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Game Theory

Applications in Logistics and Economy

Edited by Danijela Tuljak-Suban

GAME THEORY - APPLICATIONS IN LOGISTICS AND ECONOMY

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Meet the editor

Danijela Tuljak-Suban is an assistant professor at the University of Ljubljana, Faculty of Maritime Studies and Transport. She holds her BSc degree in Mathematics from the University of Trieste and her MSc degree in Mathematics from the University of Ljubljana. She obtained her doctoral degree in the field of Maritime Studies and Transport from the University of Ljubljana, Faculty of Maritime Studies and Transport. She currently lectures on subjects related to mathematics, statistics, operational research and applied mathematics in the field of maritime transport. Her research is focused on the applications of fuzzy reasoning, MCDM and optimization methods to logistics and transport problems.

Contents

Preface

“One should always play fairly when one has the winning cards.”
Oscar Wilde

Generally, it is easy to make a choice between alternatives when the final decision depends only on known facts. However, when others make decisions that could influence the results of our decisions, making a good decision becomes more complex.

Clearly most of the decisions taken in our private lives or at work belong to the second category. Formalizing the decision-making process through the definition of a game contest where the decision makers are players, the alternatives are moves, and the progress of the game is measured by a utility function could improve the chances of making a good decision.

Game theory offers decision makers mathematical models and tools that could improve their decisions.

Economy and logistics are based on making good decisions, taking into consideration one’s own point of view and environmental requirements. As such, it could be useful to know the types of games and the subsequent mathematical methods that could be used to increase the chances of success.

The aim of the book is to explain and present the features of game theory and the possible ways in which it can be applied through some practical examples.

This book presents applications of game theory to describe and solve problems connected to economy, logistics, transportation and also partnership problems. Also included are cooperative, competitive games and the free-rider problem where instances of games with the aim of maintaining friendships or team work are explained.

In the first chapter, the well-known prisoners’ dilemma is used to examine the dipping headlight problem that occurs when two cars meet on a dark road.

The next two chapters present applications of game theory to the logistics of energy and computer science, especially in the area of wireless networks.

In addition, the Stackelberg game, known as the leader-follower game, is presented. Particularly in the case of asymmetric information, where the information available to the followers is limited with respect to that which is available to the leader. Clearly, this is a situation often seen in real life.

In contrast to the previous chapter, a situation of complete cooperation or team work is proposed in the next chapter, and these situations are known as social dilemmas.

In the chapter, the passive attitude of some players who do not exert any effort toward the common goal and benefit from the efforts of others is examined. This situation in game theory is called the free-rider problem.

Relationships are explored in the last two chapters. The penultimate chapter explains marriage plans: the fully transferable case and the fully non-transferable case. Clearly, we are speaking about money. The explained cooperative marriage plan is a special case of the optimal mass transport problem, also known as Monge-Kantorovich theory.

This field of game theory has many applications especially in transportation and logistics.

The last chapter of the book is about supermodularity and preferences regarding lattices, which may seem very far from game theory, but the proposed results are widely connected to economies of scale and indicate the synergetic relationship of subsystems.

The book provides a wide range of examples of the use of game theory even in situations that may be considered unsuitable for this subject. The reader will be able to obtain a wide range of methods to be applied and used.

The distinctive features of this volume are the careful attention to methodological foundations and the critical analysis of the solution concepts. The style of the book is a mix between the informal and the exact (mathematic formulation), so the volume will be interesting both to readers who want more theoretical information on game theory and to those who already know this theory and want to deepen some aspects in a more formal (mathematical) manner.

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Dipping Headlights: An Iterated Prisoner's Dilemma or Assurance Game

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Additional information is available at the end of the chapter

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Abstract

When two cars meet in the dark on a rural road, and both drivers use their headlights on full beam, they will both be dazzled. Because of that, we usually dip the lights. However, if modeled as a finite prisoner's dilemma super-game, the standard game theoretic solution would be that both drivers defect and use full lights throughout the super-game. If modeled as an Assurance game, cooperative strategies have better chances of succeeding. Several different strategies are presented and discussed, and in particular whether the strategy most often used by drivers can be invaded by the strategy novice drivers learn in their driver education. On theoretical grounds, one can expect the cooperation to unravel because novice drivers are taught to defect both in the beginning and at the end of the super-game. However, empirical results from three different surveys reveal that novice drivers change their behavior and adapt to normal practice with experience in real traffic. Hence, cooperation has sustained because cooperative and retaliating strategies dominate in the population. In addition, when drivers retaliate by flashing their headlights, it is probably perceived as a negative penalty which causes the inexperienced drivers to adjust the behavior to normal practice.

Keywords: car driver, dipping headlights, Prisoner's dilemma, assurance game, cooperation

1. Introduction

When two cars meet in the dark on a rural road, and both drivers use their headlights on full beam, they will both be dazzled. Because of that, we usually dip the lights. This is, however, not a good explanation, because for each driver it may be individually preferable to continue with full lights, regardless of what the driver of the oncoming car does. If he dips his lights, I can proceed with full lights without being dazzled. If he does not dip his lights, the worst thing for me to do would be to dip my own lights, being both dazzled and not able to see the road ahead because of low lights.

The situation can be considered as a game of conflict and cooperation. The result might be conflict (both use full lights), cooperation (both use low lights), or a mix of conflict and cooperation. If the drivers have the aforementioned preferences, they are engaged in the most famous of all games: The prisoner's dilemma (PD). However they might value mutual cooperation as the best outcome with preferences according to the assurance game (AG) (aka stag hunt).

Normally, situations of road-user interaction that can be considered as games in a game theoretic sense will be games that consist of very few moves, or games where the actors move simultaneously, but where each game is frequently repeated with new actors [1–5]. The situation when car drivers meet in the dark on a rural road may, however, be viewed as a supergame, that is, a game stretched out in time where the same actors play several constituent or static games against each other.

In this chapter, the game theoretic model will first be outlined both as a PD game and as an AG game. Why there has been established a convention of mutual cooperation in the dipping headlights game will briefly be discussed. We will then present possible strategies in the dipping headlights game and discuss if these strategies are evolutionary stable strategies. A particularly interesting attempt of invasion is the strategy that novice drivers learn during their driver education; they are taught both to keep full lights longer than what is normal and to switch back from low lights to full lights at an earlier stage than what is normal [6]. We will discuss whether this attempt of invasion will succeed. We will proceed by presenting empirical results revealing the distribution of strategies in the driver population and how behavior in the dipping headlights game has changed over time and discuss why cooperation in this game has survived.

2. The model

We start by presenting the dipping headlights game as an ordinary two-person one-shot game. This is depicted in **Figure 1**.

		B	
		Low lights	Full lights
A	Low lights	R, R	S, T
	Full lights	T, S	P, P

Figure 1. The dipping headlights game as a two-person one-shot game.

When meeting in the dark, both actors (drivers) have two choices, either to put on full lights or to use low lights. Each of the four cells in the matrix represents an outcome of the game, giving one payoff to A and one payoff to B. The letter on the left-hand side of each cell represents the payoff to A, and the letter on right-hand side represents the payoff to B.

For each actor, there are four possible payoffs. If they have PD preferences, the ranking of the payoffs is as follows: T > R > P > S. T represents "Temptation," that is, the temptation to defect while the other cooperates, R stands for Reward, that is, the reward for mutual cooperation, S symbolizes sucker's payoff, that is, the worst payoff resulting from playing cooperatively while the opponent defects, and P represents punishment, that is, the outcome of mutual defection [7]. In addition it is often assumed that 2R > T + S, that is, that each player values the cooperative solution as better than alternatively to receive T and S in subsequent rounds (if played several times). If the drivers have AG preferences they rank the outcome as R > T > P > S.

In one-shot prisoner's dilemma (PD), the dominant strategy is to defect and hence P, P is the only Nash equilibrium. In the assurance game (AG) there are no dominant strategies and two Nash equilibria: that both cooperate (R, R) and that both defect (P, P).

When the dipping headlights game is modeled as a two-person PD or AG, the choice of full lights equals defection and the choice of low lights equals cooperation. In the dipping headlights game the dominant strategy is accordingly to use full lights if the drivers have PD preferences but not if they have AG preferences. However, also in the AG, the best option is to use full lights if the opponent uses full lights. Thus, also if the game is an AG, mutual defection may be the result.

When, however, the PD game is repeated, that is, the same two players play the same game in a number of trials, the outcome will not necessarily be mutual defection. And in the real-life dipping headlights game, when two drivers meet in the dark, the game can be viewed as an iterated PD game with a certain number of single games, that is, a PD super-game. At any point in time from the game starts (when full lights dazzle the driver of a meeting car) until it ends (when the cars pass each other), both drivers have chosen either full lights or low lights.

In such repeated games, cooperative strategies can survive if they are conditionally cooperative, that is, adapted to what the opponent has chosen in the previous sub-game of the super-game. One such strategy is tit-for-tat (TFT) which chooses to cooperate in the first game, and then continues to cooperate, if the opponent has cooperated in the previous game, and to defect if the opponent defected in the previous game. Thus TFT achieves cooperation against cooperative opponents and it avoids the sucker's payoff against defecting opponents. In a simulation of iterated PDs with a number of different strategies, Robert Axelrod [8] found that TFT was the most successful strategy in the long run.

Normally, in dynamic game theory, it is assumed that future payoffs are discounted by some factor, the logic being that the payoff you receive immediately is valued higher than the same value received later in the game. This will however not be the case in the dipping headlights game. On the contrary, as this super-game approaches its end, the payoff for each driver will normally be higher than in the beginning of the game; you will be more dazzled by an oncoming car the closer it is, and your own full lights will be the sole source of lighting for a longer stretch of road the closer you are to the oncoming car. So it is reasonable to consider the payoffs at the end of the game to be higher for both drivers, regardless of what the outcome is. Thus, in this game, instead of having a discount factor in the repeated game, we will introduce the opposite, a premium factor.

In order to analyze the dipping headlights game we must make some assumption. First we assume that the super-game starts when the cars are about 300 m apart and that the super-game ends when the cars pass one another. We assume that the players play simultaneously, that is, at each moment in time, they have both made a move, either full or low lights. Each move takes place during a short time interval, depending on how quickly the drivers are able to change between full lights and low lights. We assume that each move lasts for 0.5 s and that the game begins when the distance between the cars is 310 m. Given the distance, it will take 7 s before the cars meet, giving 14 sub-games or moves for each driver in the game.

3. Strategies in the dipping headlights game

In the dipping headlights game, there are probably different strategies in different countries, and they may also differ among drivers in each country. Below, five strategies are outlined.

3.1. Strategy 1: "Custom"

In Norway, probably the most common strategy, which we will call "Custom," is to play cooperatively and to dip the lights when driver A meets driver B in the dark (at approximately 300 m) and to keep low lights until the cars pass one another.

If driver B does not dip the lights when the game has started (according to driver A), driver A will flash his lights. The normal process will then be that driver B dips his lights and that both keep low lights until the cars pass one another.

The flashing of lights from A can be considered to play cooperatively (C) in the first game, then to defect (D) in the second, then to play C in the third game (the sequence DC being the flash), whereas B plays consistently D, D, D in these three games. Then, after the third game, if B does not dip his lights in the fourth game, A will normally play TFT for the rest of the game.

When both players play custom, B will immediately dip his lights after A has done so, and the time interval between the two drivers who have dipped their lights is microscopic. When both play custom, they will both receive R throughout the game. Custom players will not engage in any defection in the last move(s) and will receive RR also in the last game.

3.2. Strategy 2: "School-strat"

In Norway, both the road authorities, the traffic police and the driving schools, recommend using full lights more often than what is normal. Different editions of the textbook used in the Norwegian driver education state that the current practice is not optimal and they recommend to keep full lights until the distance to the oncoming car is around 200–250 m, then switch to low lights and to switch back to full lights when the distance to the oncoming car is about 10–20 m. They recommend this practice regardless of what the driver of the oncoming car does. The logic behind is that full-beam headlights do not actually dazzle oncoming drivers at distances greater than 200 m, nor when cars are 10–20 m apart. This strategy, which learner drivers are taught in the driving schools, is called school-strat.

3.3. Strategy 3: "Tit-for-tat"

We assume that tit-for-tat (TFT) equals custom in that it initiates cooperation at around 300 m by dipping the lights. If the oncoming car does not dip his lights, TFT switches to full lights and keeps full lights until the oncoming car has dipped his lights. In the last game TFT will copy the move of the opponent in the next-to-last game.

3.4. Strategy 4: "All D"

In All D, the driver chooses full lights throughout the game, regardless of what the driver of the oncoming car does.

3.5. Strategy 5: "All C"

In All C, the driver chooses low lights throughout the game, regardless of what the driver of the oncoming car does. We assume that the game starts at the similar point as for drivers playing custom and TFT, that is, when the distance between the cars is approximately 300 m, and that it ends when the cars pass one another.

4. Evolutionary stable strategies

In a population with several encounters like in road traffic, the strategies that will survive in the long run must be evolutionary stable strategies (ESS). The concept of ESS was introduced by John Maynard Smith [9] and outlined as follows by Robert Sugden (p. 27–28) [5]:

Let E(I, J) symbolize the expected utility derived by any player from a game where he plays strategy I and the opponent plays strategy J. For a strategy I to be ESS, the expected utility of playing I against itself must be at least as high or higher than any other strategy J (pure or mixed) against I:

$$E(I, I) \geq E(J, I) \tag{1}$$

If this condition holds, it follows that if every driver plays strategy I, no single driver can gain by switching to another strategy. According to this condition, there is however the possibility that there are other strategies that are equally good against I as I itself. Thus in addition to (1) the following criteria must be met:

For all strategies J (pure or mixed) where J ≠ I,

$$\text{Either } E(I, I) > E(J, I) \text{ or } E(I, J) > E(J, J) \tag{2}$$

If conditions (1) and (2) are satisfied strategy I is an ESS and a stable equilibrium.

4.1. Evolutionary stable strategies in the dipping headlights game

We have identified five different strategies in the dipping headlights game. Of these there are three that will cooperate against each other throughout the game: custom, TFT and All C. They will all receive R in every sub-game. Hence, the interesting cases are when these cooperative strategies meet school-strat and All D.

It seems quite obvious that All C will easily be invaded by both school-strat and All D. Against All D, it will lose and receive S throughout the game. All C will also loose against school-strat, who uses full lights in the beginning of the game (until the distance is 200–250 m), and also full lights at the end.

The more interesting cases are when the strategies custom and TFT meet school-strat and All D.

4.1.1. Custom against All D

When custom meets All D in one super-game, All D will win. Custom will play C, D, C, C, and then D for the rest of the game against All D. We assume that after the flashing of the light (D, C in the second and third game), custom holds low lights also in the fourth game. If the opponent has not cooperated in the fourth game, custom plays D in the fifth. Against All D custom will continue to play D from the fifth game onward. Thus, All D will receive T and custom will receive S in the three games when custom plays C (and All D plays D). In the rest of the game they will both receive P. If there is equal distance between payoffs T, R, P, and S, it is easy to see that custom will receive a better outcome against itself than All D will against custom. Against itself custom will receive R throughout, whereas All D will receive 3T + Pn (where n symbolizes the number of remaining sub-games in the sub-game). If we also include a premium that raises the value of the payoffs throughout the game, the values of the losses to custom in the beginning of the game against All D (3S) will easily be outweighed by the gain received later in the game when custom plays against itself. Thus, E(Custom, Custom) > E(All D, Custom).

4.1.2. TFT against All D

Against TFT the same logic will apply: All D will receive a higher payoff than TFT in one single super-game, since TFT starts by cooperating, but TFT is better against itself (and against custom and other cooperative strategies) than All D is. Thus, in a population where everybody else plays TFT (or custom), E(TFT, TFT) > E (All D, TFT).

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4.1.3. Custom against school-strat

The prescribed behavior in the driver education is school-strat, and this is perhaps the most interesting case to consider against custom and TFT. If we assume that player A uses custom against a player B using School-strat, the sequence may be as follows:

A. C, D, C, C, C, … C, C.

B. D, D, D, D, C, … C, D.

It is obvious that B will receive a higher payoff than A: 4T + P + Rn vs. 4S + P + Rn. Hence in one single game against custom, school-strat will perform better. School-strat will also perform better against custom than custom performs against itself (T > R).

School-strat will, however, perform worse against itself than custom does against custom. Against itself school-strat will receive 5P + Rn, whereas custom receives R throughout the game against itself. Nevertheless, school-strat may invade a population of custom-players. If school-strat meets custom it outperforms custom, if it meets itself it is better to play school-strat than Custom. Hence, E(School-strat, Custom) > E (Custom, Custom).

4.1.4. TFT against school-strat

Against TFT school-strat will not perform as well as against custom, since TFT retaliates earlier in the game than Custom does. If A plays TFT and B plays school-strat, the sequence will be as follows:

A. C, D, D, D, D, C … C, C.

B. D, D, D, D, C, C … C, D.

We see that TFT and school-strat will perform equally well during the first five moves of the game, both receiving T + 3P + S. As mentioned, in the dipping headlights game, it is reasonable to include a premium that raises the payoffs as the game progresses. Hence, the payoff of playing D against C (giving value T) in the fifth game is valued higher than the payoff of D against C in the first game. But since school-strat plays D in the last game, it nevertheless scores better than TFT when they meet. Hence E (school-strat, TFT) > E (TFT, TFT).

4.2. Developments over time

So far, we have just considered what strategies will be best against each other, without considering the developments over time. We have seen that school-strat will score better both against custom and against TFT. Against the latter, school-strat scores better since it defects in the last game. Thus, a moderated TFT (TFTm), also playing defect in the last game, would win against school-strat. Against each other both school-strat and TFTm would receive T + 4P + S. If we introduce a premium TFTm wins since the T received in the fifth game is more worth than the T school-strat receives in the first game. Even without a premium, TFTm will outperform school-strat since TFTm scores better against itself compared to school-strat against itself.

A moderated custom strategy, also defecting in the last game, will however not be able to resist invasion from school-strat.

TFTm, which uses full lights in the last game, will outperform school-strat. The reason is that it manages to cooperate with other cooperative strategies in the beginning of the game. However, such a moderated TFT strategy will lose against another cooperative strategy that uses full light also in the next-to-last game. Hence, if all players copy this strategy, using full lights in the two last rounds, a cooperative strategy defecting in the third-to-last game will outperform the existing strategies and so on.

This is the standard logic in the finite prisoner's dilemma super-game. Each actor will choose to defect in this last game because in the last game to defect will be a dominant strategy just as in a one-shot game. Given this result in the last game, the same will happen in the next-to-last game because now the last game cannot be used to sanction the other player's action in the next-to-last game. But, the same will then also be true in the third-to-last game and so on all the way to the first game. If, however, it is not known when the last game is in play, or the super-game is one of infinite horizon, conditional cooperative strategies may be successful because the actors may realize a cooperative solution, which is better for both parties than mutual defection.

However, it has been shown that even when the game has a finite number of moves, conditional cooperative strategies may be equilibrium strategies, if there is incomplete information [10]. The reasoning behind this result is that if you for some reason suspect your opponent not to play defect in every sub-game, you might be better off yourself by choosing to cooperate in one or more sub-games. Thus, if there are some drivers in the population who chooses always to cooperate, and/or there are drivers with assurance game preferences ($R > T > P > S$), conditional cooperation can be a viable strategy even in the finite number prisoner's dilemma super-game.

5. Strategies and behavior in real traffic

So far, the dipping headlights game has been discussed theoretically. In this section we will report results from three different data sets of self-reported behavior in the dipping headlights game. The data sets are all surveys to driver populations in Norway.

5.1. Survey 1 to car drivers

In 1991, Bjørnskau [1] surveyed Norwegian car drivers with a rather detailed questionnaire about behavior in the dipping headlights game. The questionnaire was administered to 12,000 car owners in the Oslo region. The reply rate was 29.2% giving a net sample of 3.505 car owners.

The drivers were asked whether they usually dipped before or after the driver of the oncoming car did so, when driving in the dark, how they reacted if the driver of the oncoming car did not dip his lights, and when they changed back to full lights. They were also asked if they had received special education in driving in the dark, in their driver education. Such special education in driving in the dark was introduced in the driver education in Norway in the late 1970s.

The results revealed that cooperative strategies were in a clear majority among drivers. The strategies most often used were even more cooperative than the strategy custom described above. Among the drivers 17% used All C, 6% used TFT, 16% used custom, and 55% used a more forgiving version of custom in the first moves of the game (either they flashed more than once and/or they did not switch to full lights after flashing if the opponent did not change to low lights). Interestingly, 72% of the drivers said they would flash the lights one or more times if the driver of the oncoming car did not dip the lights (in time).

At the end of the game, only 3% said they switched back to full lights when the cars were 10–20 m apart. Another 17% said they changed back when the distance was 5–10 m, 28% when it was 1–5 m, and 52% when cars passed one another or after passing the oncoming car.

Given these large proportions of cooperative strategies in the driver population, school-strat should not have had any difficulties in invading the population. At the time of the survey (1991), the school-strat behavior had been taught at driving schools for about 15 years. Thus, there should have been ample time for the behavior to change.

Also among drivers in the sample who had been given lessons in dark driving (21.4% in the sample), a clear majority did not behave according to the school-strat strategy, neither in the beginning nor at the end of the game. However, those who had recently conducted the driver's test and received training in dark driving behaved more in accordance with the school-strat strategy than those with longer driver experience. In general those with dark driving lessons behaved somewhat more according to the prescribed practice at the end of the game than in the beginning. There were however no tendencies toward changing back to full lights at earlier stages in the end game with time or with experience.

It was concluded that the behavior drivers are taught at the dark driving course in the driver education are abandoned in real life.

5.2. Survey 2 to car drivers

In a study of driver attitudes and behavior, Sagberg and Bjørnskau [11] sampled novice drivers with 1, 5 and 9 months of experience, respectively, together with a smaller group of experienced drivers. The drivers answered a version of the driver behavior questionnaire (DBQ) together with questions about behavior when meeting other cars in the dark. Results on the DBQ questions are given by Bjørnskau and Sagberg [12].

Figure 2 gives the distribution of answers to the question of when drivers normally change to low lights upon meeting in the dark, among drivers with dark driving education, in survey 1 and survey 2.

The results reveal clearly that there is only a very small proportion of drivers who say they normally switch to low lights after the driver of the oncoming car has switched to low lights.

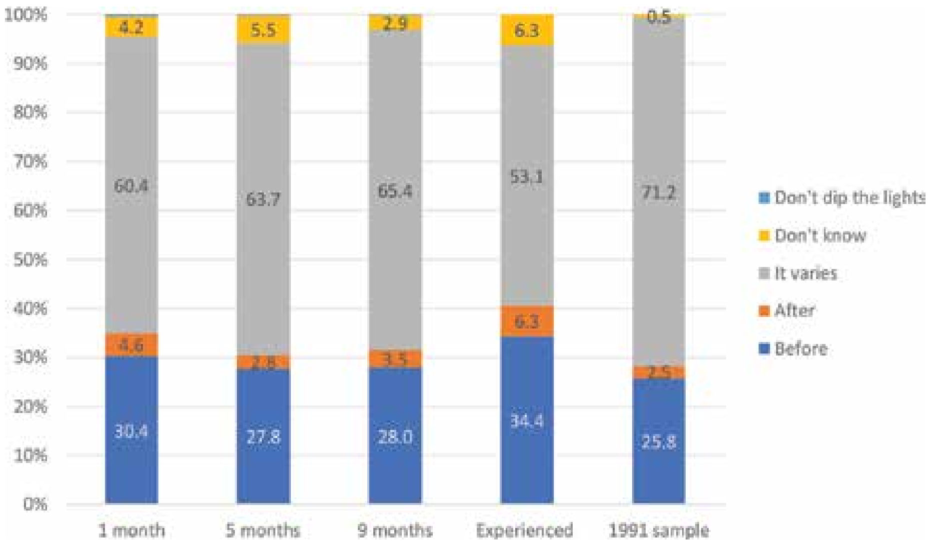

Figure 2. Drivers' self-reported answers to when they switch from full lights to low lights when meeting another car at night. Sample from 2001 to 2002 (survey 2), distributed by driving experience (N = 1433) and sample from 1991 (survey 1) (N = 3421). Percentage.

The share among the inexperienced drivers is not higher than among the experienced drivers. If school-strat had been adopted by novice drivers, we should expect these shares to be considerably higher. Most drivers say it varies whether he/she or the driver of the oncoming car changes to low lights first. However, quite a few say they normally change before the driver of the oncoming car has done so.

An interesting result is that there seems to be no clear development in behavior over the 10 years that have passed from survey 1 (1991) to survey 2 (2001–2002) nor with increased experience among novice drivers. According to the theoretical analysis we could have expected less cooperation over time. We also argued that the cooperation in the game could unravel given the fact that it is a finite super-game, but this has clearly not happened. In fact almost no one says they keep full lights throughout the game.

Figure 3 gives the results about when drivers change back from low lights to full lights upon meeting in the dark among drivers with lessons in dark driving in survey 1 and survey 2.

A very small fraction says they change back to full lights when cars are 10–20 m apart, which would be the correct behavior according to school-strat. A majority of the inexperienced drivers (1–9 months) say they switch back before the cars meet and also in the 1991 sample this is the case. An interesting result is the clear tendency to switch later as novice drivers gain experience. Thus, there seems to be a clear tendency that novice drivers change their behavior toward normal practice as they gain experience during the first months of driving. The difference between the three groups of novice drivers (1, 5 and 9 months of driving experience) is

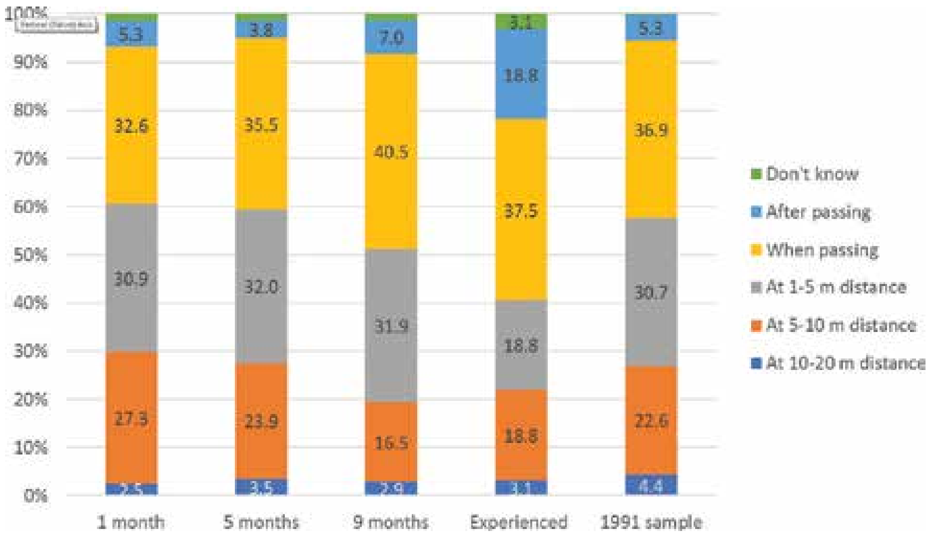

Figure 3. Drivers' self-reported answers to when they switch back from low to full lights when meeting another car at night. Sample from 2001 to 2002 (survey 2), distributed by driving experience (N = 1433) and sample from 1991 (survey 1) (N = 3435). Percentage.

statistically significant ($\chi^2 = 19.4$, df = 10, $p < 0.035$). Also the difference between all four groups in the 2001–2002 sample is statistically significant ($\chi^2 = 31.1$, df = 15, $p < 0.008$).

As already mentioned, we saw in survey 1 that a large majority of drivers said they flashed the headlights if the oncoming driver did not dip the lights (in time). Hence, we would expect novice drivers, who have recently been taught to use school-strat, to have experienced oncoming cars to flash their headlights to a greater extent than experienced drivers. We would also expect that the less experienced you are, the more often this would happen.

In survey 2, drivers were asked as to how many times during the last month they had experienced other drivers to flash the headlights against them. The results are given in **Figure 4**.

There is a quite clear tendency in the expected direction. Drivers with 1 month driving experience have been flashed at more often than drivers with 5 and 9 months experience. The most experienced drivers in the sample have experienced this even less often. There seems to be a clear tendency, but it is not statistically significant at the standard 5% level ($\chi^2 = 11.8$, df = 6, $p < 0.067$).

5.3. Survey 3 to car drivers

The third survey we report results from is a panel study where a Norwegian sample of 982 young novice drivers aged 18–20 responded to a questionnaire on attitudes, errors, behavior and so on in 2007/2008 (study A) and then again in 2010 (study B) [13]. Hence we will be able to see whether the same drivers change behavior during the two first years of driving.

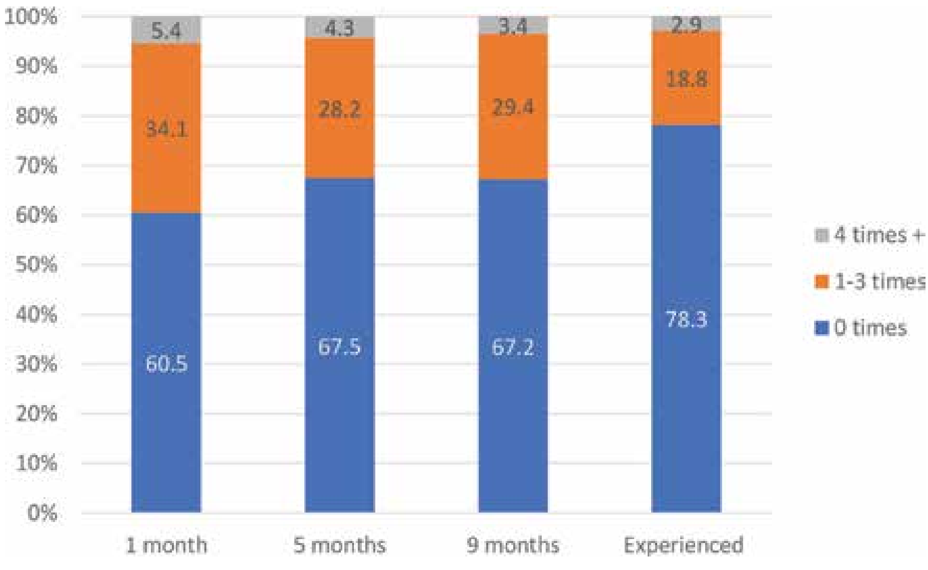

Figure 4. Drivers' self-reported answers to how many times during the last month they have experienced to be flashed at by other drivers. Sample from 2001 to 2002 (survey 2), distributed by driving experience (N = 1433). Percentage.

The question about experiencing other drivers to flash their headlights was asked in two versions, a general question like the one used in survey 2 and reported in **Figure 4** and a specific one about being flashed at by an oncoming car in the dark. On both questions, answers were given on the following scale: 1 = 0 times, 2 = 1–3 times, 3 = 4–6 times, 4 = 7–9 times, and 5 = 10 or more times. The results are presented in **Table 1**.

Both among men and women there are statistically significant changes in the expected direction over time, on both questions. In the B study, when respondents are 20–22 years old, they experience to be flashed at to a lesser extent than in the A study when they are 18–20 years old. There is also a significant difference between men and women. On both questions men are more often flashed at than women are. There is no interaction between time and sex.

	Study	Average scores		p-value		
		Women	Men	Time	Sex	Interaction
Been flashed at by other road users	A	1.23	1.42	0.001	0.001	ns
	B	1.17	1.30			
Been flashed at by an oncoming car in the dark	A	1.20	1.29	0.001	0.001	ns
	B	1.10	1.15			

Two-way ANOVA. N_{women} = 533, N_{men} = 393.
p-values for differences in time, sex, and the interaction between time and sex.

Table 1. Average scores on driver-behavior questions distributed by sex and study (A and B).

6. Discussion

We have seen that the data from three different data sets clearly indicate that the strategies taught in the Norwegian driver education is abandoned over time when drivers gain experience in traffic. Novice drivers who recently have passed they driver test use full lights more than more experienced drivers, but over time they adjust to the normal practice, which is to use low lights more than what is optimal according to the driver education textbooks. This problem is described similarly in the current edition of the driver education textbook as in the editions from 1976, 1990 and 1996 [6, 14, 15]. In this period thousands of new drivers have entered the roads, being taught the same lesson, but according to the textbooks, the problem remains the same: lights are used less than optimal.

Thus, cooperation has survived in the dipping headlights game even though we, both on theoretical grounds and due to the driver education, could expect it to unravel. Why is it so?

The analyses and empirical results presented here can provide some tentative answers to why the cooperation in the dipping headlights game has survived. There are mainly two mechanisms that may have produced the result:

a) The majority of drivers are more cooperative than assumed in the prisoners' dilemma game.

We have seen that in the driver population, there was a substantial proportion (17%) of drivers who said they would always cooperate, regardless of what the oncoming driver did. In addition, many drivers showed more cooperative behavior than prescribed by TFT and custom. If enough drivers in the population value cooperation as the better solution (e.g., they have AG preferences), and there are at the same time a substantial number who retaliate against defection early in the game, cooperative strategies can give higher payoffs. And, if many drivers indeed have AG preferences, the unraveling of cooperation from behind may not happen: the drivers value cooperation in the last game as better than to use full lights single-handedly.

b) The flashing of lights to oncoming cars who do not dip their lights is interpreted as a sanction.

Nearly 80% of the drivers participating in the first survey said they retaliated when drivers of oncoming cars did not dip the lights. A large majority flashed with the lights, and some used TFT, that is, full lights, until the oncoming driver dipped his lights. Thus, if drivers try to use the lessons learned in the driver education, and keep full lights longer than normal, they will meet strong reactions by almost every driver they meet. In the modeling of the game, we have only considered the values of full and low lights and not included any additional valuation of the sanctions the flashing headlights of an oncoming car represent.

Most novice drivers will try to adjust to normal behavior in traffic. Many studies reveal that they adapt to what they perceive to be normal practice during the first months of driving [12]. Thus, when flashed at by oncoming cars, they will probably interpret this as a sanction and hence the flashing is valued more negatively than just the glaring it provides. In addition, when heavily sanctioned at the beginning of the game for using full lights, it might also result in less use of full lights at the end of the game.

The invasion from upfront fails because the practice is heavily sanctioned by other car drivers. The invasion from behind, which could unravel the cooperation if drivers have PD preferences, fails because there are many (enough) drivers who have more cooperative preferences, for example, AG preferences.

The driver education tries to teach drivers to use full lights more than what is normal. It can be argued that this is a dangerous practice since it might give rise to defections in the last games unraveling the cooperative solution. These attempts have not succeeded because the novice drivers entering the driver population at any moment of time will constitute a small minority of drivers, and hence they will be sanctioned by flashing headlights on the roads and change their behavior over time. If large numbers of novice drivers had entered the population at the same time, constituting a large proportion of the drivers in the population, the established cooperation in the dipping headlights game might unravel.

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Chapter 2

Game Theory Application in Smart Energy Logistics and Economy

Baseem Khan

Additional information is available at the end of the chapter

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Abstract

In many parts of the world, energy sectors are transformed from conventional to the smart deregulated market structures. In such smart deregulated market environment, cooperative game theory can play a vital role for analyzing various smart deregulated market problems. As an optimization tool, cooperative game theory is very useful in smart energy logistics and economy analysis problem. The economy associated with smart deregulated structure can be better optimized and allocated with the help of cooperative game theory. Initially, due to regulated structure, there is no cooperation between different entities of energy sector. But after new market structure, all the entities are free to take their own decisions as an independent entity. Transmission open access of energy logistics is also comes into the picture, as all the generators and demands have the same right to access the transmission system. In this market situation, multiple utilities are using the same energy logistic network. This situation can be formulated as a cooperative game in which generators and demands are represented by players. This chapter deals with energy logistic cost allocation problems for a smart deregulated energy market. It is cooperative in nature as all the agents are using the same energy logistic network.

Keywords: smart grid, cooperative game theory, Shapley value, Nucleolus approach, transmission usage cost allocation, transmission loss allocation

1. Introduction

Cooperative game theory is a decision-making tool that helps game players (decision-makers) to take decisions under various strategic situations. It is a branch of applied mathematics that is utilized for a wide range of applications in the field of technology, science (social, political, behavioral), economics, biology, and philosophy. In strategic situations, game theory helps to mathematically model the behavior of various players [1].

Power sector is restructured in many parts of the world. Under this new structure, power sector is working under market forces. Its structure is transformed from regulated (government-controlled) to deregulated (market-controlled) one. The whole power sector is divided into three basic entities, i.e., generation company (GENCO), transmission company (TRANSCO), and distribution company (DISCOM). Other than these three basic entities, various new entities also emerged such as independent system operator (ISO), power pool, power exchange, etc. These entities ensure the reliable and secure operation of new restructured power sector. Private industries also participate in these sectors as independent players, such as independent power producers (IPPs) in GENCO and various distribution franchises in DISCOMs. The aim of restructuring is to bring competition and operating efficiency in power industry that result in reliable, economic, and quality power supply to consumers. Further, restructuring initiated the implementation of smart grid technology.

Conventional grid can be converted into smart grid with the incorporation of following characteristics:

- Self-healing from faults
- Incorporation of demand response programs for enabling consumer's active participation
- Robustness against any kind of cyber and physical attack
- Able to supply quality power as per the customer's requirements
- Able to incorporate all generation sources, i.e., conventional and renewable
- Enable incorporation of storage devices
- Enable restructuring to develop new markets, services, and products
- Operate economically by optimizing resources

From the abovementioned characteristics, it is clear that the smart grid system is completely working under market forces. Different entities aim to enhance their profits. Therefore, cooperative game theory can be applied by different market entities to increase their revenues.

The development of game theory and its applications also reflected in many energy market modeling and analysis problems. In 1999, for energy market modeling and analysis, IEEE Power and Energy Society published a landmark tutorial on game theory application in power systems [2]. During the past 20 years, many researchers implemented game theory for various power system problems, and this trend is also reflected in the journal and conference publications.

There are various technical and economic issues in smart grid system that requires fair and unbiased solution. Thus, cooperative game theory is utilized by various entities around the world for solving critical technical as well as economic issues.

This chapter deals with various smart energy logistics and economy problems solved by using cooperative game theory.

2. Smart grid

According to Brian Seal, senior project manager, power delivery and utilization, Electric Power Research Institute (EPRI), "Smart grid is a marketing term that is devoid of technical definition." A variety of operational and energy measures such as smart appliances, smart meters, renewable energy resources, and energy-efficient technologies are part of smart grid. There are different technologies, which are incorporated by different utilities in all the three energy sectors, i.e., generation, transmission, and distribution [3].

2.1. Smart technologies in generation sector

Various techniques incorporated for smart operation of generation system should be able to understand the unique nature of energy generation of resources. This understanding is very helpful for optimizing the energy generation. Further, multiple feedbacks from different points in the grid are helpful to maintain the desired voltage, frequency, and power factor standards.

For making generation sector smart, utilities are incorporating novel technologies in the system, continuously. Some of these technologies are as follows:

2.1.1. Incorporation of distributed generation and microgrid

As name indicates distributed generation incorporated various energy resources those are distributed in their nature. It includes technologies such as microturbines, energy storage, electric vehicles, solar energy, fuel cells, and micro wind turbine [4].

For efficiently incorporating the abovementioned distributed energy sources, microgrid technology can be utilized. It provides a better way to incorporate renewable energy sources in smart grid. **Figure 1** presents the general structure of microgrid technology [5].

As seen from **Figure 1**, a microgrid consists of photovoltaic source, wind turbine, microturbine, fuel cell, electric vehicle technology, battery energy storage, diesel generator, and electrical loads.

2.1.2. Frequency regulation management

Due to the incorporation of a large number of green and distributed energy sources in the smart grid, more fluctuation is occurred in the base load generation. Therefore, extra regulation will be required to maintain the balance in supply frequency. Flywheel plant technology is incorporated in the smart grid system to maintain the frequency regulation. In the case of excess power generation, extra energy is supplied to the flywheels for storage [6].

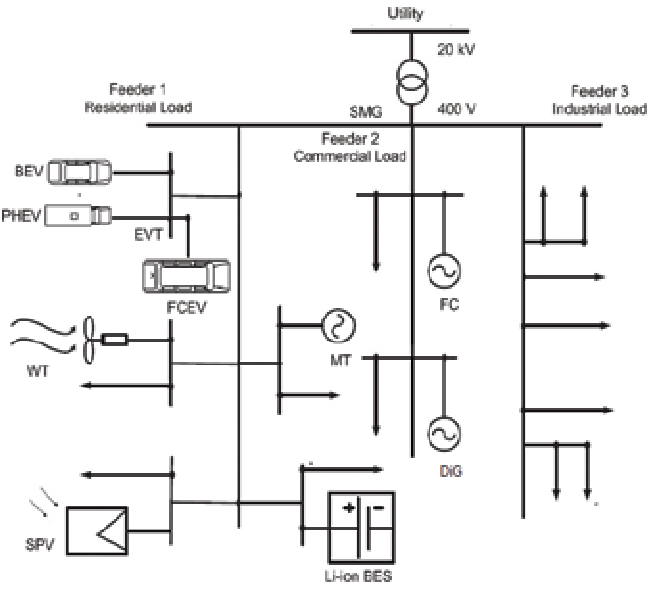

Figure 1. Microgrid [5].

2.1.3. Generation control

The time frame of the generation area under consideration is very important from the point of view of generation control. For generation and load control, supervisory control and data acquisition (SCADA) system provides the data for every second. Generation control is directly affected by the use of smart meter technology at the end-customer's location [6].

2.2. Smart technologies in transmission sector

Transmission system transfers a large amount of power at various voltage levels from generation point to the substations. For supplying resilient power supply to the energy markets, a reliable transmission system is the first requirement. To achieve this goal, under smart grid environment, synchrophasor technology is emerged as vital component. Different building blocks of a smart and reliable transmission system are as follows: wide area communication networks, phasor measurement units (PMUs), phasor data concentrators (PDCs), and smart substations [7].

2.2.1. Phasor measurement units

For enabling complete power system monitoring and control, the modern metering devices such as phasor measuring units (PMUs) are installed by different utilities. These are the most accurate and time-synchronized devices that provide voltage and current measurements. It directly measured the voltage phase of the bus at which PMU is installed. Further, it also measured the current phasor of few or all the transmission lines connected to the PMU installed bus.

2.2.2. Phasor data concentrators (PDCs)

As shown in **Figure 2**, isolated PMUs are utilized to develop a wide area monitoring system (WAMS). PMU fed Global Positioning System (GPS) time-stamped measurement signals to phasor data concentrator (PDC). The main function of PDC is to collect and sort the phasor measurements obtained by PMU. It is clear from **Figure 2** that signal processor converted PMU data into useful information, which is available on human machine interface (HMI) system. By using HMI system, an operator can easily access the important information of the system state [7].

2.2.3. Smart substation

A smart substation refers such a system, which control and monitor both critical and non-critical operational information. Information about system power factor, breaker status, and transformer operation and battery condition comes under operational information.

2.3. Smart technologies in distribution sector

For making distribution sector smart, techniques incorporated should be such that which makes distribution system self-healing, self-optimizing, and self-balancing. Further, it also includes superconducting cables and automated monitoring and control tool.

Different techniques such as smart controllable load, smart meters, electric vehicle technology, etc. are incorporated in the distribution sector to make it smart [8].

2.3.1. Smart controllable load

Variable load is the key challenge in the power system. Due to this, the large number of power system problems like generation control, frequency regulation, stability problems, etc. arises.

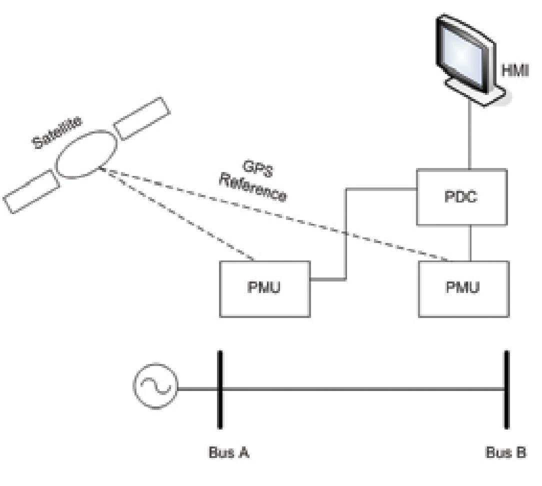

Figure 2. Layout of PMU with GPS time-stamped signal [7].

Therefore, the key aim of smart grid technology is to make variable load more controllable and flexible. For that purpose smart meter technology is very useful because it has an ability to disconnect the load remotely. It requires two basic components [9]:

1. Two-way communication system
2. Loads equipped with controls

2.3.2. Smart meters

Conventional energy meters do not have communication capability and need to be read manually. Smart meter technology equipped with two-way communication system. It measures demand on second-by-second basis and provides consumer access to their energy demands. Further, utility used smart grid technology to remotely disconnect loads. Communication system is the key for smart meter technology. Possible options for communication technology are as follows [9]:

1. Use existing customer broadband connections
2. Broadband over power line (BPL)
3. Meshed wireless

2.3.3. Electric vehicle technology

Electric vehicle technology is the key driver for widespread implementation of controllable electric load. But recharging of electric vehicles also requires load control. There are three basic types of electric vehicle [9]:

1. Battery electric vehicle
2. Hybrid electric vehicle
3. Fuel cell electric vehicle

3. Cooperative game theory

Game theory is a decision-making tool. It is the field of applied mathematics that deals with the conflicts of interests of persons or group of persons. These conflicts of interest are included in the term "Game." Game theory is broadly categorized into two sections: coalition or cooperative games and strategic or noncooperative games. In general, cooperative games are utilized for optimal allocation or assignment problems, while noncooperative games are utilized for analyzing oligopolistic models by using Nash equilibrium. Player refers to the person in both types of games. There is a key assumption in game theory that all the players behave rationally. Therefore, each player earns the profit from the game [10].

Cooperative game theory has different approaches such as core, Nucleolus, Shapley, Owen, solidarity, and Aumann Shapley.

3.1. Terminology used in cooperative game theory

Cooperative game theory provides an optimal and fair allocation between its players. An allocation game is defined by a couple (N, v). Let $N = \{1, 2, 3, \ldots\ldots n\}$ represents the set of n number of players, where n being equivalent to $|N|$. v is the characteristic value or function value. A group of players S that coordinate together are represented as a subset of $N, S \subset N$. A set N represents the grand coalition, and null set denotes an empty coalition. Players can be grouped in various ways that depends on their interest and convenience. Here, 2^N represented the collection of coalitions in N. Each player belongs to only one of the coalitions. Further, players of a certain coalition are related to each other. But these players are not related to the participants of other coalitions [11].

A real valued function $v: 2^N \rightarrow R$ is the game on N. It allots a value to each group and satisfies $v(\emptyset) = 0$. Characteristic value $v(S)$ provides the maximal value incurred by the coalition S by cooperation between coalition players [12]. A payoff vector is $\{x_1, x_2, x_3, \ldots x_n\}$, where x_n is the payoff concerned to agent n that represented the result of the game. The three rationalities represented in Eqs. (1)–(3), namely individual, group, and global rationalities, must be satisfied for fair and equitable allocation to all the players [11]:

$$x(i) \leq v(i); \quad i \in N, \tag{1}$$

$$x(S) \leq v(S); \quad S \subset N \tag{2}$$

$$x(N) = v(N); \tag{3}$$

$$\text{(With) } x(S) = \sum\nolimits_{i \in S} x(i);$$

Imputation is a payoff vector that satisfied the individual and global rationalities. Further, if imputation satisfied the group rationality, then the solution is laid in the core. Group or individual rationality verifies that a group of players or individual players must not have higher value than their stand-alone value. Global rationality ensures that the value obtained by the cooperation to all the players should be matched to the total value to be covered. It is known as break-even condition or Pareto optimum. In addition to this, if allocated value is less than the sum of individual allocations, then the solution of cooperative game is stable. As a result, all the participants are incentivized to stay in the group, leaving no significance for any participant to pull out.

3.2. Different approaches of cooperative game theory

Various cooperative game theory approaches are utilized by the researchers to find the optimal solution. Some of them are as follows:

3.2.1. The core approach

The core of the game (N, v) is the set of all the solutions presented by Eq. (4) such that

$$x(S) \leq v(S); \forall S \subset N, \quad \text{With } x(S) = \sum_{i \in S} x_i \tag{4}$$

Therefore, the core is nothing but the subset of the group of imputations. It is the simplest cooperative game theory approach. This approach is related to a group of imputations that leaves no choice for an optimal solution to its participants and does not permit any type of subsidies between coalitions.

3.2.2. Shapley value approach

Shapley value approach is a cooperative game theory approach that is utilized for fair allocation between players. This fair solution is obtained by the cooperation between different players. The basic concept utilized by Shapley value is that the allocated value to a particular player is the average of the value in all available coalitions. It fulfills all the required characteristics for fair allocation. Further, the solution is symmetric and additive in nature [13].

It is a priori value, which adds by each player to the grand coalition of cooperative game through a particular characteristic function. For calculating this value, all the available combinations should well think out. The net contribution of each player to the grand coalition is depended on the entry of that particular player. The addition of all of these contributions provides the Shapley value. It is represented by $\varnothing_i(v)$ for player i, as shown in Eq. (5):

$$\varnothing_i(v) = \sum_{s, i \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} [v(S) - v(S - \{i\})] \tag{5}$$

where

$|S|$ represents the number of players in coalition S; $|N|$ represents the total number of players; and $v(S)$ represents the characteristic function associated with coalition S.

3.2.3. Nucleolus value approach

The concept of Nucleolus, as introduced in 1969, is characterized by two features: every game has one and only one nucleolus, and unless the core is empty, the nucleolus is in the core [14]. In Nucleolus solution, the dissatisfaction for every coalition is minimized till the solution becomes fair and acceptable for all the coalitions and the players as well. A measure of inequality of an imputation x for a coalition S is defined in Eq. (6) as the excess:

$$e(x, s) = v(S) - \sum_{i \in S} x_i \tag{6}$$

This gives hint of the amount by which the coalition falls short of its potential. The largest dissatisfaction is calculated and is reduced. After this, the next largest dissatisfaction is taken up and reduced. This can be solved as a solution of a set of linear programming problem. An imputation in this case lies within the core if all the surpluses are either negative or zero. Thus, the advantage of the nucleolus solution is that it is part of the core. Thus, no other payoff vector can dominate the nucleolus over any association. When a payoff vector is not dominated, then it is more expected to be accepted by players [15].

4. Cooperative game theory application in smart energy logistics and economy

Conventionally, power sector is regulated by the government. All the three major sectors, i.e., generation, transmission, and distribution, are operated by government-owned entities. Therefore, it is impossible to implement cooperative game theory. In the 1990s, after deregulation, power sector is completely transformed. Three major sectors are transformed into private entities. Further, various private players also come in these sectors. Now, the number of players is increased, and cooperative game theory can be utilized to raise the profit of different players.

Deregulation brings transmission open access into the picture, because different entities have the same rights to access the transmission system. Therefore, the operating condition in which multiple entities utilized the same transmission network can be modeled as cooperative game theory problem. In this game, different generators and demands are represented as players. Therefore, different transmission access problems such as transmission usage and usage cost allocation, transmission loss, and loss cost allocation can be optimized with the help of cooperative game theory because all the participants utilized the common network [10]. Cooperative game theory provides the optimal allocation of all the abovementioned problems in a fair and equitable way.

4.1. Optimal transmission usage cost allocation

The smart deregulated market structure of energy sector requires economic efficiency. In this regard, the solution approaches of cooperative game theory behave well in terms of economic efficiency, fairness, and stability [16]. In [17], Shapley and Nucleolus approaches are utilized for transmission usage cost allocation. To accommodate all the loads in the pool market, Shapley value allocated the transmission usage cost to demands. Shapley value is the most preferable approach when the solution lies in the core. It uniformly and fairly allocated the transmission usage cost among the players [18]. Shapley value has a drawback that it explodes when the number of players in the game is very large. Aumann Shapley approach overcomes this drawback by reflecting the marginal contribution of a player to the cumulative system savings [19].

4.1.1. Characteristic function

There is no unique way of characterizing the cost of coalition, i.e., $v(S)$. For transmission usage cost allocation game, the characteristic value specifies the minimal cost that will be incurred by each coalition [13]. In cooperative game theory, $v(S)$ is defined as per the choice of user either on the basis of cost or on the basis of transmission usage. In [20], the basis of transmission network usage cost has been chosen. A power flow tracing method is used to evaluate characteristic value $v(S)$ as well as stand-alone cost $v(i)$ of player i of a system [15]. The work follows the ratio for cost allocation between generators and loads as 23:77% in pool market [21].

The characteristic function of the cooperative game, developed in [20], is presented in Eq. (7):

$$v(s) = \sum_{l \in N_L} (P_{m-n}) * C_{m-n} \tag{7}$$

where $v(S)$ is the fixed cost of providing transmission service to coalition S, P_{m-n} is the power flow in the line $m-n$, N_L is the number of lines, and C_{m-n} is the cost of the line $m-n$.

4.1.2. Cooperative gaming for optimal usage cost allocation in 6-bus system

The 6-bus system is considered as pool market for realizing Shapley value and Nucleolus approach of cooperative game theory. Therefore, bilateral contracts are not allowed, and the whole power is traded in a mandatory pool with the pool operator having a wide knowledge of the generator's data. In this attempt cooperative gaming is allowed among loads, and they behave as the players in the pool market.

If all the three loads are going to cooperate with each other, then the possible coalitions are 7, including the single-player coalition. The evaluated characteristic values using power flow tracing algorithm [20] for seven coalitions are presented in **Table 1**.

For power flow tracing, Newton-Raphson load flow runs with different collations. The load flow results are presented in **Table 1**. Afterward, Shapley and Nucleolus approaches are utilized for optimally allocating transmission usage cost to loads. **Table 2** presents a comparison between Shapley and Nucleolus values.

Results are obtained that satisfy all the three conditions of gaming, i.e., individual rationality, group rationality, and the global rationality of game theory. Thus, the accomplishment of group rationality proves that the solution lies in the core. As allocated payoff vector is part of the core, hence more likely to be accepted by the players.

Characteristic value of coalition in the 6-bus pool market (sr. no.)	Coalition	Characteristic value [Rs./hr.]
1	L4	161.107
2	L5	374.46
3	L6	229.04
4	L4 L5	547.069
5	L4 L6	396.05
6	L5 L6	614.79
7	L4 L5 L6	759.08

Table 1. Characteristic value of coalition in the 6-bus pool market.

Shapley value and Nucleolus value allocation for loads (Sr. no.)	Load	Stand-alone cost [Rs. /hr]	Shapley value allocation [Rs./hr.]	Nucleolus value allocation [Rs./hr.]
1	L4	161.107	158.402	158.9663
2	L5	374.46	374.45	373.5036
3	L6	229.04	226.23	226.5914

Table 2. Shapley value and Nucleolus value allocation for loads.

Individual rationality: $x(i) \le v(i)$

$$x(L4) \le v(L4), \text{i.e.}, 158.402 < 161.107$$

$$x(L5) \le v(L5), \text{i.e.}, 374.45 < 374.46$$

$$x(L6) \le v(L6), \text{i.e.}, 226.23 < 229.04$$

Group rationality: $x(S) \le v(S)$

$$x(L4L5) \le v(L4L5)$$

$$x(L4) + x(L5) \le v(L4L5)$$

$$158.402 + 374.45 < 547.069$$

$$532.852 < 547.069$$

Global rationality: $x(N) = v(N)$

$$\sum_{i=L4,L5L6} x_i = v(L4L5L6) = 759.082$$

From the above, it is clear that the results obtained from the Shapley and Nucleolus approaches lie in the core. Therefore, fair and equitable solution is obtained.

4.2. Optimal transmission loss allocation

In [22], authors developed a Shapley value and Nucleolus approach-based transmission loss allocation method under smart energy market structure. Generally, 7% transmission losses are occurred in practical power system. Therefore, in this study authors also considered total 7% transmission losses.

4.2.1. Characteristic function

For transmission loss allocation game, the characteristic value specifies the minimal loss that will be incurred by each coalition [13]. In [23], particular loss allocation index (PLAI) method is utilized to evaluate characteristic value as well as stand-alone value, i.e., transmission loss of a system. In this method authors allocated 77% losses to loads and 23% losses to generators [21].

The characteristic function of the cooperative game for loss allocation in [22] is derived by PLAI as shown below.

For loads, particular loss allocation indices (PLAI) are presented in Eq. (8):

$$\mathrm{PLAI}_{\mathrm{ln}}^{L_T} = \frac{P_{A_{\mathrm{ln}}}^{L_T}}{PF_{\mathrm{ln}}} p_{\mathrm{ln}} \tag{8}$$

where $\mathrm{PLAI}_{\mathrm{ln}}^{L_T}$ is the losses occurred in transmission line due to load L_T, $P_{A_{\mathrm{ln}}}^{L_T}$ represents the transmission line usage allocated to load L_T, PF_{ln} represents the power flow in respected transmission line calculated by load flow, and p_{ln} represents the transmission losses.

4.2.2. Cooperative gaming for optimal loss allocation in 6-bus system

An algorithm used in [24] is used for calculating transmission loss allocation by using Shapley and Nucleolus approaches. Results are shown for 6-bus system. **Table 3** provides transaction data for 6-bus system.

Now, **Table 4** presents the characteristic values for transmission loss allocation using Nucleolus approach.

Table 5 provides the transmission loss allocated to users using Nucleolus approach.

Transaction data of 6-bus system (in per unit) (trans. no.)	User	Supplier	Transaction quantity
1	D4	G1	0.731
2	D5	G2	0.725
3	D6	G3	0.714
1 and 2	D4,D5	G1,G2	1.461
1 and 3	D4,D6	G1,G3	1.448
2 and 3	D5, D6	G2,G3	1.444
1,2, and 3	D4,D5,D6	G1,G2,G3	2.184

Table 3. Transaction data of 6-bus system (in per unit).

Transaction losses of 6-bus system (in per unit) (transaction combination)	Active power losses
1	0.031
2	0.025
3	0.014
1 and 2	0.061
1 and 3	0.048
2 and 3	0.044
1,2, and 3	0.084

Table 4. Transaction losses of 6-bus system (in per unit).

Comparison between Shapley and Nucleolus approach (in per unit) loads	Stand-alone loss	Shapley value allocation	Nucleolus value allocation
L4	0.031	0.03	0.03
L5	0.025	0.03	0.0297
L6	0.014	0.0182	0.0187

Table 5. Comparison between Shapley and Nucleolus approach (in per unit).

5. Conclusion

The present energy sector involves a large number of stakeholders. Therefore, cooperative game theory application in modeling of smart energy market and economic analysis is increasing day by day. In addition to this, implementation of smart grid increased the applicability of cooperative game theory manyfolds because it is driven by the market forces. The huge amount of economy is involved in smart energy sector; thus, cooperative game theory plays a vital role to allocate this economy between various shareholders in a fair and equitable way.

This chapter provides an overview of smart grid structure along with various cooperative game theory applications in the present smart deregulated environment. For nondiscriminatory transmission open access, the problems of transmission usage, cost, and loss allocation and pricing must be dealt fairly. Due to the conflicting nature of these problems, power system becomes more complex. As a result cooperative game theory approaches such as Shapley value and Nucleolus approach are very useful to deal the abovementioned problems. This chapter discusses the transmission usage cost and loss allocation problems with the help of Shapley and Nucleolus approach. A sample 6-bus system is utilized to show the applicability of cooperative game theory approaches on the smart deregulated market structure.

The futuristic application of cooperative game theory problem may be to solve the cost optimization problem of distributed energy sources and microgrid. Further, various open access problems such as usage cost and loss allocation can be performed by incorporating the cost of smart grid technologies such as phasor measurement units. Additionally, other cooperative game theory techniques such as Aumann Shapley can also be implemented to solve large energy sector problems.

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Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this chapter.

Appendix

6-Bus system data

The data of test system, namely 6-bus system used in this work, is given below [25]. It contains three generator busses and three load busses. The data are at 100 MVA base. **Tables 6** and **7** present the line data and bus data of the 6-bus system, respectively.

Line data of 6-bus system (in per unit) (line no.)	From bus	To bus	R	X	Susceptance
1	1	2	0.1	0.2	0.02
2	1	4	0.05	0.2	0.02
3	1	5	0.08	0.3	0.03
4	2	3	0.05	0.25	0.03
5	2	4	0.05	0.1	0.01
6	2	5	0.1	0.3	0.02
7	2	6	0.07	0.2	0.025
8	3	5	0.12	0.26	0.025
9	3	6	0.02	0.1	0.01
10	4	5	0.2	0.4	0.04
11	5	6	0.1	0.3	0.03

Table 6. Line data of 6-bus system (in per unit).

Bus data of 6-bus system (in per unit) (bus no.)	Bus type	Voltage	Angle	PL	QL	PG	QG
1	1	1.05	0	0	0	0	0
2	2	1.05	0	0	0	0.5	0
3	2	1.07	0	0	0	0.6	0
4	0	1	0	0.7	0.7	0	0
5	0	1	0	0.7	0.7	0	0
6	0	1	0	0.7	0.7	0	0

Table 7. Bus data of 6-bus system (in per unit).

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Chapter 3

The Game Theory: Applications in the Wireless Networks

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Additional information is available at the end of the chapter

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Abstract

Recent years have witnessed a lot of applications in the computer science, especially in the area of the wireless networks. The applications can be divided into the following two main categories: applications in the network performance and those in the energy efficiency. The game theory is widely used to regulate the behavior of the users; therefore, the cooperation among the nodes can be achieved and the network performance can be improved when the game theory is utilized. On the other hand, the game theory is also adopted to control the media access control protocol or routing protocol; therefore, the energy exhaust owing to the data collision and long route can be reduced and the energy efficiency can be improved greatly. In this chapter, the applications in the network performance and the energy efficiency are reviewed. The state of the art in the applications of the game theory in wireless networks is pointed out. Finally, the future research direction of the game theory in the energy harvesting wireless sensor network is presented.

Keywords: game theory, wireless networks, network security, network performance, energy efficiency

1. Introduction

The game theory is the subject on constructing mathematical models concerning the conflict and cooperation among the intelligent rational decision-makers [1]. Traditionally, the game theory has been applied in some economic problems owing to its utilization in the analysis of the resource management. Although the game theory is mainly used in the economics, the political science, and the psychology, it also has gained applications in the logic and the computer science recently. One of the most frequently used game models is the zero-sum game in which someone is getting the prize while results in the loss of other participants. Since the conflict and cooperation usually coexist in the wireless networks, recent years have witnessed the applications of the game theory in the wireless networks. For instance, in the wireless communication scenario, the resources, such as the bandwidth, the media access control, the energy supply, and so on, are limited for the nodes in the same network segment. Since all the nodes in the same network segment compete for the same media resources, some regulations are required to control their behaviors. Recent years have witnessed a large number of research papers on the applications of the game theory in the wireless networks [2, 3]. In general, the game-theory-based technologies concerning the wireless networks can be divided into two main categories. One kind of the technologies adopts the game theory to improve the network performance and the other utilizes the game theory to improve the energy efficiency. In the early 1990s, the game theory was firstly used by the researchers to propose new pricing strategies for the Internet services [4]. Exactly in the same decade, the game theoretic model gained wide applications for the noneconomic problems in the wireless networks. From the late 1990s to the beginning of the 2000s, the game theory had gained applications in the wireless networks. One of the main applications of the game theory in wireless networks is to model and analyze the routing and resource allocation problems [5]. For instance, the noncooperative game theory model can be adopted to regulate the network traffic to improve the network performance.

In the wireless network scenario, there are usually many nodes existing in the same network segment, so all the nodes which intend to transmit data should share the same limited radio resources, such as the wireless channels, the transmission power, and so on. How to regulate all the nodes who share the same radio resources properly in order to achieve the optimal network performance is a critical problem. As for the multiple access wireless networks, the nodes can either cooperative or compete with each other to achieve their objectives (e.g., the optimal throughput and the quality of service). Consequently, the game theory can be a very useful mathematical tool to model and analyze the resource allocation problems in wireless networks. With the help of the game theory, the critical problems, such as the channel assignment, power control, the cooperation enforcement among the nodes, and so on, can be solved effectively.

Due to the multiple access feature existing in the wireless networks, the nodes in the same local area network (LAN) are allowed to share a set of available channels for data transmission. Therefore, the nodes can either compete or cooperate with each other to access the sharing channel for a social or an individual goal. The game theory can be applied to model and analyze the individual or the group behavior for the wireless networks owing to its ability in understanding the interactions among rational entities. Besides, the game theory can provide a distributed solution for the wireless networks owing to its solid theoretical foundations. It is the usual case that the central controlling for the wireless networks requires much information exchanges and coordination, and sometimes, it causes a huge overhead. In order to make the network more extensible, the distributed algorithm is required. However, the rational or selfish nodes tend to optimize their own payoff without considering the social performance in the absence of the central controller. Therefore, the existing centralized schemes are no longer suitable for such case, whereas the game theory has the superiority in controlling the nodes in a distributed way.

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Besides, sometimes, it is expected that the future wireless networks can support a variety of services with diverse quality of service (QoS) requirements. For instance, an application mixing of delay-sensitive and delay-tolerant requirements would coexist in the same network. The scenario with mixture requirements on QoS is called the wireless media sensor networks (WMSNs) [6]. For instance, the voice applications pay more attention to the network delay, whereas the data transmission requires more bandwidth. Therefore, it is a main challenge to exploit the optimal overall performance for the network.

The main contribution of the chapter lies as follows: the basic concept of the game theory and the three main components were introduced first. Then, the concept of wireless network was presented, and the main problems existing in the media access control level or the network level were also pointed out. The superiority of the game theory in dealing with the conflict and cooperation was stated. Finally, the applications of the game theory in the wireless network were elaborated in detail.

2. The game theory

Game theory, which was presented in 1944, is a theory concerning the decision-making. It gives some guides to the participants who face a dilemma whether to cooperate or conflict so as to obtain the maximum returns. In the game theory, an important concept Nash Equilibrium was proposed in 1950 which has promoted the research of the noncooperative game. When a game model reaches the Nash Equilibrium, it means that any players can impossibly obtain more favorable utility via other actions.

The game theory is a mathematical tool for analyzing the interactions of two or more decision-makers. The game theory is capable of stimulating the players to cooperate with each other to achieve a desirable goal. Usually, a game model consists of three main components: the player set, the strategy set, and the utility function of each player. As for the wireless networks, the nodes lie in the same network segment constitute the set of the players. The strategy set consists of the choice which is made by the nodes when deciding whether to relay messages for others or not. The utility function should be designed carefully to stimulate the players to cooperate with each other to achieve a considerable overall goal. It is worth noting that in the game theory, any actions taken by the user may affect the performance of others in the same network segment.

The classical game theory bases on the assumption that all the players are perfectly rational. The prediction about the game is only agreement with the actual results when each player has perfect rationality. To be perfectly rational, it is necessary that in the model, every node should be aware of the other node's actions as well as his characteristics. Nevertheless, this demand cannot always be met owing to some practical reasons. For instance, on account of the energy constraint, not every player is acquainted with the information of others. Besides, the individual differences in intelligence and learning capacity also lead to the differences in the rational level.

The game theoretic can be also divided into two main categories: the complete information game and the incomplete information game. In the first type, all the players have complete information about all the players' strategic spaces and the corresponding objectives, whereas the players of the later type only know a little about the strategic spaces and the corresponding objectives. For example, the incomplete game model can be applied to the problem of jamming for the wireless networks. As for the resource allocation problem, the behaviors of the nodes in the wireless networks can be modeled as a series of auctions, which take place in multiple rounds until all the users' requirements are met.

There is also a kind of game model named the evolutionary game theory. The evolutionary game theory can be applied to the situation where each player is of limited rationality. It was firstly introduced by Maynard Smith in 1974 [7]. Its development dues to the efforts which aim at explaining the evolution of genetically determined social behavior in the biological science. As we all know, in the real network environment, the assumption that all the players should be rational enough to determine their decisions is obviously not always satisfied. In the wireless sensor networks, the nodes usually have limited rationality. So, the evolutionary game theory can be utilized to solve some issues in the wireless sensor networks.

3. The problems existing in the wireless networks

A wireless network is a kind of system which consists of a number of nodes communicating with each other via wireless data connection. It is usually implemented via radio communications. The communication without cable can reduce the cost of deployment and maintenance, therefore the wireless network has gained a lot of applications. Examples of the wireless networks include the wireless local area networks (WLAN), the wireless sensor networks (WSNs), the ad hoc network, and the satellite communication networks.

In the wireless communication scenario, a large number of nodes compete with each other for the common resource, such as the wireless channel, the bandwidth, etc. When a source node need to transmit the data to the destination node, it needs the other nodes' help to relay the message. Therefore, the data transmission follows the hop-by-hop transmission pattern. However, not all the nodes are willing to relay the data for others owing to the energy or bandwidth consumption for relaying data. Sometimes, the nodes tend to struggle with each other for the limited resource; the network capacity is reduced when it happens. In the worst case, the data collision happens and it leads to the packet loss. The packet loss results in the decline of the network performance, such as the extension of the network delay. So, how to stimulate the nodes to cooperative with each other so as to improve the network performance is a problem facing the wireless network.

The wireless sensor networks (WSNs) is a kind of wireless network which consists of a huge number of tiny sensor nodes. Usually, the sensor nodes are powered by the battery and most of the WSNs are deployed in the regions out of the human's reach. Therefore, it is impossible or unpractical to recharge the sensor nodes. When the portion of the energy-exhausting nodes reaches a certain threshold, the network partition generates. For some applications, the network partition means the termination of the network life span. In order to extend the network lifetime, the energy efficiency should be improved. In general, the energy efficiency includes twofold, namely the minimization of energy consumption and the energy consumption equilibrium. Since the "hot spot problem" exists in the wireless sensor networks, some selfish nodes tend not to relay the data for others to save their energy. The cooperation among the sensor nodes can improve the energy efficiency. Therefore, some incentive strategies should be designed to promote the cooperation among the nodes.

In conclusion, there are two main problems existing in the wireless networks, namely the network performance and the energy efficiency for the WSNs. The game theory has gained wide applications in improving the network performance and the energy efficiency. The state of the art in the applications of the game theory in the wireless networks was detailed in the chapter.

4. The applications of the game theory in the network performance

Different from the traditional local area networks (LAN), the media access control for the wireless networks is more complicated owing to the openness of the media. Any node can get access to the media as long as it lies in the transmission range of another node. If two nodes which lie in their transmission range send data at the same time, the data collision happens. The data collision has a bad influence on the network performance. Usually, the network performance is evaluated by the throughput, the packet loss ratio, the network delay, and the network delay jitters. The data collision results in the packet loss and the decline of the network capacity, sometimes even the termination of the network life span. So, it is crucial to avoid the data collision in order to improve the network performance. In a distributed wireless system, a huge number of network nodes behave cooperative toward a common goal, such as environmental monitoring, emergency rescue, enemy tracking, and so on. In such a scenario, how to attain mutual cooperation is an important scheme. Sometimes, not all of the nodes are willing to cooperate because it consumes much resource to relay messages for others. For some extreme case, the task may be hardly to be completed. Recently, a lot of works have emerged concerning the network performance and they are introduced in detail in this section.

It has been proven in the recent literature that the proper pricing techniques can be deployed among a number of users to achieve various resource allocation policies. In the wireless relay networks, the relay nodes have no incentives to relay messages for the other users without an appropriate compensation mechanism, since it leads to the energy exhaust or the decline of the network capacity. So, the pricing mechanism provides a useful scheme that reimburses the relay nodes for using their resources by making some payoff [8–10]. Thereby, the payment providing for the relay nodes makes them be willing to forward the messages for other users.

In the wireless networks, the resource allocation is usually modeled as a noncooperative game theoretic framework in order to maximize each individual's utility. However, the selfishness of autonomous users may result in the throughput unfairness which only benefits certain users. Tan et al. presented a payment-based power control scheme using game theory in which each user announces a set of price coefficients that reflects different compensations paid by other users for the interference they produce to alleviate the throughput unfairness problem [11]. In their framework, the users who generate higher interference are required to pay more by transmitting at a lower power to give other users a fairer chance of sharing the throughput. The users could misbehave by broadcasting high price coefficients to force other users to transmit at a lower power without any incentive to play fairly. This problem was treated as a price game which resembles a prisoner's dilemma game.

The traditional networks are built on the assumption that all the network entities cooperate with each other to achieve the desirable network performance or scalability. However, the assumption may not always found owing to the emergence of some users who change the network behavior in a way in order to benefit themselves at the cost of others. Sometimes, the node with more ration would only act to achieve an outcome that he gets most. That case is more common in the multihop wireless networks like ad hoc network or sensor network which often consists of wireless battery-powered devices and the networks that need cooperation with each other to complete a task. However, the cooperation may be hard to achieve because of the limited resource, such as the bandwidth, the computational power, and the energy supply. Ng et al. presented a game-theoretic approach to strengthen the cooperation for the wireless multihop networks [12]. Tan et al. [11] applied the game theory to achieve collusive networking behavior in the multihop networks. Pricing, promiscuous listening, and mass punishments were avoided together via the game theory. Besides, the authors also provided a proof of the viability of the model under a theoretical wireless environment and showed the model can be applied to design a generic protocol which was called the Selfishness Resilient Resource Reservation protocol.

A cloud-assisted model for the malware detection and a dynamic differential game against malware propagation were presented by Zhou et al. [13]. An SVM-based malware detection model was constructed with data sharing at the security platform in the cloud. Besides, the number of malware-infected nodes was calculated precisely basing on the attributes of WMS transmission. A dynamic differential game and target cost function were successively derived for the Nash Equilibrium between the malware and WMS system. Finally, a saddle-point malware detection and suppression algorithm was proposed basing on the modified epidemic model and the computation of optimal strategies. Brown and Fazel proposed a game theoretic scheme to improve the energy efficiency for the cooperative wireless networks [14]. The tools from both cooperative and noncooperative game theory were utilized and the pareto-efficient cooperative energy allocation strategy was achieved to resist the selfish nodes, basing on the axiomatic bargaining technique. Besides, they developed the necessary and sufficient conditions under the nonfading channel without extrinsic incentive mechanism or altruistic node. Finally, they developed the technique to endogenously form the cooperative partnership without any central control. Sergi et al. exploited the game theory to model the problem via setting up a cluster of cooperative nodes in a wireless network as a multiplayer noncooperative game [15]. In their game model, all the nodes belonging to a potential relay cluster constitute the set of players and the set of actions for each player consists of only two options. Finally, a novel strategy for the management of node participation to a distributed cooperative link was derived. Seigi et al. adopted the game theory to derive a novel solution to manage the virtual antenna array basing on the transmissions in the ad hoc wireless network which consists of selfish nodes [16]. In their strategy, each node decides whether and when to transmit data packets over a shared wireless channel in an autonomous fashion. Simulation shows it offers a higher throughput level and a higher efficiency than other communication protocols which implement selection diversity in the distributed multiantenna system.

As a promising approach for the system-level analysis for the power control (PC) in wireless networks, Ginde et al. extended the game theory to the study of link adaptation, which involved the variation of modulation parameters, in addition to PC [17]. The game model in Ref. [6] was called the Link Adaptation Game (LAG) and a Nash Equilibrium (NE) was proven to be existing. Finally, a distributed algorithm was proposed to discover the NE, and it was analytically shown to converge to an NE via treating it as a point-to-set map.

Banchs et al. addressed the problem of selfishness in the Distributed Opportunistic Scheduling (DOS) from the game-theoretic's standpoint firstly [18]. The key idea of the algorithm is to react to a selfish station by using a more aggressive configuration that punishes the station. They designed a mechanism for the punishment that was sufficiently severe to prevent the selfish behavior and was not so severe to render the system unstable building on multivariable control theory. Finally, the algorithm was proven to be effective against selfish stations through conducting a game-theoretic analysis-based repeated games.

Ren et al. proposed a game theoretic model of the topology control to analyze the decentralized interactions among heterogeneous sensors [19]. They studied the function for the node to achieve the desirable frame success rate and the node degree, while minimizing the power consumption. Besides, they proposed a static compete-information game formulation for power scheduling and then proved the existence of the Nash Equilibrium with simultaneous move. They applied the game theory to analyze the distributed decision-making process of the individual sensor node and to analyze the desirable utilities of the heterogeneous sensor node. A new game theoretic model yields decentralized optimization for joint topology control and power management in their paper, and the global game equilibrium was iteratively reached by considering the individual node degree, the message delivery ratio, and the cost of increasing power.

Wang et al. provided a noncooperative game theoretic solution to enforce the cooperation in the wireless networks in the presence of channel noise [20]. They focused on the one-hop information exchange and modeled the packet forwarding process as a hidden action game with imperfect private monitoring. Besides, a state machine-based strategy was proposed to reach the Nash Equilibrium, and the equilibrium was proved to be a sequential one with the carefully designed system parameters. Furthermore, their discussion was extended to a general wireless network scenario by considering how cooperation can prevail over collusion via using the evolutionary game theory.

5. The applications of the wireless networks in the energy efficiency

In wireless networks, the intermediate nodes are chosen as cooperative nodes to relay packets for the source-destination pairs. However, not all the nodes are willing to relay data for others in the networks with inherent selfish nodes. Especially for the wireless sensor networks (WSNs), which consist of a large number of tiny sensor nodes, which are usually energy-limited. So, most of the nodes tend to reserve their energy to achieve the longer lifetime. As a result, the energy exhausts quickly if each source node sends data directly to the sink. The nodes far from the sink tend to use up their energy before those close to the sink. Therefore, how to stimulate the selfish nodes to relay messages is of great importance for the WSNs. The development of extrinsic incentive mechanisms is adopted to solve the problem, e.g., virtual currency, or the insertion of altruistic nodes in the networks to enforce cooperative behavior.

As a kind of autonomous system, the WSNs are becoming increasingly integrated into the daily life owing to their cheap deployment expense. However, the sensor nodes are powered by the battery usually, so the energy for them is limited. How to improve the energy efficiency and extend the network lifetime attracts many researchers' attention. The game theory was widely applied in the energy-efficient algorithm for the WSNs because of its superiority in regulating the behaviors of many players. Li et al. proposed an energy-efficient algorithm which combined the game theory and the software-defined network theory [21]. They integrated an SDN into the WSNs and presented an improved software-defined WSNs (SD-WSNs) architecture. Basing on the improved SD-WSNs architecture, they proposed an energy-efficient algorithm which introduced the game theory to extend the network lifetime. Like any other energy-efficient schemes, the residual energy and the transmission power were taken into consideration to prolong the life span as long as possible.

In addition to the energy-efficient routing protocols, the topology control was also adopted to improve the energy efficiency. According to the first-order radio model, the energy for transmitting a certain amount of data is in proportion to the square of the transmission distance, sometimes even the fourth power of the transmission distance. Therefore, the data transmission usually follows the hop-by-hop pattern to reduce the energy consumption. When the message is transmitted via the hop-by-hop pattern, the nodes near the sink exhaust the energy early this phenomenon is called the "hot spot problem". The "hot spot problem" leads to the network partition and the termination of the network lifetime. The main cause of it lies in the unequal energy exhausting. So, how to make the energy dissipation more evenly is also a factor to improve the energy efficiency. Topology control was adopted to evenly distribute the energy dissipation. Namely, deploying more nodes in the area which is close to the sink is an effective scheme to improve the energy efficiency.

D'Oro et al. proposed a computationally efficient algorithm to maximize the energy efficiency in the multicarrier wireless interference network [22]. Via suitably allocating the system radio resources, such as the transmit power and subcarrier assignment, the problem can be formulated into the maximization of the global energy efficiency with subject to both maximum power and minimum transmission rate constraints. Finally, it was converted into a challenging nonconvex fractional problem and was tackled through an interplay of fractional programming, learning, and game theory.

Zappone et al. proposed an energy-efficient power control and a receiver design in the relay-assisted DS/CDMA wireless networks via the game theory [23]. The noncooperative power strategies for the uplink of relay-assisted DS/CDMA wireless networks were considered through the game-theoretic tools. It was assumed that each user was interested in maximizing his own energy efficiency which is measured in bit/Joule and denoting the number of error-free delivered bits for each energy-unit used for transmission. Several noncooperative games were presented and analyzed, and extensive simulation was conducted to confirm the theoretical findings.

In addition to the traditional wireless sensor networks which are powered only by the battery, a kind of energy harvesting wireless sensor network (EHWSN) emerges recently. For the EHWSN, the energy harvesting technology is utilized to prolong the life span of the network. The EHWSN can recharge the nodes or the network by harvesting the renewable energy from the environmental sources, such as the sun, the wind, the vibrations, and so on [24–26]. Although the energy harvesting technology provides a feasible scheme to extend the lifetime of the sensor nodes to some extent, the intermittent as well as the random EH process and the complexity in achieving global network information need to manage the energy efficiency and optimize the resource in a distributed way. Namely, the process of energy harvesting is intermittent and random in nature owing to the uncertain and dynamically changing environmental conditions. In most existing works, it is assumed that either the transmitter possesses noncausal information on the exact data arrival or the transmitter knows the statistics of EH and the data arrival processes. However, the characteristics of energy harvesting process and the data arrival process change with time in most of the practical scenarios.

Besides, the emerging of the EH nodes makes the centralized energy management more challenging because the complexity in achieving optimal energy utilization policies increases significantly with the amount of the nodes in the network [27]. Furthermore, the global knowledge of the sensor nodes sometimes is hard to obtain or even unattainable. Thereby some distributed optimization schemes which only base on the local information are required. The game theory is a kind of promising mathematical tools which model interactive decision-making processes and can be widely utilized in a distributed way in the wireless sensor networks.

Meshkati et al. provided an overview of the game theoretic schemes for the energy-efficient resource allocation in the wireless networks [28]. A direct-sequence CDMA (DS-CDMA) network where each user tends to locally and selfishly choose its action so as to maximize its own utility and satisfy the QoS requirements was considered by the authors. The multiple access feature of the wireless channel makes it possible that any user's strategy choice will affect others' performance. For instance, the choice of the transmit power, the transmission rate, the packet rate, the modulation, the multiuser receiver, the multiantenna processing algorithm, or the allocation strategy will have a great influence on the others' energy efficiency.

Liu et al. presented a game-based coordination for the wireless sensor and actor networks basing on the assumption that the better cooperation among actors can bring a better balance between the energy consumption and the energy efficiency improvement [29]. The authors introduced a game theoretic approach called coalition game into their model which was named cooperative-game-based actor-to-actor coordination algorithm. It was a multiplayer strategy which consisted of many actors. The game performs at each time when an action needs to be executed in an event area. Finally, the task allocation problem in WSANs was converted into a utility assignment in the actor-alliance. Besides, the game theory strategy was evaluated through the network simulator NS2, and the results show a better energy efficiency.

In the wireless networks scenario, there is a lack of any central controllers. Thus, usually, the node makes its own decision independently. Therefore, the fully cooperative behaviors are encouraged to increase the system capacity at the given energy budget. It has been proven that power control is an efficient scheme to meet the quality of service request. On the other hand, in some wireless networks scenario, several service providers coexist to offer multiple access for the customer. Multihop routes exist in that case. If the providers cooperate with each other via jointly deploying and pooling their resources, such as the spectrum and infrastructure, and agree to serve each other's customers, their aggregate payoffs, and individual shares, may substantially increase through opportunistic utilization of resources.

Long et al. designed a noncooperative power control algorithm without pricing mechanism [30]. The interaction among the users' decision on power control was viewed as a repeated game. A reinforcement learning algorithm to properly schedule the user's power level was proposed by the authors. The potential of the cooperation can be achieved when each service provider intelligently decides with whom it would cooperate, when to cooperate, and how to deploy as well as share the resources during the cooperation. Singh et al. modeled the cooperation via the theory of transferable payoff coalitional games [31]. The cooperation strategy involved the acquisition, deployment, and allocation of the channels and the base stations. The optimum strategy can be computed as the solution of a concave or an integer optimization. It was also shown that the optimal cooperation strategy and the stabilizing payoff shares can be obtained in polynomial time via respectively solving the primals and the duals of the above optimization.

Ren et al. proposed a pricing and distributed power control in the wireless relay networks to stimulate the nodes to cooperate with each other [32]. A wireless network with amplify-and-forward relay was taken into consideration and a pricing framework that enables the relay to set the proper price to maximize either its revenue or any desirable system performance was also presented. Specifically, the relay nodes set price and correspondingly charge the users basing on the quality of the received signals. Provided the specified price, the users compete with each other to employ the relay nodes to forward their messages. Each user was modeled as a rational player and he can maximize his own net utility through the proper power allocation. The competition among all the nodes was analyzed within the framework of a noncooperative game theory, and there always exists a unique pure Nash Equilibrium point that can be achieved via distributed iterations. Finally, a low-complexity uniform pricing algorithm and an optimal differentiated pricing algorithm were proposed. It has been shown that any system utility can be maximized via applying the differentiated pricing algorithm which enforces the users to transmit at a certain power level.

In most of the existing literature, the works were related to the distributed game theory-based strategies which were usually deployed for the battery-powered WSNs. Only a few game theory-based works focused on the energy harvesting wireless sensor networks. The game theory-based energy-efficient strategies for the EHWSN were introduced in detail.

The complex interactions among the individual sensor nodes were taken into consideration, and the game theory was utilized to optimize the general multichannel multiaccess problem in an EHWSN in a distributed way [33]. In their work, the strict delay constraints were imposed for the process of data transmission. The struggle for the common channel access among the sensor nodes were formulated into a noncooperative game, and it was proved to be an ordinal potential game which has at least one Nash Equilibrium (NE).

6. The future research direction

For many wireless networks, such as the wireless sensor networks, the ad hoc networks, the transmission between two different nodes has to be accomplished with the help of an intermediate node due to the transmit power or other constraints [34]. The traditional network resource allocation problem mainly depends on the centralized control scheme, which requires all the users to cooperate with each other, whereas in many scenario, the nodes are not willing to work together owing to the consideration of reserving their own resource. In the wireless networks, the resource is limited for the nodes lying in the same network segment; therefore, the nodes will not relay messages for others for the sake of saving the channel or the energy resource. With the development of the energy harvesting technology for the wireless sensor networks, the new challenge emerges for the wireless sensor networks to keep it cooperating. So, the new strategy should be proposed to improve the network performance and the energy efficiency. Besides, the development of the wireless energy transfer has induced a new kind of wireless transmission technology, namely the Simultaneous Wireless Information and Power Transfer (SWIPT) technology [35], the new features need novel strategy to be designed to promote the cooperation among the nodes. It gives new challenge for the game theory application in the wireless networks.

7. Conclusion

In this chapter, the game theory was introduced first. The components of the game theoretic model were discussed in detail. As a mathematical tool, the game theory model has gained a lot of applications in the economic, the biology, and the telecommunication, and so on. Recent years have witnessed a lot of applications of the game theory in the wireless networks owing to its superiority in stimulating cooperation of the individual. The applications of the game theoretic can be divided into two main categories: the applications in network performance and that in the energy efficiency improving. Therefore, in this chapter, the state of the art of the game theoretic applications in the wireless networks was discussed in detail. Finally, the future research direction of the game theoretic applications in wireless networks was also provided.

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Conflict of interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work; there is no professional or other personal interest of any nature or kind in any product, service and/or company that could be construed as influencing the position presented in, or the review of, the manuscript entitled "The game theory: applications in computer science".

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Chapter 4

Infinite Supermodularity and Preferences

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Additional information is available at the end of the chapter

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Abstract

This chapter studies the ordinal content of supermodularity on lattices. This chapter is a generalization of the famous study of binary relations over finite Boolean algebras obtained by Wong, Yao and Lingras. We study the implications of various types of supermodularity for preferences over finite lattices. We prove that preferences on a finite lattice merely respecting the lattice order cannot disentangle these usual economic assumptions of supermodularity and infinite supermodularity. More precisely, the existence of a supermodular representation is equivalent to the existence of an infinitely supermodular representation. In addition, the strict increasingness of a complete preorder on a finite lattice is equivalent to the existence of a strictly increasing and infinitely supermodular representation. For wide classes of binary relations, the ordinal contents of quasisupermodularity, supermodularity and infinite supermodularity are exactly the same. In the end, we extend our results from finite lattices to infinite lattices.

Keywords: supermodularity, ∞-supermodularity, lattice

JEL Classifications: D11, D12, C65

1. Introduction

The aim of this chapter is mainly twofold. It intends first to emphasize that on finite lattices, preferences merely respecting the lattice order cannot disentangle the usual economic hypothesis of supermodularity representations from the much stronger ∞− supermodular representations. Thus, we complement the work of Chambers and Echenique [1, 2] who nicely prove under the assumption of weak monotonicity that supermodularity is equivalent to the notion of quasisupermodularity introduced by Milgrom and Shannon [3].

Second, we aim at offering simple constructive proofs for the existence of ∞-supermodular representations on finite lattices, hence generalizing to finite lattice the characterization obtained on finite Boolean Algebras by Wong, Yao and Lingras [4] of complete preorders representable by belief functions.

It is well known that supermodularity is a concept widely used in relation with economies of scale. It indicates a synergy relationship of subsystems, so that the marginal returns to the marginal element are closely related to the size of the existing elements. This creates nonlinear expectations that could have wide potential applications in social sciences. For example, we might see their applications in nonlinear pricing models as well as product bundling models.

This chapter is organized as follows. Section 2 introduces several notions of supermodularity. Then we propose our main result Theorem 1 over ∞-supermodularity representations and underline through Proposition 1 that, in a finite lattice, quasisupermodularity is a very weak assumption, since for weakly increasing preference relations which are complete pre-orders, it cannot be distinguished from weak quasisupermodularity but also from what we call strong quasisupermodularity. Section 3 finally shows that complete preorders merely requiring strong monotonicity for preferences lead to the existence of $\infty-$ supermodular representations.

2. Infinite Supermodularity

2.1. $\infty-$ supermodularity and preference

Definition 1 *Let* $(X, \leq)$ *be a finite lattice and* $\succeq$ *a preference relation* $\succeq$ *on X* (*i.e. a binary relation* $\succeq$ *on X with asymmetric part* $\succ$ *and symmetric part* $\sim$).

$\succeq$ is said to be weakly increasing if $x \leq y \Rightarrow x \preceq y$.

$\succeq$ is said to be strictly increasing if it is weakly increasing and $x < y \Rightarrow x \prec y$.

$\succeq$ is said to be weakly quasisupermodular if $x \vee y \sim y \Rightarrow x \wedge y \sim x$.

$\succeq$ is said to be quasisupermodular if $x \succeq x \wedge y \Rightarrow x \vee y \succeq y$ and $x \succ x \wedge y \Rightarrow x \vee y \succ y$.

$\succeq$ is said to be strongly quasisupermodular if $x \sim x \wedge y \Rightarrow x \wedge z \sim x \wedge y \wedge z$.

Note that what we call strong quasisupermodularity is dual (with $\wedge$ instead of $\vee$) of what Chambers and Echenique [1] called *modularity,* a property referred to as *Generalized Kreps* by Epstein and Marinacci [5].

Definition 2 *A function* $u : X \to \mathbb{R}$ *is said to be quasisupermodular if, for any* $x, y \in X$, $u(x) \geq u(x \wedge y)$ *implies* $u(x \vee y) \geq u(y)$ *and* $u(x) > u(x \wedge y)$ *implies* $u(x \vee y) > u(y)$. *It is said to be supermodular if, for any* $x, y \in X$, $u(x \wedge y) + u(x \vee y) \geq u(x) + u(y)$.[1]

[1] Clearly, u supermodular implies u quasisupermodular.

Definition 3 *$\succeq$ is said to be supermodular if it allows a supermodular representation $u : X \to \mathbb{R}$, i.e. there exists $u : X \to \mathbb{R}$ supermodular such that: for all $x, y \in X$, if $x \succeq y$ then $u(x) \geq u(y)$ and if $x \succ y$ then $u(x) > u(y)$. Furthermore, the representation u is weakly increasing if $x \geq y \Rightarrow u(x) \geq u(y)$.*

Definition 4 *Let $(X, \leq)$ be a finite lattice then $v : X \to \mathbb{R}$ is said to be $\infty-$ supermodular if*

$$v\left(\vee_{k=1}^{n} x_k\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} v\left(\underset{i \in I}{\wedge} x_i\right), \quad \forall n \geq 2, \quad x_k \in X, k = 1, \dots, n$$

The following simple example illustrates that indeed supermodularity and $\infty-$ supermodularity are two different notions for weakly increasing functions on a finite lattice $(X, \leq)$.

Example 5 *let $S = \{s_1, s_2, s_3\}$ and consider $(X, \leq)$ where X consists of the following partitions of S :*

$$X = \{0 := \{\{s_1\}, \{s_2\}\{s_3\}\}, \quad x_i = \{S \setminus \{s_i\}, \{s_i\}\} \;\; i = 1, 2, 3; \quad 1 := \{s_1, s_2, s_3\}\}$$

and $\leq$ is defined by $x \leq y$ if partition y is a refinement of partition x. It is straightforward to see that $x_i \wedge x_j = 0$ and $x_i \vee x_j = 1$ for $i \neq j$. Furthermore let $u : X \to R$ be defined by $u(0) = 0, u(x_i) = 1$, $\forall 1 \leq i \leq 3$ and $u(1) = 2$, clearly u is supermodular but not $\infty-$supermodular as just proved now. In actual fact, for a function $v : X \to R$ defined by $v(0) = 0$, $v(x_i) = 1$, $\forall i = 1, 2, 3$, then v is $\infty-$supermodular if and only if $v(1) \geq 3$.

Chambers and Echenique [2] have shown that a preference relation $\succeq$ on a lattice has a weakly increasing supermodular representation if and only if it has a weakly increasing and quasisupermodular representation. Now, we show that this is also equivalent to $\succeq$ allowing a weakly increasing $\infty-$ supermodular representation.

Theorem 1: A binary relation $\succeq$ on X has a weakly increasing and quasisupermodular representation if and only if it has a weakly increasing and $\infty-$ supermodular representation.

Proof. If part: Since an $\infty-$ supermodular representation is always super modular, the if part is immediate.

Only if: This part of the proof is highly inspired by the paper of David Kreps, A representation theorem for preference for flexibility, and especially by the proof of Lemma 3, p.572.

From Theorem 1 of Chambers and Echenique [2], we know that there is a weakly increasing supermodular $u : X \to \mathbb{R}$ which represents $\succeq$.

Let R be the total preorder induced on X by u, i.e. $xRy \Leftrightarrow u(x) \geq u(y)$. Let P be the asymmetric part of R. Clearly, R agrees with $\succeq$, i.e. $x \succeq y \Rightarrow xRy$ and $x \succ y \Rightarrow xPy$. Therefore, the proof will be complete if one shows that R can be represented by a ∞-supermodular function v which, by construction, will be necessarily weakly increasing.

Henceforth, to simplify the exposition, we will abuse of notation, letting R be denoted again by $\succeq$. Note that this new $\succeq$ is again monotone, i.e. $x \geq y \Rightarrow x \succeq y$, since $x \geq y \Rightarrow u(x) \geq u(y)$.

Let $\tilde{x}$ denote the equivalence class of any $x \in X$. There is a finite number of equivalence classes, $\tilde{x}_n \succ \dots \succ \tilde{x}_1$. Note that since u is supermodular: $x \succ x \wedge y \Rightarrow x \vee y \succ y$. This will be very useful later on. At last, for any $x \in X$, define $\overset{\leq}{x} = \{y \in X \mid y \leq x\}$. The following lemma will be crucial [8, 9].

Lemma 1. For any $x \in X$, there exists a unique $x^* \in X$ such that $\overset{\leq}{x} \cap \widetilde{x} = \{y \in X \mid x^* \leq y \leq x\}$.

Proof. Let us first show uniqueness. Suppose that x^* and y^* satisfy the property of Lemma 1. Then, $x^* \leq y^*$ and $y^* \leq x^*$ so $x^* = y^*$.

Since $\widetilde{x}$ is finite, there exists at least one minimal element for $\leq$ in $\overset{\leq}{x} \cap \widetilde{x}$, which is denoted by x^*. The proof will be completed if we show that $\overset{\leq}{x} \cap \widetilde{x} = \{y \in X \mid x^* \leq y \leq x\}$.

Note that $\{y \in X \mid x^* \leq y \leq x\} \subseteq \overset{\leq}{x} \cap \widetilde{x}$. Actually, if $y \in X$ and $x^* \leq y \leq x$, then weak increasingness of $\succeq$ implies $x \succeq y \succeq x^*$, hence $y \sim x$ since $x \sim x^*$.

It remains to prove that $\overset{\leq}{x} \cap \widetilde{x} \subseteq \{y \in X \mid x^* \leq y \leq x\}$, or, equally, that $y \in X$, $y \leq x$, $y \sim x$ implies $y \geq x^*$. So let us show that if $y \in X$, $y \leq x$, $y \sim x$, then not($y \geq x^*$) is impossible.

If not ($y \geq x^*$), then $x^* > x^* \wedge y$. Actually, one has always $x^* \geq x^* \wedge y$ and $y \geq x^* \wedge y$, so if $x^* = x^* \wedge y$, then we would get $y \geq x^*$, a contradiction.

Let us see now that, from the definition of x^*, $x^* > x^* \wedge y$ implies $x^* \succ x^* \wedge y$. Actually, if $x^* \wedge y \succeq x^*$, since $x^* \geq x^* \wedge y$ and $\succeq$ is monotone, it turns out that $x^* \succeq x^* \wedge y \succeq x^*$, which entails $x^* \wedge y \sim x^*$. Therefore, $x^* \wedge y \in \widetilde{x}$, but, since $x^* > x^* \wedge y$, this contradicts the fact that x^* is a minimal element of $\widetilde{x}$ for $\leq$.

So $x^* \succ x^* \wedge y$ and, by supermodularity, $x^* \vee y \succ y$. But $y \leq x$ and $x^* \leq x$, hence $x^* \vee y \leq x$. So, by monotonicity, $x \succeq x^* \vee y \succ y \sim x$. Therefore, $x \succ x$, a contradiction, which completes the proof of Lemma 1. ■

We can now turn to finishing the proof of Theorem 1. We intend to define, for any $y \in X$, $n(y) \in \mathbb{R}^+$ in a consistent way such that the function v, defined by $v(x) = \sum_{y \leq x} n(y)$ for any $x \in X$, represents $\succeq$.

Let 0_X be the minimal element of X for $\leq$. Since $x \geq 0_X$ for any $x \in X$ and $x \geq 0_X$ implies $x \succeq 0_X$, one has $\widetilde{x}_1 = \widetilde{0}_X$. For any $y \in \widetilde{x}_1$, let $n(y) = 0$. Therefore, for any $x \in \widetilde{x}_1$, $v(x) = \sum_{y \leq x} n(y) = 0$.

Let us now show by induction that the $n(y)$'s can be defined in such a way that there exists $\alpha_1 = 0 < \alpha_2 < \ldots < \alpha_i < \ldots < \alpha_n$, satisfying $v(x) = \alpha_i$, $\forall x \in \widetilde{x}_i$ and $n(y) \geq 0$ $\forall y \in X$, $y \leq x$ where $x \in \widetilde{x}_i$.

This is true for $\widetilde{x}_1$. Suppose that this has been done up to $i-1$, $1 \leq i-1 \leq n-1$ and let us prove the result for index i.

Let $\widetilde{x}_i = \left\{y_1, \ldots, y_j, \ldots, y_m\right\}$. Let us first show that we can suitably obtain $v(y_j^*) = \alpha_i > \alpha_{i-1}$, for $1 \leq j \leq m$. Note that $v\left(y_j^*\right) = n\left(y_j^*\right) + \sum_{y < y_j^*} n(y)$.

Since by monotonicity $y < y_j^*$ implies $y \preceq y_j^*$, hence $y \preceq x_i$, it comes from the definition of y_j^* that $y \prec x_i$. Therefore, at step i, $\sum_{y < y_j^*} n(y)$ is already defined, so since there is a finite number of y_j^*'s, one can choose α_i such that $\alpha_i > \alpha_{i-1}$ and such that $n(y_j^*) = \alpha_i - \sum_{y < y_j^*} n(y)$ be positive. For such an α_i, we consequently get $v(y_j^*) = \alpha_i > \alpha_{i-1}$.

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It thus remains to see that we can choose suitably the $n(.)$ values of the remaining y's satisfying $y \leq x$ for $x \in \tilde{x}_i$. So for any given y_j, $j = 1 \dots m$, we need to show that it is possible to get $v(y_j) = \alpha_i$, where $v(y_j) = \sum_{y \leq y_j} n(y)$. Let y be such that $y \leq y_j$, then by monotonicity $y \preceq y_j$. If $y \sim y_j$, then $y_j^* \leq y \leq y_j$, and if $y \prec y_j$, then necessarily $y < y_j$ and, therefore, by definition of y_j^*, necessarily $y < y_j^*$, indeed $y < y_j^*$ implies $y < y_j$. So $\{y,\ y \leq y_j\} = \{y,\ y < y_j^*\} \cup \left\{y_j^*\right\} \cup \{y,\ y_j^* < y \leq y_j\}$. Since any y such that $y_j^* < y \leq y_j$ has not yet been attributed a value $n(.)$, we can state $n(y) = 0$ for such y's. It comes that $v(y_j) = n(y_j^*) + \sum_{y < y_j^*} n(y)$, that is $v(y_j) = \alpha_i$.

So finally we get $n(z)$'s satisfying the required condition of representation: $x \succeq y \Leftrightarrow \sum_{z \leq x} n(z) \geq \sum_{z \leq y} n(z)$ with $n(z) \geq 0$, $\forall z \in X$. It remains to show that v defined this way is indeed ∞-supermodular. While we might involve Möbius inversion as in the seminal book of Rota [6]), we choose for sake of self completion to propose the following direct proof. Let $x_k \in X$, $k = 1, \dots, n$, $n \geq 2$, and let us prove that

$$v\left(\vee_{k=1}^{n} x_k\right) \geq \sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} v\left(\underset{i \in I}{\wedge} x_i\right)$$

For $x \in X$, let $I(x) = \{k \mid 1 \leq k \leq n,\ \ x \leq x_k\}$, then:

$$\begin{aligned}
\sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} v\left(\underset{i \in I}{\wedge} x_i\right) = \\
&= \sum_{\varnothing \neq I \subseteq \{1,\dots,n\}} (-1)^{|I|+1} \sum_{x \leq \underset{i \in I}{\wedge} x_i} n(x) \\
&= \sum_{I(x) \neq \varnothing} n(x) \sum_{\varnothing \neq I \subseteq I(x)} (-1)^{|I|+1} \\
&= \sum_{I(x) \neq \varnothing} n(x)(-1) \left[\sum_{I \subseteq I(x)} (-1)^{|I|} - (-1)^{|\varnothing|} \right] \\
&= \sum_{I(x) \neq \varnothing} n(x)
\end{aligned}$$

But $\sum_{I(x) \neq \varnothing} n(x) \leq \sum_{x \leq \underset{k \in \{1,..,m\}}{\vee} x_k} n(x) = v\left(\vee_{k=1}^{n} x_k\right)$ since $n(x) \geq 0\ \ \forall x \in X$.

Hence, v is an $\infty-$ supermodular and weakly increasing by construction. ∎

The following corollary shows that, as soon as the binary relation $\succeq$ on X is a complete preorder, that is, reflexive, transitive and complete (i.e. $\forall (x, y) \in X^2$, $x \succeq y$ or $y \succeq x$ or both), one can obtain a much more general result.

Corollary 1: For a complete preorder $\succeq$ on a lattice $(X, \leq)$, the following assertions are equivalent:

i. $\succeq$ is weakly increasing and quasisupermodular.

ii. $\succeq$ has a weakly increasing and quasisupermodular representation.

iii. $\succeq$ has a weakly increasing and $\infty-$ supermodular representation.

Proof. (i) $\Rightarrow$ (iii): Starting the proof of only if of Theorem 1 at the point where we were considering the finite equivalent classes of X for $\sim$ gives the result, taking into account the fact that by hypothesis $\succeq$ is monotone, or, in other words, weakly increasing, and that $\succeq$ quasisupermodular implies: $x \succ x \wedge y \Rightarrow x \vee y \succ y$. (iii) $\Rightarrow$ (ii) is immediate. (ii) $\Rightarrow$ (i): Let u be a weakly increasing and quasisupermodular representation of $\succeq$. Let $x \geq y$, then u weakly increasing implies $u(x) \geq u(y)$ and u representation of the complete preorder $\succeq$ implies $x \succeq y$, therefore $\succeq$ is weakly increasing. It remains to prove that $\succeq$ is quasisupermodular. Since $\succeq$ is weakly increasing and since $x \geq x \wedge y$ and $x \vee y \geq y$, one gets $x \succeq x \wedge y$ and $x \vee y \succeq y$ so indeed $x \succeq x \wedge y \Rightarrow x \vee y \succeq y$. Let us now show $x \succ x \wedge y \Rightarrow x \vee y \succ y$: $x \succ x \wedge y$ and u represents $\succeq$ implies $u(x) > u(x \wedge y)$, u quasisupermodular then implies $u(x \vee y) > u(y)$, and u represents the complete preorder $\succcurlyeq$ finally implies $x \vee y \succ y$, which completes the proof of the corollary. ■

Remark and example: The following example illustrates that indeed even a complete preorder $\succeq$ on a finite lattice $(X, \leq)$ may possess both a weakly increasing super modular representation which is not $\infty-$ supermodular and also a weakly increasing $\infty-$ supermodular representation.

Consider a finite set $S = \{s_1, s_2, s_3, s_4\}$ and the finite lattice $(X, \leq) = (\mathcal{P}(S), \subseteq)$. Let $u : X \to \mathbb{R}$ be defined by $u(\varnothing) = 0 = u(x)$ if the cardinal of x, denoted $|x|$, equals 1, $u(x) = \frac{1}{6}$ if $|x| = 2$, $u(x) = \frac{1}{3}$ if $|x| = 3$ and $u(x) = 1$ if $|x| = 4$.

As proved in Chateauneuf and Jaffray [7] in Example 4, u is supermodular but not $\infty-$ supermodular. Let $\succeq$ be the complete preorder on $(X, \leq)$ defined by $x \succeq y \Leftrightarrow u(x) \geq u(y)$. Clearly, u is a weakly increasing supermodular representation of $\succeq$ which is not $\infty-$ supermodular. Hence, $\succeq$ has a weakly increasing and quasisupermodular representation, and therefore from Corollary 1 $\succeq$ has a weakly increasing and $\infty-$ supermodular representation.

For instance, setting $n(\varnothing) = 0$, $n(y) = 0$ if $|y| \in \{1, 3, 4\}$ and $n(y) = \frac{1}{6}$ if $|y| = 2$, defining $v(x) = \sum_{y \leq x} n(y)$ does the job, since $v(\varnothing) = v(x) = 0$ if $|x| = 1$, $v(x) = \frac{1}{6}$ if $|x| = 2$, $v(x) = \frac{3}{6}$ if $|x| = 3$ and $v(x) = 1$ if $|x| = 4$. Hence, v is a weakly increasing and $\infty-$ supermodular representation of $\succeq$.

2.2. Weak quasisupermodularity, quasisupermodularity and strong quasisupermodularity

Now we show, for a weakly increasing complete preorder $\succeq$ on a finite lattice $(X, \leq)$, the equivalence of the different notions of quasisupermodularity defined in Definition 1.

Proposition 1: Let $(X, \leq)$ be a finite lattice, if $\succeq$ is a weakly increasing complete preorder, then the following statements are equivalent.

1. $\succeq$ is weakly quasisupermodular.

2. $\succeq$ is quasisupermodular.

3. $\succeq$ is strongly quasisupermodular.

Proof. (2) $\Rightarrow$ (1): We need to show $x \vee y \sim y \Rightarrow x \wedge y \sim x,\ \forall x, y \in X$. Suppose not, then by monotonicity, we must have $x \succ x \wedge y$, quasisupermodularity implies $x \vee y \succ y$, a contradiction. (1) $\Rightarrow$ (3): Suppose $x \wedge y \sim x$, we need to show that $x \wedge z \sim x \wedge y \wedge z$. Since $x \geq (x \wedge y) \vee (x \wedge z)$, thus if $x \wedge y \sim x$, then $x \wedge y \succeq (x \wedge y) \vee (x \wedge z)$, hence $x \wedge y \sim (x \wedge y) \vee (x \wedge z)$. By weak quasisupermodularity, we have $(x \wedge y) \wedge (x \wedge z) \sim x \wedge z$, i.e.$(x \wedge y) \wedge z \sim x \wedge z$. (3) $\Rightarrow$ (2): We need to show that $x \succeq x \wedge y \Rightarrow x \vee y \succeq y$ and $x \succ x \wedge y \Rightarrow x \vee y \succ y$. Since $\succeq$ is weakly increasing it is clear that $x \succeq x \wedge y \Rightarrow x \vee y \succeq y$ is always true. Now we prove the second statement: $x \succ x \wedge y \Rightarrow x \vee y \succ y$. First, $\succeq$ is weakly increasing implies $x \vee y \succeq y$, thus if it is not $x \vee y \succ y$ then it must be the case that $x \vee y \sim y = (x \vee y) \wedge y$, strong quasisupermodularity implies $x = (x \vee y) \wedge x \sim (x \vee y) \wedge y \wedge x = y \wedge x$, which says $x \sim y \wedge x$ contradicting $x \succ x \wedge y$. ■

Thus, we have shown that, for a weakly increasing complete preorder $\succeq$ over a finite lattice $(X, \leq)$, *Weakquasisupermodularity* $\Leftrightarrow$ *quasisupermodularity* $\Leftrightarrow$ *strong quasisupermodularity.*

3. ∞-Supermodular representation for strictly monotone preference on a lattice

Definition 6 *A function* $f : X \to \mathbb{R}$ *is strictly increasing if* $x < y \Rightarrow f(x) < f(y)$.

Theorem 2 below shows that if the preference relation $\succeq$ on $(X, \leq)$ i a complete preorder, then strict monotonicity of $\succeq$ is not only necessary but also sufficient in order to get a strictly increasing $\infty-$ supermodular function u representing $\succeq$. Moreover, the proof offers a simple constructive way to build such a representation.

Theorem 2: Let $\succeq$ be a complete preorder on $(X, \leq)$, then the following statements are equivalent:

i. $\succeq$ is strictly increasing.

ii. $\succeq$ has a strictly increasing and quasisupermodular representation.

iii. $\succeq$ has a strictly increasing and $\infty-$ supermodular representation.

Proof. (i) $\Rightarrow$ (iii): Let $\tilde{x}$ denote the equivalence class of $x \in X$ for $\sim$, and let us consider the finite number of equivalence classes $\tilde{x}_1 \prec \ldots \prec \tilde{x}_i \prec \ldots \prec \tilde{x}_n$. As in the proof of Theorem 1, it is enough to show that there exist $n(z) \geq 0\ \forall z \in X$ such that, setting $u(x) = \sum_{z \leq x} n(z)\ \forall x \in X$, one gets $x \succeq y$ if and only if $u(x) \geq u(y)$. Actually, such an u will indeed represent $\succeq$ and be $\infty-$ supermodular. Moreover, since $\succeq$ is strictly increasing, $x < y$ implies $x \prec y$ and we will get $x < y$ implies $u(x) < u(y)$ so u will be strictly increasing. So let us define inductively the $n(z)$'s in order that the function u defined by $u(x) = \sum_{z \leq x} n(z)\ \forall x \in X$ represents $\succeq$.

Let 0_X stands for the minimal element in X. Note that $\tilde{x}_1 = \{0_X\}$. Actually, $\forall x \in X$, $x \geq 0_X$, so if $x = 0_X$, indeed $x \sim 0_X$ by reflexivity of $\succeq$, and if $x > 0_X$, then $x \succ 0_X$ since $\succeq$ is strictly increasing. It turns out that, letting $n(0_X) = \alpha_1 \geq 0$ (eventually $\alpha_1 > 0$), one gets $u(x) = \alpha_1\ \forall x \in \tilde{x}_1$.

Let us now consider $\widetilde{x}_2$. For any $x \in X$, $z < x$ implies $z \prec x$ by strict monotonicity of $\succeq$. So for any given $x \in \widetilde{x}_2$, one gets $z < x$ if and only if $z = 0_X$. Actually: $z < x \Rightarrow z \prec x_2 \Rightarrow z \in \widetilde{x}_1 \Rightarrow z = 0_X$ and, conversely, given $0_X \prec x$, $0_X = x$ is impossible, and, since $0_X \leq x$, one gets $0_X < x$. Therefore, defining $n(x) = \beta_1 > 0 \;\; \forall x \in \widetilde{x}_2$, one gets for $x \in \widetilde{x}_2$ that $u(x) = \sum_{z \leq x} n(z) = \alpha_2 > \alpha_1$ where $\alpha_2 = \alpha_1 + \beta_1$.

Consider now $\widetilde{x}_3$. The same reasoning as before shows that for any $x \in \widetilde{x}_3$, $\{z, z \leq x\} = \{x\} \cup \{z, z < x\}$ and $z < x$ implies $z \prec x_3$. Since the x's belonging to $\widetilde{x}_3$ are finite, let $\overline{x} \in \widetilde{x}_3$ be such that $\sum_{z<\overline{x}} n(z) = \max_{x \in \widetilde{x}_3} \sum_{z<x} n(z)$. Note that this quantity is well defined since $n(z)$ has already been defined for $z \prec x_3$. Choose $n(\overline{x}) = \beta(\overline{x}) > 0$ sufficiently great in order that $\alpha_3 := n(\overline{x}) + \sum_{z<\overline{x}} n(z) > \alpha_2$. Choose now the remaining $n(x)$'s where $x \in \widetilde{x}_3$ such that $n(x) + \sum_{z<x} n(z) = \alpha_3$. Then, necessarily $n(x) \geq n(\overline{x}) > 0$. So we get $u(x) = \sum_{z \leq x} n(z) = \alpha_3 \;\; \forall x \in \overline{x}_3$.

Indeed, this process applies step by step along increasing rank of the classes, and thus gives the searched for result.

(iii) $\Rightarrow$ (ii) is immediate. (ii) $\Rightarrow$ (i) is immediate since $x < y \Rightarrow u(x) < u(y) \Rightarrow x \prec y$ because u represents the complete preorder. ■

As an immediate consequence, we obtain a stronger form of Corollary 5 of Chambers and Echenique [2].

Corollary 2 Let $(X, \leq)$ be a finite lattice. If a binary relation $\succeq$ on X has a strictly increasing representation, then it has a strictly increasing supermodular representation and even a strictly increasing $\infty-$ supermodular representation.

Proof. Let u be a strictly increasing representation of $\succeq$ and define the complete preorder R on X by $xRy \Leftrightarrow u(x) \geq u(y)$. Then, $x > y \Rightarrow u(x) > u(y) \Rightarrow xRy$ and not(yRx). Hence, R is a strictly increasing complete preorder on $(X, \leq)$. From Theorem 2, R, hence $\succeq$, has a strictly increasing $\infty-$ supermodular representation and, therefore, has a strictly increasing supermodular representation. This indeed implies that $\succeq$ has a supermodular representation as it is proved in Corollary 5 of Chambers and Echenique [2]. ■

4. Extensions to infinite lattices

We shall extend our major result to infinite lattices.

First it should noted that when we consider infinite lattices, we would need a separability in order represent the given preference. The following is a counter example.

Example 7 *Let L be the standard Borel $\sigma-$algebra on $[0,1]$, $\mu(x)$ the std. Lebesgue measure, $\nu(x)$ be a distribution that with mass points on all the rational numbers of $[0,1]$. We define an order on L to be $x < y$ if $\mu(x) < \mu(y)$ or $\mu(x) = \mu(y)$, and $\nu(x) < \nu(y)$. Clearly this is the induced lexicographic order on L. It is well known that the lexicographic order is not separable and does not allow a representation, thus the defined order would not allow any representation.*

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Definition 8 *A preference* $\preccurlyeq$ *on* $(X, \leq)$ *is said to be lower finitely separable if there is a countable set C such that 1.* $C \cap \{x | x \preccurlyeq y\}$ *is finite for all* $y \in X$, *and 2.* $\forall x \prec y, \quad C \cap [z | x \prec z \preccurlyeq y] \neq \emptyset$.

A differential operator on a ordered lattice can be introduced in the following manner.

Definition 9 *The difference operator on lattice* $(X, \leq)$ *is defined recursively as* $\nabla^0 f(x) = f(x)$,

$$\nabla_{a_1} f(x) = f(x) - f(x \wedge a_1), \ldots, \nabla_{a_1, \ldots, a_k} f(x) = \nabla_{a_k} \nabla_{a_1, \ldots, a_{k-1}} f(x)$$
$$= f(x) - \sum f(x \wedge a_i) + , \ldots, + (-1)^k f(x \wedge a_1 \wedge, \ldots, \wedge a_k).$$

The following proposition is well known for supermodular functions on finite lattices, one can see for example the work by J. P. Barthelemy (2000) p. 199–200.

Proposition 10 *an increasing function is* ∞*-supermodular if and only if* $\nabla_{a_1, \ldots, a_k} f(x) \geq 0$, $\forall a_1, \ldots a_k, k = 1, 2, \ldots . |X| - 2$.

Now we are ready to extend our result that strictly increasing preference must allow strictly increasing ∞-supermodular representation on *any infinite lattices with lower finite separability.*

Proposition 11 *Let* $(X, \leq)$ *be a lattice with lower finite separability then the following two statements are equivalent.*

i. $\succeq$ is a strictly increasing preference relation.

ii. There exists an strictly increasing $\infty-$ supermodular f representing $\succeq$.

Proof. Recall that a subset C is said to separate $\succcurlyeq$ if $\forall x \prec y, \exists c \in C$ such that $x \preccurlyeq c \preccurlyeq y$.

Let $C = \{c_0 \prec c_1 \prec \ldots \prec c_m \ldots\}$ be a chain relative to $\succcurlyeq$ separating the preference $\succcurlyeq$, since $\succcurlyeq$ is strictly increasing, we know that C will also separate the lattice order $\geq$. Denote x_* the maximal element of $\{c \in C | c \preccurlyeq x\}$. It is well defined due to lower finiteness.

Given any weight function assigned to the separating chain $w() : C \to R$, Denote $\int_x^y w(c) dc = \sum_{x \prec c \preccurlyeq y} w(c)$, if $x \prec y$, and $= 0$ otherwise.

Clearly given any positive function $w(c) : C \to R_{++}$, the function $f(x) = \int_{c_0}^{x} w(c) dc$ represents $\succcurlyeq$.

Now we claim that we can properly choose $w(c)$ in such a way that $f(x) = \int_{c_0}^{x} w(c) dc$ is infinitely supermodular.

Actually, we choose $w(c_0) > 0$, and $w(c) \geq 2^{|X|} \left[w(c_0) + \int_{c_0}^{c} w(s) ds \right]$, note this this is possible because $(X, \leq)$ is a finite lattice.

We claim if we choose $w(c)$ as in above then $f(x) = \int_{c_0}^{x} w(c) dc$ will infinitely supermodular.

In fact, the first difference: $\nabla_y f(x) = f(x) - f(x \wedge y) = \int_{x \wedge y}^{x} w(c)dc > 0$ if $x \wedge y \prec x,$ and $= 0$ otherwise.

$$\nabla_{y,z} f(x) = f(x) - f(x \wedge y) - [f(x \wedge z) - f(x \wedge z \wedge y)]$$
$$= \int_{x \wedge y}^{x} w(c)dc - \int_{x \wedge y \wedge z}^{x \wedge z} w(c)dc > 0$$

if $y \wedge x \prec x$ or $z \wedge x \prec x$ by our choice of $w(c),$ and $= 0$ otherwise.

It can be easily checked that if there is some $a_i \wedge x \succcurlyeq x,$ $\nabla_{a_1,..,a_k} f(x) = 0,$ if not, then $a_i \wedge x \prec x$ $\forall i = 1, 2, .., k$

$$\nabla_{a_1,..a_k} f(x) = f(x) - \sum_i f(x \wedge a_i) + .. + (-1)^k f(x \wedge a_1 \wedge .. \wedge a_m)$$
$$\geq w(x_*) - 2^{|X|} \int_{c_0}^{x_*} w(c)dc \geq 0$$

This completes the proof. ∎

5. Conclusion

In this chapter, we have explored the ordinal content of supermodularity on lattices. We studied the implications of various types of supermodularity for preferences over lattices. Especially we show that preferences on a lattice merely respecting the lattice order cannot disentangle these usual economic assumptions of supermodularity and infinite supermodularity. In addition, the strict increasingness of a complete preorder on a lattice is equivalent to the existence of a strictly increasing and infinitely supermodular representation. For wide classes of binary relations, the ordinal contents of quasisupermodularity, supermodularity and infinite supermodularity are exactly the same.

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Coordination Concerns: Concealing the Free Rider Problem

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Abstract

In our daily routine, we face many situations where we need to coordinate in cooperating, such as maintaining friendships or team working. If we all put all our efforts in doing so, we will efficiently get the best possible outcome for everybody. However, we are occasionally tempted to look the other way and let the rest solve the problem. These situations are called social dilemmas and this passive attitude of not exerting any effort in the common goal and benefitting of others doing so is denominated the free rider problem. The purpose of this chapter is to present this issue by means of a public goods game and propose the different mechanisms experimental literature has demonstrated to conceal the free rider problem. Examples of this nature are maintaining relationships over prolonged periods of time, determining a minimum threshold for the common project rewards to be enjoyed by everybody or enabling transparent communication. Additionally, sanctioning opportunities promote cooperation even further. Besides economic penalizations, punishment can occur at a social domain through hostility, ostracism or bond breaking. Moreover, it can be implemented either from an individual approach or through the use of centralized institutions endowed with sanctioning power.

Keywords: public goods game, coordination, free riding, heterogeneity, social preferences, punishment

1. Introduction

Coordination is a key element in most of our day-to-day interactions with other individuals. Think about the chief executive officer (CEO) who has to bring together different working units, the managers at each of those units organizing their teams or the workers in each of those teams trying to work together with a common goal. But coordination is not only crucial during working hours; think about reaching an agreement at your neighborhood community about setting up a new elevator, renewing the contract to the maintenance staff or modernizing the almost torn-down façade. Remember having tried to organize a meeting with your classmates to reminisce and catch up or just think about how instruments synchronize in a sonata.

Regardless of it being a working or a social environment, coordination is pursued as a guarantee of efficiency: maximizing utility using the minimum resources for it. Recall the firm example, for instance. Any profit-oriented firm will try to get the most out of its profits with the least assets and productive factors possible, where time is one of the most valuable assets in a competitive context in which it is standard to see how rivals sprint to be original and inventive. In this setting, coordinated teams will work faster and will avoid duplication and shortages, common in teams with a lack of organization. Think about going to a restaurant and receiving your drinks twice or not receiving them at all.

From a social perspective, if neighbors propose the modernization of that façade, it may be moved by their esthete self but there is probably also a component of wanting to appreciate their property. Evidently, the upgrading should imply the minimum cost that indeed causes the expected revalue.

In the field of game theory, the public goods game (PGG) has been the baseline to reproduce any of the situations previously described. This simple game, to be explained in the following section, clearly captures the importance of coordinated actions in terms of efficiency along with the associated issues that coordination raises: if coordination is so beneficial, why is it sometimes difficult to achieve? Intuitively, coordination is costly. Coordination requires effort, time and resources. And more importantly, given its social benefits, you cannot avoid that somebody that does not put in those ingredients takes advantage of the outcomes. Recall the elevator example formerly presented: you cannot prevent a neighbor who has not paid for the installation from using the elevator. The fact that coordination is costly and that its outcomes are non-excludable tempts selfish individuals to free ride from coordinating. Either way, they are going to take the elevator.

In the following section, we formally describe the PGG and the theoretical predictions for it, illustrating the free rider issue. Furthermore, we present a comparative statics analysis using recent scientific findings regarding the different elements that define a PGG. After that, we describe the four main mechanisms the literature has proposed to face the free riding matter. From these, we select sanctioning as the one with most potential and devote the last section to the detailed description of the different types of punishment schemes that can be implemented.

2. Coordination issue

The PGG, in its simplest form, is a 2×2 game where two players must simultaneously decide whether to contribute or not to contribute to a public good. The best outcome, where both players receive the highest payoffs, is reached if both of them contribute. However, if a

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player believes the other one is going to contribute, then he receives a higher payoff by not contributing, given it is a costly action, and the public good is going to be funded, thanks to the other player's contribution. Finally, if none of them contribute, the public good's costs are not covered. In a normal form, a PGG would look like **Table 1**, where the row-player is Player 1, the column-player is Player 2 and contribute (C) and not contribute (NC) are the possible actions for each player. Each payoff cell contains the payoff for Player 1, the payoff for Player 2 for every possible combination of actions. Notice that when both players coordinate in contributing, they both receive a payoff of 2. However, if one of them contributes and the other one does not, the player who has contributed bears all the costs and is left with a payoff of 0, whereas the player who has free ridden by not contributing benefits from the public good without engaging in the costly action, that is, he receives a payoff of 3. Finally, if none of them contribute, they both receive 1, which is a worse outcome than both of them contributing and earning 2. Both players have perfect information about the payoffs for each possible scenario.

	C	**NC**
C	2,2	0,3
NC	3,0	1,1

Table 1. Classic public goods game.

In game theory, the standard solution concept of a simultaneous game with perfect information is the Nash Equilibrium (NE), named after John Forbes Nash Jr. It states that a pair of actions is an NE if no player has a profitable unilateral deviation from it. Assuming both players are rational and have selfish preferences, that is, they maximize their material payoff, the NE of this game is that both players choose not to contribute (NC, NC) receiving a payoff of 1 each. One could think that the solution of this game is that both players contribute to the public good, as they are both better off than not contributing (2 > 1). However, notice that if any player believes the other player is going to contribute, they have incentives to free ride by not contributing and make a payoff of 3 instead of 2. Therefore, (C,C) cannot be an NE. However, if both players are intelligent, they can both apply this reasoning, such that they both end up in NC, NC with a payoff of 1 each. This pair of actions is an NE because no player has incentives to deviate to contribution and make a payoff of 0.

We can generalize this simple game to a broader situation that can be applied to, for instance, a firm facing this social problem. Let us consider a group of n workers who receive an endowment in effort. From this endowment, they must decide how much effort to keep for their own interest and how much to destine to the team. The sum of all of the efforts the workers invest in the firm project is then multiplied by a multiplier and equally divided among all the workers, regardless of their contribution. The material payoffs of any player would be given by Eq. (1), where ω is the endowment in effort every subject receives, g_i is the individual contribution of subject i, and α is the marginal per capita return of the project.

$$\pi_i = \omega - g_i + \alpha \sum_{i=1}^{n} g_i \tag{1}$$

The NE in this game is that every individual chooses $g_i = 0$, even though the social optimum is reached if $g_i = \omega$. For this to be true, the marginal per capita return (MPCR) must be $\alpha < 1$.

Consider, for instance, a task where a working team of three salesmen must work jointly to reach a particular goal in sales. Every salesman will get an increase in their salary, proportional to the aggregate sales achieved by the group, which makes the goal beneficial for all of them. However, achieving that goal implies a substantial level of effort. The social optimum would be reached if all of them cooperated to achieve that task, attaining the maximum salary increase possible. However, they are benefitting from the aggregate sales, that is, from the same social project. Thus, any of them could decide to free ride on effort and take advantage of the sales the other two members achieve. However, if the three of them think in the same way, nobody would devote any effort and there would be no salary increase.

Despite these predictions, experimental evidence has repeatedly and extensively proven deviations from the NE. In particular, positive levels of cooperation to public goods are usually achieved. Subjects contribute with some amount between 40 and 60% of their endowment.

In the following, and before moving on to how the coordination issue can be concealed, let us see what recent experimental evidence has proven about variations in the presented setup. There are four elements that we consider homogenous for all of the subjects in this standard game: endowment, MPCR, information, and preferences. In other words, we consider that all subjects have the same effort capabilities, that the return of the public good is common from everybody, that all of them are provided the same information about the project and that they all want to maximize their material payoff. In this section, we aim to see how the violation of these homogeneity assumptions changes the outcome of the game in an experimental setting.

2.1. Wealth heterogeneity

Wealth homogeneity is one of the first assumptions that raises suspicions. Considering equally rich societies is an unrealistic and rather utopian assumption. If we accept that we have different levels of wealth but that we are able to group ourselves into homogeneous groups, there are significant differences in what low-endowment and high-endowment groups contribute to public goods. In particular, low-income groups tend to over-contribute while high-income groups under-contribute. In other words, the proportion of their endowment that poorer groups destine to public goods is higher than that of richer groups [1–4]. Additionally, as it has been proven in [5], this result is robust to the origin of the endowment. In other words, it does not matter whether subjects have had to work for that endowment and it is in fact an income or whether it has just been given to them effortlessly as wealth. In either case, those whose endowment is lower contribute to a larger extent than those whose endowment is higher.

Nevertheless, one could argue that we usually face coordination issues in heterogeneous groups. In other words, we sometimes are not able to classify ourselves into low- and high-wealth groups and we, in fact, belong to unequal communities. In this case, heterogeneous groups contribute less than homogeneous groups [5]. This accounts for an inequality issue, where having variety could be detrimental for group performance.

These results highlight the importance of working in homogeneous groups. Going back to the salesman example, perhaps not all of them have the same time availability or the same effort capability. If we take these features as given, the unit manager should try and form homogeneous groups according to the workers' characteristics in order to maximize the total sales.

2.2. Productivity heterogeneity

The second dubious assumption is the fact that the public good's productivity, captured by the MPCR, is common for everybody. This implies that everybody values the public good in the same way and, therefore, obtains the same return from it. If we consider a neighborhood community discussing about the elevator, it is comprehensible that the return that somebody living on the first floor receives from having an elevator is not the same as the return of somebody living on the last floor. If instead, we consider a working team incentivized with a salary increase, some of them could argue that devoting that extra effort is too time-consuming and that they prefer to spend that time with their families rather than earning more money. Furthermore, personal circumstances affect our daily attitude, concentration and productivity at work, and they do not affect all of us equally.

In this line, literature has demonstrated that when endowed with different productivity levels, low-MPCR subjects contribute less than high-MPCR subjects. This holds even in heterogeneous groups with different productivity levels [6]. However, heterogeneous groups contribute less than homogeneous ones, analogously to the case of wealth heterogeneity. Moreover, as [7] stresses, these lower contributions are not a consequence of the heterogeneity itself but of the nature of such asymmetry.

This implies that teams should share interests, motivations, and goals. Likewise, different team performance-related bonuses should be avoided among identical workers. This way, coordination will be higher and so will efficiency.

Finally, let us make a brief comment about productivity related to group size. The MPCR, which we saw in the model as α, is, in fact, the result of a multiplier representing each individual's valuation of the public good divided by the number of individuals among which the public good is going to be shared. Possibly, what we expect is that increasing group size reduces cooperation given that it requires a higher degree of coordination. As shown in [8], if the larger group size entails a decline of the MPCR, the effect will indeed be negative on cooperation. Nonetheless, for the same MPCR, large groups contribute more, on average, to public goods than smaller ones [9, 10], despite the potential coordination issues. This positive effect is called group size effect and rules out the common belief of small groups being superior.

An example of this could be a logistics manager coordinating different working divisions of a supply chain. Most of the times, firms commit to handing in the final product before a particular date. In this case, where coordination is fundamental to meet such a deadline, the logistics manager should not be afraid of working with large groups in each of the chain links, as long as their productivity is similar. They will coordinate more to complete their part of the process such that the customer has his product at the right time.

2.3. Information

The classic PGG and its solution concept assume that information is perfect and symmetric. In other words, everybody has costless access to the details about the game's evolution and outcomes and this information is the same for everybody. In our daily interactions, however, we sometimes fall short of information.

An aspect in which literature has focused on during the last decades is the feedback subjects receive after playing the PGG and how they receive such information. In this line, knowing what peers have contributed increases contributions in future rounds, an effect that is detrimental if instead of knowing the level of contributions, they are informed about the earnings [11–13]. Other factors that have turned up to increase contributions are providing feedback about virtuous behavior in the group, that is, the higher-group contributions [14] or identity revealing [15].

Behavioral economics has also spotted that the way in which information is disclosed also causes a significant impact on decision making. This result is called the framing effect and is widely used in behavioral economics, especially for marketing and public policy purposes. Regarding this, [16] carries out a meta-analysis of framing effects on PGG. This study claims that if the PGG is played in phases of several rounds each after which they receive results summary, a re-start effect is triggered increasing the contributions at the beginning of the next phase. Moreover, subjects contribute more when the public good payoffs are presented in terms of gifts instead of private and public investments. Finally, comprehension tasks significantly enhance cooperation.

Information is, therefore, a potential tool for coordination issues. In working or social situations where coordination is required, transparency is always going to improve cooperation. These findings point out how information develops trust and how trust increases cooperation toward a common objective. In this line, many firms choose to make their workers more conscious of the whole process they are involved in. Likewise, many researchers ease their survey participants a copy of the final research outcome they have participated in.

2.4. Social preferences

The last underlying assumption of the standard PGG is the preferences individuals have. Following the model, we are purely selfish individuals, only concerned about the material payoffs of our actions. This also entails that we are always absolutely capable of measuring and balancing monetary costs and benefits associated with each possible action. Some game theorists have criticized this consideration and have introduced concepts inherent to behavioral economics, in particular, ideas that have to do with social preferences.

The most well-known social preference model is the inequality aversion model [17]. This model states that only a proportion of the population has selfish preferences, while the rest of the individuals dislike inequitable outcomes. If their material payoff is lower than their peers, they suffer a disutility proportional to the distance between the payoffs (disadvantageous inequality). Additionally, they also experience disutility if their payoff is higher than that of their peers' (advantageous inequality). Nevertheless, the first disutility is stronger than the second one: you do not like being worse off, you do not like being better off but if you had to choose, you would prefer to be better off. Individuals with inequality aversion will contribute more than what the NE for selfish individuals predicts if this can reduce the inequality between them. This model was proposed as an explanation to the cooperative behavior constantly observed in laboratory experiments and has proven to be fairly explanatory for most of them.

A social preference alternative to explain human behavior is reciprocity. Reciprocal agents are friendly to friendly peers and hostile to hostile peers [18]. Notice reciprocity is, therefore, positive and negative. It is a tit for tat, an eye for an eye. Many supermarkets expect reciprocity by offering free samples or discounts of their products. Moreover, in our social relationships, we are usually willing to return favors to those who have been kind to us at some point in time.

Finally, the most endpoint case of social preferences is altruism. Altruism or selflessness is the complete opposite of selfishness: instead of maximizing your own material payoff, you maximize the welfare of others. This kind of preference is harder to see at a working level but is commonly used to describe paternal love.

Social preferences are important because they exist in social relationships and actually explain why we behave as we do. In predesigned settings, like working environments, social concerns are fundamental in team working. If the manager can design working teams, he should have a deep knowledge of each person's ethics when working together. An individual with high disadvantageous inequality concerns could feel highly frustrated if working with purely selfish colleagues.

3. Mechanisms to address the coordination issue

At this point, the reader should understand the relevance and problematics of coordination as well as the effects that variations in the basic assumptions of the model have. These variations, however, are usually endogenous features of the game rather than aspects we can influence on. In this section, we present exogenous mechanisms that change the game's rules pursuing an increase in coordination and, consequently, in efficiency.

3.1. Reputation

Up until now, predictions have been made for games where interactions are unique, rather than prolonged over time. If we think about a working team, a social event with our family and friends or any kind of community we belong to, it is reasonable to assume that we will meet those people in the future and we may face similar situations where coordination becomes crucial again.

In this respect, game theory makes a clear distinction between games that are played only once (one-shot games) and games that are played for several periods of time (repeated games). Repeated games, at the same time, can also be divided into finitely repeated games and infinitely repeated games.

If a relationship is maintained over a predetermined period of time (finitely repeated games), the theoretical prediction for the PGG is the same as the one of the one-shot game. Notice that if individuals cooperate in this context it is because they expect to maintain this friendly relationship in the future by building a reputation. However, if there is a last period, a last day, a last meeting and a last task, there are no incentives to be friendly for tomorrow. Would you strive in your last day of work? This phenomenon is named the end-of-the-world effect, where cooperation drastically falls in the last period. Now, if there is going to be full free riding in the last day, your incentives to cooperate in the second-to-last day disappear, so do your incentives in the third-to-last day and so do your incentives in the first period.

The workaround for this is for there to be no last day or alternatively (given vital restrictions) individuals do not know when the game is going to end. Using game theory terminology, the game has an infinite horizon or there is a positive probability of the game ending. In this case, positive levels of cooperation can effectively be achieved.

In experiments carried out in the laboratory, the existence of repeated interactions is common and is combined with the social concerns inherent in each subject. This combination leads to positive levels of cooperation that decay as time goes by, leading to an inverse U shape (see **Figure 1**). If the number of periods they are going to interact with is known, an end-of-the-world effect is always noticeable. This implies that reputation is necessary but it is not sufficient as a mechanism to sustain cooperation over time.

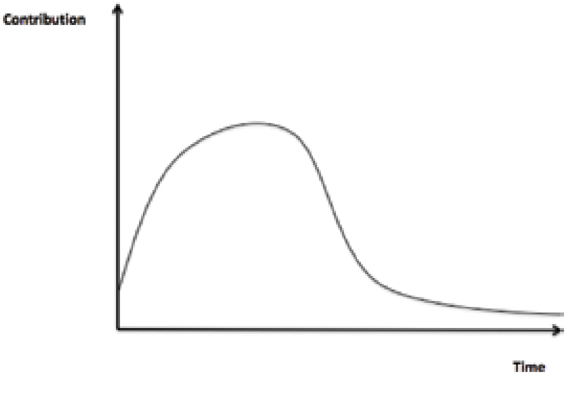

Figure 1. Standard result of public goods game experiments.

3.2. Step-level PGG

Another basic rule that can be changed to enhance cooperation is setting a particular threshold level necessary for the public good to be shared [19]. In other words, unless a minimum level of aggregate contribution is achieved, there are no public good advantages. For instance, recall the salesmen example seeking for a salary increase of Section 2. The unit manager could ask for a minimum sales target necessary for any salary increase to happen. If that threshold is sufficiently high, and the maximum effort of every member is necessary to achieve it, free riding is no longer attractive for them.

Obviously, a guaranteed success would be to set the threshold at maximum contribution such that there are no advantages on free riding. If we talk about an investment, this is feasible, as money is quantifiable. However, cases requiring human effort are more difficult to measure and compute.

Targets are natural in firm environments, where employees must meet a monthly, weekly or even daily goal. However, occasionally, these targets are settled too low such that part of the staff can achieve them by themselves without the need of high levels of coordination. Thus, it may incentivize the over-effort of some employees given that it is a take-it or leave-it approach. The definition of the correct targets is therefore essential in concealing the free rider problem.

3.3. Communication

The standard PGG is based on the premise of individuals deciding independently and simultaneously on their cooperation to the public good. However, it is ordinary to see, especially in working environments, how coworkers communicate among themselves. This communication opportunity has been repeatedly ascertained to increase the level of cooperation in PGG. Multiple communication mechanisms have been tested in the laboratory, such as non-binding face-to-face communication, audiovisual conferences, audio communication or e-mail communication, among others [20–22]—face-to-face communication being the most efficient mechanism. Nevertheless, this is not due to the loss of anonymity: verbal communication through an anonymous chat room has been demonstrated to be almost as efficient [23].

This result pinpoints the importance of enhancing a friendly environment where communication flows as part of a firm's corporate culture. In this line, many companies implement regular informal meetings or outdoor activities as part of their staff's routine for employee engagement. Furthermore, in order for it to be as binding as possible, the barriers between the interlocutors should be minimized.

3.4. Sanctioning

The most common mechanism to conceal the free rider issue and which counts with a vast theoretical and experimental literature is sanctioning. Sanctioning can be understood in many different ways: formal economic punishment in the form of a penalty, social punishment related to hostility or even breaking bonds in the working or personal domain. It is used with the purpose of smoothening the contribution nosedive in repeated PGG and, for some cases, reverts it.

According to purely selfish preferences, if punishment is costly, nobody should engage in such action. However, as social agents that we are, we do implement punishment, even in one-shot situations. The next question we propose is related to how punishment, as a matter of fact, is implemented. Should coworkers have the power to sanction each other? Should there be a responsible person in charge of doing so? Can coworkers then return the hostile behavior somehow? Should there be a certain level of agreement in a punishment decision?

The following section in this chapter tackles this issue by presenting different types of sanctioning schemes.

4. Sanctioning

4.1. Peer punishment

Peer punishment is the most standard way of implementing a mechanism to address coordination dilemma. Peer punishment consists of the opportunity for each individual to penalize, at the end of the game, those participants who have been free riders at a cost. This would be comparable to endowing coworkers with the possibility of punishing each other at the end of the day. Notice that this type of punishment is a public good itself, as everybody is better off if free riders are sanctioned, but they prefer the rest to undertake the cost of doing so.

Consider, as an example of a social setup, a group of friends meeting for dinner, where each one of them is expected to bring a dish and a beverage so that there is a variety of food and drink for dinner. If somebody free rides from their part of the contribution, but indeed benefits from what others have prepared, he could possibly not be invited again by his friends for future dinner parties as a form of social punishment. In a working scenario, coworkers could ostracize employees who free ride on effort exertion after an important task carried out by the team. At an economic level, that free riding employee could be penalized by the firm in terms of salary or even fired.

Experimental works have extensively proven that a combination of peer punishment, social preferences, and long-term interactions leads to higher contributions. The key result in this field is that peer punishment can indeed raise contributions to levels above those attainable in the absence of such punishments [18]. Furthermore, these improvements in terms of efficiency are also valued by individuals, who, if allowed to choose between a sanctioning environment and a sanction-free environment, establish themselves in the former one after a learning process [24]. Regarding the long-run effects, contributions reach significantly higher levels the greater as the number of periods subjects interact increases [25].

Many authors suggest that the ability of costly punishments to sustain high contributions to the public good depends crucially on the effectiveness of that punishment, that is, the factor by which each punishment point reduces the recipient's payoff [26]. According to the seminal work in this area [18], the cost of peer punishment should follow an exponential trend. For low levels of punishment, it should be a 1-1 relationship, but as the impact of punishment increases, this relation becomes a 3-1, that is, the cost the punisher bears is thrice the impact the punished undertakes. Other studies, however, argue that for punishment to make a difference, it must inflict a penalty that is substantially higher than the cost of meting out that punishment. In particular, they assert that the only punishment treatment that succeeds in sustaining cooperation over time is the low-cost high-impact treatment [27]. In particular, following [28], the cost-effectiveness ratio should be no less than 1-3 [28]. That is, in fact, the inverse of the seminal punishment model in [18]: for every unit of utility that is deducted from the punisher's payoff, the punished individual should have his utility reduced in three units.

What we pick up from this is that peer punishment has the power of smoothening the coordination issue as long as the cost-effectiveness ratio is suitable and relations are maintained over time. Regarding the dinner party, the free rider will have higher incentives to bring a dish if they have scheduled more dinner parties for the next months and he identifies the risk of not being invited anymore.

4.2. Counter-punishment

Counter-punishment, also known as perverse punishment, is a second-round punishment phase, where sanctioned free riders can penalize their punishers back. If one allows the possibility of counter-punishment by punished free riders, cooperators will be less willing to punish in first instance [29]. While with peer punishment, contributors use punishment as a signal of not accepting low contributions in the future, counter-punishment is used to strategically signal that future sanctions will not be tolerated. This way, peer punishment is reduced and contributions show a decaying pattern. However, counter-punishment also has its bright side if used in a good way. On the one hand, it can be used to sanction those who fail to sanction free riders, in other words, those who have free-ridden on punishment. On the other hand, it can also be used to penalize those who have exerted coercive punishment by sanctioning high contributions [30]. However, fairness concerns are necessary for this type of punishment to be used in this way.

In a company setting, counter-punishment could occur if a group of coworkers believed that somebody is being too hostile with the novice who did not rise to the challenge in a first attempt. This way, they could also decide to exclude the punitive coworker when organizing the next outdoor activity.

4.3. Coordinated punishment

Individual effective punishment is sometimes not very truthful. In real life, it is usual that a certain number of individuals are needed to effectively sanction opportunistic behavior. Everyday examples of this condition are worker strikes, a state coup or any kind of boycott. In this sense, coordinated punishment is implemented in the following way: at the end of the game, players individually decide whether to punish or not to punish opportunists, forwarding that if they succeed in reaching a threshold in the number of punishers, the damage inflicted can be very large and the individual cost of coordinated punishment can be relatively low.

Following this approach, coordinated punishment can be effective if the threshold that must be reached is sufficiently high. According to [31], coordinated punishment performs remarkably better than peer punishment when the requirement to punish a person is the emergence of a coalition of at least 40% of the group members. Authors associate the effectiveness of coordinated punishment with its ability to censor coercive punishment of the higher contributors, which, as a matter of fact, was relatively frequent in their experiment.

Coordinated punishment has also been revealed to be effective in other kinds of social dilemmas like team trust games, also called team investment games. The common method of these kinds of games is as follows. Subjects are assorted into groups of three, from which two are assigned the role of investors and one is assigned the role of the allocator. In the first stage of the game, the two investors must decide whether to invest or not in a common project, which is only successful if both of them invest. Such project generates a surplus, from which the allocator decides how much to return to each investor and how much to keep for himself, in the second stage of the game. A punishment stage could be added to this standard team trust game in next place. If this punishment scheme follows the basics of coordinated punishment such that both investors must coordinate to sanction the allocator for there to be an effective punishment, cooperation can be maintained [32].

Around us, there are many situations where coordinated punishment occurs, situations where a certain level of agreement must be attained for punishment to be effective. Think about any type of social community, where one of the members has misbehaved and the community is considering expelling the mischievous member. In this case, communities frequently undertake some kind of voting procedure for such decisions.

4.4. Pool punishment

There are numerous situations where punishment is not individually decided once the outcomes are observed but must be agreed upon before the game even starts. This way, individuals commit to punishment actions and there is no place for any kind of renegotiation of the conditions or of backing down. This reflects how investments in monitoring and sanctioning institutions to uphold the common interests are made.

In this line, different studies have explored how individuals indeed implement institutions of this type, if offered such possibility. The credible threat of this institution sanctioning opportunistic behavior at the end of the day enhances individual cooperation, which in turn has positive effects on group cooperation [33, 34]. This effect is even more pronounced with the option of counter-punishment [35].

In the last years, the comparison between peer and pool punishment has caught attention. With the purpose of overcoming difficulties and inefficiencies related to individual punishment (like the coercive punishment of high contributors we saw before), groups have continuously developed forms of self-regulation, where sanctioning is delegated to a central authority [36, 37]. Examples are specialized law forces such as the police, courts, state and non-state institutions. Hence, it could be said that it is the punishment option preferred by individuals [35, 38].

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At a firm level, the organizational hierarchy limits the sanctioning power. Besides all the examples we have provided about social punishment between coworkers, actual penalizing decisions come from higher bodies. A coworker can never fire you, a CEO can. In this sense, the commitment of the application of sanctions is usually regulated by a series of protocols and internal regulation specifying the consequences of unruly behavior in detriment of the firm. At a societal level, the same applies; you cannot economically sanction your neighbor for tax payment default or illegal parking. The most you can do is to report it to the relevant authorities for them to make use of their power.

The objective of all of these pre-designed rules that surround us is exactly to increase cooperation and avoid free riding. If the threat that we are going to be certainly caught and punished were large enough, prisons would be empty.

5. Conclusions

When we interact with other people we constantly face coordination dilemmas: in our neighborhoods, with our families, with our friends or with the people we work with. We should all put the best of ourselves so that everything works properly but there is always somebody who decides to free ride on others' money or effort. Think about a supply chain selling a product you want to buy through different channels. You can go to a local retailer to see the product, obtain information about it or even test it. However, when you get home, you are going to buy it online. The online supplier is indirectly benefiting from the service of the local retailer. We are all free riders at some point.

However, this opportunistic response is not associated with any kind of particular mischief; it is just a selfish reaction to a cooperative situation with a non-excludable outcome. Who is willing to organize the next family trip? Fortunately, the world population is not composed uniquely of selfish individuals who look the other way; most of us have social concerns for inequality, reciprocity or even altruism. This conditions how we behave in all of the described situations and brings to the surface at least someone willing to cooperate by taking the lead in planning the next trip.

Nonetheless, this is not enough. If we want to conceal the free rider problem and enhance further cooperation, we should try and form groups of people that share the same capabilities, interests, motivations, and ethics. Relationships should be maintained for as long as possible so that there is a better tomorrow for which everybody wants to fight today. Considering the multichannel supply chain example presented before, if you trust your local retailer, you could prefer to buy the product from him than from the unknown online supplier. Additionally, a sanctioning mechanism would also be helpful.

Punishment opportunities are present in most of the interactions we talk about. Punishing must not necessarily be an economic action, it can just be a social response of hostility, ostracism or bond breaking. Generally speaking, if any sort of sanctioning is at reach for the group members, cooperation significantly increases. In more detail, punishment can either be an individual decentralized decision or it can be a power endowed to a centralized authority, like a government. For interactions involving numerous agents, the establishment of hierarchies with different responsibility levels is a more feasible way of ensuring collective cooperation. But we do not need to go to massive populations to find such hierarchies: any task that requires a team working at any small-scale enterprise will already need a "good" manager.

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Chapter 6

Happy Family of Stable Marriages

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we study some aspects of the problem of stable marriage. There are two distinguished marriage plans: the fully transferable case, where money can be transferred between the participants, and the fully nontransferable case where each participant has its own rigid preference list regarding the other gender. We continue to discuss intermediate partial transferable cases. Partial transferable plans can be approached as either special cases of cooperative games using the notion of a core or as a generalization of the cyclical monotonicity property of the fully transferable case (fake promises). We introduce these two approaches and prove the existence of stable marriage for the fully transferable and nontransferable plans. The marriage problem is a special case of more general assignment problems, which has many application in mathematical economy and logistics, in particular, the assignment of employees to hiring firms. The fully cooperative marriage plan is also a special case of the celebrated problem of optimal mass transport, which is also known as Monge-Kantorovich theory. Optimal transport problem has countless applications in many fields of mathematics, physics, computer science and, of course, economy, transportation and traffic control.

Keywords: cyclic monotonicity, core, cooperative games, Monge-Kantorovich

1. Introduction

Consider two sets $\mathcal{I}_m, \mathcal{I}_w$ of N elements each. We may think about $\mathcal{I}_m$ as a set of men and $\mathcal{I}_w$ as a set of women. We denote a man in $\mathcal{I}_m$ by i and a woman in $\mathcal{I}_w$ by i'.

A *marriage plan* (MP) is a bijection which assign to each man in $\mathcal{I}_m$ a unique woman in $\mathcal{I}_w$ (and v.v). A matching of a man $i \in \mathcal{I}_m$ to a woman $j' \in \mathcal{I}_w$ is denoted by ij'. The set of all such matchings is isomorphic to the set of permutations on $\{1, \ldots N\}$. Evidently, we can arrange the order according to a given marriage plan and represent this plan as $\{ii'\}; i = 1 \ldots N$.

The MP $\{ii'\}$ is called *stable* if and only if there are no *blocking pairs*. A blocking pair is composed of a man i and a woman $j' \neq i'$ such that *both* i prefers j' over his assigned woman i' and j' prefers i over her assigned man j.

To complete this definition, we have to establish a criterion of preferences over the possible matchings in $\mathcal{I}_m \times \mathcal{I}_w$.

Let us consider two extreme cases. The first is the *fully transferable* (FT) case [1–4]. Here we assume a *utility value* $\theta_{ij'}$ for a potential matching ij'. If ij' are matched, they can split this reward $\theta_{ij'}$ between themselves as they wish.

The second case is fully non-transferable (FNT) [5–7]. This involves no utility value (and no money reward). Each participant (man or woman) lists the set of participants of the other gender according to a preference list: For each man $i \in \mathcal{I}_m$, there exists an order relation $\succ_i$ on $\mathcal{I}_m$, such that $j' \succ_i k'$ means that the man i will prefer the woman j' over the woman k'. Likewise, each woman $i' \in \mathcal{I}_w$ have its own order relation $\succ_{i'}$ over $\mathcal{I}_m$.

These two notions seem very different, and indeed they are, not only because the first one seems to defines the preference in materialistic terms and the second hints on "true love." In fact, we can quantify the nontransferable case as well: There may be a reward $\theta^m_{ij'}$ for a man i marrying a woman j', such that $j' \succ_i k'$ iff $\theta^m_{ij'} > \theta^m_{ik'}$. Likewise, $\theta^w_{ij'}$ quantifies the reward the the woman j' obtains while marrying the man i.

Given a matching $\{ii'\}$, a blocking pair in the FNT case is a pair ij', $j' \neq i'$ such that the man i prefers the woman j' over his matched woman i' (i.e., $j' \succ_i i'$, or $\theta^m_{ij'} > \theta^m_{ii'}$) *and* the woman j' prefers i over her matched man j ($j \succ_{i'} i$, or $\theta^w_{ij'} > \theta^w_{jj'}$). Thus, a blocking pair ij' is defined by

$$\min\left\{\theta^m_{ij'} - \theta^m_{ii'},\ \theta^w_{ij'} - \theta^w_{jj'}\right\} > 0 \tag{1}$$

Definition 1.1. *The matching $\{ii'\}$ is stable if and only if*

$$\min\left\{\theta^m_{ij'} - \theta^m_{ii'},\ \theta^w_{ij'} - \theta^w_{jj'}\right\} \leq 0$$

for any $i, j \in \mathcal{I}_m$ and $i', j' \in \mathcal{I}_w$.

Let

$$\theta_{ij'} := \theta^m_{ij'} + \theta^w_{ij'}. \tag{2}$$

Definition 1.1 implies that the condition

$$\theta_{ii'} + \theta_{jj'} \geq \theta_{ij'} + \theta_{ji'} \tag{3}$$

is *necessary* for all i, j for the stability of $\{ii'\}$ in the FNT case.

Let us consider now the fully transferable (FT) case. Here a married pair ii' can share the rewards $\theta_{ii'}$ for their marriage. Suppose the man i cuts u_i and the woman i' cuts $v_{i'}$ form their mutual reward $\theta_{ii'}$. Evidently, $u_i + v_{i'} = \theta_{ii'}$. If

$$u_i + v_{j'} < \theta_{ij'} \tag{4}$$

for some $j' \neq i'$ then ij' is a blocking pair, since both i and j' can increase their cuts to match the mutual reward $\theta_{ij'}$. Hence

$$\theta_{ij'} + \theta_{ji'} > u_i + v_{j'} + u_j + v_{i'} = \theta_{ii'} + \theta_{jj'}$$

so (3) is a necessary condition for the stability in the FT case as well.

Evidently, condition (3) is *not* a sufficient one, unless $N = 2$ in the FT case.

A simple example ($N = 2$):

$$\begin{array}{ccc} \theta^m & w_1 & w_2 \\ m_1 & 1 & 0 \\ m_2 & 0 & 1 \end{array} \quad ; \quad \begin{array}{ccc} \theta^w & w_1 & w_2 \\ m_1 & 1 & 5 \\ m_2 & 0 & 1 \end{array}$$

The matching $\{11', 22'\}$ is FNT stable. Indeed $\theta^m_{11'} = 1 > \theta^m_{12'} = 0$ while $\theta^m_{22'} = 1 > \theta^m_{21'} = 0$, so both men are happy, and this is enough for FNT stability, since that neither $\{12'\}$ nor $\{21'\}$ is a blocking pair. On the other hand, if the married pairs share their rewards $\theta_{ij'} = \theta^m_{ij'} + \theta^w_{ij'}$ we get

$$\begin{array}{ccc} \theta & w_1 & w_2 \\ m_1 & 2 & 5 \\ m_2 & 0 & 2 \end{array}$$

so

$$\theta_{11'} + \theta_{22'} = 4 < 5 = \theta_{12'} + \theta_{21'} \quad ,$$

thus $\{21', 12'\}$ is the stable marriage in the FT case.

However, we may extend the necessary condition (3) in the FT case as follows:

Consider the couples $i_1 i'_1, \ldots i_k i'_k$, $k \geq 2$. The sum of the rewards for these couples is $\sum_{l=1}^{k} \theta_{i_l i'_l}$. Suppose they perform a" chain deal" such that man i_l marries woman i'_{l+1} for $1 \leq l \leq k-1$, and the last man i_k marries the first woman i'_1. The net reward for the new matching is $\sum_{l=1}^{k-1} \theta_{i_l i'_{l+1}} + \theta_{i_k i'_1}$.

This leads to a definition of a *blocking chain*:

Definition 1.2. *A chain $i_1 i'_1, \ldots i_k i'_k$ of married couples forms a blocking chain iff*

$$\sum_{l=1}^{k} \left(\theta_{i_l i'_{l+1}} - \theta_{i_l i'_l} \right) > 0 \tag{5}$$

where $i'_{k+1} := i'_1$. If there are no blocking chains then the matching $\{ii'\}$ is called cyclically monotone [8].

The notion of a blocking chain extends the condition (4) from $k = 2$ to $k \geq 2$. It turns that it is also necessary condition for the stability in the fully transferable case:

Proposition 1.1. *If a marriage $\{ii'\}$ is a stable one for the FT case then it is cyclically monotone.*

Proof. Let $\{ii'\}$ be a matching, such that u_i is the cut of man i marrying i' and $v_{i'}$ the cut of the woman i' marrying i. Suppose by negation that $i_1 i'_1 \ldots i_k i'_k$ is a blocking chain. Since $u_i + v_{i'} \leq \theta_{ii'}$ we obtain

$$\sum_{l=1}^{k} \theta_{i_l i'_{l+1}} > \sum_{l=1}^{k} \theta_{i_l i'_l} \geq \sum_{l=1}^{k} \left(u_{i_l} + v_{i'_l} \right) = \sum_{l=1}^{k} \left(u_{i_l} + v_{i'_{l+1}} \right)$$

so, in particular, there exists a pair $i_l i'_{l+1}$ for which $\theta_{i_l i'_{l+1}} > u_{i_l} + v_{i'_{l+1}}$. Hence $i_l i'_{l+1}$ is a blocking pair via (4). □

We shall see later on that cyclical monotonicity is, actually, an *equivalent definition* to stability in the FT case.

The notion of cyclical monotonicity implies an additional level of cooperation for the marriage game. Not only the married pair share their utility between themselves via (2), but also different couples are ready to share their reward via a chain deal according to Definition 1.2. If the total reward after the chain exchange exceeds their reward prior to this deal, the lucky ones are ready to share their reward with the unlucky and compensate their losses.

What about the FNT case? Of course there is no point talking about a "chain deal" in that case. However, we may define a "FNT blocking chain" $i_1 i'_1 \ldots i_k i'_k$ by

$$\max_{1 \leq l \leq k} \min \left\{ \theta^m_{i_l i'_{l+1}} - \theta^m_{i_l i'_l}, \theta^w_{i_l i'_{l+1}} - \theta^w_{i_l i'_l} \right\} > 0 \tag{6}$$

where, again, $i'_{k+1} \equiv i'_1$. Definition 1.1 is analogs to the statement that that there are no blocking chains of this form. Thus, a marriage $\{ii'\}$ is stable in the FNT case if and only if

$$\max_{1 \leq l \leq k} \min \left\{ \theta^m_{i_l i'_{l+1}} - \theta^m_{i_l i'_l}, \theta^w_{i_l i'_{l+1}} - \theta^w_{i_l i'_l} \right\} \leq 0 \tag{7}$$

for any chain deal $i_1 i'_1 \ldots i_k i'_k$.

At the first sight, definition (7) seems redundant, since it provides no further information. However, we can observe the analogy between (5) and (7). In fact, (7) and (5) are obtained from each other by the exchanges

$$\theta_{i_l i'_{l+1}} - \theta_{i_l i'_l} \Leftrightarrow \min\left\{\theta^m_{i_l i'_{l+1}} - \theta^m_{i_l i'_l}, \theta^w_{i_l i'_{l+1}} - \theta^w_{i_l i'_l}\right\} \quad \text{and} \quad \sum_1^k \Leftrightarrow \max_{1 \le i \le k} \tag{8}$$

In Section 2.2, we take advantage on this representation.

2. Partial sharing

Here we present two possible definitions of intermediate marriage game which interpolate between the fully transferable and the non transferable case. The first is based on the notion of core of a cooperative game, and the second is based on cyclic monotonicity.

2.1. Stable marriage as a cooperative game

This part follows some of the ideas in Galichon et al. and references therein[1] [9]. See also [10].

Assume that we can guarantee a cut u_i for each married man i, and a cut $v_{j'}$ for each married woman j'. In order to define a stable marriage we have to impose some conditions which will guarantee that no man or woman can increase his or her cut by marrying a different partner. For this let us define, for each pair ij', a *pairwise bargaining set* $\mathcal{F}(ij') \subset \mathbb{R}^2$ which contains all possible cuts $\left(u_i, v_{j'}\right)$ for a matching of man i with woman j'.

Assumption 2.1

i. For each $i \in \mathcal{I}_m$ and $j' \in \mathcal{I}_w$, $\mathcal{F}(ij')$ are closed sets in $\mathbb{R}^2$, equal to the closure of their interior. Let $\mathcal{F}_0(ij')$ the interior of $\mathcal{F}(ij')$.

ii. $\mathcal{F}(ij')$ is monotone in the following sense: If $(u, v) \in \mathcal{F}(ij')$ then $(u', , v') \in \mathcal{F}(ij')$ whenever $u' \le u$ and $v' \le v$.

iii. There exist $C_1, C_2 \in \mathbb{R}$ such that

$$\{(u, v); \max(u, v) \le C_2\} \subset \mathcal{F}(ij') \subset \{(u, v); u + v \le C_1\}$$

for any $i \in \mathcal{I}_m, j \in \mathcal{I}_w$.

[1] which was turned to my attention by R. McCann.

The meaning of the feasibility set is as follows:

> Any married couple $ij' \in \mathcal{I}_m \times \mathcal{I}_w$ can guarantee the cut u for i and v for j', provided $(u, v) \in \mathcal{F}(ij')$.

Definition 2.1. *The* feasibility set $V(\mathcal{F}) \subset \mathbb{R}^{2N}$ *is composed of all vectors* $(u_1, \ldots u_N, v_1, \ldots v_N)$ *which satisfies*

$$\left(u_i, v_{j'}\right) \in \mathbb{R}^2 - \mathcal{F}_0(ij')$$

for any $ij' \in \mathcal{I}_m \times \mathcal{I}_w$

The marriage plan $\{ii'\}$ *is stable if and only if there exists* $(u_1, \ldots v_N) \in V(\mathcal{F})$ *such that* $(u_i, v_{i'}) \in \mathcal{F}(ii')$ *for any* $i \in \{1, \ldots N\}$.

The FNT case is contained in definition 2.1, where

$$\mathcal{F}(ij') := \left\{u \leq \theta^m_{ij'}; \quad v \leq \theta^w_{ij'}\right\}. \tag{9}$$

Indeed, if $\{ii'\}$ is a stable marriage plan let $u_i = \theta^m_{ii'}$ and $v_{i'} = \theta^w_{ii'}$. Then $(u_1, \ldots v_N)$ satisfies $(u_i, v_{i'}) \in \mathcal{F}$ for any $i \in \{1 \ldots N\}$. Since there are no blocking pairs if follows that for any $j' \neq i'$, either $\theta^m_{ij'} > \theta_{ii'} = u_i$ or $\theta^w_{ij'} > \theta^w_{jj'} = v_{j'}$, hence $\left(u_i, v_{j'}\right) \in \mathbb{R}^2 - \mathcal{F}_0(ij')$ so $(u_1 \ldots v_N) \in V(\mathcal{F})$ (**Figure 1a**).

The FT case (**Figure 1b**) is obtained by

$$\mathcal{F}(ij') := \left\{(u, v); u + v \leq \theta_{ij'}\right\}. \tag{10}$$

Indeed, if $\{ii'\}$ is a stable marriage plan and $(u_1, \ldots v_N)$ are the corresponding cuts satisfying $u_i + v_{j'} = \theta_{ij'}$, then for each $j' \neq i'$ we obtain $u_i + v_{j'} \geq \theta_{ij'}$ (otherwise ij' is a blocking pair). This implies that $\left(u_i, v_{j'}\right) \in \mathbb{R}^2 - \mathcal{F}_0(ij')$.

There are other sensible models of *partial transfers* which fit into the formalism of Definition 2.1 and Theorem 3.1. Let us consider several examples:

1. *Transferable marriages restricted to non-negative cuts*: In the transferable case, the feasibility sets may contain negative cuts for the man u or for the woman v (even though not for both, if it is assumed $\theta_{ij'} > 0$). To avoid the undesired stable marriages were one of the partners get a negative cut, we may replace the feasibility set (10) by

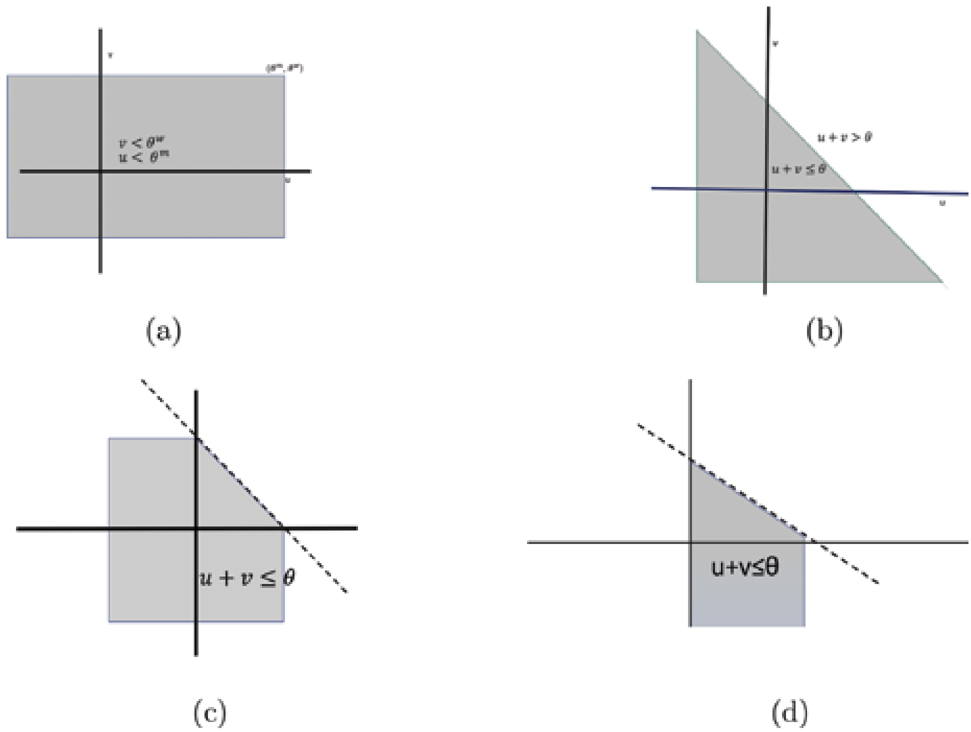

Figure 1. Pairwise bargaining sets.

$$\mathcal{F}(ij') := \left\{ (u, v) \in \mathbb{R}^2; u + v \le \theta_{ij'}, \max(u, v) \le \theta_{ij'} \right\},$$

see **Figure 1c**. It can be easily verified that if $(u_1, \ldots v_N) \in V(\mathcal{F})$ contains negative components, then $([u_1]_+, \ldots [v_N]_+)$, obtained by replacing the negative components by 0, is in $V(\mathcal{F})$ as well. Thus, the core of this game contains vectors in $V(\mathcal{F})$ of non-negative elements.

2. In the transferable case (10), we allowed both men and women to transfer money to their partner. Indeed, we assumed that the man's i cut is $\theta^m_{ij'} - w$ and the woman's j cut is $\theta^w_{ij'} + w$, where $w \in \mathbb{R}$. Suppose we wish to allow only transfer between men to women, so we insist on $w \ge 0$. In that case, we choose (**Figure 1d**)

$$\mathcal{F}(ij') := \left\{ (u, v) \in \mathbb{R}^2; u + v \le \theta_{ij'}; \quad u \le \theta^m_{ij'} \right\}. \tag{11}$$

3. Let us assume that the transfer w from man i to woman j' is taxed, and the tax depends on i, j'. Thus, if man i transfers $w > 0$ to a woman j' he reduces his cut by w, but the woman cut is increased by an amount $\beta_{i,j} w$, were $\beta_{i,j} \in [0, 1]$. Here $1 - \beta_{i,j}$ is the tax implied for this transfer. It follows that

$$u_i \le \theta^m_{ij'} - w; \qquad v_{j'} \le \theta^w_{ij'} + \beta_{i,j} w, \quad w \ge 0$$

Hence

$$\mathcal{F}(ij') := \left\{(u,v) \in \mathbb{R}^2; u_i + \beta_{i,j}^{-1} v_{j'} \le \theta_{ij'}^{\beta}, u_i \le \theta_{ij'}^{m}\right\},$$

where $\theta_{ij'}^{\beta} := \theta_{ij'}^{m} + \beta_{i,j}^{-1}\theta_{ij'}^{w}$. The geometrical description of $\mathcal{F}$ us as in **Figure 1d**, where the dashed line is tilted.

2.2. Stability by fake promises

Suppose a man can make a promise to a married woman (which is not his wife), and vice versa. The principle behind it is that each of them does not intend to honor his/her own promise, but, nevertheless, believes that the other party will honor hers/his. It is also based on both partial sharing inside a married pair, as well as some collaboration between the pairs.

Define

$$\Delta^{(q)}(i,j') := \min\left\{ \begin{array}{l} q\left(\theta_{ij'}^{m} - \theta_{ii'}^{m}\right) + \theta_{ij'}^{w} - \theta_{jj'}^{w} \\ q(\theta_{ij'}^{w} - \theta_{jj'}^{w} + \theta_{ij'}^{m} - \theta_{ii'}^{m} \end{array} \right\}, \tag{12}$$

where $0 \le q \le 1$. In particular

$$\Delta^{(0)}(i,j') := \min\left\{\theta_{ij'}^{m} - \theta_{ii'}^{m}, \theta_{ij'}^{w} - \theta_{j'j'}^{w}\right\}$$
$$\Delta^{(1)}(i,j') := \theta_{ij'}^{m} - \theta_{ii'}^{m} + \theta_{ij'}^{w} - \theta_{ii'}^{w} \equiv \theta_{ij'} - \theta_{ii'}.$$

The value of q represents the level of *internal sharing* inside the couple. Thus, $q = 0$ means there is no sharing whatsoever, and the condition $\Delta^{(0)}(i,j') > 0$ for a blocking pairs implies that both i and j' gains from the exchange, is displayed in (6).

On the other hand, $\Delta^{(1)}(i,j') + \Delta^{(1)}(j,i') > 0$, namely

$$\theta_{ii'} + \theta_{jj'} < \theta_{ij'} + \theta_{ji'}$$

is, as we argued, a necessary condition for a blocking pair in FT case, where θ represents the sum of the rewards to of the pair via (2).

We now consider an additional parameter $p \in [0,1]$ and define the real valued function on $\mathbb{R}$:

$$x \mapsto [x]_p := [x]_+ - p[x]_- \tag{13}$$

Note that $[x]_p = x$ for any p if $x \ge 0$, while $[x]_1 = x$ for any real x. The parameter p represents the level of sharing *between the pairs*.

Definition 2.2. *Let* $0 \le p, q \le 1$. *The matching* $\{ii'\}$ *is* $(p,q)-$ *stable if for any* $k \in \mathbb{N}$ *and* $i_1, i_2, \ldots i_k \in \{1, \ldots N\}$

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$$\sum_{l=1}^{k} \left[\Delta^{(q)}\left(i_l, i'_{l+1}\right)\right]_p \leq 0 where \quad i_{k+1} = i_1$$

where $i_{k+1} := i_1$.

Note that $p = 0$ implies that $\Delta^{(q)}(i, j') \leq 0$ for *any* $j' \neq i'$. If, in addition, $q = 0$ then this inequality implies that i′j′ is not a blocking pair in the FNT case.

On the other hand, $p = 1$ implies

$$\sum_{l=1}^{k} \Delta^{(q)}\left(i_l, i'_{l+1}\right) \leq 0 \text{where}$$

which is reduced to (5) if $q = 1$ as well.

Let us interpret the meaning of q, p in the context of utility exchange. A man $i \in \mathcal{I}_m$ can offer some bribe w to any other women j' he might be interested in (except his own wife, so $j' \neq i'$). His cut for marrying j' is now $\theta^m_{ij'} - w$. The cut of the woman j' should have been $\theta^w_{ij'} + w$. However, the happy woman has to pay some tax for accepting this bribe. Let $q \in [0, 1]$ be the fraction of the bribe she can get (after paying her tax). Her supposed cut for marrying i is just $\theta^w_{ij'} + qw$. Woman j' will believe and accept offer from man i if two conditions are satisfied: the offer should be both

1. *Competitive*, namely $\theta^w_{ij'} + qw \geq \theta^w_{j'j'}$.
2. *Trusted*, if woman j' believes that man i is motivated. This implies $\theta^m_{ij'} - w \geq \theta^m_{ii'}$.

The two conditions above can be satisfied, and the offer is *acceptable*, only if

$$q\left(\theta^m_{ij'} - \theta^m_{ii'}\right) + \theta^w_{ij'} - \theta^w_{jj'} > 0. \tag{14}$$

Symmetrically, man i will accept an offer from a woman $j' \neq i'$ only if

$$q\left(\theta^w_{ij'} - \theta^w_{ii'}\right) + \theta^m_{ij'} - \theta^m_{jj'} > 0. \tag{15}$$

The *utility* of the exchange ii' to ij' is, then defined by the minimum $\Delta^{(q)}(i, j')$ of (14, 15) via (12).

To understand the role of p, consider the chain of pairs exchanges

$$\left(i_1 i'_1 \to i_1 i'_2\right), \ldots \left(i_{k-1} i'_{k-1} \to i_{k-1} i'_k\right), \left(i_k i'_k\right) \to \left(i_k i'_1\right).$$

Each of the pair exchange $(i_l, i_l) \to (i_l, i_{l+1})$ yields a utility $\Delta^{(q)}\left(i_l, i'_{l+1}\right)$ for the new pair. The lucky new pairs in this chain of couples exchange are those who makes a positive reward. The unfortunate new pairs are those who suffer a loss (negative reward). The lucky pairs, whose interest is to activate this chain, are ready to compensate the unfortunate ones by contributing some of their gained utility. The chain will be activated (and the original marriages will break down) if the mutual contribution of the fortunate pairs is enough to cover *at least* the $p-$ part of the mutually loss of utility of the unfortunate pairs. This is the condition

$$\sum_{\Delta^{(q)}\left(i_l,i'_{l+1}\right)>0} \Delta^{(q)}\left(i_l,i'_{l+1}\right) + p \sum_{\Delta^{(q)}\left(i_l,i'_{l+1}\right)<0} \Delta^{(q)}\left(i_l,i'_{l+1}\right) \equiv \sum_{l=1}^{k} \left[\Delta^{(q)}\left(i_l,i'_{l+1}\right)\right]_p > 0.$$

Stability by Definition 2.2 grantees that no such chain is activated.

3. Existence of stable marriage plans

In the general case of Assumption 2.1, the existence of a stable matching follows from the following Theorem:

Theorem 3.1. *Let $W(\mathcal{F}) \subset \mathbb{R}^{2N}$ defined as follows:*

$$(u_1, \ldots u_N, v_1, \ldots v_N) \in W(\mathcal{F}),$$

$\Leftrightarrow \exists$ an injection $\tau : \mathcal{I}_m \to \mathcal{I}_w$ such that $(u_i, v_{i'}) \in \mathcal{F}(ii')$ where $i' = \tau(i)$, $\forall i \in \mathcal{I}_m$. Then there exists $(u_1, \ldots u_N, v_1, \ldots v_N) \in W(\mathcal{F})$ such that

$$\left(u_i, v_{j'}\right) \in \mathbb{R}^2 - \mathcal{F}_0(ij') \tag{16}$$

for any $(i, j') \in \mathcal{I}_m \times \mathcal{I}_w$.

The set of vectors in $W(\mathcal{F})$ satisfying (16) is called *the core*. Note that the core is identified with the set of $\mathbb{R}^{2N}$ vector in $V(\mathcal{F})$ which satisfy the condition $(u_i, v_{i'}) \in \mathcal{F}(ii')$. Hence Definition 2.1 can be recognized as the nonemptiness of the core, which is equivalent to the existence of a stable matching.

Theorem 3.1 is, in fact, a special case of the celebrated Theorem of Scarf [11] for cooperative games, tailored to the marriage scenario (see also [12, 13]). As we saw, it can be applied to the fully nontransferable case (9), as well as to the fully transferable case (10).

Theorem 3.1 implies, in particular, the existence of stable marriage in the FNT case corresponding to $p = q = 0$ or (9), as well as for the FT case corresponding to $p = q = 1$ or (10).

3.1. Gale-Shapley algorithm in the non-transferable case

Here we describe the celebrated, constructive algorithm due to Gale and Shapley [5].

1. At the first stage, each man $i \in \mathcal{I}_m$ proposes to the woman $j \in \mathcal{I}_w$ at the top of his list. At the end of this stage, some women got proposals (possibly more than one), other women may not get any proposal.
2. At the second stage, each woman who got more than one proposal binds the man whose proposal is most preferable according to her list (who is now engaged). She releases all the other men who proposed. At the end of this stage, the men's set $\mathcal{I}_m$ is composed of two parts: engaged and released.
3. At the next stage, each *released* man makes a proposal to the *next* woman in his preference list (whenever she is engaged or not).
4. Back to stage 2.

It is easy to verify that this process must end at a finite number of steps. At the end of this process, all women and men are engaged. This is a stable matching!

Of course, we could reverse the role of men and women in this algorithm. In both cases, we get a stable matching. The algorithm we indicated is the one which is best from the men's point of view. In the case where the women propose, the result is best for the women. In fact.

Theorem 3.2. [14] *For any NT stable matching $\{ii'\}$, the rank of the woman i' according to man i is at most the rank of the woman matched to i by the above, men proposing algorithm.*

3.2. Variational formulation in the fully transferable case

There are several equivalent definitions of stable marriage plan in the FT case. Here we introduces two of these.

Recall that if $\mathcal{F}$ is given by (11) the feasibility set $V(\mathcal{F})$ (Definition 2.1) takes the form

$$V(\mathcal{F}) := \left\{ (u_1, \dots v_N) \in \mathbb{R}^{2N}; \quad u_i + v_{j'} \geq \theta_{ij'} \quad \forall ij' \in \mathcal{I}_m \times \mathcal{I}_w \right\}. \tag{17}$$

Recall also Definition 1.2 for cyclical monotonicity.

Theorem 3.3 *$\{ii'\}$ is a stable marriage plan in the FT case if and only if one of the following equivalent conditions is satisfied:*

- Efficiency (or *maximal public utility*): $\sum_{i=1}^N \theta_{ii'} \geq \sum_{i=1}^N \theta_{i\sigma(i)}$ for any marriage plans $\sigma : \mathcal{I}_m \to \mathcal{I}_w$.
- $\{ii'\}$ is cyclically monotone.
- Optimality: The minimal sum $\sum_1^N u_i^0 + v_i^0$ of cuts in the feasibility set (17) satisfies $u_i^0 + v_{i'}^0 = \theta_{ii'}$ (i.e., $\{u_1^0, \dots v_N^0\}$ is in the core).

The efficiency characterization of stable marriage connects this notion with optimal transport and the celebrated *Monge Kantorovich* theory [15–17]. See also [18].

Since the set of all bijections is finite and the maximum on a finite set is always achieved, we obtain from the efficiency characterization.

Corollary 3.1. *There always exists a stable marriage plan in the FT case.*

Remark 3.1 *As far as we know, the fully transferable case (17) is the only case whose stable marriages are obtained by a variational argument.*

Proof. (of theorems 3.3) In Proposition 1.1, we obtained that FT stability implies cyclical monotonicity. We now prove that cyclical monotonicity implies efficiency. The proof follows the idea published originally by Afriat [19] and was introduced recently in a much simpler form by Brezis [20].

Let

$$-u_i^0 := \inf_{k-chains,\, k\in\mathbb{N}} \left(\sum_{l=1}^{k-1} \theta_{i_l i'_l} - \theta_{i_l i'_{l+1}} \right) + \theta_{i_k i'_k} - \theta_{i_k i'}. \tag{18}$$

Let $\alpha > -u_i^0$ and consider a $k-$ chain realizing

$$\alpha > \left(\sum_{l=1}^{k-1} \theta_{i_l i'_l} - \theta_{i_l i'_{l+1}} \right) + \theta_{i_k i'_k} - \theta_{i_k i'} \tag{19}$$

By cyclic monotonicity, $\sum_{l=1}^{k} \theta_{i_l i'_l} - \theta_{i_l i'_{l+1}} \geq 0$. Since $i'_{k+1} = i'_1$,

$$\sum_{l=1}^{k-1} \theta_{i_l i'_l} - \theta_{i_l i'_{l+1}} \geq \theta_{i_k i'_1} - \theta_{i_k i'_k}$$

so (19) implies

$$\alpha > \theta_{i_k, i'_1} - \theta_{i_k, i'} \geq 0,$$

in particular $u_i^0 < \infty$.

Hence, for any $j \in \mathcal{I}_m$

$$\begin{aligned} \alpha + \theta_{ii'} - \theta_{ij'} > & \left(\sum_{l=1}^{k-1} \theta_{i_l, i'_l} - \theta_{i_l, i'_{l+1}} \right) \\ & + \theta_{i_k i'_k} - \theta_{i_k i'} + \theta_{ii'} - \theta_{ij'} \geq -u_j^0 \end{aligned} \tag{20}$$

where the last inequality follows by the substitution of the $k+1-$ cycle $i_1 = i, i_2, \ldots, i_k, i_{k+1} = i$ in (18). Since α is any number bigger than $-u_i^0$ it follows

$$-u_i^0 + \theta_{ii'} - \theta_{ij'} \geq -u_j^0, \tag{21}$$

for any pair $i, j \in \mathcal{I}_m$. Now, let σ be any permutation in $\mathcal{I}_m$ and let $j = \sigma(i)$. Then

$$-u_i^0 + \theta_{ii'} - \theta_{i\sigma(i')} \geq -u_{\sigma(i)}^0. \tag{22}$$

Since σ is a bijection on $\mathcal{I}_m$ as well, so $\Sigma_{i=1}^N u_i^0 = \sum_{i=1}^N u_{\sigma(i)}^0$. Then, sum (22) over $1 \leq i \leq N$ to obtain

$$\sum_{i=1}^N \theta_{ii'} \geq \sum_{i=1}^N \theta_{i\sigma(i')},$$

so $\{ii'\}$ is an efficient marriage plan.

To prove that any efficient solution is stable, we define $v_j^0 := \theta_{jj'} - u_j^0$ so

$$u_j^0 + v_{j'}^0 = \theta_{jj'}. \tag{23}$$

Then (21) implies

$$u_i^0 + v_{j'}^0 = u_i^0 + \theta_{jj'} - u_j^0 \geq u_i^0 - u_i^0 + \theta_{ij'} = \theta_{ij'} \tag{24}$$

for any i, j. Thus, (23, 24) establish that $\{ii'\}$ is a stable marriage via Definition 2.1.

Finally, the optimality condition follows immediately from the definition of the feasibility set

$$\sum_1^N u_i + v_{i'} = \sum_1^N u_i + v_{\sigma(i)} \geq \sum_1^N \theta_{i\sigma(i)}$$

for any bijection $\sigma : \mathcal{I}_m \to \mathcal{I}_w$ and from (23). □

3.3. On existence and nonexistence of stable fake promises

Theorem 3.4 *If the matching $\{ii'\}$ is $(p, q)-$ stable, then it is also $(p', q')-$ stable for $p' \geq p$ and $q' \leq q$.*

The proof of this Theorem follows from the definitions (12, 13) and the following.

Lemma 3.1. *For any, $i \neq j$ and $1 \geq q > q' \geq 0$,*

$$(1+q)^{-1}\Delta^{(q)}(i,j) > (1+q')^{-1}\Delta^{(q')}(i,j).$$

Proof. For $a, b \in \mathbb{R}$ and $r \in [0, 1]$ define

$$\Delta_r(a,b) := \frac{1}{2}(a+b) - \frac{r}{2}|a-b|.$$

Observe that $\Delta_1(a,b) \equiv \min(a,b)$. In addition, $r \mapsto \Delta_r(a,b)$ is monotone not increasing in r. A straightforward calculation yields

$$\min(qa+b, qb+a) = \Delta_1(qa+b, qb+a) = (q+1)\Delta_{\frac{1-q}{1+q}}(a,b),$$

and the Lemma follows from the above observation, upon inserting $a = \theta_m(i,j) - \theta_m(i,i)$ and $b = \theta_w(i,j) - \theta_w(j,j)$. □

What can be said about the existence of s $(p,q)-$ stable matching in the general case? Unfortunately, we can prove now only a negative result:

Proposition 3.1. *For any* $1 \geq q > p \geq 0$*, a stable marriage does not exist unconditionally.*

Proof. We only need to present a counter-example. So, let $N = 2$. To show that the matching $11', 22'$ is not stable we have to show

$$\left[\Delta^{(q)}(1,2')\right]_p + \left[\Delta^{(q)}(2,1')\right]_p > 0 \tag{25}$$

while, to show that $12', 21'$ is not stable we have to show

$$\left[\Delta^{(q)}(1,1')\right]_p + \left[\Delta^{(q)}(2,2')\right]_p > 0. \tag{26}$$

By definition (12) and Lemma 3.1

$$\begin{aligned}\Delta^{(q)}(1,2') &= (q+1)\Delta_r\left(\theta^m_{12'} - \theta^m_{11'}, \theta^w_{12'} - \theta^w_{22'}\right)\\ \Delta^{(q)}(2,1') &= (q+1)\Delta_r\left(\theta^m_{21'} - \theta^m_{22'}, \theta^w_{21'} - \theta^w_{11'}\right)\end{aligned}$$

where $r = \frac{1-q}{1+q}$. To obtain $\Delta^{(q)}(1,1')$, $\Delta^{(q)}(2,2')$ we just have to exchange man 1 with man 2, so

$$\begin{aligned}\Delta^{(q)}(2,2') &= (q+1)\Delta_r\left(\theta^m_{22'} - \theta^m_{21'}, \theta^w_{22'} - \theta^w_{12'}\right)\\ \Delta^{(q)}(1,1') &= (q+1)\Delta_r\left(\theta^m_{11'} - \theta^m_{12'}, \theta^w_{11'} - \theta^w_{21'}\right).\end{aligned}$$

All in all, we only have four parameters to play with:

$$\begin{aligned}&a_1 := \theta^m_{12'} - \theta^m_{11'}, \quad a_2 = \theta^w_{12'} - \theta^w_{22'},\\ &b_1 = \theta^m_{21'} - \theta^m_{22'}, b_2 = \theta^w_{21'} - \theta^w_{11'},\end{aligned}$$

so the two conditions to be verified are

$$[\Delta_r(a_1,a_2)]_p + [\Delta_r(b_1,b_2)]_p > 0; \quad [\Delta_r(-a_1,-b_2)]_p + [\Delta_r(-b_1,-a_2)]_p > 0.$$

Let us insert $a_1 = a_2 := a > 0$. $b_1 = b_2 := -b$ where $b > 0$. So

$$[\Delta_r(a_1,a_1)]_p = a, \qquad [\Delta_r(b_1,b_2)]_p = -pb,$$

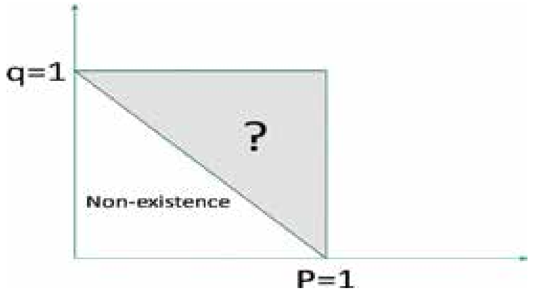

Figure 2. Conjecture: Is there an unconditional existence of stable marriages in the gray area?

while $\Delta_r(-a_1, -b_2) = \Delta_r(-b_1, -a_2) = \frac{b-a}{2} - \frac{r}{2}(a+b)$. In particular, the condition $\frac{a}{b} < \frac{1-r}{1+r}$ implies $[\Delta_r(-a_1, -b_2)]_p = [\Delta_r(-b_1, -a_2)]_p > 0$ which verifies (26). On the other hand, if $a - pb > 0$ then (25) is verified. Both conditions can be verified if $\frac{1-r}{1+r} > p$. Recalling $q = \frac{1-r}{1+r}$ we obtain the result.

Conjecture 1 *If* $0 < p < q < 1$ *then there always exists a* (p, q) *stable marriage* (c.f. **Figure 2**).

4. Conclusions

We considered several paradigms of marriage plans between two sets of different genders and the same cardinality. In particular, the extreme cases of completely transferable and completely nontransferable marriage plans. In the completely transferable case, we proved that all stable matching are obtained by an optimization which maximizes the sum of the rewards of the participants. In the completely nontransferable case, the stable marriage plane is obtained as a result of a constructive algorithm due to Gale and Shapley.

We also introduced two paradigms for partially transferable marriage plans. The first paradigm is based on a special case of cooperative coalition games, and quoted (without a proof) the theorem on existence of a stable marriage plan in that setting. The second paradigm is based on extending the notion of cyclical monotonicity which characterizes the fully transferable case. The existence of stable marriage plan in the intermediate cases of the second paradigm is still an open problem.

The marriage problem is a special case of more general assignment problems which has many application in mathematical economy and logistics. In general, the two sets of men and women can be replaced by two sets of any number of agents (e.g., firms and employees), and the 1–1 assignment in the marriage case be replaced by any number to one assignments (e.g., several employees to a given firm), allowing also the possibility of unemployment. Both paradigms introduced in this paper can be extended to include these more general cases.

From another point of view, the fully cooperative marriage plan is also a special case of the celebrated problem of optimal mass transport, also known as Monge-Kantorovich theory, after the French mathematician Monge who lived in Napoleon's time, and the soviet mathematician Kantorovich who won the Nobel prize in Economics in 1975. Optimal transport problem has countless applications in many fields such as mathematics, physics, computer science and, of course, economy, transportation, and traffic control.

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Chapter 7

Stochastic Leader-Follower Differential Game with Asymmetric Information

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we discuss a leader-follower (also called Stackelberg) stochastic differential game with asymmetric information. Here the word "asymmetric" means that the available information of the follower is some sub-σ-algebra of that available to the leader, though they play as different roles in the classical literatures. Stackelberg equilibrium is represented by the stochastic versions of Pontryagin's maximum principle and verification theorem with partial information. A linear-quadratic (LQ) leader-follower stochastic differential game with asymmetric information is studied as applications. If some system of Riccati equations is solvable, the Stackelberg equilibrium admits a state feedback representation.

Keywords: backward stochastic differential equation (BSDE), leader-follower stochastic differential game, asymmetric information, stochastic filtering, linear-quadratic control, Stackelberg equilibrium

1. Introduction

Throughout this chapter, we denote by $\mathbb{R}^n$ the Euclidean space of n-dimensional vectors, by $\mathbb{R}^{n\times d}$ the space of $n\times d$ matrices, by $\mathcal{S}^n$ the space of $n\times n$ symmetric matrices. $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the scalar product and norm in the Euclidean space, respectively. T appearing in the superscripts denotes the transpose of a matrix. f_x, f_{xx} denote the partial derivative and twice partial derivative with respect to x for a differentiable function f.

1.1. Motivation

In practice, there are many problems which motivate us to study the leader-follower stochastic differential games with asymmetric information. Here we present two examples.

Example 1.1: (Continuous time principal-agent problem) The principal contracts with the agent to manage a production process, whose cumulative proceeds (or output) Y_t evolve on $[0, T]$ as follows:

$$dY_t = Be_t dt + \sigma dW_t + \tilde{\sigma} d\tilde{W}_t, \quad Y_0 = Y_0 \in \mathbb{R}, \tag{1}$$

where $e_t \in \mathbb{R}$ is the agent's effort choice, B represents the productivity of effort, and there are two additive shocks (due to the two independent Brownian motions $W, \tilde{W}$) to the output. The proceeds of the production add to the principal's asset y_t, which earns a risk free return r, and out of which he pays the agent $s_t \in \mathbb{R}$ and withdraws his own consumption $d_t \in \mathbb{R}$. Thus the principal's asset evolves as

$$dy_t = \left[ry_t + Be_t - s_t - d_t \right] dt + \sigma dW_t + \tilde{\sigma} d\tilde{W}_t, \quad y_0 = y_0 \in \mathbb{R}, \tag{2}$$

where y_0 is the initial asset. The agent has his own wealth m_t, out of which he consumes c_t, thus

$$dm_t = [rm_t + s_t - c_t] dt + \overline{\sigma} dW_t + \tilde{\overline{\sigma}} d\tilde{W}_t, \quad m_0 = m_0 \in \mathbb{R}, \tag{3}$$

Thus, the agent earns the same rate of return r on his savings, gets income flows due to his payment s_t, and draws down wealth to consume. In the above $\sigma, \tilde{\sigma}, \overline{\sigma}, \tilde{\overline{\sigma}}$ are all constants. At the terminal time T, the principal makes a final payment s_T and the agent chooses consumption based on this payment and his terminal wealth m_T. In the above, we restrict y_t, s_t, d_t to be nonnegative.

We consider an optimal implementable contract problem in the so-called "hidden savings" information structure (Williams [1], also in Williams [2]). In this problem, the principal can observe his asset y_t and the agent's initial wealth m_0 but cannot monitor the agent's effort e_t, consumption c_t, and wealth m_t for $t > 0$. The principal must provide incentives for the agent to put forth the desired amount of the effort. For any s_t, d_t, the agent first chooses his effort e_t^* and consumption c_t^*, such that his exponential preference

$$J_1(e, c, s, d) = \mathbb{E}\left[-\int_0^T e^{-\rho t} \exp\left[-\lambda \left(c_t - \frac{1}{2} e_t^2 \right) \right] dt + e^{-\rho T}(s_T + m_T) \right] \tag{4}$$

is maximized. Here $\rho > 0$ is the discount rate and $\lambda > 0$ denotes the risk aversion parameter. The above $\left(e_t^*, c_t^* \right)$ is called an implementable contract if it meets the recommended actions of the principal's, which is based on the principal's observable wealth y_t. Then, the principal selects his payment s_t^* and consumption d_t^* to maximize his exponential preference

$$J_2(e^*, c^*, s, d) = \mathbb{E}\left[-\int_0^T e^{-\rho t}\exp\left(-\lambda d_t\right)dt + e^{-\rho T}\left(y_T - s_T\right)\right]. \tag{5}$$

Let $\mathcal{F}_t$ denote the σ-algebra generated by Brownian motions $W_s, \tilde{W}_s, 0 \le s \le t$. Intuitively, $\mathcal{F}_t$ contains all the information up to time t. Let $\mathcal{G}_{1,t}$ contains the information available to the agent, and $\mathcal{G}_{2,t}$ contains the information available to the principal, up to time t respectively. Moreover, $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$. In the game problem, first the agent solves the following optimization problem:

$$J_1(e^*, c^*, s, d) = \max_{e,\, c} J_1(e, c, s, d), \tag{6}$$

where (e^*, c^*) is a $\mathcal{G}_{1,t}$-adapted process pair. And then the principal solves the following optimization problem:

$$J_2(e^*, c^*, s^*, d^*) = \max_{s,\, d} J_2(e^*, c^*, s, d), \tag{7}$$

where (s^*, d^*) is a $\mathcal{G}_{2,t}$-adapted process pair. This formulates a stochastic Stackelberg differential game with asymmetric information. In this setting, the agent is the follower and the principal is the leader. Any process quadruple (e^*, c^*, s^*, d^*) satisfying the above two equalities is called a Stackelberg equilibrium. In Williams [1], a solvable continuous time principal-agent model is considered under three information structures (full information, hidden actions, and hidden savings) and the corresponding optimal contract problems are solved explicitly. But it can not cover our model.

Example 1.2: (Continuous time manufacturer-newsvendor problem) Let $D(\cdot)$ be the demand rate for a product in the market, which satisfies

$$dD(t) = a\left(\mu - D(t)\right)dt + \sigma dW(t) + \tilde{\sigma}\tilde{W}(t), \quad D(0) = d_0 \in \mathbb{R}, \tag{8}$$

where $a, \mu, \sigma, \tilde{\sigma}$ are constants. Suppose that the market is consisted with a manufacturer selling the product to end users through a retailer. At time t, the retailer chooses an order rate $q(t)$ for the product and decides its retail price $R(t)$, and is offered a wholesale price $w(t)$ by the manufacturer. We assume that items can be salvaged at unit price $S \ge 0$, and that items cannot be stored, that is, they must be sold instantly or salvaged. The retailer will obtain an expected profit

$$J_1(q(\cdot), R(\cdot), w(\cdot)) = \mathbb{E}\int_0^T \left[(R(t) - S)\min[D(t), q(t)] - (w(t) - S)q(t)\right]dt. \tag{9}$$

When the manufacturer has a fixed production cost per unit $M \ge 0$, he will get an expected profit

$$J_2(q(\cdot), R(\cdot), w(\cdot)) = \mathbb{E}\int_0^T \left[(w(t) - M)q(t) - S\max\left[q_t - D_t, 0\right]\right] dt. \tag{10}$$

In the above, we assume that $S < M \le w(t) \le R(t)$.

Let $\mathcal{F}_t$ denote the σ-algebra generated by $W(s), \tilde{W}(s), 0 \le s \le t$, which contains all the information up to time t. At time t, let the information $\mathcal{G}_{1,t}$, $\mathcal{G}_{2,t}$ available to the retailer and the manufacturer, respectively, are both sub-σ-algebras of $\mathcal{F}_t$. Moreover, $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$. This can be explained from the practical application's aspect. Specifically, the manufacturer chooses a wholesale price $w(t)$ at time t, which is a $\mathcal{G}_{2,t}$-adapted stochastic process. And the retailer chooses an order rate $q(t)$ and a retail price $R(t)$ at time t, which are $\mathcal{G}_{1,t}$-adapted stochastic processes. For any $w(\cdot)$, to select a $\mathcal{G}_{1,t}$-adapted process pair $(q^*(\cdot), R^*(\cdot))$ for the retailer such that

$$J_1(q^*(\cdot), R^*(\cdot), w(\cdot)) \equiv J_1(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot)) = \max_{q(\cdot), R(\cdot)} J_1(q(\cdot), R(\cdot), w(\cdot)), \tag{11}$$

and then to select a $\mathcal{G}_{2,t}$-adapted process $w^*(\cdot)$ for the manufacturer such that

$$J_2(q^*(\cdot), R^*(\cdot), w^*(\cdot)) \equiv J_2(q^*(\cdot; w^*(\cdot)), R^*(\cdot; w^*(\cdot)), w^*(\cdot)) = \max_{w(\cdot)} J_2(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot)), \tag{12}$$

formulates a leader-follower stochastic differential game with asymmetric information. In this setting, the manufacturer is the leader and the retailer is the follower. Any process triple $(q^*(\cdot), R^*(\cdot), w^*(\cdot))$ satisfying the above is called a Stackelberg equilibrium. In Øksendal et al. [3], a time-dependent newsvendor problem with time-delayed information is solved, based on stochastic differential game (with jump-diffusion) approach. But it cannot cover our model.

1.2. Problem formulation

Motivated by the examples earlier, in this chapter we study the leader-follower stochastic differential games with asymmetric information. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. $(W(\cdot), \tilde{W}(\cdot))$ is a standard $\mathbb{R}^2$-valued Brownian motion and $\{\mathcal{F}_t\}_{0\le t\le T}$ be its natural augmented filtration and $\mathcal{F}_T = \mathcal{F}$ where $T > 0$ is a finite time duration. Let the state satisfy the *stochastic differential equation* (SDE)

$$\begin{cases} dx^{u_1,u_2}(t) = b(t, x^{u_1,u_2}(t), u_1(t), u_2(t))dt + \sigma(t, x^{u_1,u_2}(t), u_1(t), u_2(t))dW(t) \\ \qquad\qquad + \tilde{\sigma}(t, x^{u_1,u_2}(t), u_1(t), u_2(t))d\tilde{W}(t), \quad x^{u_1,u_2}(0) = x_0, \end{cases} \tag{13}$$

where $u_1(\cdot)$ and $u_2(\cdot)$ are control processes taken by the two players in the game, labeled 1 (the follower) and 2 (the leader), with values in nonempty convex sets $U_1 \subseteq \mathbb{R}$, $U_2 \subseteq \mathbb{R}$, respectively. $x^{u_1,u_2}(\cdot)$, the solution to SDE Eq. (13) with values in $\mathbb{R}$, is the state process with initial state $x_0 \in \mathbb{R}^n$. Here $b(t, x, u_1, u_2) : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$, $\sigma(t, x, u_1, u_2) : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$, $\tilde{\sigma}(t, x, u_1, u_2) : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$ are given $\mathcal{F}_t$-adapted processes, for each (x, u_1, u_2).

Let us now explain the asymmetric information character between the follower (player 1) and the leader (player 2) in this chapter. Player 1 is the follower, and the information available to him at time t is based on some sub-σ-algebra $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t}$, where $\mathcal{G}_{2,t}$ is the information available to the leader. We assume in this and next sections that $\mathcal{G}_{1,t} \subseteq \mathcal{G}_{2,t} \subseteq \mathcal{F}_t$. We define the admissible control sets of the follower and the leader, respectively, as follows.

$$\mathcal{U}_k := \left\{ u_k | u_k : \Omega \times [0,T] \to U_k \text{ is } \mathcal{G}_{k,t} - \text{adapted and } \sup_{0 \le t \le T} \mathbb{E}|u_k(t)|^i < \infty, \ i = 1, 2, \cdots \right\}, \ k = 1, 2. \quad (14)$$

The game initiates with the announcement of the leaders control $u_2(\cdot) \in \mathcal{U}_2$. Knowing this, the follower would like to choose a $\mathcal{G}_{1,t}$-adapted control $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot))$ to minimize his cost functional

$$J_1(u_1(\cdot), u_2(\cdot)) = \mathbb{E}\left[\int_0^T g_1(t, x^{u_1,u_2}(t), u_1(t), u_2(t))dt + G_1(x^{u_1,u_2}(T))\right]. \quad (15)$$

Here $g_1(t, x, u_1, u_2) : \Omega \times [0,T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$ is an $\mathcal{F}_t$-adapted process, and $G_1(x) : \Omega \times \mathbb{R} \to \mathbb{R}$ is an $\mathcal{F}_T$-measurable random variable, for each (x, u_1, u_2). Now the follower encounters a stochastic optimal control problem with partial information.

SOCPF. For any chosen $u_2(\cdot) \in \mathcal{U}_2$ by the leader, choose a $\mathcal{G}_{1,t}$-adapted control $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot)) \in \mathcal{U}_1$, such that

$$J_1\left(u_1^*(\cdot), u_2(\cdot)\right) \equiv J_1\left(u_1^*(\cdot; u_2(\cdot)), u_2(\cdot)\right) = \inf_{u_1 \in \mathcal{U}_1} J_1(u_1(\cdot), u_2(\cdot)), \quad (16)$$

subject to Eqs. (13) and (15). Such a $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot))$ is called an optimal control, and the corresponding solution $x^{u_1^*,u_2}(\cdot)$ to Eq. (13) is called an optimal state.

In the following step, once knowing that the follower will take such an optimal control $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot))$, the leader would like to choose a $\mathcal{G}_{2,t}$-adapted control $u_2^*(\cdot)$ to minimize his cost functional

$$J_2\left(u_1^*(\cdot), u_2(\cdot)\right) = \mathbb{E}\left[\int_0^T g_2\left(t, x^{u_1^*,u_2}(t), u_1^*(t; u_2(t)), u_2(t)\right)dt + G_2\left(x^{u_1^*,u_2}(T)\right)\right]. \quad (17)$$

Here $g_2(t, x, u_1, u_2) : \Omega \times [0,T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}, G_2(x) : \Omega \times \mathbb{R} \to \mathbb{R}$ are given $\mathcal{F}_t$-adapted processes, for each (x, u_1, u_2). Now the leader encounters a stochastic optimal control problem with partial information.

SOCPL. Find a $\mathcal{G}_{2,t}$-adapted control $u_2^*(\cdot) \in \mathcal{U}_2$, such that

$$J_2\left(u_1^*(\cdot), u_2^*(\cdot)\right) = J_2\left(u_1^*\left(\cdot; u_2^*(\cdot)\right), u_2^*(\cdot)\right) = \inf_{u_2 \in \mathcal{U}_2} J_2\left(u_1^*(\cdot; u_2(\cdot)), u_2(\cdot)\right), \quad (18)$$

subject to Eqs. (13) and (17). Such a $u_2^*(\cdot)$ is called an optimal control, and the corresponding solution $x^*(\cdot) \equiv x^{u_1^*,u_2^*}(\cdot)$ to Eq. (13) is called an optimal state. We will rewrite the problem for the leader in more detail in the next section. We refer to the problem mentioned above as a *leader-follower stochastic differential game with asymmetric information*. If there exists a control process pair $\left(u_1^*(\cdot), u_2^*(\cdot)\right) = \left(u_1^*\left(\cdot; u_2^*(\cdot)\right), u_2^*(\cdot)\right)$ satisfying Eqs. (16) and (18), we refer to it as a *Stackelberg equilibrium*.

In this chapter, we impose the following assumptions.

(A1.1) *For each $\omega \in \Omega$, the functions $b, \sigma, \tilde{\sigma}, g_1$ are twice continuously differentiable in (x, u_1, u_2). For each $\omega \in \Omega$, functions g_2 and G_1, G_2 are continuously differentiable in (x, u_1, u_2) and x, respectively. Moreover, for each $\omega \in \Omega$ and any $(t, x, u_1, u_2) \in [0, T] \times \mathbb{R} \times U_1 \times U_2$, there exists $C > 0$ such that*

$$\begin{aligned}&(1 + |x| + |u_1| + |u_2|)^{-1}|\phi(t, x, u_1, u_2)| + |\phi_x(t, x, u_1, u_2)| + |\phi_{u_1}(t, x, u_1, u_2)| \\ &+|\phi_{u_2}(t, x, u_1, u_2)| + |\phi_{xx}(t, x, u_1, u_2)| + |\phi_{u_1u_1}(t, x, u_1, u_2)| + |\phi_{u_2u_2}(t, x, u_1, u_2)| \leq C,\end{aligned} \tag{19}$$

for $\phi = b, \sigma, \tilde{\sigma}$, and

$$\begin{aligned}&\left(1 + |x|^2\right)^{-1}|G_1(x)| + (1 + |x|)^{-1}|G_{1x}(x)| + \left(1 + |x|^2\right)^{-1}|G_2(x)| + (1 + |x|)^{-1}|G_{2x}(x)| \leq C, \\ &\left(1 + |x|^2 + |u_1|^2 + |u_2|^2\right)^{-1}|g_1(t, x, u_1, u_2)| + (1 + |x| + |u_1| + |u_2|)^{-1}(|g_{1x}(t, x, u_1, u_2)| + |g_{1u_1}(t, x, u_1, u_2)| \\ &\quad + |g_{1u_2}(t, x, u_1, u_2)|) + |g_{1xx}(t, x, u_1, u_2)| + |g_{1u_1u_1}(t, x, u_1, u_2)| + |g_{1u_2u_2}(t, x, u_1, u_2)| \leq C, \\ &\left(1 + |x|^2 + |u_1|^2 + |u_2|^2\right)^{-1}|g_2(t, x, u_1, u_2)| + (1 + |x| + |u_1| + |u_2|)^{-1}(|g_{2x}(t, x, u_1, u_2)| \\ &\quad + |g_{2u_1}(t, x, u_1, u_2)| + |g_{2u_2}(t, x, u_1, u_2)|) \leq C.\end{aligned} \tag{20}$$

1.3. Literature review and contributions of this chapter

Differential games are initiated by Issacs [4], which are powerful in modeling dynamic systems where more than one decision-makers are involved. Differential games have been researched by many scholars and have been applied in biology, economics, and finance. Stochastic differential games are differential games for stochastic systems involving noise terms. See Basar and Olsder [5] for more information about differential games. Recent developments for stochastic differential games can be seen in Hamadène [6], Wu [7], An and Øksendal [8], Wang and Yu [9, 10], and the references therein.

Leader-follower stochastic differential game is the stochastic and dynamic formulation of the Stackelberg game, which was introduced by Stackelberg [11] in 1934, when the concept of a hierarchical solution for markets where some firms have power of domination over others, is defined. This solution concept is now known as the Stackelberg equilibrium, which in the context of two-person nonzero-sum games, involves players with asymmetric roles, one leader and one follower. Pioneer study for stochastic Stackelberg differential games can be seen in Basar [12]. Specifically, a leader-follower stochastic differential game begins with the follower aims at minimizing his cost functional in response to the leader's decision on the whole duration of the game. Anticipating the follower's optimal decision depending on his entire strategy, the leader selects an optimal strategy in advance to minimize his cost functional, based on the stochastic Hamiltonian system satisfied by the follower's optimal decision. The pair of the leader's optimal strategy and the follower's optimal response is known as the Stackelberg equilibrium.

A linear-quadratic (LQ) leader-follower stochastic differential game was studied by Yong [13] in 2002. The coefficients of the the cost functionals and system are random, the diffusion term of the state equation contain the controls, and the weight matrices for the controls in the cost functionals are not necessarily positive definite. The related Riccati equations are derived to give a state feedback representation of the Stackelberg equilibrium in a nonanticipating way. Bensoussan et al. [14] obtained the global maximum principles for both open-loop and closed-loop stochastic Stackelberg differential games, whereas the diffusion term does not contain the controls.

In this chapter, we study a leader-follower stochastic differential game with asymmetric information. Our work distinguishes itself from these mentioned above in the following aspects. (1) In our framework, the information available to the follower is based on some sub-σ-algebra of that available to the leader. Moreover, both information filtration available to the leader and the follower could be sub-σ-algebras of the complete information filtration naturally generated by the random noise source. This gives a new explanation for the asymmetric information feature between the follower and the leader, and endows our problem formulation more practical meanings in realty. (2) Our work is established in the context of partial information, which is different from that of partial observation (see e.g., Wang et al. [15]) but related to An and Øksendal [8], Huang et al. [16], Wang and Yu [10]. (3) An important class of LQ leader-follower stochastic differential game with asymmetric information is proposed and then completely solved, which is a natural generalization of that in Yong [13]. It consists of a stochastic optimal control problem of SDE with partial information for the follower, and followed by a stochastic optimal control problem of *forward-backward stochastic differential equation* (FBSDE) with complete information for the leader. This problem is new in differential game theory and have considerable impacts in both theoretical analysis and practical meaning with future application prospect, although it has intrinsic mathematical difficulties. (4) The Stackelberg equilibrium of this LQ problem is characterized in terms of the *forward-backward stochastic differential filtering equations* (FBSDFEs) which arises naturally in our setup. These FBSDFEs are new and different from those in [10, 16]. (5) The Stackelberg equilibrium of this LQ problem is explicitly given, with the help of some new Riccati equations.

The rest of this chapter is organized as follows. In Section 2, we solve our problem to find the Stackelberg equilibrium. In Section 3, we apply our theoretical results to an LQ problem. Finally, Section 4 gives some concluding remarks.

2. Stackelberg equilibrium

2.1. The Follower's problem

In this subsection, we first solve **SOCPF**. For any chosen $u_2(\cdot) \in \mathcal{U}_2$, let $u_1^*(\cdot)$ be an optimal control for the follower and the corresponding optimal state be $x^{u_1^*, u_2}(\cdot)$. Define the Hamiltonian function $H_1 : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$H_1\left(t,x,u_1,u_2;q,k,\tilde{k}\right) = qb(t,x,u_1,u_2) + k\sigma(t,x,u_1,u_2) + \tilde{k}\tilde{\sigma}(t,x,u_1,u_2) - g_1(t,x,u_1,u_2). \quad (21)$$

Let an $\mathcal{F}_t$-adapted process triple $\left(q(\cdot),k(\cdot),\tilde{k}(\cdot)\right) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ satisfies the adjoint BSDE

$$\begin{cases} -dq(t) = \Big\{b_x(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))q(t) + \sigma_x(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))k(t) \\ \qquad +\tilde{\sigma}_x(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))\tilde{k}(t) - g_{1x}(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t))\Big\}dt \\ \qquad - k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \ \ q(T) = -G_{1x}\left(x^{u_1^*,u_2}(T)\right). \end{cases} \quad (22)$$

Proposition 2.1 *Let* **(A1.1)** *hold. For any given* $u_2(\cdot)\in\mathcal{U}_2$, *let* $u_1^*(\cdot)$ *be the optimal control for* **SOCPF**, *and* $x^{u_1^*,u_2}(\cdot)$ *be the corresponding optimal state. Let* $\left(q(\cdot),k(\cdot),\tilde{k}(\cdot)\right)$ *be the adjoint process triple. Then we have*

$$\mathbb{E}\left[\left\langle H_{1u_1}\left(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t);q(t),k(t),\tilde{k}(t)\right),u_1-u_1^*(t)\right\rangle\Big|\mathcal{G}_{1,t}\right]\geq 0, \quad a.e. t\in[0,T], a.s., \quad (23)$$

holds, for any $u_1\in U_1$.

Proof Similar to the proof of Theorem 2.1 of [10], we can get the result.

Proposition 2.2 *Let* **(A1.1)** *hold. For any given* $u_2(\cdot)$, *let* $u_1^*(\cdot)\in\mathcal{U}_1$ *and* $x^{u_1^*,u_2}(\cdot)$ *be the corresponding state. Let* $\left(q(\cdot),k(t),\tilde{k}(\cdot)\right)$ *be the adjoint process triple. For each* $(t,\omega)\in[0,T]\times\Omega$, $H_1\left(t,\cdot,\cdot,u_2(t);q(t),k(t),\tilde{k}(t)\right)$ *is concave,* $G_1(\cdot)$ *is convex, and*

$$\mathbb{E}\left[H_1(t,x^{u_1^*,u_2}(t),u_1^*(t),u_2(t);q(t),k(t),\tilde{k}(t))|\mathcal{G}_{1,t}\right] = \max_{u_1\in U_1}\mathbb{E}\left[H_1(t,x^{u_1^*,u_2}(t),u_1,u_2(t);q(t),k(t),\tilde{k}(t))|\mathcal{G}_{1,t}\right], \quad (24)$$

holds for $a.e. t\in[0,T]$, a.s. Then $u_1^*(\cdot)$ is an optimal control for **SOCPF**.

Proof Similar to the proof of Theorem 2.3 of [10], we can obtain the result.

2.2. The Leader's problem

In this subsection, we first state the **SOCPL**. Then, we give the maximum principle and verification theorem. For any $u_2(\cdot)\in\mathcal{U}_2$, by Eq. (23), we assume that a functional $u_1^*(t) = u_1^*\left(t;\widehat{x}^{u_1^*,\hat{u}_2}(t),\widehat{u}_2(t),\widehat{q}(t),\widehat{k}(t),\widehat{\tilde{k}}(t)\right)$ is uniquely defined, where

$$\widehat{x}^{u_1^*,\hat{u}_2}(t):=\mathbb{E}\left[x^{u_1^*,u_2}(t)|\mathcal{G}_{1,t}\right], \widehat{u}_2(t):=\mathbb{E}[u_2(t)|\mathcal{G}_{1,t}], \widehat{q}(t):=\mathbb{E}[q(t)|\mathcal{G}_{1,t}], \widehat{k}(t):=\mathbb{E}[k(t)|\mathcal{G}_{1,t}], \widehat{\tilde{k}}(t):=\mathbb{E}\left[\tilde{k}(t)|\mathcal{G}_{1,t}\right]. \quad (25)$$

For the simplicity of notations, we denote $x^{u_2}(\cdot)\equiv x^{u_1^*,u_2}(\cdot)$ and define ϕ^L on $\Omega\times[0,T]\times\mathbb{R}\times U_2$ as $\phi^L(t,x^{u_2}(t),u_2(t)):=\phi\left(t,x^{u_1^*,u_2}(t),u_1^*\left(t;\widehat{x}_2^{u_1^*,\hat{u}}(t),\widehat{u}_2(t),\widehat{q}(t),\widehat{k}(t),\tilde{k}(t)\right),u_2(t)\right)$, for $\phi = b,\sigma,\tilde{\sigma}, g_1$, respectively. Then after substituting the above control process $u_1^*(\cdot)$ into Eq. (22), the leader encounters the controlled FBSDE system

$$\begin{cases} dx^{u_2}(t) = b^L(t, x^{u_2}(t), u_2(t))dt + \sigma^L(t, x^{u_2}(t), u_2(t))dW(t) + \tilde{\sigma}^L(t, x^{u_2}(t), u_2(t))d\tilde{W}(t), \\ -dq(t) = \Big\{ b_x^L(t, x^{u_2}(t), u_2(t))q(t) + \sigma_x^L(t, x^{u_2}(t), u_2(t))k(t) + \tilde{\sigma}_x^L(t, x^{u_2}(t), u_2(t))\tilde{k}(t) \\ \qquad -g_{1x}^L(t, x^{u_2}(t), u_2(t)) \Big\} dt - k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \quad x^{u_2}(0) = x_0, \quad q(T) = -G_{1x}(x^{u_2}(T)). \end{cases} \tag{26}$$

Note that Eq. (26) is a controlled *conditional mean-field FBSDE*, which now is regarded as the "state" equation of the leader. That is to say, the state for the leader is the quadruple $\left(x^{u_2}(\cdot), q(\cdot), k(\cdot), \tilde{k}(\cdot)\right)$.

Remark 2.1 The equality $u_1^*(t) = u_1^*\left(t; \widehat{x}^{u_1^*, \hat{u}_2}(t), \widehat{u}_2(t), \widehat{q}(t), \widehat{k}(t), \widehat{\tilde{k}}(t)\right)$ does not hold in general. However, for LQ case, it is satisfied and we will make this point clear in the next section.

Define

$$\begin{aligned} J_2^L(u_2(\cdot)) &:= J_2\big(u_1^*(\cdot), u_2(\cdot)\big) = \mathbb{E}\left[\int_0^T g_2(t, x^{u_1^*, u_2}(t), u_1^*(t), u_2(t))dt + G_2\big(x^{u_1^*, u_2}(T)\big)\right] \\ &\equiv \mathbb{E}\left[\int_0^T g_2(t, x^{u_1^*, u_2}(t), u_1^*(t; \widehat{x}^{u_1^*, \hat{u}_2}(t), \widehat{u}_2(t), \widehat{q}(t), \widehat{\tilde{k}}(t), \tilde{k}(t)), u_2(t))dt + G_2\big(x^{u_1^*, u_2}(T)\big)\right] \\ &:= \mathbb{E}\left[\int_0^T g_2^L(t, x^{u_2}(t), u_2(t))dt + G_2(x^{u_2}(T))\right], \end{aligned} \tag{27}$$

where $g_2^L : \Omega \times [0, T] \times \mathbb{R} \times U_2 \to \mathbb{R}$. Note the cost functional of the leader is also conditional mean-field's type. We propose the stochastic optimal control problem with partial information of the leader as follows.

SOCPL. Find a $\mathcal{G}_{2,t}$-adapted control $u_2^*(\cdot) \in \mathcal{U}_2$, such that

$$J_2^L\big(u_2^*(\cdot)\big) = \inf_{u_2 \in \mathcal{U}_2} J_2^L(u_2(\cdot)), \tag{28}$$

subject to Eqs. (26) and (27). Such a $u_2^*(\cdot)$ is called an optimal control, and the corresponding solution $x^*(\cdot) \equiv x^{u_2^*}(\cdot)$ to Eq. (26) is called an optimal state process for the leader.

Let $u_2^*(\cdot)$ be an optimal control for the leader, and the corresponding state $\left(x^*(\cdot), q^*(\cdot), k^*(\cdot), \tilde{k}^*(\cdot)\right)$ is the solution to Eq. (26). Define the Hamiltonian function of the leader $H_2 : \Omega \times [0, T] \times \mathbb{R}^n \times U_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} &H_2\left(t, x^{u_2}, u_2, q, k, \tilde{k}; y, z, \tilde{z}, p\right) = y b^L(t, x^{u_2}, u_2) + z\sigma^L(t, x^{u_2}, u_2) + \tilde{z}\tilde{\sigma}^L(t, x^{u_2}, u_2) \\ &+ g_2^L(t, x^{u_2}, u_2) - p\Big[b_x^L(t, x^{u_2}, u_2)q + \sigma_x^L(t, x^{u_2}, u_2)k + \tilde{\sigma}_x^L(t, x^{u_2}, u_2)\tilde{k} - g_{1x}^L(t, x^{u_2}, u_2)\Big]. \end{aligned} \tag{29}$$

Let $\phi^{L*}(t) \equiv \phi^L\big(t, x^*(t), \widehat{x}^*(t), u_2^*(t), \widehat{u}_2^*(t)\big)$ for $\phi = b, \sigma, \tilde{\sigma}, g_1, g_2$ and all their derivatives. Suppose that $(y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the unique $\mathcal{F}_t$-adapted solution to the adjoint conditional mean-field FBSDE of the leader

$$
\begin{cases}
dp(t) = \Big\{b_x^{L*}(t)p(t) + \mathbb{E}[b_{\widehat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}]\Big\}dt + \Big\{\sigma_x^{L*}(t)p(t) + \mathbb{E}[\sigma_{\widehat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}]\Big\}dW(t) \\
\qquad + \Big\{\tilde{\sigma}_x^{L*}(t)p(t) + \mathbb{E}[\tilde{\sigma}_{\widehat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}]\Big\}d\tilde{W}(t), \;\; p(0)=0, \\
-dy(t) = \Big\{b_x^{L*}(t)y(t) + \mathbb{E}[b_{\widehat{x}}^{L*}(t)y(t)|\mathcal{G}_{1,t}] + \sigma_x^{L*}(t)z(t) + \mathbb{E}[\sigma_{\widehat{x}}^{L*}(t)z(t)|\mathcal{G}_{1,t}] \\
\qquad + \tilde{\sigma}_x^{L*}(t)\tilde{z}(t) + \mathbb{E}\Big[\tilde{\sigma}_{\widehat{x}}^{L*}(t)\tilde{z}(t)|\mathcal{G}_{1,t}\Big] - b_{xx}^{L*}(t)q^*(t)p(t) - \mathbb{E}\Big[b_{x\widehat{x}}^{L*}q^*(t)p(t)|\mathcal{G}_{1,t}\Big] \\
\qquad - \sigma_{xx}^{L*}(t)k(t)p(t) - \mathbb{E}\Big[\sigma_{x\widehat{x}}^{L*}(t)k(t)p(t)|\mathcal{G}_{1,t}\Big] - \tilde{\sigma}_{xx}^{L*}(t)\tilde{k}(t)p(t) - \mathbb{E}\Big[\sigma_{x\widehat{x}}^{L*}(t)\tilde{k}(t)p(t)|\mathcal{G}_{1,t}\Big] \\
\qquad + g_{1xx}^{L*}(t)p(t) + \mathbb{E}[g_{1x\widehat{x}}^{L*}(t)p(t)|\mathcal{G}_{1,t}] + g_{2x}^{L*}(t) + \mathbb{E}[g_{2\widehat{x}}^{L*}(t)|\mathcal{G}_{1,t}]\Big\}dt - z(t)dW(t) - \tilde{z}(t)d\tilde{W}(t), \\
y(T) = G_{1xx}(x^*(T))p(T) + G_{2x}(x^*(T)).
\end{cases}
\tag{30}
$$

Now, we have the following two results.

Proposition 2.3 *Let* **(A1.1)** *hold. Let* $u_2^*(\cdot) \in \mathcal{U}_2$ *be an optimal control for* **SOCPL** *and* $\left(x^*(\cdot), q^*(\cdot), k^*(\cdot), \tilde{k}^*(\cdot)\right)$ *be the optimal state. Let* $(y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot))$ *be the adjoint quadruple, then*

$$
\begin{aligned}
&\mathbb{E}\Big[\Big\langle H_{2u_2}(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t)), u_2 - u_2^*(t)\Big\rangle \\
&\quad + \Big\langle \mathbb{E}[H_{2\hat{u}_2}(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t))|\mathcal{G}_{1,t}], \widehat{u}_2 - \widehat{u}_2^*(t)\Big\rangle\Big|\mathcal{G}_{2,t}\Big] \\
&\quad \geq 0, \;\; a.e. t \in [0,T], a.s., \quad for \;\; any \;\; u_2 \in U_2.
\end{aligned}
\tag{31}
$$

Proof The maximum condition Eq. (31) can be derived by convex variation and adjoint technique, as Anderson and Djehiche [17]. We omit the details for saving space. See also Li [18], Yong [19] and the references therein for mean-field stochastic optimal control problems. □

Proposition 2.4 *Let* **(A1.1)** *hold. Let* $u_2^*(\cdot) \in \mathcal{U}_2$ *and* $\left(x^*(\cdot), q^*(\cdot), k^*(\cdot), \tilde{k}^*(\cdot)\right)$ *be the corresponding state, with* $G_{1xx}(x) \equiv G_1 \in \mathcal{S}^n$. *Let* $(y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot))$ *be the adjoint quadruple. For each* $(t, \omega) \in [0,T] \times \Omega$, *suppose that* $H_2(t, \cdot, \cdot, \cdot, \cdot, \cdot; y(t), z(t), \tilde{z}(t), p(t))$ *and* $G_2(\cdot)$ *are convex, and*

$$
\begin{aligned}
&\mathbb{E}\Big[H_2(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t)) \\
&\quad + \mathbb{E}[H_2(t, x^*(t), u_2^*(t), q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t))|\mathcal{G}_{1,t}]\Big|\mathcal{G}_{2,t}\Big] \\
&\quad = \max_{u_2 \in U_2} \mathbb{E}\Big[H_2(t, x^*(t), u_2, q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t)) \\
&\quad + \mathbb{E}[H_2(t, x^*(t), u_2, q^*(t), k^*(t), \tilde{k}^*(t); y(t), z(t), \tilde{z}(t), p(t))\Big|\mathcal{G}_{1,t}]\Big|\mathcal{G}_{2,t}\Big], \;\; a.e. t \in [0,T], a.s.
\end{aligned}
\tag{32}
$$

Then $u_2^*(\cdot)$ is an optimal control for **SOCPL**.

Proof This follows similar to Shi [20]. We omit the details for simplicity. □

3. Applications to LQ case

In order to illustrate the theoretical results in Section 2, we study an LQ leader-follower stochastic differential game with asymmetric information. In this section, we let $\mathcal{G}_{1,t} := \sigma\{\tilde{W}_s; 0 \le s \le t\}$ and $\mathcal{G}_{2,t} = \mathcal{F}_t$. This game is a special case of the one in Section 2, but the resulting deduction is very technically demanding. We split this section into two subsections, to deal with the problems of the follower and the leader, respectively.

3.1. Problem of the follower

Suppose that the state $x^{u_1,u_2} \in \mathbb{R}$ satisfies a linear SDE

$$\begin{cases} dx^{u_1,u_2}(t) = [Ax^{u_1,u_2}(t) + B_1u_1(t) + B_2u_2(t)]dt + [Cx^{u_1,u_2}(t) + D_1u_1(t) + D_2u_2(t)]dW(t) \\ \qquad\qquad + \tilde{C}x^{u_1,u_2}(t) + \tilde{D}_1u_1(t) + \tilde{D}_2u_2(t)]d\tilde{W}(t), \\ x^{u_1,u_2}(0) = x_0. \end{cases} \tag{33}$$

Here, u_1 is the follower's control process and u_2 is the leader's control process, which take values both in $\mathbb{R}$; $A, C, \tilde{C}, B_1, D_1, \tilde{D}_1, B_2, D_2, \tilde{D}_2$ are constants. In the first step, for announced u_2, the follower would like to choose a $\mathcal{G}_{1,t}$-adapted, square-integrable control u_1^* to minimize the cost functional

$$J_1(u_1, u_2) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(Q_1|x^{u_1,u_2}(t)|^2 + N_1|u_1(t)|^2\right)dt + G_1|x^{u_1,u_2}(T)|^2\right]. \tag{34}$$

In the second step, knowing that the follower would take u_1^*, the leader wishes to choose an $\mathcal{F}_t$-adapted, square-integrable control u_2^* to minimize

$$J_2(u_1^*, u_2) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(Q_2\left|x^{u_1^*,u_2}(t)\right|^2 + N_2|u_2(t)|^2\right)dt + G_2\left|x^{u_1^*,u_2}(T)\right|^2\right], \tag{35}$$

where $Q_1, Q_2, G_1, G_2 \ge 0, N_1 \ge 0, N_2 > 0$ are constants. This is an LQ leader-follower stochastic differential game with asymmetric information. We wish to find its Stackelberg equilibrium (u_1^*, u_2^*).

Define the Hamiltonian function of the follower as

$$\begin{aligned} &H_1\left(t, x, u_1, u_2, q, k, \tilde{k}\right) \\ &= q(Ax + B_1u_1 + B_2u_2) + k(Cx + D_1u_1 + D_2u_2) + \tilde{k}\left(\tilde{C}x + \tilde{D}_1u_1 + \tilde{D}_2u_2\right) - \frac{1}{2}Q_1x^2 - \frac{1}{2}N_1u_1^2. \end{aligned} \tag{36}$$

For given control u_2, suppose that there exists a $\mathcal{G}_{1,t}$-adapted optimal control u_1^* of the follower, and the corresponding optimal state is $x^{u_1^*,u_2}$. By Proposition 2.1, Eq. (36) yields that

$$0 = N_1 u_1^*(t) - B_1 \widehat{q}(t) - D_1 \widehat{k}(t) - \tilde{D}_1 \widehat{\tilde{k}}(t), \tag{37}$$

where the $\mathcal{F}_t$-adapted process triple $\left(q, k, \tilde{k}\right) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfies the BSDE

$$\begin{cases} -dq(t) = \left[Aq(t) + Ck(t) + \tilde{C}\tilde{k}(t) - Q_1 x^{u_1^*, u_2}(t)\right]dt - k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \\ q(T) = -G_1 x^{u_1^*, u_2}(T). \end{cases} \tag{38}$$

We wish to obtain the state feedback form of u_1^*. Noting the terminal condition of Eq. (38) and the appearance of u_2, we set

$$q(t) = -P(t)x^{u_1^*, u_2}(t) - \varphi(t), \quad t \in [0, T], \tag{39}$$

for some deterministic and differentiable $\mathbb{R}$-valued function $P(t)$, and $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted process φ which admits the BSDE

$$d\varphi(t) = \alpha(t)dt + \beta(t)d\tilde{W}(t), \quad \varphi(T) = 0. \tag{40}$$

In the above equation, $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$ are $\mathcal{F}_t$-adapted processes, which are to be determined later. Now, applying Itô's formula to Eq. (39), we have

$$\begin{aligned} -dq(t) = {} & \left[\dot{P}(t)x^{u_1^*, u_2}(t) + P(t)Ax^{u_1^*, u_2}(t) + \alpha(t) + P(t)B_1 u_1^*(t) + P(t)B_2 u_2(t)\right]dt \\ & + P(t)\left[Cx^{u_1^*, u_2}(t) + D_1 u_1^*(t) + D_2 u_2(t)\right]dW(t) \\ & + \left\{P(t)\left[\tilde{C}x^{u_1^*, u_2}(t) + \tilde{D}_1 u_1^*(t) + \tilde{D}_2 u_2(t)\right] + \beta(t)\right\}d\tilde{W}(t). \end{aligned} \tag{41}$$

Comparing Eq. (41) with Eq. (38), we arrive at

$$\begin{aligned} k(t) &= -P(t)\left[Cx^{u_1^*, u_2}(t) + D_1 u_1^*(t) + D_2 u_2(t)\right], \\ \tilde{k}(t) &= -P(t)\left[\tilde{C}x^{u_1^*, u_2}(t) + \tilde{D}_1 u_1^*(t) + \tilde{D}_2 u_2(t)\right] - \beta(t), \end{aligned} \tag{42}$$

and

$$\alpha(t) = -\left[\dot{P}(t) + 2AP(t) + Q_1\right]x^{u_1^*, u_2}(t) - A\varphi(t) - P(t)B_1 u_1^*(t) - P(t)B_2 u_2(t) + Ck(t) + \tilde{C}\tilde{k}(t), \tag{43}$$

respectively. Taking $\mathbb{E}[\cdot|\mathcal{G}_{1,t}]$ on both sides of Eqs. (39) and (42), we get

$$\widehat{q}(t) = -P(t)\widehat{x}^{u_1^*, \hat{u}_2}(t) - \widehat{\varphi}(t), \tag{44}$$

and

$$\begin{aligned} \widehat{k}(t) &= -P(t)\left[C\widehat{x}^{u_1^*, \hat{u}_2}(t) + D_1 u_1^*(t) + D_2 \widehat{u}_2(t)\right], \\ \widehat{\tilde{k}}(t) &= -P(t)\left[\tilde{C}\widehat{x}^{u_1^*, \hat{u}_2}(t) + \tilde{D}_1 u_1^*(t) + \tilde{D}_2 \widehat{u}_2(t)\right] - \widehat{\beta}(t), \end{aligned} \tag{45}$$

respectively. Applying Lemma 5.4 in [21] to Eqs. (33) and (38) corresponding to u_1^*, we derive the optimal filtering equation

$$\begin{cases} d\hat{x}^{u_1^*,\hat{u}_2}(t) = \left[A\hat{x}^{u_1^*,\hat{u}_2}(t) + B_1u_1^*(t) + B_2\hat{u}_2(t)\right]dt + \left[\tilde{C}\hat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{D}_1u_1^*(t) + \tilde{D}_2\hat{u}_2(t)\right]d\tilde{W}(t), \\ -d\hat{q}(t) = \left[A\hat{q}(t) + C\hat{k}(t) + \tilde{C}\hat{\tilde{k}}(t) - Q_1\hat{x}^{u_1^*,\hat{u}_2}(t)\right]dt - \tilde{k}(t)d\tilde{W}(t), \\ \hat{x}^{u_1^*,\hat{u}_2}(0) = x_0, \quad \hat{q}(T) = -G_1\hat{x}^{u_1^*,\hat{u}_2}(T). \end{cases} \tag{46}$$

Note that Eq. (46) is not a classical FBSDFE, since the generator of the BSDE depends on an additional process $\hat{k}$. For given u_2, it is important if Eq. (46) admits a unique $\mathcal{G}_{1,t}$-adapted solution $\left(\hat{x}^{u_1^*,\hat{u}_2}, \hat{q}, \hat{k}, \hat{\tilde{k}}\right)$. We will make it clear soon. For this target, first, by Eq. (37) and supposing that.

(A2.1) $\tilde{N}_1(t) := N_1 + D_1^2P(t) + \tilde{D}_1^2P(t) > 0$, $\forall t \in [0, T]$,

we immediately arrive at

$$u_1^*(t) = -\tilde{N}_1^{-1}(t)\left[\tilde{S}_1(t)\hat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{S}(t)\hat{u}_2(t) + B_1\hat{\varphi}(t) + \tilde{D}_1\hat{\beta}(t)\right], \tag{47}$$

where $\tilde{S}_1(t) := \left(B_1 + CD_1 + \tilde{C}\tilde{D}_1\right)P(t)$, $\tilde{S}(t) := \left(D_1D_2 + \tilde{D}_1\tilde{D}_2\right)P(t)$. Substituting Eq. (47) into Eq. (43), we can obtain that if

$$\begin{cases} \dot{P}(t) + \left(2A + C^2 + \tilde{C}^2\right)P(t) - \left(B_1 + CD_1 + \tilde{C}\tilde{D}_1\right)^2\left[N_1 + D_1^2P(t) + \tilde{D}_1^2P(t)\right]^{-1}P(t)^2 + Q_1 = 0, \\ P(T) = G_1, \end{cases} \tag{48}$$

admits a unique differentiable solution $P(t)$, then

$$\begin{aligned} \alpha(t) = & -\tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)x^{u_1^*,u_2}(t) + \tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)\hat{x}^{u_1^*,\hat{u}_2}(t) - A\varphi(t) + \tilde{S}_1(t)\tilde{N}_1^{-1}(t)B_1\hat{\varphi}(t) \\ & -\tilde{S}_2(t)u_2(t) + \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{S}(t)\hat{u}_2(t) - \tilde{C}\beta(t) + \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{D}_1\hat{\beta}(t), \end{aligned} \tag{49}$$

where $\tilde{S}_2(t) := \left(B_2 + CD_2 + \tilde{C}\tilde{D}_2\right)P(t)$. By **(A2.1)**, we know that Eq. (48) admits a unique solution $P(t) > 0$ from standard Riccati equation theory [22]. In particular, if $\tilde{C} = \tilde{D}_1 = 0$, Eq. (48) reduces to

$$\begin{cases} \dot{P}(t) + \left(2A + C^2\right)P(t) - \left(B_1 + CD_1\right)^2\left[N_1 + D_1^2P(t)\right]^{-1}P(t)^2 + Q_1 = 0, \\ P(T) = G_1, \quad N_1 + D_1^2P(t) > 0, \end{cases} \tag{50}$$

which recovers the standard one in [22]. With Eq. (49), the BSDE Eq. (40) takes the form

$$\begin{cases} -d\varphi(t) = [\tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)x^{u_1^*,u_2}(t) - \tilde{S}_1^2(t)\tilde{N}_1^{-1}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + A\varphi(t) - \tilde{S}_1(t)\tilde{N}^{1^{-1}}(t)B_1\widehat{\varphi}(t) \\ \qquad + \left(\tilde{C} - \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{D}_1\right)\beta(t) + \tilde{S}_2(t)u_2(t) - \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{S}(t)\widehat{u}_2(t)]dt - \beta(t)d\tilde{W}(t), \\ \varphi(T) = 0. \end{cases} \tag{51}$$

Moreover, for given u_2, plugging Eq. (47) into the forward equation of Eq. (46), and letting

$$\begin{cases} \tilde{A}(t) := A - B_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t), \tilde{\tilde{C}}(t) := \tilde{C} - \tilde{D}_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t), \tilde{B}_2(t) := B_2 - B_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t), \\ \tilde{F}_1(t) := -B_1\tilde{N}_1^{-1}(t)B_1,\ \ \tilde{B}_1(t) := -B_1\tilde{N}_1^{-1}(t)\tilde{D}_1, \tilde{F}_3(t) := -\tilde{D}_1\tilde{N}_1^{-1}(t)\tilde{D}_1,\ \ \tilde{\tilde{D}}_2(t) := \tilde{D}_2 - \tilde{D}_1\tilde{N}_1^{-1}(t)\tilde{S}(t), \end{cases} \tag{52}$$

we have

$$\begin{cases} d\widehat{x}^{u_1^*,\hat{u}_2}(t) = \left[\tilde{A}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{F}_1(t)\widehat{\varphi}(t) + \tilde{B}_1(t)\widehat{\beta}(t) + \tilde{B}_2(t)\widehat{u}_2(t)\right]dt \\ \qquad + \left[\tilde{\tilde{C}}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{B}_1(t)\widehat{\varphi}(t) + \tilde{F}_3(t)\widehat{\beta}(t) + \tilde{\tilde{D}}_2(t)\widehat{u}_2(t)\right]d\tilde{W}(t),\ \ \widehat{x}^{u_1^*,\hat{u}_2}(0) = x_0, \end{cases} \tag{53}$$

which admits a unique $\mathcal{G}_{1,t}$-adapted solution $\widehat{x}^{u_1^*,\hat{u}_2}$, for given $\left(\widehat{\varphi},\widehat{\beta}\right)$. Applying Lemma 5.4 in [21] to Eq. (51) again, we have

$$-d\widehat{\varphi}(t) = \left[\tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\widehat{\beta}(t) + \tilde{F}_4(t)\widehat{u}_2(t)\right]dt - \widehat{\beta}(t)d\tilde{W}(t),\ \ \widehat{\varphi}(T) = 0, \tag{54}$$

where $\tilde{F}_4(t) := \tilde{S}_2(t) - \tilde{S}_1(t)\tilde{N}_1^{-1}(t)\tilde{S}(t)$. For given $\widehat{u}_2$, Eq. (54) admits a unique solution $\left(\widehat{\varphi},\widehat{\beta}\right)$ from standard BSDE theory. Putting Eqs. (53) and (54) together, we get

$$\begin{cases} d\widehat{x}^{u_1^*,\hat{u}_2}(t) = \left[\tilde{A}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{F}_1(t)\widehat{\varphi}(t) + \tilde{B}_1(t)\widehat{\beta}(t) + \tilde{B}_2(t)\widehat{u}_2(t)\right]dt \\ \qquad + \left[\tilde{\tilde{C}}(t)\widehat{x}^{u_1^*,\hat{u}_2}(t) + \tilde{B}_1(t)\widehat{\varphi}(t) + \tilde{F}_3(t)\widehat{\beta}(t) + \tilde{\tilde{D}}_2(t)\widehat{u}_2(t)\right]d\tilde{W}(t), \\ -d\widehat{\varphi}(t) = \left[\tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\beta(t) + \tilde{F}_4(t)\widehat{u}_2(t)\right]dt - \widehat{\beta}(t)d\tilde{W}(t),\ \ \widehat{x}^{u_1^*,\hat{u}_2}(0) = x_0,\ \ \widehat{\varphi}(T) = 0, \end{cases} \tag{55}$$

which admits a unique $\mathcal{G}_{1,t}$-adapted solution $\left(\widehat{x}^{u_1^*,\hat{u}_2},\widehat{\varphi},\widehat{\beta}\right)$. By Eqs. (55), (44), (45), and (47), we can uniquely obtain the solvability of Eq. (46). Moreover, we can check that the convexity/concavity conditions in Proposition 2.2 hold, and u_1^* given by Eq. (47) is really optimal. We summarize the above procedure in the following theorem.

Theorem 3.1 *Let* **(A2.1)** *hold,* $P(t)$ *satisfy Eq.* (48). *For chosen* u_2 *of the leader,* u_1^* *given by Eq.* (47) *is the optimal control of the follower, where* $\left(\widehat{x}^{u_1^*,\hat{u}_2},\widehat{\varphi},\widehat{\beta}\right)$ *is the unique* $\mathcal{G}_{1,t}$*-adapted solution to Eq.* (55).

3.2. Problem of the leader

Since the leader knows that the follower will take u_1^* by Eq. (47), the state equation of the leader writes

$$\begin{cases} dx^{u_2}(t) = \Big[Ax^{u_2}(t) + \Big(\tilde{A}(t) - A\Big)\widehat{x}^{\hat{u}_2}(t) + \tilde{F}_1(t)\widehat{\varphi}(t) + \tilde{B}_1(t)\widehat{\beta}(t) + B_2u_2(t) + \big(\tilde{B}_2(t) - B_2\big)\widehat{u}_2(t)\Big]dt \\ \qquad + \Big[Cx^{u_2}(t) + \tilde{F}_5(t)\widehat{x}^{\hat{u}_2}(t) + \tilde{\tilde{B}}_1(t)\widehat{\varphi}(t) + \tilde{\tilde{D}}_1(t)\widehat{\beta}(t) + D_2u_2(t) + \tilde{F}_2(t)\widehat{u}_2(t)\Big]dW(t) \\ \qquad + \Big[\tilde{C}x^{u_2}(t) + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\widehat{x}^{\hat{u}_2}(t) + \tilde{B}_1(t)\widehat{\varphi}(t) + \tilde{F}_3(t)\widehat{\beta}(t) + \tilde{D}_2u_2(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\widehat{u}_2(t)\Big]d\tilde{W}(t), \\ -d\widehat{\varphi}(t) = \Big(\tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\widehat{\beta}(t) + \tilde{F}_4(t)\widehat{u}_2(t)\Big)dt - \widehat{\beta}(t)d\tilde{W}(t), \quad x^{u_2}(0) = x_0, \quad \widehat{\varphi}(T) = 0, \end{cases} \tag{56}$$

where $x^{u_2} \equiv x^{u_1^*, u_2}$, $\widehat{x}^{\hat{u}_2} \equiv \widehat{x}^{u_1^*, \hat{u}_2}$ and $\tilde{\tilde{B}}_1(t) := -B_1\tilde{N}_1^{-1}(t)D_1$, $\tilde{\tilde{D}}_1(t) := -D_1\tilde{N}_1^{-1}(t)\tilde{D}_1$, $\tilde{F}_5(t) := -D_1\tilde{N}_1^{-1}(t)\tilde{S}_1(t)$, $\tilde{F}_2(t) := -D_1\tilde{N}_1^{-1}(t)\tilde{S}(t)$. Noting that Eq. (56) is a decoupled conditional mean-field FBSDE, its solvability for $\mathcal{F}_t$-adapted solution $\left(x^{u_2}, \widehat{\varphi}, \widehat{\beta}\right)$ can be easily guaranteed.

The problem of the leader is to choose an $\mathcal{F}_t$-adapted optimal control u_2^* such that the cost functional

$$J_2(u_2) = \frac{1}{2}\mathbb{E}\left[\int_0^T \Big(Q_2|x^{u_2}(t)|^2 + N_2|u_2(t)|^2\Big)dt + G_2|x^{u_2}(T)|^2\right] \tag{57}$$

is minimized. Define the Hamiltonian function of the leader as

$$\begin{aligned} H_2\Big(t, x^{u_2}, u_2, \widehat{\varphi}, \widehat{\beta}; y, z, \tilde{z}, p\Big) &= \frac{1}{2}\Big(Q_2|x^{u_2}|^2 + N_2|u_2|^2\Big) \\ &+ y\Big[Ax^{u_2} + \Big(\tilde{A}(t) - A\Big)\widehat{x}^{\hat{u}_2} + \tilde{F}_1(t)\widehat{\varphi} + \tilde{B}_1(t)\widehat{\beta} + B_2u_2 + \Big(\tilde{B}_2(t) - B_2\Big)\widehat{u}_2\Big] \\ &+ p\Big(\tilde{A}(t)\widehat{\varphi} + \tilde{\tilde{C}}(t)\widehat{\beta} + \tilde{F}_4(t)\widehat{u}_2\Big) + z\Big(Cx^{u_2} + \tilde{F}_5(t)\widehat{x}^{\hat{u}_2} + \tilde{\tilde{B}}_1(t)\widehat{\varphi} + \tilde{\tilde{D}}_1(t)\widehat{\beta} + D_2u_2 + \tilde{F}_2(t)\widehat{u}_2\Big) \\ &+ \tilde{z}\Big[\tilde{C}x^{u_2} + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\widehat{x}^{\hat{u}_2} + \tilde{B}_1(t)\widehat{\varphi} + \tilde{F}_3(t)\widehat{\beta} + \tilde{D}_2u_2 + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\widehat{u}_2\Big]. \end{aligned} \tag{58}$$

Suppose that there exists an $\mathcal{F}_t$-adapted optimal control u_2^* of the leader, and the corresponding optimal state is $\left(x^*, \widehat{\varphi}^*, \widehat{\beta}^*\right) \equiv \left(x^{u_2^*}, \widehat{\varphi}^*, \widehat{\beta}^*\right)$. Then by Propositions 2.3, 2.4, Eq. (58) yields that

$$0 = N_2u_2^*(t) + \tilde{F}_4(t)\widehat{p}(t) + B_2y(t) + \big(\tilde{B}_2(t) - B_2\big)\widehat{y}(t) + D_2z(t) + \tilde{F}_2(t)\widehat{z}(t) + \tilde{D}_2(t)\tilde{z}(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\tilde{z}(t), \tag{59}$$

where the $\mathcal{F}_t$-adapted process $(p, y, z, \tilde{z})$ satisfies

$$\begin{cases} dp(t) = \Big[\tilde{A}(t)p(t) + \tilde{F}_1(t)y(t) + \tilde{\tilde{B}}_1(t)z(t) + \tilde{B}_1(t)\tilde{z}(t)\Big]dt \\ \qquad\qquad + \Big[\tilde{\tilde{C}}(t)p(t) + \tilde{B}_1(t)y(t) + \tilde{\tilde{D}}_1(t)z(t) + \tilde{F}_3(t)\tilde{z}(t)\Big]d\tilde{W}(t), \\ -dy(t) = \Big[Ay(t) + \Big(\tilde{A}(t) - A\Big)\widehat{y}(t) + Cz(t) + \tilde{F}_5(t)\widehat{z}(t) + \tilde{C}\tilde{z}(t) + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\tilde{z}(t) + Q_2x^*(t)\Big]dt \\ \qquad\qquad - z(t)dW(t) - \tilde{z}(t)d\tilde{W}(t), \ \ p(0) = 0, \ \ y(T) = G_2x^*(T). \end{cases} \tag{60}$$

In fact, the problem of the leader can also be solved by a direct calculation of the derivative of the cost functional. Without loss of generality, let $x_0 \equiv 0$, and set $u_2^* + \varepsilon u_2$ for $\epsilon > 0$ sufficiently small, with $u_2 \in \mathbb{R}$. Then it is easy to see from the linearity of Eqs. (56) and (60), that the solution to Eq. (56) is $x^* + \epsilon x^{u_2}$. We first have

$$\begin{aligned} \tilde{J}(\epsilon) &:= J_2\big(u_2^* + \epsilon u_2\big) = \frac{1}{2}\mathbb{E}\int_0^T [\langle Q_2(x^*(t) + \epsilon x^{u_2}(t)), x^*(t) + \epsilon x^{u_2}(t)\rangle \\ &+ \big\langle N_2\big(u_2^*(t) + \epsilon u_2(t)\big), u_2^*(t) + \epsilon u_2(t)\big\rangle]dt + \frac{1}{2}\mathbb{E}\langle G_2(x^*(T) + \epsilon x^{u_2}(T)), x^*(T) + \epsilon x^{u_2}(T)\rangle. \end{aligned} \tag{61}$$

Hence

$$0 = \frac{\partial \tilde{J}(\epsilon)}{\partial \epsilon}\bigg|_{\varepsilon=0} = \mathbb{E}\int_0^T \big[\langle Q_2x^*(t), x^{u_2}(t)\rangle + \big\langle N_2u_2^*(t), u_2(t)\big\rangle\big]dt + \mathbb{E}\langle G_2x^*(T), x^{u_2}(T)\rangle. \tag{62}$$

Let the $\mathcal{F}_t$-adapted process quadruple $(p, y, z, \tilde{z})$ satisfy Eq. (60). Then we have

$$0 = \mathbb{E}\int_0^T \big[\langle Q_2x^*(t), x^{u_2}(t)\rangle + \big\langle N_2u_2^*(t), u_2(t)\big\rangle\big]dt + \mathbb{E}\langle y(T), x^{u_2}(T)\rangle. \tag{63}$$

Applying Itô's formula to $\langle x^{u_2}(t), y(t)\rangle - \langle p(t), \widehat{\varphi}(t)\rangle$, noting Eqs. (56) and (60), we derive.

$$\begin{aligned} 0 &= \mathbb{E}\int_0^T \big\langle Q_2x^*(t) + Ay(t) + Cz(t) + \tilde{C}\tilde{z}(t), x^{u_2}(t)\big\rangle dt \\ &+ \mathbb{E}\int_0^T \Big\langle \Big(\tilde{A}(t) - A\Big)y(t) + \tilde{F}_5(t)z(t) + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\tilde{z}(t), \widehat{x}^{\widehat{u}_2}(t)\Big\rangle dt \\ &+ \mathbb{E}\int_0^T \big\langle N_2u_2^*(t) + B_2y(t) + D_2z(t) + \tilde{D}_2\tilde{z}(t), u_2(t)\big\rangle dt \\ &+ \mathbb{E}\int_0^T \Big\langle (\tilde{B}_2 - B_2)y(t) + \tilde{F}_2(t)z(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\tilde{z}(t), \widehat{u}_2(t)\Big\rangle dt \\ &- \mathbb{E}\int_0^T \Big\langle Q_2x^*(t) + Ay(t) + \Big(\tilde{A}(t) - A\Big)\widehat{y}(t) + Cz(t) + \tilde{C}\tilde{z}(t) + \tilde{F}_5(t)\widehat{z}(t) + \Big(\tilde{\tilde{C}}(t) - \tilde{C}\Big)\tilde{z}(t), x^{u_2}(t)\Big\rangle dt \\ &+ \mathbb{E}\int_0^T \Big\langle \tilde{F}_1(t)y(t) + \tilde{\tilde{B}}_1(t)z(t) + \tilde{B}_1(t)\tilde{z}(t), \widehat{\varphi}(t)\Big\rangle dt + \mathbb{E}\int_0^T \Big\langle \tilde{B}_1(t)y(t) + \tilde{\tilde{D}}_1(t)z(t) + \tilde{F}_3(t)\tilde{z}(t), \widehat{\beta}(t)\Big\rangle dt \\ &+ \mathbb{E}\int_0^T \Big\langle p(t), \tilde{A}(t)\widehat{\varphi}(t) + \tilde{\tilde{C}}(t)\widehat{\beta}(t)\Big\rangle dt - \mathbb{E}\int_0^T \Big\langle \widehat{\varphi}(t), \tilde{A}(t)p(t) + \tilde{F}_1(t)y(t) + \tilde{\tilde{B}}_1(t)z(t) + \tilde{B}_1(t)\tilde{z}(t)\Big\rangle dt \\ &- \mathbb{E}\int_0^T \Big\langle \widehat{\beta}(t), \tilde{\tilde{C}}(t)p(t) + \tilde{B}_1(t)y(t) + \tilde{\tilde{D}}_1(t)z(t) + \tilde{F}_3(t)\tilde{z}(t)\Big\rangle dt + \mathbb{E}\int_0^T \Big\langle p(t), \tilde{F}_4(t)\widehat{u}_2(t)\Big\rangle dt \\ &= \mathbb{E}\int_0^T \big\langle N_2u_2^*(t) + B_2y(t) + D_2z(t) + \tilde{D}_2\tilde{z}(t), u_2(t)\big\rangle dt \\ &+ \mathbb{E}\int_0^T \Big\langle (\tilde{B}_2(t) - B_2)y(t) + \tilde{F}_2(t)z(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\tilde{z}(t), \widehat{u}_2(t)\Big\rangle dt + \mathbb{E}\int_0^T \Big\langle p(t), \tilde{F}_4(t)\widehat{u}_2(t)\Big\rangle dt \\ &= \mathbb{E}\int_0^T \langle N_2u_2^*(t) + \tilde{F}_4(t)\widehat{p}(t) + B_2y(t) + (\tilde{B}_2(t) - B_2)\widehat{y}(t) + D_2z(t) + \tilde{F}_2(t)\widehat{z}(t) \\ &+ \tilde{D}_2(t)\tilde{z}(t) + \Big(\tilde{\tilde{D}}_2(t) - \tilde{D}_2\Big)\tilde{z}(t), u_2(t)\rangle dt. \end{aligned} \tag{64}$$

This implies Eq. (59).

In the following, we wish to obtain a "nonanticipating" representation for the optimal controls u_2^* and u_1^*. For this target, let us regard $(x^*, p)^T$ as the optimal state, put

$$X = \begin{pmatrix} x^* \\ p \end{pmatrix}, \quad Y = \begin{pmatrix} y^* \\ \widehat{\varphi} \end{pmatrix}, Z = \begin{pmatrix} z \\ 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} \tilde{z} \\ \widehat{\beta}^* \end{pmatrix}, \tag{65}$$

and (suppressing some t below)

$$\begin{cases} \mathcal{A}_1 := \begin{pmatrix} A & 0 \\ 0 & \tilde{A}(t) \end{pmatrix}, \mathcal{A}_2 := \begin{pmatrix} \tilde{A}(t) - A & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{B}}_1 := \begin{pmatrix} 0 & \tilde{B}_1(t) \\ \tilde{B}_1(t) & 0 \end{pmatrix}, \tilde{\tilde{\mathcal{B}}}_1 := \begin{pmatrix} 0 & 0 \\ \tilde{\tilde{B}}_1(t) & 0 \end{pmatrix}, \\ \mathcal{B}_2 := \begin{pmatrix} B_2 \\ 0 \end{pmatrix}, \tilde{\mathcal{B}}_2 := \begin{pmatrix} \tilde{B}_2(t) - B_2 \\ 0 \end{pmatrix}, \mathcal{C}_1 := \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{C}}_1 := \begin{pmatrix} \tilde{C} & 0 \\ 0 & \tilde{\tilde{C}}(t) \end{pmatrix}, \\ \tilde{\mathcal{C}}_2 := \begin{pmatrix} \tilde{\tilde{C}}(t) - \tilde{C} & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\tilde{\mathcal{D}}}_1 := \begin{pmatrix} 0 & \tilde{\tilde{D}}_1(t) \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{D}}_2 := \begin{pmatrix} \tilde{D}_2 \\ 0 \end{pmatrix}, \tilde{\tilde{\mathcal{D}}}_2 := \begin{pmatrix} \tilde{\tilde{D}}_2(t) - \tilde{D}_2 \\ 0 \end{pmatrix}, \\ \mathcal{D}_2 := \begin{pmatrix} D_2 \\ 0 \end{pmatrix}, \mathcal{G}_2 := \begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix}, \tilde{\mathcal{F}}_1 := \begin{pmatrix} 0 & \tilde{F}_1(t) \\ \tilde{F}_1(t) & 0 \end{pmatrix}, \tilde{\mathcal{F}}_2 := \begin{pmatrix} \tilde{F}_2(t) \\ 0 \end{pmatrix}, \quad X_0 := \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ \tilde{\mathcal{F}}_3 := \begin{pmatrix} 0 & \tilde{F}_3(t) \\ \tilde{F}_3(t) & 0 \end{pmatrix}, \tilde{\mathcal{F}}_4 := \begin{pmatrix} 0 \\ \tilde{F}_4(t) \end{pmatrix}, \tilde{\mathcal{F}}_5 := \begin{pmatrix} \tilde{F}_5(t) & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{Q}_2 := \begin{pmatrix} Q_2 & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{66}$$

With the notations, Eq. (56) with Eq. (60) is rewritten as

$$\begin{cases} dX(t) = \Big(\mathcal{A}_1 X(t) + \mathcal{A}_2 \widehat{X}(t) + \tilde{\mathcal{F}}_1 Y(t) + \tilde{\tilde{\mathcal{B}}}_1 Z(t) + \tilde{\mathcal{B}}_1 \tilde{Z}(t) + \mathcal{B}_2 w^*(t) + \tilde{\mathcal{B}}_2 \widehat{u}_2^*(t) \Big) dt \\ \qquad + \Big(\mathcal{C}_1 X(t) + \tilde{\mathcal{F}}_5 \widehat{X}(t) + \tilde{\tilde{\mathcal{B}}}_1^T Y(t) + \tilde{\tilde{\mathcal{D}}}_1 \tilde{Z}(t) + \mathcal{D}_2 w^*(t) + \tilde{\mathcal{F}}_2 \widehat{u}_2^*(t) \Big) dW(t) \\ \qquad + \Big(\tilde{\mathcal{C}}_1 X(t) + \tilde{\mathcal{C}}_2 \widehat{X}(t) + \tilde{\mathcal{B}}_1^T Y(t) + \tilde{\tilde{\mathcal{D}}}_1^T Z(t) + \tilde{\mathcal{F}}_3 \tilde{Z}(t) + \tilde{\mathcal{D}}_2 w^*(t) + \tilde{\tilde{\mathcal{D}}}_2 \widehat{u}_2^*(t) \Big) d\tilde{W}(t), \\ -dY(t) = \Big(\mathcal{Q}_2 X(t) + \mathcal{A}_1^T Y(t) + \mathcal{A}_2^T \widehat{Y}(t) + \mathcal{C}_1^T Z(t) + \tilde{\mathcal{F}}_5^T \widehat{Z}(t) + \tilde{\mathcal{C}}_1^T \tilde{Z}(t) + \tilde{\mathcal{C}}_2^T \tilde{Z}(t) + \tilde{\mathcal{F}}_4 \widehat{u}_2^*(t) \Big) dt \\ \qquad - Z(t) dW(t) - \tilde{Z}(t) d\tilde{W}(t), \quad X(0) = X_0, \quad Y(T) = \mathcal{G}_2 X(T). \end{cases} \tag{67}$$

Noting Eq. (59), we have

$$\begin{aligned} u_2^*(t) &= -N_2^{-1} \Big[\tilde{\mathcal{F}}_4^T \widehat{X}(t) + \mathcal{B}_2^T Y(t) + \tilde{\mathcal{B}}_2^T \widehat{Y}(t) + \mathcal{D}_2^T Z(t) + \tilde{\mathcal{F}}_2^T \widehat{Z}(t) + \tilde{\mathcal{D}}_2^T \tilde{Z}(t) + \tilde{\tilde{\mathcal{D}}}_2^T \tilde{Z}(t) \Big], \\ \widehat{u}_2^*(t) &= -N_2^{-1} \Big[\tilde{\mathcal{F}}_4^T \widehat{X}(t) + \left(\mathcal{B}_2 + \tilde{\mathcal{B}}_2 \right)^T \widehat{Y}(t) + \left(\mathcal{D}_2 + \tilde{\mathcal{F}}_2 \right)^T \widehat{Z}(t) + \left(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2 \right)^T \tilde{Z}(t) \Big]. \end{aligned} \tag{68}$$

Inserting Eq. (68) into Eq. (67), we get

$$
\begin{cases}
dX(t) = \Big(\mathcal{A}_1 X(t) + \overline{\mathcal{A}}_2 \widehat{X}(t) + \overline{\tilde{\mathcal{F}}}_1 Y(t) + \overline{\mathcal{B}}_2 \widehat{Y}(t) + \mathcal{B}_3 Z(t) + \overline{\overline{\mathcal{B}}}_2 \widehat{Z}(t) + \overline{\tilde{\mathcal{B}}}_1 \tilde{Z}(t) + \overline{\tilde{\mathcal{B}}}_2 \tilde{Z}(t)\Big) dt \\
\qquad + \Big(\mathcal{C}_1 X(t) + \overline{\tilde{\mathcal{F}}}_5 \widehat{X}(t) + \tilde{\mathcal{B}}_3^T Y(t) + \overline{\mathcal{D}}_2 \widehat{Y}(t) + \tilde{\tilde{\mathcal{D}}}_2 Z(t) + \overline{\overline{\mathcal{D}}}_2 \widehat{Z}(t) + \mathcal{D}_3 \tilde{Z}(t) + \overline{\tilde{\mathcal{D}}}_2 \tilde{Z}(t)\Big) dW(t) \\
\qquad + \Big(\tilde{\mathcal{C}}_1 X(t) + \overline{\tilde{\mathcal{C}}}_2 \widehat{X}(t) + \overline{\tilde{\mathcal{B}}}_1^T Y(t) + \overline{\mathcal{D}}_3 \widehat{Y}(t) + \mathcal{D}_3^T Z(t) + \overline{\overline{\mathcal{D}}}_3^T \widehat{Z}(t) + \overline{\tilde{\mathcal{F}}}_3 \tilde{Z}(t) + \overline{\tilde{\mathcal{D}}}_3 \tilde{Z}(t)\Big) d\tilde{W}(t), \\
-dY(t) = \Big(\mathcal{Q}_2 X(t) + \overline{\tilde{\mathcal{F}}}_4 \widehat{X}(t) + \mathcal{A}_1^T Y(t) + \overline{\mathcal{A}}_2^T \widehat{Y}(t) + \tilde{\mathcal{C}}_1^T \tilde{Z}(t) + \mathcal{C}_1^T Z(t) + \overline{\tilde{\mathcal{F}}}_5^T \widehat{Z}(t) + \overline{\tilde{\mathcal{C}}}_2^T \tilde{Z}(t)\Big) dt \\
\qquad - Z(t) dW(t) - \tilde{Z}(t) d\tilde{W}(t), \quad X(0) = X_0, \quad Y(T) = \mathcal{G}_2 X(T),
\end{cases}
\tag{69}
$$

where

$$
\begin{cases}
\overline{\mathcal{A}}_2 := \mathcal{A}_2 - (\mathcal{B}_2 + \tilde{\mathcal{B}}_2) N_2^{-1} \tilde{\mathcal{F}}_4^T, \overline{\tilde{\mathcal{B}}}_1 := \tilde{\mathcal{B}}_1 - \mathcal{B}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T, \overline{\mathcal{B}}_2 := -\mathcal{B}_2 N_2^{-1} \tilde{\mathcal{B}}_2^T - \tilde{\mathcal{B}}_2 N_2^{-1} (\mathcal{B}_2 + \tilde{\mathcal{B}}_2)^T, \\
\overline{\overline{\mathcal{B}}}_2 := -\mathcal{B}_2 N_2^{-1} \tilde{\mathcal{F}}_2^T - \tilde{\mathcal{B}}_2 N_2^{-1} (\mathcal{D}_2 + \tilde{\mathcal{F}}_2)^T, \overline{\tilde{\mathcal{B}}}_2 := -\mathcal{B}_2 N_2^{-1} \tilde{\tilde{\mathcal{D}}}_2^T - \tilde{\mathcal{B}}_2 N_2^{-1} \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big)^T, \\
\mathcal{B}_3 := \tilde{\tilde{\mathcal{B}}}_1 - \mathcal{B}_2 N_2^{-1} \mathcal{D}_2^T, \overline{\tilde{\mathcal{C}}}_2 := \tilde{\mathcal{C}}_2 - \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big) N_2^{-1} \tilde{\mathcal{F}}_4^T, \overline{\tilde{\mathcal{D}}}_2 := -\mathcal{D}_2 N_2^{-1} \tilde{\tilde{\mathcal{D}}}_2^T - \tilde{\mathcal{F}}_2 N_2^{-1} \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big)^T, \\
\tilde{\tilde{\mathcal{D}}}_2 := -\mathcal{D}_2 N_2^{-1} \mathcal{D}_2^T, \overline{\mathcal{D}}_2 := -\mathcal{D}_2 N_2^{-1} \tilde{\mathcal{B}}_2^T - \tilde{\mathcal{F}}_2 N_2^{-1} (\mathcal{B}_2 + \tilde{\mathcal{B}}_2)^T, \overline{\overline{\mathcal{D}}}_2 := -\mathcal{D}_2 N_2^{-1} \tilde{\mathcal{F}}_2^T - \tilde{\mathcal{F}}_2 N_2^{-1} (\mathcal{D}_2 + \tilde{\mathcal{F}}_2)^T, \\
\mathcal{D}_3 := \tilde{\tilde{\mathcal{D}}}_1 - \mathcal{D}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T, \overline{\mathcal{D}}_3 := -\tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\mathcal{B}}_2^T - \tilde{\tilde{\mathcal{D}}}_2 N_2^{-1} (\mathcal{B}_2 + \tilde{\mathcal{B}}_2)^T, \\
\overline{\overline{\mathcal{D}}}_3 := -\tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\mathcal{F}}_2^T - \tilde{\tilde{\mathcal{D}}}_2 N_2^{-1} (\mathcal{D}_2 + \tilde{\mathcal{F}}_2)^T, \overline{\tilde{\mathcal{D}}}_3 := -\tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\tilde{\mathcal{D}}}_2^T - \tilde{\tilde{\mathcal{D}}}_2 N_2^{-1} \Big(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\Big)^T, \\
\overline{\tilde{\mathcal{F}}}_1 := \tilde{\mathcal{F}}_1 - \mathcal{B}_2 N_2^{-1} \mathcal{B}_2^T, \overline{\tilde{\mathcal{F}}}_3 := \tilde{\mathcal{F}}_3 - \tilde{\mathcal{D}}_2 N_2^{-1} \tilde{\mathcal{D}}_2^T, \tilde{\tilde{\mathcal{F}}}_4 := -\tilde{\mathcal{F}}_4 N_2^{-1} \tilde{\mathcal{F}}_4^T, \overline{\tilde{\mathcal{F}}}_5 := \tilde{\mathcal{F}}_5 - (\mathcal{D}_2 + \tilde{\mathcal{F}}_2) N_2^{-1} \tilde{\mathcal{F}}_4^T.
\end{cases}
\tag{70}
$$

We need to decouple Eq. (69). Similar to Eq. (39), put

$$
Y(t) = \mathcal{P}_1(t) X(t) + \mathcal{P}_2(t) \widehat{X}(t), \quad t \in [0, T], \tag{71}
$$

where $\mathcal{P}_1(t), \mathcal{P}_2(t)$ are differentiable, deterministic 2×2 matrix-valued functions with $\mathcal{P}_1(T) = \mathcal{G}_2, \mathcal{P}_2(T) = 0$. Applying Lemma 5.4 in [21] to the forward equation in Eq. (35), we obtain

$$
\begin{cases}
d\widehat{X}(t) = \Big[(\mathcal{A}_1 + \overline{\mathcal{A}}_2) \widehat{X}(t) + \Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big) \widehat{Y}(t) + \Big(\mathcal{B}_3 + \overline{\overline{\mathcal{B}}}_2\Big) \widehat{Z}(t) + \Big(\overline{\tilde{\mathcal{B}}}_1 + \overline{\tilde{\mathcal{B}}}_2\Big) \tilde{Z}(t)\Big] dt \\
\quad + \Big[\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big) \widehat{X}(t) + \Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big) \widehat{Y}(t) + \Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big) \widehat{Z}(t) + \Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big) \tilde{Z}(t)\Big] d\tilde{W}(t), \\
\widehat{X}(0) = X_0.
\end{cases}
\tag{72}
$$

Applying Itô's formula to (3.31), we get

$$
\begin{aligned}
dY(t) = &\Big\{\Big(\dot{\mathcal{P}}_1 + \mathcal{P}_1\mathcal{A}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_1\mathcal{P}_1\Big)X(t) + \Big[\dot{\mathcal{P}}_2 + \mathcal{P}_1\overline{\mathcal{A}}_2 + \mathcal{P}_1\overline{\mathcal{B}}_2\mathcal{P}_1 + \mathcal{P}_2\big(\mathcal{A}_1 + \overline{\mathcal{A}}_2\big) \\
&+\mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)\mathcal{P}_1, +\mathcal{P}_1\Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)\mathcal{P}_2, +\mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)\mathcal{P}_2\Big]\widehat{X}(t) + \mathcal{P}_1\mathcal{B}_3 Z(t) + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1\tilde{Z}(t) \\
&+\Big[\mathcal{P}_1\overline{\overline{\mathcal{B}}}_2 + \mathcal{P}_2\Big(\mathcal{B}_3 + \overline{\overline{\mathcal{B}}}_2\Big)\Big]\widehat{Z}(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{B}}}_2 + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1 + \overline{\tilde{\mathcal{B}}}_2\Big)\Big]\tilde{Z}(t)\Big\}dt \\
&+\Big\{\big(\mathcal{P}_1\mathcal{C}_1 + \mathcal{P}_1(t)\mathcal{B}_3^T\mathcal{P}_1\big)X(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{F}}}_5 + \mathcal{P}_1\mathcal{B}_3^T\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2)\Big]\widehat{X}(t) + \mathcal{P}_1\tilde{\tilde{\mathcal{D}}}_2 Z(t) \\
&+\mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\widehat{Z}(t) + \mathcal{P}_1\mathcal{D}_3\tilde{Z}(t) + \mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{Z}(t)\Big\}dW(t) \\
&+\Big\{\Big(\mathcal{P}_1\tilde{\mathcal{C}}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1\Big)X(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2 + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1 + \mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 \\
&+\mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 + \mathcal{P}_2\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big), +\mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1\Big]\widehat{X}(t) + \mathcal{P}_1\mathcal{D}_3^T Z(t) + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_3\tilde{Z}(t) \\
&+\Big[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big)\Big]\widehat{Z}(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big)\Big]\tilde{Z}(t)\Big\}d\tilde{W}(t) \\
= &-\Big\{\big(\mathcal{Q}_2 + \mathcal{A}_1^T\mathcal{P}_1\big)X(t) + \Big(\overline{\tilde{\mathcal{F}}}_4 + \overline{\mathcal{A}}_2^T\mathcal{P}_1 + \mathcal{A}_1^T\mathcal{P}_2 + \overline{\mathcal{A}}_2^T\mathcal{P}_2\Big)\widehat{X}(t) + \mathcal{C}_1^T Z(t) + \overline{\tilde{\mathcal{F}}}_5^T\widehat{Z}(t) \\
&+\tilde{\mathcal{C}}_1^T\tilde{Z}(t) + \overline{\tilde{\mathcal{C}}}_2^T\tilde{Z}(t)\Big\}dt + Z(t)dW(t) + \tilde{Z}(t)d\tilde{W}(t).
\end{aligned}
\tag{73}
$$

Comparing $dW(t)$ and $d\tilde{W}(t)$ on both sides of Eq. (73), we have

$$
\begin{aligned}
Z(t) = &\big(\mathcal{P}_1\mathcal{C}_1 + \mathcal{P}_1\mathcal{B}_3^T\mathcal{P}_1\big)X(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{F}}}_5 + \mathcal{P}_1\mathcal{B}_3^T\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2)\Big]\widehat{X}(t) \\
&+ \mathcal{P}_1\tilde{\tilde{\mathcal{D}}}_2 Z(t) + \mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\widehat{Z}(t) + \mathcal{P}_1\mathcal{D}_3\tilde{Z}(t) + \mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{Z}(t), \\
\tilde{Z}(t) = &\Big(\mathcal{P}_1\tilde{\mathcal{C}}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1\Big)X(t) + \Big[\mathcal{P}_2\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big) + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1, +\mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 \\
&+ \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 + \mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2, +\mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1\Big]\widehat{X}(t) + \mathcal{P}_1\mathcal{D}_3^T Z(t) \\
&+ \Big[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big)\Big]\widehat{Z}(t) + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_3\tilde{Z}(t) + \Big[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big)\Big]\tilde{Z}(t).
\end{aligned}
\tag{74}
$$

Taking $\mathbb{E}[\cdot|\mathcal{G}_{1,t}]$, we derive

$$
\begin{aligned}
\widehat{Z}(t) = &\Big[\mathcal{P}_1\Big(\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5\Big) + \mathcal{P}_1\big(\mathcal{B}_3^T + \overline{\mathcal{D}}_2\big)\mathcal{P}_1 + \mathcal{P}_1\big(\mathcal{B}_3^T + \overline{\mathcal{D}}_2\big)\mathcal{P}_2\Big]\widehat{X}(t) \\
&+ \mathcal{P}_1\Big(\tilde{\tilde{\mathcal{D}}}_2 + \overline{\overline{\mathcal{D}}}_2\Big)\widehat{Z}(t) + \mathcal{P}_1\Big(\mathcal{D}_3 + \overline{\tilde{\mathcal{D}}}_2\Big)\tilde{Z}(t), \\
\widehat{\tilde{Z}}(t) = &\Big[\mathcal{P}_1\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big) + \mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1, +\mathcal{P}_2\Big(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\Big), +\mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_1 \\
&+ \mathcal{P}_1\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2 + \mathcal{P}_2\Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)\mathcal{P}_2\Big]\widehat{X}(t) \\
&+ (\mathcal{P}_1 + \mathcal{P}_2)\Big(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\Big)\widehat{Z}(t) + (\mathcal{P}_1 + \mathcal{P}_2)\Big(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\Big)\tilde{Z}(t).
\end{aligned}
\tag{75}
$$

Supposing that (I_2 denotes the 2×2 unit matrix)

$$
\begin{aligned}
&(\mathbf{A2.2}) \quad \tilde{\mathcal{N}}_2^{-1} := \left[I_2 - \mathcal{P}_1\left(\tilde{\tilde{\mathcal{D}}}_2 + \overline{\overline{\mathcal{D}}}_2\right)\right]^{-1} \quad \text{and} \\
&\qquad \tilde{\tilde{\mathcal{N}}}_2^{-1} := \left[I_2 - (\mathcal{P}_1 + \mathcal{P}_2)\left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\tilde{\mathcal{N}}_2^{-1}\mathcal{P}_1\left(\mathcal{D}_3 + \overline{\tilde{\mathcal{D}}}_2\right) - (\mathcal{P}_1 + \mathcal{P}_2)\left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\right)\right]^{-1} \quad \text{exist,}
\end{aligned} \tag{76}
$$

we get

$$
\widehat{Z}(t) = \Sigma_0(\mathcal{P}_1, \mathcal{P}_2)\widehat{X}(t), \tilde{Z}(t) = \tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)\widehat{X}(t), \tag{77}
$$

where

$$
\begin{cases}
\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) := \tilde{\mathcal{N}}_2^{-1}\Big\{\mathcal{P}_1\left(\mathcal{D}_3 + \overline{\tilde{\mathcal{D}}}_2\right)\tilde{\tilde{\mathcal{N}}}_2^{-1}(\mathcal{P}_1 + \mathcal{P}_2)\Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2 + \left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)(\mathcal{P}_1 + \mathcal{P}_2) + \left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\tilde{\mathcal{N}}_2^{-1} \\
\qquad \mathcal{P}_1\left[\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5 + \left(\mathcal{B}_3^T + \overline{\mathcal{D}}_2\right)(\mathcal{P}_1 + \mathcal{P}_2)\right]\Big] + \mathcal{P}_1\left[\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5 + \left(\mathcal{B}_3^T + \overline{\mathcal{D}}_2\right)(\mathcal{P}_1 + \mathcal{P}_2)\right]\Big\}, \\
\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) := \tilde{\tilde{\mathcal{N}}}_2^{-1}(\mathcal{P}_1 + \mathcal{P}_2)\Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2 + \left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)(\mathcal{P}_1 + \mathcal{P}_2) \\
\qquad + \left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\tilde{\mathcal{N}}_2^{-1}\mathcal{P}_1\left[\mathcal{C}_1 + \overline{\tilde{\mathcal{F}}}_5 + \left(\mathcal{B}_3^T + \overline{\mathcal{D}}_2\right)(\mathcal{P}_1 + \mathcal{P}_2)\right]\Big].
\end{cases} \tag{78}
$$

Inserting Eq. (77) into Eq. (74), we have

$$
\begin{aligned}
Z(t) = {} & \left(\mathcal{P}_1\mathcal{C}_1 + \mathcal{P}_1\mathcal{B}_3^T\mathcal{P}_1\right)X(t) + \mathcal{P}_1\Big[\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T\mathcal{P}_2, + \overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2), + \overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \\
& + \overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)\Big]\widehat{X}(t) + \mathcal{P}_1\tilde{\tilde{\mathcal{D}}}_2 Z(t) + \mathcal{P}_1\mathcal{D}_3\tilde{Z}(t), \\
\tilde{Z}(t) = {} & \left(\mathcal{P}_1\tilde{\mathcal{C}}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1\right)X(t) + \Big[\mathcal{P}_2\left(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\right) + \mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_1 + \mathcal{P}_1\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 \\
& + \mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 + \left[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \\
& + \left[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)\Big]\widehat{X}(t) + \mathcal{P}_1\mathcal{D}_3^T Z(t) + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_3\tilde{Z}(t).
\end{aligned} \tag{79}
$$

Supposing that

$$
\begin{aligned}
&(\mathbf{A2.3}) \quad \overline{\mathcal{N}}_2^{-1} := \left(I_2 - \mathcal{P}_1\tilde{\tilde{\mathcal{D}}}_2\right)^{-1} := \left[I_{2n} + \mathcal{P}_1\mathcal{D}_2 N_2^{-1}\mathcal{D}_2^T\right]^{-1} \\
&\text{and} \quad \overline{\overline{\mathcal{N}}}_2^{-1} := \left(I_2 - \mathcal{P}_1\mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\mathcal{D}_3 - \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_3\right)^{-1} := \Big[I_{2n} - \mathcal{P}_1\left(\tilde{\tilde{\mathcal{D}}}_1 - \mathcal{D}_2 N_2^{-1}\tilde{\mathcal{D}}_2^T\right)^T \\
&\qquad \times \left[I_{2n} + \mathcal{P}_1\mathcal{D}_2 N_2^{-1}\mathcal{D}_2^T\right]^{-1}\mathcal{P}_1\left(\tilde{\tilde{\mathcal{D}}}_1 - \mathcal{D}_2 N_2^{-1}\tilde{\mathcal{D}}_2^T\right) - \mathcal{P}_1\left(\tilde{\mathcal{F}}_3 - \tilde{\mathcal{D}}_2 N_2^{-1}\tilde{\mathcal{D}}_2^T\right)\Big]^{-1} \quad \text{exist,}
\end{aligned} \tag{80}
$$

we get

$$Z(t) = \Sigma_1(\mathcal{P}_1, \mathcal{P}_2)X(t) + \Sigma_2(\mathcal{P}_1, \mathcal{P}_2)\widehat{X}(t), \quad \tilde{Z}(t) = \tilde{\Sigma}_1(\mathcal{P}_1, \mathcal{P}_2)X(t) + \tilde{\Sigma}_2(\mathcal{P}_1, \mathcal{P}_2)\widehat{X}(t), \tag{81}$$

where.

$$\begin{cases} \Sigma_1(\mathcal{P}_1, \mathcal{P}_2) := \overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left[\mathcal{C}_1 + \mathcal{B}_3^T\mathcal{P}_1 + \mathcal{D}_3\mathcal{P}_1\left(\mathcal{C}_1 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1\right) + \mathcal{D}_3\overline{\overline{\mathcal{N}}}^{2-1}\mathcal{P}_1\mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left(\mathcal{C}_1 + \mathcal{B}_3^T\mathcal{P}_1\right)\right], \\ \tilde{\Sigma}_1(\mathcal{P}_1, \mathcal{P}_2) := \overline{\overline{\mathcal{N}}}_2^{-1}\mathcal{P}_1\left[\mathcal{C}_1 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1 + \mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left(\mathcal{C}_1 + \mathcal{B}_3^T\mathcal{P}_1\right)\right], \\ \Sigma_2(\mathcal{P}_1, \mathcal{P}_2) := \overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\Big\{\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) \\ \quad + \mathcal{D}_3\overline{\overline{\mathcal{N}}}_2^{-1}\left[\mathcal{P}_2\left(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\right) + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_1 + \mathcal{P}_1\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 \right. \\ \quad \left. + \mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2 + \left[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) + \left[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) \right. \\ \quad \left. + \mathcal{P}_1\mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left[\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)\right]\right]\Big\}, \\ \tilde{\Sigma}_2(\mathcal{P}_1, \mathcal{P}_2) := \overline{\overline{\mathcal{N}}}_2^{-1}\Big\{\mathcal{P}_2\left(\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2\right) + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_1 + \mathcal{P}_1\overline{\mathcal{D}}_3\mathcal{P}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{C}}}_2 + \mathcal{P}_1\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 \\ \quad + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\right)\mathcal{P}_2 + \left[\mathcal{P}_1\overline{\overline{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\mathcal{D}_3^T + \overline{\overline{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) + \left[\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_3^T + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_3 + \overline{\tilde{\mathcal{D}}}_3\right)\right]\mathcal{P}_1\overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2) \\ \quad + \mathcal{P}_1\mathcal{D}_3^T\overline{\mathcal{N}}_2^{-1}\mathcal{P}_1\left[\overline{\tilde{\mathcal{F}}}_5 + \mathcal{B}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1 + \mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)\right]\Big\}. \end{cases} \tag{82}$$

Comparing the coefficients of dt in Eq. (73) and putting Eqs. (77) and (81) into them, we get

$$\begin{cases} 0 = \dot{\mathcal{P}}_1 + \mathcal{P}_1\mathcal{A}_1 + \mathcal{A}_1^T\mathcal{P}_1 + \mathcal{P}_1\overline{\tilde{\mathcal{F}}}_1\mathcal{P}_1 + \mathcal{Q}_2 + (\mathcal{C}_1 + \mathcal{P}_1\mathcal{B}_3)\Sigma_1(\mathcal{P}_1, \mathcal{P}_2) + \left(\tilde{\mathcal{C}}_1^T + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1\right)\tilde{\Sigma}_1(\mathcal{P}_1, \mathcal{P}_2), \\ 0 = \dot{\mathcal{P}}_2 + \mathcal{P}_2\left(\mathcal{A}_1 + \overline{\mathcal{A}}_2\right) + \left(\mathcal{A}_1 + \overline{\mathcal{A}}_2\right)^T\mathcal{P}_2 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\right)\mathcal{P}_1 + \mathcal{P}_1\left(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\right)\mathcal{P}_2 \\ \quad + \mathcal{P}_2\left(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\right)\mathcal{P}_2 + \mathcal{P}_1\overline{\mathcal{A}}_2 + \overline{\mathcal{A}}_2^T\mathcal{P}_1 + \mathcal{P}_1\overline{\mathcal{B}}_2\mathcal{P}_1 + \overline{\tilde{\mathcal{F}}}_4 + (\mathcal{C}_1 + \mathcal{P}_1\mathcal{B}_3)\Sigma_2(\mathcal{P}_1, \mathcal{P}_2) \\ \quad + \left(\tilde{\mathcal{C}}_1^T + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_1\right)\tilde{\Sigma}_2(\mathcal{P}_1, \mathcal{P}_2) + \left[\overline{\overline{\mathcal{F}}}_5^T + \mathcal{P}_1\overline{\overline{\mathcal{B}}}_2 + \mathcal{P}_2\left(\mathcal{B}_3 + \overline{\overline{\mathcal{B}}}_2\right)\right]\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \\ \quad + \left[\overline{\tilde{\mathcal{C}}}_2^T + \mathcal{P}_1\overline{\tilde{\mathcal{B}}}_2 + \mathcal{P}_2\left(\overline{\tilde{\mathcal{B}}}_1 + \overline{\tilde{\mathcal{B}}}_2\right)\right]\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2), \quad \mathcal{P}_1(T) = \mathcal{G}_2, \quad \mathcal{P}_2(T) = 0. \end{cases} \tag{83}$$

Note that the system of Riccati equations (83) is not standard, and its solvability is open. Due to some technical reason, we can not obtain the solvability of it now. However, in some special case, $\mathcal{P}_1(t)$ and $\mathcal{P}_2(t)$ are not coupled. Then we can first solve the first equation of $\mathcal{P}_1(t)$, then that of $\mathcal{P}_2(t)$ by standard Riccati equation theory. We will not discuss for the space limit. And we will consider the general solvability of Eq. (83) in the future.

Instituting Eqs. (77) and (81) into Eq. (68), we obtain

$$
\begin{aligned}
u_2^*(t) = -N_2^{-1}\Big[&\mathcal{B}_2^T\mathcal{P}_1 + \mathcal{D}_2^T\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\mathcal{D}}_2^T\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\tilde{\mathcal{F}}_4^T + \mathcal{B}_2^T\mathcal{P}_2 + \tilde{\mathcal{B}}_2^T(\mathcal{P}_1+\mathcal{P}_2) \\
&+ \mathcal{D}_2^T\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\mathcal{F}}_2^T\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\mathcal{D}}_2^T\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \tilde{\tilde{\mathcal{D}}}_2^T\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t),
\end{aligned}
\tag{84}
$$

and the optimal "state" $X = (x^*, p)^T$ of the leader satisfies

$$
\begin{cases}
dX(t) = \Big\{\Big[\mathcal{A}_1 + \overline{\tilde{\mathcal{F}}}_1\mathcal{P}_1 + \mathcal{B}_3\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\overline{\mathcal{A}}_2 + \overline{\tilde{\mathcal{F}}}_1\mathcal{P}_2 + \overline{\mathcal{B}}_2(\mathcal{P}_1+\mathcal{P}_2) \\
\qquad + \mathcal{B}_3\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{B}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)\Big\}dt \\
\qquad + \Big\{\Big[\mathcal{C}_1 + \tilde{\mathcal{B}}_3^T\mathcal{P}_1 + \tilde{\tilde{\mathcal{D}}}_2\Sigma_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\overline{\tilde{\mathcal{F}}}_5 + \tilde{\mathcal{B}}_3^T\mathcal{P}_2 + \overline{\mathcal{D}}_2(\mathcal{P}_1+\mathcal{P}_2) \\
\qquad + \tilde{\tilde{\mathcal{D}}}_2\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)\Big\}dW(t) \\
\qquad + \Big\{\Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_1 + \mathcal{D}_3^T\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2)\Big]X(t) + \Big[\overline{\tilde{\mathcal{C}}}_2 + \overline{\tilde{\mathcal{B}}}_1^T\mathcal{P}_2 + \overline{\mathcal{D}}_3(\mathcal{P}_1+\mathcal{P}_2) \\
\qquad + \mathcal{D}_3^T\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_3^T\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_3\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)\Big\}d\tilde{W}(t), \\
X(0) = X_0,
\end{cases}
\tag{85}
$$

where $\widehat{X}$ is governed by

$$
\begin{cases}
d\widehat{X}(t) = \Big[\mathcal{A}_1 + \overline{\mathcal{A}}_2 + \Big(\overline{\tilde{\mathcal{F}}}_1 + \overline{\mathcal{B}}_2\Big)(\mathcal{P}_1+\mathcal{P}_2) + \mathcal{B}_3\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2) \\
\quad + \mathcal{B}_3\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{B}}}_2\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_1\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{B}}}_2\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)dt \\
\qquad + \Big[\tilde{\mathcal{C}}_1 + \overline{\tilde{\mathcal{C}}}_2 + \Big(\overline{\tilde{\mathcal{B}}}_1^T + \overline{\mathcal{D}}_3\Big)(\mathcal{P}_1+\mathcal{P}_2) + \mathcal{D}_3^T\Sigma_1(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_1(\mathcal{P}_1,\mathcal{P}_2) \\
\quad + \mathcal{D}_3^T\Sigma_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\overline{\mathcal{D}}}_3^T\Sigma_0(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{F}}}_3\tilde{\Sigma}_2(\mathcal{P}_1,\mathcal{P}_2) + \overline{\tilde{\mathcal{D}}}_3\tilde{\Sigma}_0(\mathcal{P}_1,\mathcal{P}_2)\Big]\widehat{X}(t)d\tilde{W}(t), \\
\widehat{X}(0) = X_0.
\end{cases}
\tag{86}
$$

We summarize the above analysis in the following theorem.

Theorem 3.2 *Let* **(A2.1)∼(A2.3)** *hold,* $(\mathcal{P}_1(t), \mathcal{P}_2(t))$ *satisfy Eq.* (83), $\widehat{X}$ *be the* $\mathcal{G}_{1,t}$*-adapted solution to Eq.* (86), *and* X *be the* $\mathcal{F}_t$*-adapted solution to Eq.* (85). *Define* $\left(Y, Z, \tilde{Z}\right)$ *by Eqs.* (71) *and* (81), *respectively. Then Eq.* (69) *holds, and* u_2^* *given by Eq.* (84) *is a feedback optimal control of the leader.*

Finally, the optimal control u_1^* of the follower can also be represented in a nonanticipating way. In fact, by Eq. (47), noting Eqs. (68), (71), and (77), we have

$$\begin{aligned}
u_1^*(t) &= -\tilde{N}_1^{-1}(t)\left(\tilde{S}_1^T(t)\widehat{x}^*(t) + \tilde{S}(t)\widehat{u}_2^*(t) + B_1^T\widehat{\varphi}^*(t) + \tilde{D}_1^T\beta^*(t)\right) \\
&= -\tilde{N}_1^{-1}(t)\left[\begin{pmatrix} \tilde{S}_1^T(t) & 0\end{pmatrix}\widehat{X}(t) + \tilde{S}(t)\widehat{u}_2^*(t) + \begin{pmatrix} 0 & B_1^T\end{pmatrix}\widehat{Y}(t) + \begin{pmatrix} 0 & \tilde{D}_1^T\end{pmatrix}\tilde{Z}(t)\right] \\
&= -\tilde{N}_1^{-1}(t)\Bigg[\begin{pmatrix} \tilde{S}_1^T(t) & 0\end{pmatrix} - \tilde{S}(t)N_2^{-1}[\tilde{\mathcal{F}}_4^T + (\mathcal{B}_2 + \tilde{\mathcal{B}}_2)^T(\mathcal{P}_1 + \mathcal{P}_2) + (\mathcal{D}_2 + \tilde{\mathcal{F}}_2)^T\Sigma_0(\mathcal{P}_1, \mathcal{P}_2) \\
&\quad + \left(\tilde{\mathcal{D}}_2 + \tilde{\tilde{\mathcal{D}}}_2\right)^T\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)] + \begin{pmatrix} 0 & B_1^T\end{pmatrix}(\mathcal{P}_1 + \mathcal{P}_2) + \begin{pmatrix} 0 & \tilde{D}_1^T\end{pmatrix}\tilde{\Sigma}_0(\mathcal{P}_1, \mathcal{P}_2)\Bigg]\widehat{X}(t),
\end{aligned} \tag{87}$$

which is observable for the follower.

Remark 3.3 When we consider the complete information case, that is, $\tilde{W}(\cdot)$ disappears and $\mathcal{G}_{1,t} = \mathcal{F}_t$, Theorems 3.1 and 3.2 coincide with Theorems 2.3 and 3.3 in Yong [13].

4. Concluding remarks

In this chapter, we have studied a leader-follower stochastic differential game with asymmetric information. This kind of game problem possesses several attractive features. First, the game problem has the Stackelberg feature, which means the two players play as different roles during the game. Thus the usual approach to deal with game problems, such as [6–8, 10], where the two players act as equivalent roles, does not apply. Second, the game problem has the asymmetric information between the two players, which was not considered in [3, 13, 14]. In detail, the information available to the follower is based on some sub-σ-algebra of that available to the leader. Stochastic filtering technique is introduced to compute the optimal filtering estimates for the corresponding adjoint processes, which act as the solution to some FBSDFE. Third, the Stackelberg equilibrium is represented in its state feedback form for the LQ problem under some appropriate assumptions. Some new conditional mean-field FBSDEs and system of Riccati equations are introduced to deal with the leader's LQ problem.

In principle, Theorems 3.1 and 3.2 provide a useful tool to seek Stackelberg equilibrium. As a first step in this direction, we apply our results to the LQ problem to obtain explicit solutions. We hope to return to the more general case in our future research. It is worthy to study the closed-loop Stackelberg equilibrium for our problem, as well as the solvability of the system of Riccati equations. These challenging topics will be considered in our future work.

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Notes

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