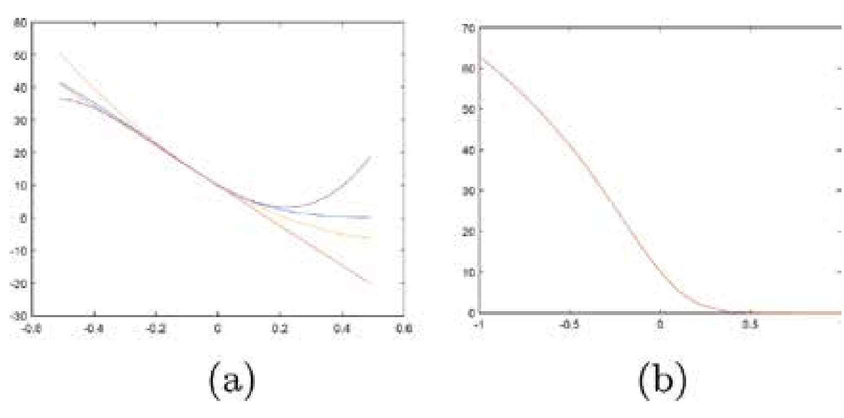

Figure 1.
(a) Conditional price (blue curve) as function of log-price values and its Taylor approximations up to third order around the average. (b) Conditional price vs. its cubic spline approximation.

is shown. The remaining three curves represent the first-order (green), second-order (red), and third-order (magenta) Taylor polynomials around the average value $y^* = E_{\mathbb{Q}}\left(Y_T^{(2)}\right)$. In **Figure 1b**, conditional prices and its cubic spline approximation are shown. At this scale both are indistinguishable. Notice that, although the Taylor approximation is excellent in a neighborhood of the expansion point, there are significant deviations for values far from the mean. These deviations, under the assumption of normality of the jump sizes, result to be infrequent; therefore, they do not impact the global error, but might be significant when other probability distributions, in particular heavy-tailed ones, or even normal jumps with higher volatilities, are taken into account. Instead of local approximations, as the case of Taylor polynomial expansion, uniform approximations on a given interval may reduce the error. Expansions based on orthogonal basis, e.g., Chebyshev or varying coefficients as in the case of cubic splines, are suggested. Notice that the function $C(y, k)$ is continuous in y for any value of k; therefore, Weierstrass' theorem of uniform convergence applies. Curiously, the convergence of Bernstein polynomials, applied in the original proof of the theorem, is remarkably slow.

Figure 2 shows the differences between the conditional price and the cubic spline for different values of the underlying price. Truncation values were selected as $a = -1$ and $b = 1$. Generally speaking the choice of these values depends on the probability distribution of the underlying asset. In practice it requires an exploratory study of the available data. On the other hand, the larger the interval, the more accurate is the approximation but also is the computational effort. Moreover, we have found that the results are sensible to this choice, though rather robust to the number of splines and the truncation values.

Truncation values for the number of jumps, denoted in the paper by M_0, M_1 and M_2, should cover most of the jump probability distribution $\{p_k, k \in \mathbb{N}^3\}$. An efficient way of choosing these values consists in starting to evaluate the sum at a point close to where the maximum value of the p_k's is attained, namely,

$$k = (k_0, k_1, k_2) = \left([\lambda_0 T - 1]_+, [\lambda_1 T - 1]_+, [\lambda_2 T - 1]_+\right)$$

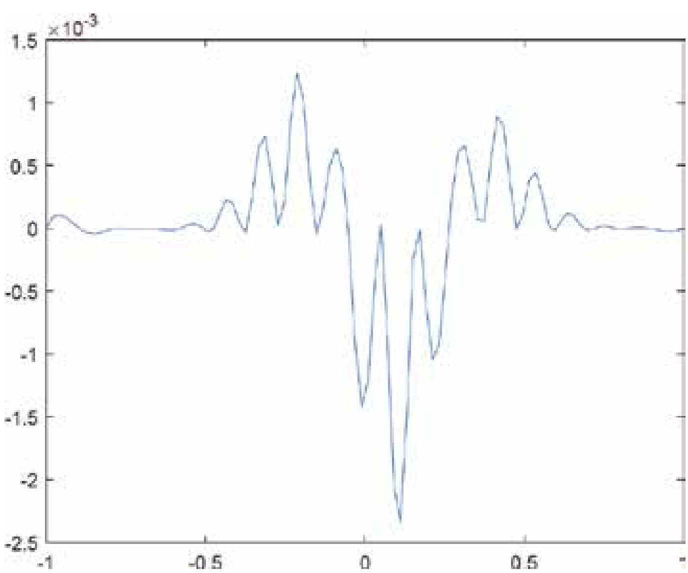

Figure 2.
Curve representing the difference between conditional price and cubic spline approximation for the benchmark parameters.

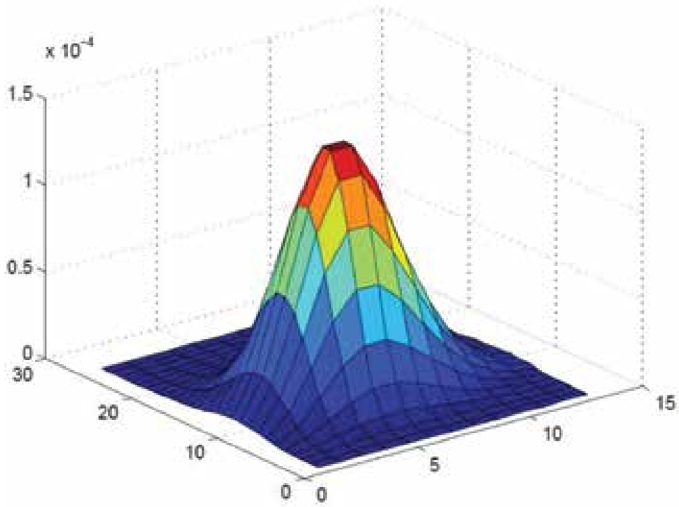

Figure 3.
Probabilities p_k to observe $k = (k_0, k_1)$ jumps when $k_2 = 5$. Truncation values $M_0 = 15, M_1 = 10, M_2 = 10$ capture 99.67% of the probability distribution in the number of jumps.

where $[x]_+$ represents the maximum of the integer part of x and zero, then adding expression (18) for points $j = (j_0, j_1, j_2)$ over the set

$$N_M(k) = \left\{ (k_0 + j_0, k_1 + j_1, k_2 + j_2) \in \mathbb{N}^3 / \left(k_l - \frac{M_l}{2} \right)_+ \le j_l \le k_l + \frac{M_l}{2}, l = 0, 1, 2 \right\}$$

until $\sum_k p_k \ge \delta$, where δ is a predetermined value close to one.

In **Figure 3** we show the probability distribution $\{p_k, k \in \mathbb{N}^3\}$, for $k_2 = 5$ varying k_0 and k_1. We observe probabilities become negligible after certain values of (k_0, k_1) with a peak around the center of the distribution. For the benchmark parameter set truncation values $M_0 = 15, M_1 = 10, M_2 = 10$ capture 99.67% of the probability mass.

5. Conclusions and future developments

The paper establishes a methodology over the use of polynomial approximations based on Taylor, Chebyshev, and cubic splines to the price of basket contracts. This approach produces accurate results at a lesser computational effort than a standard Monte Carlo technique. The claim is supported by numerical evidence in the case of spread options, under a bivariate jump-diffusion model with a complex Gaussian jump structure that allows to capture the dependence between assets.

The study needs to be extended to different parameter values to corroborate the results in a wider scope. Moreover, optimal choices in the numerical implementation, for example, the order of the polynomials, the number of points in the grid, and truncation levels, require a further study.

Sensitivities with respect to the parameters in the model and the contract, i.e., maturity, strike, interest rate, correlation, etc., can be easily calculated with a straightforward adaptation of the current method. It is enough to approximate the corresponding derivatives instead.

A natural question is how to adapt our method when a non-Gaussian joint distribution of the jump sizes is considered. In this setting, the conditional probability distribution is generally unknown; nonetheless, the use of a copula approach to capture the dependence may provide some insight.

Acknowledgements

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A. Appendix

A.1 Taylor implementation up to third order

After computing the second and third derivatives of $C(y,k)$ and the corresponding derivatives of the moment-generating function of Z, we can compute Taylor approximations up to third order around the point y^* as

$$
\begin{aligned}
C_{JD}^{T}(y^*,1) &= w_1 \sum_{k\in\mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T+\overline{\mu}_1(k)+\frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right)\\
&\qquad \left[C(y^*,k)+D^{(1)}C(y^*,k)\left((\overline{\mu}_2(k)-y^*)+\sqrt{\sigma_{11}(k)\sigma_{22}(k)}\overline{\rho}(k)\right)\right]p_k\\
C_{JD}^{T}(y^*,2) &= C_{JD}^{T}(y^*,1)+w_1 \sum_{k\in\mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T+\overline{\mu}_1(k)+\frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right)D^{(2)}\\
&\qquad C(y^*,k)\left[\frac{1}{2}(\overline{\mu}_2(k)-y^*)^2+(\overline{\mu}_2(k)-y^*)\sqrt{\sigma_{11}(k)\sigma_{22}(k)}\overline{\rho}(k)\right.\\
&\qquad \left.+\frac{1}{2}\left(\sigma_{22}(k)(1+\sigma_{11}(k))(\overline{\rho}(k))^2\right)\right]p_k\\
C_{JD}^{T}(y^*,3) &= C_{JD}^{T}(y^*,2)+\ w_1 \sum_{k\in\mathbb{N}_M^3} \exp\left(\frac{1}{2}\sigma(k)T+\overline{\mu}_1(k)+\frac{1}{2}\sigma_{11}(k)(\overline{\rho}(k))^2\right)D^{(3)}C(y^*,k)\\
&\qquad \left[\frac{1}{6}(\overline{\mu}_2(k)-y^*)^3+\frac{1}{2}(\overline{\mu}_2(k)-y^*)^2\sqrt{\sigma_{11}(k)\sigma_{22}(k)}\overline{\rho}(k)\right.\\
&\qquad +\frac{1}{2}(\overline{\mu}_2(k)-y^*)\sigma_{22}(k)\left(1+\sigma_{11}(k)(\overline{\rho}(k))^2\right)\\
&\qquad \left.+\frac{1}{6}\sigma_{22}(k)^{\frac{3}{2}}\sqrt{\sigma_{11}(k)}\overline{\rho}(k)\left(\sigma_{11}(k)(\overline{\rho}(k))^2+3\right)\right]p_k
\end{aligned}
$$

A.2 Proof of Theorem 1

From Eq. (2) written in its integral form

$$Y_T=\mu T+\Sigma^{\frac{1}{2}}B_T+Z_T$$

it is easy to see that

$$E_Q(Y_T/N_T) = \mu T + \tilde{N}_T \circ \mu_J + N_T^{(0)}\mu_{0,J} = \overline{\mu}_j\left(\tilde{N}_T\right)$$

$$\Sigma_Y(N_T) := Var(Y_T/N_T) = \Sigma^{\frac{1}{2}} Var(B_T)\Sigma^{\frac{1}{2}} + Var(Z_T/N_T)$$

$$= \Sigma T + Var(Z_T/N_T)$$

From the expression above, in the case of $j \neq l$, we have

$$\mathrm{cov}\left(\left(Z_T^{(j)}, Z_T^{(l)}\right)/N_T\right) = E_Q\left(\sum_{k=1}^{N_T^{(0)}}\sum_{k'=1}^{N_T^{(0)}}\left(X_{0,k}^{(j)} - E_Q X_{0,k}^{(j)}\right)\left(\tilde{X}_{k'}^{(l)} - E_Q X_{0,k}^{(l)}\right)/N_T\right)$$

$$= N_T^{(0)}\mathrm{cov}\left(X_{0,k}^{(j)}, X_{0,k}^{(l)}/N_T\right) = N_T^{(0)}\sigma_0^{j,l}$$

Similarly, for $j = l$

$$\mathrm{cov}\left(Z_T^{(j)}, Z_T^{(l)}/N_T\right) \quad = \quad N_T^{(j)}\left(\sigma_J^{(j)}\right)^2 + N_T^{(0)}\sigma_0^{j,j}$$

Then, conditionally on N_T, we have

$$Y_T \sim N\left(\mu T + N_T \circ \mu_J + N_T^{(0)}\mu_{0,J}, \Sigma_Y(N_T)\right) \tag{19}$$

Hence, the price is expressed as

$$C_{JD} = e^{-rT}\sum_{k\in\mathbb{N}^{d+1}} E_Q(h(S_T)/N_T = k)p_k = \sum_{k\in\mathbb{N}^{d+1}} C(k)p_k \tag{20}$$

where $C(k) = e^{-rT}E_Q[h(S_T)/N_T = k]$.
On the other hand, conditioning on $[N_T = k] \cap \tilde{Y}_T$:

$$\begin{aligned}
&C(k) := e^{-rT}\\
&E_Q[h(S_T)/N_T = k] = e^{-rT}E_Q\left[E_Q\left(h(S_T)/N_T = k, \tilde{Y}T\right)/N_T = k\right]\\
&\quad = w_1 e^{-rT}\\
&E_Q\left[E_Q\left[\left(S_0^{(1)}\exp\left(Y_T^{(1)}\right) - \left(\frac{K}{w_1} - \sum_{j=2}^{d}\frac{w_j}{w_1}S_0^{(j)}\exp\left(Y_T^{(j)}\right)\right)\right)_+ /N_T = k, \tilde{Y}_T\right]/N_T = k\right]
\end{aligned}$$

$$= w_1 e^{-rT}E_Q\left[E_Q\left[\left(S_0^{(1)}\exp\left(Y_T^{(1)}\right) - K_1\left(\tilde{Y}_T\right)\right)_+ /N_T = k, \tilde{Y}_T\right]/N_T = k\right] \tag{21}$$

where $K_1(y) = \frac{K}{w_1} - \sum_{j=2}^{d}\frac{w_j}{w_1}S_0^{(j)}e^{y^{(j)}}$.

Taking into account Eq. (19), again conditioning on the events $\tilde{Y}_T = y$ and $N_T = k$, it is well known that $Y_T^{(1)}$ has a univariate normal distribution with mean and variance given, respectively, by $\mu(y, k)$ and $\sigma^2(k)T$. See, for example, [14].

Hence, we can write, on the set $\left[\tilde{Y} = y \cap N_T = k\right]$:

$$Y_T^{(1)} = \mu(y, k) + \sigma(k)\sqrt{T}Z$$

Then, replacing the expression above in Eq. (21), we have

$$\begin{aligned} C(k) &= w_1 e^{-rT} E_{\mathcal{Q}}\Big[E_{\mathcal{Q}}\Big[\Big(S_0^{(1)}\exp\left(\mu\big(\tilde{Y}_T,N_T\big)+\sigma(N_T)\sqrt{T}Z\right)-K_1\big(\tilde{Y}_T\big)\Big)_+/\mathcal{F}^{\tilde{Y}_T},N_T=k\Big]/N_T=k\Big] \\ &= w_1 e^{-rT} E_{\mathcal{Q}}\Big[\exp\left(-\left(r-\frac{1}{2}\sigma(k)\right)T+\mu(\tilde{Y}_T,k)\right)E_{\mathcal{Q}}\Big[\Big(S_0^{(1)}\exp\left(\left(r-\frac{1}{2}\sigma^2(N_T)\right)T+\sigma(N_T)\sqrt{T}Z\right) \\ &\quad -\exp\left(\left(r-\tfrac{1}{2}\sigma^2(N_T)\right)T-\mu(\tilde{Y}_T,N_T)\right)K_1\big(\tilde{Y}_T\big)\Big)_+/\mathcal{F}^{\tilde{Y}_T},N_T\Big]/N_T=k\Big] \\ &= w_1\exp\left(\frac{1}{2}\sigma^2(k)T\right)E_{\mathcal{Q}}\left[\exp\left(\mu\big(\tilde{Y}_T,N_T\big)\right)C\big(\tilde{Y}_T,N_T\big)/N_T=k\right] \end{aligned} \tag{22}$$

Eq. (4) easily follows after replacing Eq. (22) into Eq. (20).

A.3 Proof of Theorem 4

In Eq. (6) we replace the function $C(y,k)$ by its Taylor expansion given in Eq. (7).

Then, the Taylor approximation of $C(k)$ is

$$\begin{aligned} C^T(y^*,k) &= w_1\exp\left(\frac{1}{2}\sigma^2(k)T\right)E_{\mathcal{Q}}\left[\exp\left(\mu\big(\tilde{Y}_T,N_T\big)\right)C^T\big(\tilde{Y}_T,y^*,N_T\big)/N_T=k\right] \\ &= w_1\sum_{l=0}^{n}\sum_{L\in R_l}\frac{D^L C(y^*,k)}{L!}\,exp\left(\frac{1}{2}\sigma^2(k)T+\overline{\mu}_1(k)-\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k)\tilde{\mu}(k)'\right) \\ &\quad E_{\mathcal{Q}}\left[\exp\left(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\right)\big(\tilde{Y}_T-y^*\big)^L/N_T=k\right] \\ &= w_1\exp\left(A_2(k)\right)\sum_{l=0}^{n}\sum_{R_l}\frac{D^L C(y^*,k)}{L!}D^L M_{\tilde{Y}_T-y^*}\left(\Sigma_{1\tilde{Y}}(k)\Sigma_{\tilde{Y}}^{-1}(k),k\right) \end{aligned}$$

Eq. (11) follows after replacing $C(k)$ in Eq. (20) by the expression above and truncating at point M.

After replacing Eq. (9) into Eq. (22), we have

$$\begin{aligned} C^{Ch}(k) &= \frac{w_1}{2}\hat{c}_0(k)\exp\left(A_1(k)\right)E_{\mathcal{Q}}\left[\exp\left(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\right)1_D\big(\tilde{Y}_T\big)/N_T=k\right] \\ &\quad +w_1\exp\left(A_1(k)\right)\sum_{l\in B_n}\hat{c}_l(k)E_{\mathcal{Q}}\left[\exp\left(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\right)T_l^D\big(\tilde{Y}_T\big)/N_T=k\right] \\ &= \frac{w_1}{2}\hat{c}_0(k)K_1(k,a,b)+w_1\exp\left(A_1(k)\right)\sum_{l\in B_n}\sum_{m\in C_l}\hat{c}_l(k)b_{m,l} \\ &\quad E_{\mathcal{Q}}\left[\exp\left(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\right)\left(-1+2\frac{\tilde{Y}_T-a}{b-a}\right)^{l-2m}1_D\big(\tilde{Y}_T\big)\right] \end{aligned}$$

Eq. (12) easily follows.

Finally, by similar arguments,

$$\begin{aligned} C_{JD}^{spl}(k) &= w_1\exp\left(\frac{1}{2}\sigma^2(k)T\right) \\ &\quad \sum_{j=1}^{N}\sum_{l\in B_3}\alpha_{j,l}(k)E_{\mathcal{Q}}\left(\exp\left(\Sigma_{1\tilde{Y}}(N_T)\Sigma_{\tilde{Y}}^{-1}(N_T)\tilde{Y}_T\right)\big(\tilde{Y}_T-c_j\big)^l 1_{D_j}\big(\tilde{Y}_T\big)/N_T=k\right) \end{aligned}$$

$$
\begin{aligned}
&= w_1 \exp\left(\frac{1}{2}\sigma^2(k)T\right)\\
&\sum_{j=1}^{N}\sum_{l\in B_3} \exp\left(\Sigma_{1\tilde{Y}}(k)\Sigma\tilde{Y}^{-1}(k)c_j\right)\alpha_{j,l}(k)D^l M_{\tilde{Y}-c_j}\left(\Sigma_{1\tilde{Y}}(k)\Sigma\tilde{Y}^{-1}(k),k,D_j\right)
\end{aligned}
$$

from which (13) follows.

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Chapter 8

The Orthogonal Expansion in Time-Domain Method for Solving Maxwell Equations Using Paralleling-in-Order Scheme

Zheng-Yu Huang, Zheng Sun and Wei He

Abstract

The orthogonal expansion in time-domain method is a new kind of unconditionally stable finite-difference time-domain (FDTD) method for solving the Maxwell equation efficiently. Generally, it can be implemented by two schemes: marching-on-in-order and paralleling-in-order, which, respectively, use weighted Laguerre polynomials and associated Hermite functions as temporal expansions and testing functions. This chapter summarized paralleling-in-order-based FDTD method using associated Hermite functions and Legendre polynomials. And a comparison from theoretical analysis to numerical examples is shown. The LD integral transfer matrix can be considered as a "dual" transformation for AH differential matrix, which gives a possible way to find more potential orthogonal basis function to implement a paralleling-in-order scheme. In addition, the differences with these two orthogonal functions are also analyzed. From the numerical results, we can see their agreements in some general cases while differing in some cases such as shielding analysis with the long-time response requirement.

Keywords: associated Hermite, finite-difference time-domain (FDTD), Legendre polynomials, paralleling-in-order, unconditionally stable

1. Introduction

To overcome the numerical stability constraints of conventional finite-difference time-domain (FDTD) method [1, 2], many unconditionally stable methods to reduce or eliminate requirements of the stability condition have been proposed and developed, such as alternating-direction implicit method [2, 3] and locally one-dimensional schemes [3], explicit and unconditionally stable FDTD method [4], and orthogonal expansions in time domain [5–8]. For the orthogonal expansions schemes, field-versus-time variations in the FDTD space lattice are expanded using an appropriate set of orthogonal temporal basis and testing functions, such as weighted Laguerre polynomials (WLP) and associated Hermite (AH) functions, which leads to two different solution schemes: marching-on-in-order and paralleling-in-order, respectively. Both of them appear to be promising according to the reported work where the computational time can be reduced to at least 10% of the conventional FDTD scheme [1]. Recently, the Legendre (LD) polynomials are explored as another possible orthogonal expansion incorporated with FDTD to form a paralleling-in-order-based unconditionally stable FDTD method. Based on it, in this chapter, we made a comparison investigation for these two new methods, which are AH FDTD method and LD FDTD method, especially focused on their differences. Through a numerical example, we validate their effectiveness when compared with the conventional FDTD method and summarized the characteristics of the two methods.

2. Formulation for paralleling-in-order scheme: AH and LD functions

2.1 2D Maxwell's equations in time domain

The 2D time-domain Maxwell's equations with the TEz wave case in lossy medium are considered:

$$\varepsilon \frac{\partial E_x(r,t)}{\partial t} + \sigma_e E_x(r,t) = \frac{\partial H_z(r,t)}{\partial y} - J_x(r,t) \tag{1}$$

$$\mu \frac{\partial H_z(r,t)}{\partial t} + \sigma_m H_z(r,t) = \frac{\partial E_x(r,t)}{\partial y} - \frac{\partial E_y(r,t)}{\partial x} - M_z(r,t) \tag{2}$$

$$\varepsilon \frac{\partial E_y(r,t)}{\partial t} + \sigma_e E_y(r,t) = -\frac{\partial H_z(r,t)}{\partial x} - J_y(r,t) \tag{3}$$

where ε, μ, σ_e, and σ_m are the permittivity, the permeability, the electric conductivity, and the magnetic loss of the medium, respectively. $E_\xi(r,t)$ and $J_\xi(r,t)$ $(\xi = x, y)$ are the electric field component and the electric current densities, respectively. $H_z(r,t)$ and $M_z(r,t)$ are the magnetic field component and magnetic current densities, respectively.

2.2 The differential and integral transfer matrices to deal with the partial differential term in Maxwell's equations

2.2.1 The associated Hermite function

Associated Hermite function is defined as

$$\left\{ \phi_n(t) = \left(2^n n! \pi^{1/2}\right)^{-1/2} e^{-t^2/2} H_n(t) \right\}, \text{(n = 0, 1...)} \tag{4}$$

where $H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right)$ is Hermite polynomials. Although it is not causal, it can be transformed into causal form by virtue of a proper translating and scaling parameters and then used to span the causal electromagnetic responses. The transformed basis function is $\left\{\overline{\phi}_n(\tilde{t}) = \left(2^n n! \sigma \pi^{1/2}\right)^{-1/2} e^{-\tilde{t}^2/2} H_n(\tilde{t})\right\}$, where transformed time variable $\tilde{t} = \left(t - T_f\right)/\sigma$. And T_f is a translating parameter and σ is a scaling parameter. By controlling these two parameters, the time-frequency support of the AH functions $\{\overline{\phi}_n(\tilde{t})\}$ space can be changed flexibly. So, arbitrary locally time-supported functions can be spanned by these transformed basis functions, including the causal electromagnetic responses.

From [7], if a causal function $u(r,t)$, such as the electric or magnetic field function, can be expanded by

$$u(r,t) = \sum_{n=0}^{\infty} u_n(r)\overline{\phi}_n(\tilde{t}) \tag{5}$$

we can deduce the first derivative of $u(x, t)$ with respect to

$$\frac{\partial}{\partial t}u(r,t) = \frac{1}{\sigma}\sum_{n=0}^{\infty}\left(u_{n+1}(r)\sqrt{\frac{n+1}{2}} - u_{n-1}(r)\sqrt{\frac{n}{2}}\right)\overline{\phi}_n(\tilde{t}) \tag{6}$$

Then, the Q-tuple AH domain coefficients for $u(r, t)$ and $\dot{u}(r, t)$ from (5) and (6) can be obtained as $U = \left[U^0 \cdots U^{Q-1}\right]^T$ and $\dot{U} = \left[\dot{U}^0 \cdots \dot{U}^{Q-1}\right]^T$. And, we can readily obtain the relationship between $U$and $\dot{U}$ as

$$\dot{U} = \alpha U \tag{7}$$

where

$$\alpha = \frac{\sqrt{2}}{2\lambda}\begin{bmatrix} \sqrt{1} & & & & \\ -\sqrt{1} & \sqrt{2} & & & \\ & -\sqrt{2} & \ddots & & \\ & & \ddots & \sqrt{Q-1} & \\ & & & -\sqrt{Q-1} & \end{bmatrix}_{Q\times Q} \tag{8}$$

By using (8), the partial differential term in Maxwell's equations can readily be dealt with, and finally, a five-point banded matrix equation for Hz component can be obtained [9].

2.2.2 The associated Legendre polynomial

We expand all the temporal quantities in terms of the associated Legendre polynomial given by [10]:

$$P_q(t) = \sqrt{\frac{2q+1}{2}}L_q\left(2\frac{t}{l} - 1\right), \quad t \in [0, l] \tag{9}$$

where l is the time support for analyzing a causal response and Lq is the Legendre polynomial with order q, which are orthogonal in the interval $[-1,1]$ satisfying the following recurrence relation:

$$L_{q+1}(t) = \frac{2q+1}{q+1}tL_q(t) - \frac{q}{q+1}L_{q-1}(t), \tag{10}$$

and $L_0(t) = 0$, $L_1(t) = t$. Given a time-support field function $u(r, t)$, it can be expanded by (9) as

$$u(r,t) = \sum_{q=0}^{\infty} u_q(r)P_q(t) \tag{11}$$

where $u_q(r)$ is the q-th expanding coefficients, and it can be calculated by

$$u_q(r) = \int_{-\infty}^{+\infty} u(r,t)P_q(t)dt \tag{12}$$

From the intrinsic features of Legendre function, the differential relationship can be described as

$$P_q(t) = \frac{1}{2\sqrt{(2q+3)(2q+1)}} P'_{q+1}(t) - \frac{1}{2\sqrt{(2q+1)(2q-1)}} P'_{q-1}(t) \tag{13}$$

If the field derivative of $u(r, t)$ to t is expanded as

$$u'(r, t) = \sum_{q=0}^{\infty} u_q^{(1)}(r) P_q(t) \tag{14}$$

where $u_q^{(1)}(r)$ is q-th expanding coefficients for $u'(r, t)$, then incorporated with (13), it can be deduced as

$$u'(r, t) = \left(\sum_{q=0}^{\infty} \left(\frac{l}{2\sqrt{(2q+1)(2q-1)}} u_{q-1}^{(1)}(r) - \frac{l}{2\sqrt{(2q+3)(2q+1)}} u_{q+1}^{(1)}(r) \right) P_q(t) \right)' \tag{15}$$

Connecting (15) and (11), we can get

$$u_q(r) = \frac{l}{2\sqrt{(2q+1)(2q-1)}} u_{q-1}^{(1)}(r) - \frac{l}{2\sqrt{(2q+3)(2q+1)}} u_{q+1}^{(1)}(r) \tag{16}$$

When assembling $\{u_q(r)\}_{q=0,1\cdots Q-1}$ as a Q-tuple U and $\left\{u_q^{(1)}(r)\right\}_{q=0,1\cdots Q-1}$ as $U^{(1)}$, a matrix-multiply relationship can be obtained from (16) as the following:

$$U = \alpha_L U^{(1)} \tag{17}$$

where α_L is integral matrix.

$$\alpha_L = \frac{l}{2} \begin{bmatrix} & \text{-}1/\sqrt{1\cdot 3} & & & \\ 1/\sqrt{1\cdot 3} & & \text{-}1/\sqrt{3\cdot 5} & & \\ & 1/\sqrt{3\cdot 5} & & \ddots & \\ & & \ddots & & \text{-}1/\sqrt{(2Q-3)(2Q-1)} \\ & & & 1/\sqrt{(2Q-3)(2Q-1)} & \end{bmatrix}_{Q\times Q} \tag{18}$$

Alternatively, Eq. (17) can be rewritten as.

$$U^{(1)} = \alpha_L^{-1} U \tag{19}$$

2.3 From time domain to orthogonal domain and reconstruction

When the differential or integral transfer matrices are obtained, the time-domain Maxwell equation can be transformed directly into AH or LD domain. Here, let us set LD as an example to illustrate the later formulation.

Similar to the paralleling-in-order-based AH FDTD method, we can apply a Q-tuple-domain transformation for LD FDTD method to (1)–(3) and discretize them as the following:

$$\alpha_{e(i,j)} E_x|_{i,j} = \left(H_z|_{i,j} - H_z|_{i,j-1} \right) / \Delta\bar{y}_j - J_x|_{i,j} \tag{20}$$

$$\alpha_{e(i,j)} E_y\big|_{i,j} = -\left(H_z|_{i,j} - H_z|_{i-1,j}\right)/\Delta\overline{x}_i - J_y\Big|_{i,j} \tag{21}$$

$$\alpha_{m(i,j)} H_z\big|_{i,j} = \left(E_x|_{i,j+1} - E_x|_{i,j}\right)/\Delta y_j - \left(E_y\big|_{i+1,j} - E_y\big|_{i,j}\right)/\Delta x_i - M_z|_{i,j} \tag{22}$$

where

$$\alpha_{e(i,j)} = \varepsilon|_{i,j}\alpha_L^{-1} + \sigma_e|_{i,j} I \tag{23}$$

$$\alpha_{m(i,j)} = \mu_m|_{i,j}\alpha_L^{-1} + \sigma_m|_{i,j} I \tag{24}$$

where $E_x|_{i,j}$, $E_y\big|_{i,j}$, $H_z|_{i,j}$, $J_x|_{i,j}$, $J_y\Big|_{i,j}$, and $M_z|_{i,j}$ are Q-tuple representations of fields and sources, respectively. And, I is the Q-dimensional identity matrix. By assembling (20)–(22) and eliminating the electric field components, a five-diagonal banded matrix equation for Hz component can be obtained:

$$a_{l(i,j)} H_z|_{i-1,j} + a_{r(i+1,j)} H_z|_{i+1,j} + a_{m(i,j)} H_z|_{i,j} + a_{d(i,j)} H_z|_{i,j-1} + a_{u(i,j+1)} H_z|_{i,j+1} = b_{i,j} \tag{25}$$

where

$$a_{u(i,j+1)} = -\alpha_{e(i,j+1)}^{-1}/\Delta\overline{y}_{j+1}/\Delta y_j \tag{26}$$

$$a_{d(i,j)} = -\alpha_{e(i,j)}^{-1}/\Delta\overline{y}_j/\Delta y_j \tag{27}$$

$$a_{l(i,j)} = -\alpha_{e(i,j)}^{-1}/\Delta\overline{x}_i/\Delta x_i \tag{28}$$

$$a_{r(i+1,j)} = -\alpha_{e(i+1,j)}^{-1}/\Delta\overline{x}_{i+1}/\Delta x_i \tag{29}$$

$$a_{m(i,j)} = -\left(a_{r(i+1,j)} + a_{l(i,j)} + a_{u(i,j+1)} + a_{d(i,j)} + \alpha_{m(i,j)}\right) \tag{30}$$

$$b_{i,j} = -\left(\alpha_{m(i,j+1)}^{-1} J_x\Big|_{i,j+1} - \alpha_{m(i,j)}^{-1} J_x\Big|_{i,j}\right)/\Delta y + \left(\alpha_{m(i+1,j)}^{-1} J_y\Big|_{i+1,j} - \alpha_{m(i,j)}^{-1} J_y\Big|_{i,j}\right)/\Delta x - M_z|_{i,j} \tag{31}$$

By using eigenvalue transformation from $\alpha_L X = XV$, where X and V are the eigenvector matrix and diagonal matrix composed of eigenvalues $\{\lambda_q\}$, respectively, Eq. (25) can be changed to the paralleling-in-order solution. For the q-th decoupled equation, we have

$$A\left(1/\lambda_q\right) H_z^*|^q = b^*|^q \tag{32}$$

where $A(\cdot)$ is a banded sparse matrix, with the similar form as from AH FDTD method, and $b^*|^q$is the transformed variables from $b|_{i,j} = Xb^*|_{i,j}$. Finally, we can obtain a paralleling-in-order scheme to calculate all of the expanding coefficients of electromagnetic fields, and then the time-domain responses can be reconstructed from (11).

3. Comparison for the two methods

The above formula can be regarded and classified as a uniform OF differential transfer matrix transformation. Therefore, as long as the LD differential matrix is replaced by the AH domain differential transfer matrix, the FDTD algorithm based

AH FDTD	LD FDTD
Differential transfer matrix	Integral transfer matrix
$U^{(1)} = \alpha U$	$U = \alpha_L U^{(1)}$
$\alpha_{(l)} = \frac{\sqrt{2}}{2l}\begin{bmatrix} & \sqrt{1} & & & \\ -\sqrt{1} & & \sqrt{2} & & \\ & -\sqrt{2} & & \ddots & \\ & & \ddots & & \sqrt{Q-1} \\ & & & -\sqrt{Q-1} & \end{bmatrix}_{Q\times Q}$	$\alpha_{L(l)} = \frac{l}{2}\begin{bmatrix} & -1/\sqrt{1\cdot 3} & & & \\ 1/\sqrt{1\cdot 3} & & -1/\sqrt{3\cdot 5} & & \\ & 1/\sqrt{3\cdot 5} & & \ddots & \\ & & \ddots & & -1/\sqrt{(2Q-3)(2Q-1)} \\ & & & 1/\sqrt{(2Q-3)(2Q-1)} & \end{bmatrix}_{Q\times Q}$
$\frac{\partial}{\partial t} \leftarrow \alpha \rightarrow j\omega$	$\int dt \leftarrow \alpha_L \rightarrow \frac{1}{j\omega}$
$\begin{cases} T_Q \approx 2l\left(\sqrt{\pi Q/1.7} + 1.8\right) \\ F_Q \approx \dfrac{\sqrt{\pi Q/1.7} + 1.8}{2\pi l} \end{cases} \rightarrow (l,\ Q)$	Scale factor l = TQ Finite order of Q
With time-frequency Homomorphism	Without time-frequency Homomorphism
Antisymmetry Eigenvalue conjugate symmetry	Antisymmetry Eigenvalue conjugate symmetry
$A(\lambda_q)H^q = J^q$	$A(1/\lambda_q)H^q = J^q$

Table 1.
LD comparison of LD FDTD method and AH FDTD method.

on the LD orthogonal basis function, LD FDTD, including the parallel solution AH FDTD algorithm [9], and the alternate direction efficient calculation [11] can be easily realized. The implementation of the program only requires a simple modification.

Table 1 gives a comparison of the relevant properties of the LD FDTD method and the AH FDTD method. It can be seen that the two methods can be considered as a "dual" system, because the AH differential matrix is the basic element of the AH FDTD method and the LD integration matrix is also the basic element of the LD FDTD method. This gives us a revelation that is it possible that any orthogonal basis function can construct a differential or integral transfer matrix and then easily implement a paralleling-in-order scheme similar like AH FDTD algorithm? The answer might be *NOT*. Such as the Laguerre FDTD method, as introduced before, cannot be calculated in parallel. However, it is undeniable that there may be more basis functions that can implement the paralleling-in-order scheme. If any, we can collectively call these methods as the AH series unconditionally stable FDTD method.

4. Numerical verification

4.1 An infinitely large lossy dielectric plate

As AH or LD FDTD method shares with almost the same program, a 1-D program is set for a general verification. **Figure 1** shows the simulation results when a uniform plane wave penetrates an infinitely large lossy dielectric plate. The figure includes the electric field waveforms calculated by the AH FDTD method and the LD FDTD method and their relative errors with respect to the conventional FDTD method. It can be seen that the time-domain waveforms of both can be consistent with the results of the FDTD method and the relative errors are basically the same,

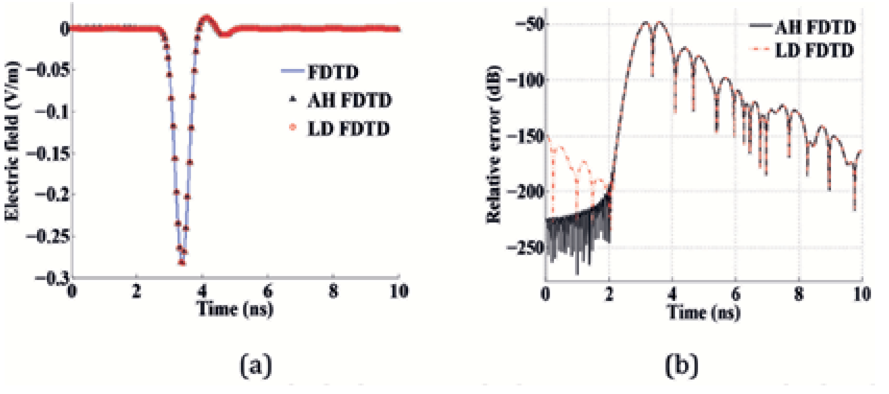

Figure 1.
Comparison of calculation results between AH FDTD method and HR FDTD method when simulating an infinitely large lossy dielectric plate. (a) Time-domain waveform. (b) Relative error.

only differing in the initial part. Therefore, in general, when the order of the two basic functions is the same and the parameters are selected reasonably, the accuracy is basically the same, and the efficiency is almost the same.

4.2 An nonuniform parallel plate waveguide with a slot

However, the two methods also have the differences when simulating the long-time response applications, such as the example in [12]. The numerical example is set as a TEz wave propagation in a parallel plate waveguide, as shown in **Figure 2**. It is with a PEC slot of the thickness 0.2 mm and the distance 0.2 mm and a partly filled dielectric material of the thickness 0.8 mm with the dielectric medium parameters given as two cases: case I, $\varepsilon = 11\,\varepsilon_0$, $\mu = \mu_0$, $\sigma_e = 0.003$ S/m, and $\sigma_m = 0$ Ω/m; case II, $\varepsilon = 2\,\varepsilon_0$, $\mu = \mu_0$, $\sigma_e = 30{,}000$ S/m, and $\sigma_m = 0$ Ω/m. There are 140×8 uniform cells ($\Delta_x = \Delta_y = 0.1$ mm) in the computational domain. A Gaussian pulse sinusoidally modulated is used as the electric current source profile:

$$J_y(t) = \exp\left(-((t-t_c)/t_d)^2\right)\sin\left(2\pi f_c(t-t_c)\right) \tag{33}$$

where $t_d = 1/(2f_c)$, $t_c = 4t_d$, and $f_c = 12$ GHz. And the total simulation time is set as $l = 1.28$ ns for case I and $l = 12.8$ ns for case II; then it leads to the marching-in-on-time steps for N = 6000 and N = 60,000, respectively. And the number of orders for LD functions is chosen as 80 and 300, respectively, to obtain a good approximation of field components.

The Ey electric field responses at measurement point p1 and p2, located at the center of the slot and behind the medium, respectively, are calculated, which are both in agreement with the conventional FDTD method as shown in **Figures 2** and **3**. For comparison, the AH FDTD method is also used in these two cases. One can find the good results in **Figure 3**, but the errors come out in **Figure 4** for AH FDTD method when the same number of orthogonal functions (Q = 80 for case I or 300 for case II) is used as LD FDTD method. However, when Q reaches 800, the results from AH FDTD method can achieve a comparable accuracy with the ones from LD FDTD method. One should note that for case II the waveform at point p2 has larger amplitude attenuation and longer delay than the result at point p1 due to the high dielectric medium located between them.

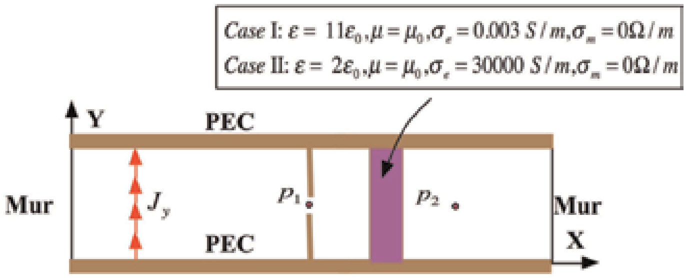

Figure 2.
The geometry configuration for a 2D parallel plate waveguide with a PEC slot and a partly filled dielectric medium [12].

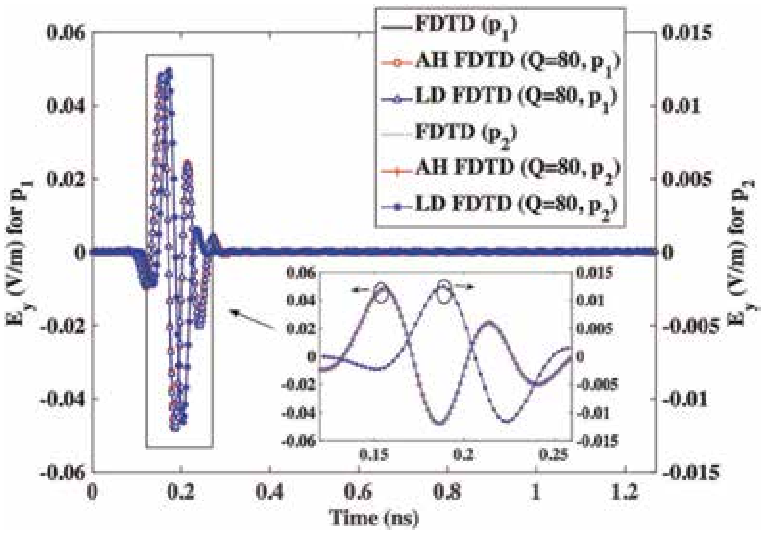

Figure 3.
The calculated results of transient electric field Ey for the case of I [12].

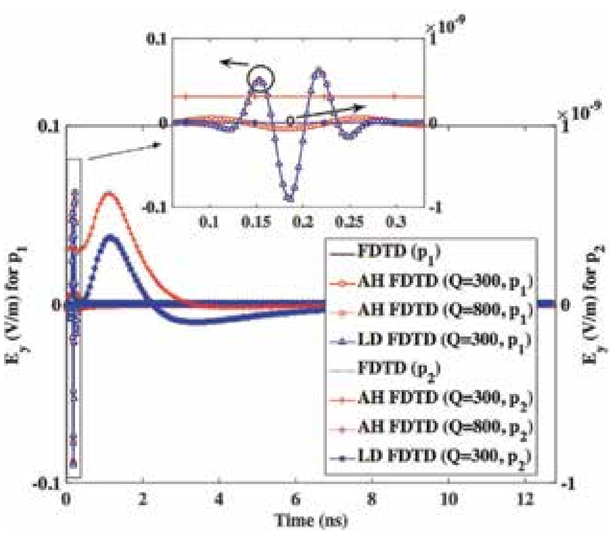

Figure 4.
The calculated results of transient electric field Ey for the case of II [12].

	Δt (ps)	Memory (MB)	CPU time (s)
FDTD (N = 6000)	0.21	1.8	2.97
AH FDTD (Q = 80)	21	2.9	1.32
LD FDTD (Q = 80)	21	2.9	1.32

Table 2.
The comparison of computational resources for the case of I [12].

	Δt (ps)	Memory (MB)	CPU time (s)
FDTD (N = 60,000)	0.21	1.8	30.8
AH FDTD (Q = 300)	21	11.8	1.55
AH FDTD (Q = 800)	21	28.9	1.95
LD FDTD (Q = 300)	21	11.8	1.55

Table 3.
The comparison of computational resources for the case of II [12].

Tables 2 and **3** show the comparison of the computational resources. We can see that the simulation takes much more time for the FDTD method compared with proposed method, especially for the case of II, while the trade-off for the proposed method is that it consumes more memory than conventional FDTD method, which is similar to the AH FDTD method. In addition, from **Table 3**, we can find the advantages compared with AH FDTD method that the proposed method can use relative smaller memory storage and slightly fewer CPU times to get a readily results.

5. Conclusions and future developments

The paralleling-in-order-based unconditionally stable FDTD methods are introduced using associated Hermite and Legendre polynomials in this chapter. The direct Q-tuple-domain transformation for time-domain Maxwell equation is guaranteed by using the integral matrix and differential matrix for Legendre function and associated Hermite functions that are introduced from the intrinsic integral or differential features for these orthogonal functions. Normally, the integral matrix of Legendre function can be considered as an inverse relationship from the differential operator, similar to the AH differential matrix. From this view, we can consider them as a uniform algorithm organized from the paralleling-in-order solution scheme. In addition, this chapter also detailed the different properties and the formula with these two methods theoretically and tested by numerical examples. Numerical examples for 1D and 2D cases validate their effectiveness and show LD FDTD with a better performance than AH FDTD method, in long-time simulation applications. In the next step, the more general paralleling-in-order scheme should be summarized, and then find or construct other possible orthogonal functions for their specific applications.

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Polynomials are well known for their ability to improve their properties and for their applicability in the interdisciplinary fields of engineering and science. Many problems arising in engineering and physics are mathematically constructed by differential equations. Most of these problems can only be solved using special polynomials. Special polynomials and orthonormal polynomials provide a new way to analyze solutions of various equations often encountered in engineering and physical problems. In particular, special polynomials play a fundamental and important role in mathematics and applied mathematics. Until now, research on polynomials has been done in mathematics and applied mathematics only. This book is based on recent results in all areas related to polynomials. Divided into sections on theory and application, this book provides an overview of the current research in the field of polynomials. Topics include cyclotomic and Littlewood polynomials; Descartes' rule of signs; obtaining explicit formulas and identities for polynomials defined by generating functions; polynomials with symmetric zeros; numerical investigation on the structure of the zeros of the q-tangent polynomials; investigation and synthesis of robust polynomials in uncertainty on the basis of the root locus theory; pricing basket options by polynomial approximations; and orthogonal expansion in time domain method for solving Maxwell's equations using paralleling-in-order scheme.

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