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# Differential Equations
## Theory and Current Research

*Edited by Terry E. Moschandreou*

# DIFFERENTIAL EQUATIONS - THEORY AND CURRENT RESEARCH

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# Meet the editor

Dr. Terry E. Moschandreou is a professor in Applied Mathematics at the University of Western Ontario in the School of Mathematical and Statistical Sciences where he has taught for several years. He received his PhD degree in Applied Mathematics from the University of Western Ontario in 1996. The greater part of his professional life was spent at the University of Western Ontario and Fanshawe College in London, Ontario, Canada. For a short period, he worked at the National Technical University of Athens, Greece. Dr. Moschandreou is the author of several research articles in blood flow and oxygen transport in the microcirculation, general fluid dynamics and theory of differential equations. Also, he has contributed in the field of finite element modelling of the upper airways in sleep apnea and surgical brain deformation modelling. More recently, he has been working with the partial differential equations of multiphase flow and level set methods as used in fluid dynamics.

# Contents

# Preface

Differential equations are mathematical equations that relate some functions with their derivatives. The functions usually represent some physical quantities and their derivatives represent their rates of change and the equation relates the two together. For example, in fluid dynamics, the Navier-Stokes equations are a system of mathematical equations that relate the velocities of the fluid to partial derivatives of velocity and pressure. The editor of the present book has worked on solving the Navier-Stokes equations in cylindrical coordinates for multiphase flows where the equations are coupled to the continuity and level set distance function equations. Such work now in press has revealed an analytical procedure to solve this system of equations by defining a composite velocity formulation for the sum of three principal directions of flow and connecting this to the level set function and its derivatives. It has been shown that it is possible to solve analytically multiphase flow using level set methods for vertical and horizontal tubes. It has been shown that in this pursuit the structure of the governing equations for multiphase flow has some interesting symmetries, which reduce the composite formulation above ordinary differential equations. Further analysis using pseudo-exact differential equations results in Abel-type equations emerging in the analysis. It is a worthy exercise to correctly reduce a system of partial differential equations to ordinary differential equations and hence prove the existence and uniqueness of solutions to such mathematical problems. For this reason, the editor of this book has been motivated to introduce various topics welcomed from an international audience of mathematicians and researchers to contribute various aspects of the theory and application of differential equations to the current project.

The editor has incorporated contributions from a diverse group of leading researchers in the field of differential equations. This book aims to provide an overview of the current knowledge in the field of differential equations. The main subject areas are divided into general theory and applications. These include fixed point approach to solution existence of differential equations, existence theory of differential equations of arbitrary order, topological methods in the theory of ordinary differential equations, impulsive fractional differential equations with finite delay and integral boundary conditions, an extension of Massera's theorem for n-dimensional stochastic differential equations, phase portraits of cubic dynamic systems in a Poincare circle, differential equations arising from the three-variable Hermite polynomials and computation of their zeros and reproducing kernel method for differential equations. Applications include local discontinuous Galerkin method for nonlinear Ginzburg-Landau equation, general function method in transport boundary value problems of theory of elasticity and solution of nonlinear partial differential equations by new Laplace variational iteration method.

Existence/uniqueness theory of differential equations is presented in this book with applications that will be of benefit to mathematicians, applied mathematicians and researchers in the field. The book is written primarily for those who have some knowledge of differential equations and mathematical analysis. The authors of each section bring a strong emphasis on theoretical foundations to the book.

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Section 1

# Theory of Differential Equations

Chapter 1

# Fixed Point Theory Approach to Existence of Solutions with Differential Equations

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Poom Kumam

Additional information is available at the end of the chapter

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**Abstract**

In this chapter, we introduce a generalized contractions and prove some fixed point theorems in generalized metric spaces by using the generalized contractions. Moreover, we will apply the fixed point theorems to show the existence and uniqueness of solution to the ordinary difference equation (ODE), Partial difference equation (PDEs) and fractional boundary value problem.

**Keywords:** fixed point, contraction, generalized contraction, differential equation, partial differential equation, fractional difference equation

## 1. Introduction

The study of differential equations is a wide field in pure and applied mathematics, chemistry, physics, engineering and biological science. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics investigated the existence and uniqueness of solutions, but applied mathematics focuses on the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modeling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods.

Following the ordinary differential equations with boundary value condition

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

where $y(x_0) = 0,\ \ y'(x_1) = c_1, \ldots, y^{(n-1)}(x_{n-1}) = c_{n-1}$ the positive integer $n$ (the order of the highest derivative). This will be discussed. Existence and uniqueness of solution for initial value problem (IVP).

$$u'(t) = f(t, u(t))$$
$$u(t_0) = u_0.$$

Differential equations contains derivatives with respect to two or more variables is called a *partial differential equation* (PDEs). For example,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where $u$ is dependent variable and $A, B, C, D, E, F$ and $G$ are function of $x, y$ above equation is classified according to discriminant $\left(B^2 - 4AC\right)$ as follows,

**1.** Elliptic equation if $\left(B^2 - 4AC\right) < 0$,

**2.** Hyperbolic equation if $\left(B^2 - 4AC\right) > 0$,

**3.** Parabolic equation if $\left(B^2 - 4AC\right) = 0$.

This will be discussed. Existence of solution for semilinear elliptic equation. Consider a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ that solves,

$$-\Delta u = f(u) \quad \text{in} \quad \Omega$$
$$u = u_0 \qquad \text{on} \quad \partial\Omega$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ is a typically nonlinear function. And fractional differential equations. This will be discussed. Fractional differential equations are of two kinds, they are Riemann-Liouville fractional differential equations and Caputo fractional differential equations with boundary value.

$${}^c D_t^\alpha u(t) = Bu(t);\, t > 0$$
$$u(0) = u_0 \in X$$

where ${}^c D_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, and $t \in [0, \tau]$, for all $\tau > 0$.

The following fractional differential equation will boundary value condition.

$$D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2$$
$$u(0) = 0, \quad u(1) = \int_0^1 u(s)ds,$$

where $f:[0,1]\times[0,\infty)\to[0,\infty)$ is a continuous function and $D^{\alpha}_{0+}$ is the standard Riemann-Liouville fractional derivative.

One method for existence and uniqueness of solution of difference equation due to fixed point theory. The primary result in fixed point theory which is known as *Banach's contraction principle* was introduced by Banach [1] in 1922.

**Theorem 1.1.** *Let $(X,d)$ be a complete metric spaces and $T:X\to X$ be a contraction mapping (that is, there exists $0\le\alpha<1$) such that*

$$d(Tx,Ty)\le\alpha d(x,y)$$

for all $x,y\in X$, then $T$ has a unique fixed point.

Since Banach contraction is a very popular and important tool for solving many kinds of mathematics problems, many authors have improved, extended and generalized it (see in [2–4]) and references therein.

In this chapter, we discuss on the existence and uniqueness of the differential equations by using fixed point theory to approach the solution.

## 2. Basic results

Throughout the rest of the chapter unless otherwise stated $(X,d)$ stands for a complete metric space.

### 2.1. Fixed point

**Definition 2.1.** Let $X$ be a nonempty set and $T:X\to X$ be a mapping. A point $x^*\in X$ is said to be a *fixed point* of $T$ if $T(x^*)=x^*$.

**Definition 2.2.** Let $(X,d)$ be a metric space. The mapping $T:X\to X$ is said to be Lipschitzian if there exists a constant $\alpha>0$ (called Lipschitz constant) such that

$$d(Tx,Ty)\le\alpha d(x,y)\quad\text{for all }\ x,y\in X.$$

A mapping $T$ with a Lipschitz constant $\alpha<1$ is called contraction.

**Definition 2.3.** Let $F$ and $X$ be normed spaces over the field $\mathbb{K}$, $T:F\to X$ an operator and $c\in F$. We say that $T$ is continuous at $c$ if for every $\varepsilon>0$ there exists $\delta>0$ such that $\|T(x)-T(c)\|<\epsilon$ whenever $\|x-c\|<\delta$ and $x\in F$. If $T$ is continuous at each $x\in F$, then $T$ is said to be continuous on $T$.

**Definition 2.4.** Let $X$ and $Y$ be normed spaces. The mapping $T:X\to Y$ is said to be completely continuous if $T(C)$ is a compact subset of $Y$ for every bounded subset $C$ of $X$.

**Definition 2.5.** Compact operator is a linear operator $L$ form a Banach space $X$ to another Banach space $Y$, such that the image under $L$ of any bounded subset of $X$ is a relatively compact subset (has compact closure) of $Y$ such an operator is necessarily a bounded operator, and so continuous.

**Definition 2.6.** A subset $C$ of a normed linear space $X$ is said to be convex subset in $X$ if $\lambda x + (1-\lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0,1]$.

**Definition 2.7.** $v$ is called the $\alpha^{th}$ weak derivative of $u$

$$D^{\alpha}u = v$$

if

$$\int_{\Omega} uD^{\alpha}\psi dx = (-1)^{|\alpha|}\int_{\Omega} v\psi dx$$

for all test function $\psi \in C_c^{\infty}(\Omega)$.

**Theorem 2.8.** *(Schauder's Fixed Point Theorem) Let X be a Banach space, $M \subset X$ be nonempty, convex, bounded, closed and $T : M \subset X \to M$ be a compact operator. Then T has a fixed point.*

**Lemma 2.9.** ref. [5] Given $f \in C(\mathbb{R})$ such that $|f(t)| \leq a = b|t|^r$ where $a > 0, b > 0$ and $r > 0$ are positive constants. Then the map $u \mapsto f(u)$ is continuous for $L^p(\Omega)$ to $L^{\frac{p}{r}}(\Omega)$ for $p \geq \max(1, r)$ and maps bounded subset of $L^p(\Omega)$ to bounded subset of $L^{\frac{p}{r}}(\Omega)$.

*Proof.* Form to Jensen's inequality

$$(a + b|t|^r)^{\left(\frac{p}{r}\right)} \leq 2^{\frac{p}{r}-1}a^{\frac{p}{r}} + 2^{\frac{p}{r}-1}b^{\frac{p}{r}}|t|^p \leq C(1 + |t|^p)$$

where $C$ is a positive constant depending on $a, b, p$ and $r$ only, since $u \in L^p(\Omega)$, we have

$$\int_{\Omega} |f(u)|^{\frac{p}{r}}dx \leq C(a, b, p, r)\left(|\Omega| + \int_{\Omega} u^p dx\right) < \infty$$

therefore $f(u) \in L^{\frac{p}{r}}(\Omega)$. Let $u_n$ be a sequence converging to $u$ in $L^p(\Omega)$. There exists a subsequence $u_n$, and a function $g \in L^p(\Omega)$ such that set, $u_{n'} \to u(x)$, and $|u_{n'}(x)| \leq g(x)$, almost everywhere. This is sometimes called the generalized DCT, or the partial converse of the DCT, or the Riesz-Fisher Theorem. From the continuity of $f$, $|f(u(x)) - f(u_{n'})| \to 0$ on $\Omega \setminus \mathbb{N}$, and

$$|f(u(x)) - f(u_{n'})|^{\frac{p}{r}} \leq C(1 + g(x)^p + |f(u)|^p)$$

where $C$ is another positive constant depending on $a, b, p$ and $r$ only, the left-hand-side is independent of $n'$ and is in $L^1(\Omega)$. We can apply the Dominated Convergence Theorem to conclude the

$$\int_{\Omega} |f(u(x)) - f(u_{n'})|^{\frac{p}{r}}dx \to 0$$

or in other words, $\|f(u(x)) - f(u_{n'})\|_{L^p(\Omega)} \to 0$. Since the limit does not depend on the subsequence this convergence $u$ holds for $u_n$. □

**Corollary 2.10.** ref. [5] Let $\mu \geq 0$. Then the map $g \mapsto \left(-\Delta + \mu I_d\right)^{-1} g$ is

**i.** continuous as map from $L^2(\Omega)$ to $H_0^1(\Omega)$ in other words

$$\|v\|_{H_0^1(\Omega)} \leq C(\Omega) \|g\|_{L^2(\Omega)}.$$

**ii.** compact as map form $L^2(\Omega)$ to $L^2(\Omega)$.

*Proof.* The first part is due to the fact that $L^2(\Omega)$ is continuously in $H^{-1}(\Omega)$. The second part follows as $\left(-\Delta + \mu I_d\right)^{-1} : L^2(\Omega) \to L^2(\Omega)$ can be viewed as composition of the continuous map $\left(-\Delta + \mu I_d\right)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ and the compact embedding $H_0^1(\Omega) \rightarrowtail L^2(\Omega)$ and as the composition of a compact linear operator a continuous linear operator is again compact.

**Theorem 2.11.** *(Poincare) For* $p \in [1, \infty)$*, there exists a constant* $C = C(\Omega, p)$ *such that* $\forall \in W_0^{1,p}(\Omega)$*;* $\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega:\mathbb{R}^n)}$. *A key tool to obtain the compactness of the fixed point maps.*

### 2.2. Fuzzy

A fuzzy set in $X$ is a function with domain X and values in $[0, 1]$. If $A$ is a fuzzy set on $X$ and $x \in X$, then the functional value $Ax$ is called the grade of membership of $x$ in $A$. The $\alpha-$ level set of A, denoted by $A_\alpha$ is defined by

$$A_\alpha = \{x : Ax \geq \alpha\} \quad \text{if} \ \ \alpha \in (0, 1], \quad A_0 = \overline{\{x : Ax > 0\}},$$

where denotes by $\overline{A}$ the closure of the set $A$. For any $A$ and $B$ are subset of $X$ we denote by $H(A, B)$ the Huasdorff distance.

**Definition 2.12.** A fuzzy set $A$ in a metric linear space is called an approximate quantity if and only if $A_\alpha$ is convex and compact in $X$ for each $\alpha \in [0, 1]$ and $\sup_{x \in X} Ax = 1$.

Let $I = [0, 1]$ and $W(X) \subset I^X$ be the collection of all approximate in $X$. For $\alpha \in [0, 1]$, the family $W_\alpha(X)$ is given by $\{A \in I^X : A_\alpha \text{ is nonempty and compact}\}$.

For a metric space $(X, d)$ we denoted by $V(X)$ the collection of fuzzy sets $A$ in $X$ for which $A_\alpha$ is compact and $\sup Ax = 1$ for all $\alpha \in [0, 1]$. Clearly, when $X$ is a metric linear space $W(X) \subset V(X)$.

**Definition 2.13.** Let $A, B \in V(X)$, $\alpha \in [0, 1]$. Then

$$p_\alpha(A, B) = \inf_{x \in A_\alpha,\, y \in B_\alpha} d(x, y), \quad D_\alpha(A, B) = H(A_\alpha, B_\alpha)$$

where $H$ is the Hausdorff distance.

**Definition 2.14.** Let $A, B \in V(X)$. Then $A$ is said to be more accurate than $B$ (or $B$ includes $A$), denoted by $A \subset B$, if and only if $Ax \leq Bx$ for each $x \in X$.

Denote with $\Phi$, the family of nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.

**Theorem 2.15.** ref. [6] Let $(X, d, \preccurlyeq)$ *be a complete ordered metric space and* $T_1, T_2 : X \to W_\alpha(X)$ *be two fuzzy mapping satisfying*

$$D_\alpha(T_1x, T_2y) \leq \phi(M(x, y)) + L \min\{p_\alpha(x, T_1x), p_\alpha(y, T_2y), p_\alpha(x, T_2y), p_\alpha(y, T_1x)\}$$

for all comparable element $x, y \in X$, where $L \geq 0$ and

$$M(x, y) = \max\left\{d(x, y), p_\alpha(x, T_1x), p_\alpha(y, T_2y), \frac{1}{2}\left[p_\alpha(x, T_2y) + p_\alpha(y, T_1x)\right]\right\}.$$

*Also suppose that*

**i.** if $y \in (T_1x_0)_\alpha$, then $y, x_0 \in X$ are comparable,

**ii.** if $x, y \in X$ are comparable, then every $u \in (T_1x)_\alpha$ and every $v \in (T_2y)_\alpha$ are comparable,

**iii.** if a sequence $\{x_n\}$ in $X$ converges to $x \in X$ and its consecutive terms are comparable, then $x_n$ and $x$ are comparable for all $n$.

*Then there exists a point* $x \in X$ *such that* $x_\alpha \subset T_1x$ *and* $x_\alpha \subset T_2x$.

*Proof.* See in [6].

**Corollary 2.16.** ref. [6] Let $(X, d, \preccurlyeq)$ *be a complete ordered metric space and* $T_1, T_2 : X \to W_\alpha(X)$ *be two fuzzy mappings satisfying*

$$D_\alpha(T_1x, T_2y) \leq q \max\left\{d(x, y), p_\alpha(x, T_1x), p_\alpha(y, T_2y), \frac{1}{2}\left[p_\alpha(x, T_2y) + p_\alpha(y, T_1x)\right]\right\}$$

for all comparable elements $x, y \in X$. Also suppose that

**i.** if $y \in (T_1x_0)_\alpha$, then $y, x_0 \in X$ are comparable,

**ii.** if $x, y \in X$ are comparable, then every $u \in (T_1x)_\alpha$ and every $v \in (T_2y)_\alpha$ are comparable,

**iii.** if a sequence $\{x_n\}$ in $X$ converges to $x \in X$ and its consecutive terms are comparable, then $x_n$ and $x$ are comparable for all $n$.

*Then there exists a point* $x \in X$ *such that* $x_\alpha \subset T_1x$ *and* $x_\alpha \subset T_2x$.

### 2.3. Metric-like space

**Definition 2.17.** [7] Let $X$ be nonempty set and function $p : X \times X \to \mathbb{R}^+$ be a function satisfying the following condition: for all $x, y, z \in X$,

$(p_1)$ $p(x,x) = p(x,y) = p(y,y)$ if and only if $x = y$,

$(p_2)$ $p(x,x) \leq p(x,y)$,

$(p_3)$ $p(x,x) = p(y,x)$,

$(p_4)$ $p(x,y) = p(x,z) + p(z,y) - p(z,z)$.

Then $p$ is called a partial metric on $X$, so a pair $(X,p)$ is said to be a partial metric space.

**Definition 2.18.** [8] A metric-like on nonempty set $X$ is a function $\sigma : X \times X \to \mathbb{R}^+$. If for all $x, y, z \in X$, the following conditions hold:

$$(\sigma_1)\ \sigma(x,y) = 0 \Rightarrow x = y;$$

$$(\sigma_2)\ \sigma(x,y) = \sigma(y,x);$$

$$(\sigma_3)\ \sigma(x,y) = \sigma(x,z) + \sigma(z,y).$$

Then a pair $(X,\sigma)$ is called a metric-like space.

It is easy to see that a metric space is a partial metric space and each partial metric space is a metric-like space, but the converse is not true but the converse is not true as in the following examples:

**Example 2.19.** [8] Let $X = \{0,1\}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2, & \text{if } x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then $(X,\sigma)$ is a metric-like space, but it is not a partial metric space, cause $\sigma(0,0) \nleq \sigma(0,1)$.

**Lemma 2.20.** ref. [9] Let $(X,p)$ be a partial metric space. Then

**i.** $\{x_n\}$ is a Cauchy sequence in $(X,p)$ if and only if it is a Cauchy sequence in the metric space $(X,d_p)$,

**ii.** $X$ is complete if and only if the metric space $(X,d_p)$ is complete.

**Definition 2.21.** [8, 10] Let $(X,\sigma)$ be a metric-like space. Then:

**i.** A sequence $\{x_n\}$ in $X$ converges to a point $x \in X$ if $\lim_{n\to\infty}\sigma(x_n,x) = \sigma(x,x)$. The sequence $\{x_n\}$ is said to be $\sigma-$ Cauchy if $\lim_{n,m\to\infty}\sigma(x_n,x_m)$ exists and is finite. The space $(X,\sigma)$ is called complete if for every $\sigma-$ Cauchy sequence in $\{x_n\}$, there exists $x \in X$ such that

$$\lim_{n\to\infty}\sigma(x_n,x) = \sigma(x,x) = \lim_{n,\, m\to\infty}\sigma(x_n,x_m).$$

**ii.** A sequence $\{x_n\}$ in $(X,\sigma)$ is said to be a $0-\sigma-$ Cauchy sequence if $\lim_{n,m\to\infty}\sigma(x_n,x_m) = 0$. The space $(X,\sigma)$ is said to be $0-\sigma-$ complete if every $0-\sigma-$ Cauchy sequence in $X$ converges (in $\tau_\sigma$) to a point $x \in X$ such that $\sigma(x,x) = 0$.

**iii.** A mapping $T : X \to X$ is continuous, if the following limits exist (finite) and

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(Tx, x).$$

Following Wardowski [11], we denote by $\mathcal{F}$ the family of all function, $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:
($F1$) $F$ is strictly increasing on $\mathbb{R}^+$,

($F2$) for every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$,

($F3$) there exists a number $k \in (0, 1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

**Example 2.22.** The following function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$:
**i.** $F(s) = \ln s$, with $s > 0$,

**ii.** $F(s) = \ln s + s$, with $s > 0$.

**Definition 2.23.** [11] Let $(X, d)$ be a metric space. A self-mapping $T$ on $X$ is called an $F$-contraction mapping if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\forall x, y \in X, \quad [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{2.1}$$

**Definition 2.24.** [12] Let $(X, \sigma)$ be a metric-like space. A mapping $T : X \to X$ is called a generalized Roger Hardy type $F-$ contraction mapping, if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\begin{aligned} \sigma(Tx, Ty) > 0 \Rightarrow \tau + F(\sigma(Tx, Ty)) \quad &\leq \quad F(\alpha\sigma(x, y) + \beta\sigma(x, Tx) + \gamma\sigma(y, Ty) \\ &\quad\quad + \eta\sigma(x, Ty) + \delta\sigma(y, Tx)) \end{aligned} \tag{2.2}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, \eta, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\eta + 2\delta < 1$.

**Theorem 2.25.** ref. [12] Let $(X, \sigma)$ be $0 - \sigma-$ complete metric-like spaces and $T : X \to X$ be a generalized Roger Hardy type $F-$ contraction. Then $T$ has a unique fixed point in $X$, either $T$ or $F$ is continuous.

*Proof.* See in [12]. □

### 2.4. Modular metric space

Let $X$ be a nonempty set. Throughout this paper, for a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, we write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y)$$

for all $\lambda > 0$ and $x, y \in X$.

**Definition 2.26** [13, 14] Let $X$ be a nonempty set. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is called a metric modular on $X$ if satisfying, for all $x, y, z \in X$ the following conditions hold:

**i.** $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$,

**ii.** $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$,

**iii.** $\omega_{\lambda+\mu}(x,y) \le \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$.

If instead of (i) we have only the condition (i')

$$\omega_\lambda(x,x) = 0 \text{ for all } \lambda > 0, x \in X,$$

then $\omega$ is said to be a pseudomodular (metric) on $X$. A modular metric $\omega$ on $X$ is said to be regular if the following weaker version of (i) is satisfied:

$x = y$ if and only if $\omega_\lambda(x,y) = 0$ for some $\lambda > 0$.

Note that for a metric (pseudo)modular $\omega$ on a set $X$, and any $x, y \in X$, the function $\lambda \mapsto \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

Note that every modular metric is regular but converse may not necessarily be true.

**Example 2.27.** Let $X = \mathbb{R}$ and $\omega$ is defined by $\omega_\lambda(x,y) = \infty$ if $\lambda < 1$, and $\omega_\lambda(x,y) = |x-y|$ if $\lambda \ge 1$, it is easy to verify that $\omega$ is regular modular metric but not modular metric.

**Definition 2.28.** [13, 14] Let $X_\omega$ be a (pseudo)modular on $X$. Fix $x_0 \in X$. The two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$$

are said to be modular spaces (around $x_0$).

Throughout this section we assume that $(X,\omega)$ is a modular metric space, $D$ be a nonempty subset of $X_\omega$ and $\mathcal{G}$:=$\{G_\omega$ is a directed graph with $V(G_\omega) = D$ and $\Delta \subseteq E(G_\omega)\}$.

**Definition 2.29.** [15, 16] The pair $(D, G_\omega)$ has Property (A) if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in $D$, with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G_\omega)$, then $(x_n, x) \in E(G_\omega)$, for all $n$.

**Definition 2.30.** ref. [17] Let $F \in \mathcal{F}$ and $G_\omega \in \mathcal{G}$. A mapping $T : D \to D$ is said to be $F$-$G_\omega$-contraction with respect to $R : D \to D$ if

**i.** $(Rx, Ry) \in E(G_\omega) \Rightarrow (Tx, Ty) \in E(G_\omega)$ for all $x, y \in D$, i.e. $T$ preserves edges w.r.t. $R$,

**ii.** there exists a number $\tau > 0$ such that

$$\omega_1(Tx,Ty) > 0 \Rightarrow \tau + F(\omega_1(Tx,Ty)) \le F(\omega_1(Rx,Ry))$$

for all $x, y \in D$ with $(Rx, Ry) \in E(G_\omega)$.

**Example 2.31.** ref. [17] Let $F \in \mathcal{F}$ be arbitrary. Then every $F$-contractive mapping w.r.t. $R$ is an $F$-$G_\omega$-contraction w.r.t. $R$ for the graph $G_\omega$ given by $V(G_\omega) = D$ and $E(G_\omega) = D \times D$.

We denote $C(T,R){:=}\{x \in D : Tx = Rx\}$ the set of all coincidence points of two self-mappings $T$ and $R$, defined on $D$.

**Theorem 2.32.** ref. [17] Let $(X,\omega)$ be a regular modular metric space with a graph $G_\omega$. Assume that $D = V(G_\omega)$ is a nonempty $\omega$-bounded subset of $X_\omega$ and the pair $(D, G_\omega)$ has property (A) and also satisfy $\Delta_M$-condition. Let $R, T : D \to D$ be two self mappings satisfying the following conditions:

**i.** there exists $x_0 \in D$ such that $(Rx_0, Tx_0) \in E(G_\omega)$,

**ii.** $T$ is an $F$-$G_\omega$-contraction w.r.t $R$,

**iii.** $T(D) \subseteq R(D)$,

**iv.** $R(D)$ is $\omega$ complete.

Then $C(R,T) \neq \emptyset$.

*Proof.* See in [17]. □

## 3. Fixed point approach to the solution of differential equations

Next, we will show a differential equation which solving by fixed point theorem in suitable spaces.

### 3.1. Ordinary differential equation

**Lemma 3.1.** ref. [18] $u$ is a solution of $u'(t) = f(t,u(t))$ satisfying the initial condition $u(t_0) = u_0$ if and only if $u(t) = u_0 + \int_{t_0}^{t} f(s,u(s))ds$.

*Proof.* Suppose that $u$ is a solution of $u'(t) = f(t,u(t))$ defined on an interval $I$ and satisfying $u(t_0) = u_0$. We integrate both sides of the equation $u'(t) = f(t,u(t))$ from $t_0$ to $t$, where $t$ is in $I$

$$\int_{t_0}^{t} u'(s)ds = \int_{t_0}^{t} f(s,u(s))ds$$

$$u(t) - u(t_0) = \int_{t_0}^{t} f(s,u(s))ds.$$

Since $u(t_0) = u_0$, we have

$$u(t) = u_0 + \int_{t_0}^{t} f(s,u(s))ds, \quad t \in I. \tag{3.1}$$

We will show that, conversely, any function which satisfies this integral equation satisfies both the differential equation and the initial condition. Suppose that $u$ is a function defined on an interval $I$ and satisfies (3.1). Setting $t = t_0$ yields $u(t_0) = u_0$, so that $u$ satisfies the initial condition. Next, we note that an integral is always a continuous function, so that a solution of (3.1) is automatically continuous. Since both $u$ and $f$ are continuous, it follows that the integrand $f(s, u(s))$ is continuous. We may therefore apply the fundamental theorem of calculus to (3.1) and conclude that $u$ is differentiable, and that is $u'(t) = f(t, u(t))$. □

The contraction mapping theorem may by used to prove the existence and uniqueness of the initial problem for ordinary differential equations. We consider a first-order of ODEs for a function $u(t)$ that take value in $\mathbb{R}^n$

$$u'(t) = f(t, u(t)) \tag{3.2}$$

$$u(t_0) = u_0. \tag{3.3}$$

The function $f(t, u(t))$ also take value in $\mathbb{R}^n$ and is assumed to be a continuous function of $t$ and a Lipschitz continuous function of $u$ on suitable domain.

**Definition 3.2.** Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ where $I$ is on interval in $\mathbb{R}$. We say that $f(t, u(t))$ is a globally Lipschitz continuous function of $u$ uniformly in t if there is a constant $C > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq C\|u - v\| \tag{3.4}$$

for all $x, y \in \mathbb{R}^n$ and all $t \in I$.

The initial value problem can be reformulated as an integral equation.

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds. \tag{3.5}$$

By the fundamental theorem of calculus, a continuous solution of (3.5) is a continuously differentiable solution of (3.2). Eq. (3.5) may by written as fixed point equation.

$$u = Tu$$

for the map $T$ defined by

$$Tu(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds.$$

**Theorem 3.3.** ref. [19] Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ where $I$ is on interval in $\mathbb{R}$ and $t_0$ is a point in the interior of $I$. If $f(t, u)$, is a continuous function of $(t, u)$ and a globally Lipschitz continuous function of $u$ uniformly in $t$ on $I \times \mathbb{R}^n$, then there is a unique continuously differentiable function $u : I \to \mathbb{R}^n$ that satisfies (3.2).

*Proof.* We will show that $T$ is a contraction on the space of continuous function defined on a time interval $t_0 \leqslant t \leqslant t_0 + \delta$, for sufficiently small $\delta$.

Suppose that $u, v : [t_0, t_0 + \delta] \to \mathbb{R}^n$ are two continuous function. Then, form (3.4), (3.5) we estimate,

$$\begin{aligned}
|Tu - Tv|_\infty &= \sup_{t_0 \leqslant t \leqslant t_0+\delta} |Tu(t) - Tv(t)| \\
&= \sup_{t_0 \leqslant t \leqslant t_0+\delta} |\int_{t_0}^{t} f(s,u(s)) - f(s,v(s))ds| \\
&\leq \sup_{t_0 \leqslant t \leqslant t_0+\delta} \int_{t_0}^{t} |f(s,u(s)) - f(s,v(s))|ds \\
&\leq \sup_{t_0 \leqslant t \leqslant t_0+\delta} C|u(s) - v(s)| \int_{t_0}^{t} ds \\
&\leq C\delta|u - v|_\infty.
\end{aligned}$$

If follow that if $\delta \leq \frac{1}{c}$ then $T$ is contraction on $C([t_0, t_0 + \delta])$. Therefore, there is a unique solution $u : [t_0, t_0 + \delta] \to \mathbb{R}^n$.

Let $f(x,y)$ be a continuous real-valued function on $[a,b] \times [c,d]$. The Cauchy initial value problem is to find a continuous differentiable function $y$ on $[a,b]$ satisfying the differential equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{3.6}$$

Consider the Banach space $C[a,b]$ of continuous real-valued functions with supremum norm defined by $\|y\| = \sup\{y(x)| : x \in [a,b]\}$.

Integrating (3.6), that yield an integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt. \tag{3.7}$$

The problem (3.6) is equivalent the problem solving the integral Eq. (3.7).

We define an integral operator $T : C[a,b] \to C[a,b]$ by

$$Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Therefore, a solution of Cauchy initial value problem (3.6) corresponds with a fixed point of $T$. One may easily check that if $T$ is contraction, then the problem (3.6) has a unique solution.

**Theorem 3.4.** ref. [20] Let $f(x,y)$ be a continuous function of $Dom(f) = [a,b] \times [c,d]$ such that $f$ is Lipschitzian with respect to $y$, i.e., there exists $k > 0$ such that

$$|f(x,u) - f(x,v)| \leq k|u - v| \text{ for all } u, v \in [c,d] \text{ and for } x \in [a,b].$$

Suppose $(x_0, y_0) \in \mathrm{int}(Dom(f))$. Then for sufficiently small $h > 0$, there exists a unique solution of the problem (3.6).

*Proof.* Let $M = \sup\{|f(x,y)| : x, y \in Dom(f)\}$ and choose $h > 0$ such that

$$C:=\{y\in C[x_0-h,x_0+h]:|y(x)-y_0|\le Mh\}.$$

Then $C$ is a closed subset of the complete metric space $C[x_0-h,x_0+h]$ and hence $C$ is complete. Note $T:C\to C$ is a contraction mapping. Indeed, for $x\in[x_0-h,x_0+h]$ and two continuous functions $y_1,y_2\in C$, we have

$$\begin{aligned}\|Ty_1-Ty_2\| &= \|\textstyle\int_{x_0}^{x} f(x,y_1)-f(x,y_2)dt\|\\ &\le |x-x_0| \sup_{s\in[x_0-h,x_0+h]} k|y_1(s)-y_2(s)|\\ &\le kh\|y_1-y_2\|.\end{aligned}$$

Therefore, $T$ has a unique fixed point implying that the problem (3.6) has a unique fixed point.

### 3.2. Ordinary fuzzy differential equation

Now, we consider the existence of solution for the second order nonlinear boundary value problem:

$$\begin{cases} x''(t)=k(t,x(t),x'(t)), & t\in[0,\Lambda],\ \ \Lambda>0,\\ x(t_1)=x_1, & \\ x(t_2)=x_2, & t_1,t_2\in[0,\Lambda,] \end{cases} \tag{3.8}$$

where $k:[0,\Lambda]\times W(X)\times W(X)\to W(X)$ is a continuous function. This problem is equivalent to the integral equation

$$x(t)=\int_{t_1}^{t_2} G(t,s)k(s,x(s),x'(s))ds+\beta(t),\quad t\in[0,\Lambda], \tag{3.9}$$

where the Green's function $G$ is given by

$$G(t,s)=\begin{cases}\dfrac{(t_2-t)(s-t_1)}{t_2-t_1} & \text{if } t_1\le s\le t\le t_2,\\ \dfrac{(t_2-s)(t-t_1)}{t_2-t_1} & \text{if } t_1\le t\le s\le t_2,\end{cases}$$

and $\beta(t)$ satisfies $\beta''=0,\beta(t_1)=x_1,\beta(t_2)=x_2$. Let us recall some properties of $G(t,s)$, precisely we have

$$\int_{t_1}^{t_2}|G(t,s)|ds\le\frac{(t_2-t_1)^2}{8}$$

and

$$\int_{t_1}^{t_2}|G_t(t,s)|ds\le\frac{(t_2-t_1)}{2}.$$

If necessary, for a more detailed explanation of the background of the problem, the reader can refer to the reference [21, 22]. Here, we will prove our results, by establishing the existence of a common fixed point for pair of integral operators defined as

$$T_i(x)(t) = \int_{t_1}^{t_2} G(t,s)k_i(s,x(s),x'(s))ds + \beta(t), \quad t \in [0,\Lambda], \ \ i \in \{1,2\} \tag{3.10}$$

where $k_1, k_2 \in C([0,\Lambda] \times W(X) \times W(X), W(X))$, $x \in C^1([0,\Lambda], W(X))$, and $\beta \in C([0,\Lambda], W(X))$.

**Theorem 3.5** ref. [6] *Assume that the following conditions are satisfied:*

**i.** $k_1, k_2 : [0,\Lambda] \times W(X) \times W(X) \to W(X)$ *are increasing in its second and third variables,*

**ii.** *there exists* $x_0 \in C^1([0,\Lambda], W(X))$ *such that, for all* $t \in [0,\Lambda]$, *we have*

$$x_0(t) \le \int_{t_1}^{t_2} G(t,s)k_1\big(t, x_0(s), x_0'(s)\big)ds + \beta(t),$$

where $t_1, t_2 \in [0,\Lambda]$,

**iii.** *there exist constants* $\gamma, \delta > 0$ *such that, for all* $t \in [0,\Lambda]$, *we have*

$$|k_1(t,x(t),x'(t)) - k_2(t,y(t),y'(t))| \le \gamma|x(t) - y(t)| + \delta|x'(t) - y'(t)|$$

for all comparable $x, y \in C^1([0,\Lambda], W(X))$,

**iv.** *for* $\gamma, \delta > 0$ *and* $t_1, t_2 \in [0,\Lambda]$ *we have*

$$\gamma\frac{(t_2-t_1)^2}{8} + \delta\frac{(t_2-t_1)}{2} < 1,$$

**v.** *if* $x, y \in C^1([0,\Lambda], W(X))$ *are comparable, then every* $u \in (T_1x)_1$ *and every* $v \in (T_2y)_1$ *are comparable.*

*Then the pair of nonlinear integral equations*

$$x(t) = \int_{t_1}^{t_2} G(t,s)k_i(s,x(s),x'(s))ds + \beta(t) \quad t \in [0,\Lambda], \ \ i \in \{1,2\} \tag{3.11}$$

has a common solution in $C^1([t_1,t_2], W(X))$.

*Proof.* Consider $\mathcal{C} = C^1([t_1,t_2], W(X))$ with the metric

$$D(x,y) = \max_{t_1 \le t \le t_2} (\gamma|x(t) - y(t)| + \delta|x'(t) - y'(t)|).$$

The $(\mathcal{C}, D)$ is a complete metric space, which can also be equipped with the partial ordering given by

$$x, y \in \mathcal{C}, \quad \Leftrightarrow x(t) \le y(t) \ \text{ for all } \ t \in [0,\Lambda].$$

In [23], it is proved that $(\mathcal{C}, \preccurlyeq)$ satisfies the following condition:

(r) for every nondecreasing sequence $\{x_n\}$ in $\mathcal{C}$ convergent to some $x \in \mathcal{C},$ we have $x_n \preccurlyeq x$ for all $n \in \mathbb{N} \cup \{0\}$.

Let $T_1, T_2 : \mathcal{C} \to \mathcal{C}$ be two integral operators defined by (3.10); clearly, $T_1, T - 2$ are well defined since $k_1, k_2,$ and $\beta$ are continuous functions. Now, $x^*$ is a solution of (3.9) if and only if $x^*$ is a common fixed point of $T_1$ and $T_2$.

By hypothesis (a), $T_1, T_2$ are increasing and, by hypothesis (b), $x_0 \preccurlyeq T_1(x_0)$. Consequently, in view of condition (r), hypothesis (i)-(iii) of Corollary 2.16 hold true.

Next, for all comparable $x, y \in \mathcal{C},$ From hypothesis (c) we obtain successively

$$\begin{aligned} |T_1(x)(t) - T_2(y)(t)| &\leq \int_{t_1} t_2 |G(t,s)||k_1(s, x(s), x'(s)) - k_1(s, y(s), y'(s))| ds \\ &\leq D(x,y) \int_{t_1}^{t_2} |G(t,s)| ds \\ &\leq \frac{(t_2 - t_1)^2}{8} D(x,y) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} |(T_1(x))^{'}(t) - (T_2(y))^{'}(t)| &\leq \int_{t_1} t_2 |G_t(t,s)||k_1\left(s, x(s), x^{'}(s)\right) - k_1\left(s, y(s), y^{'}(s)\right)| ds \\ &\leq D(x,y) \int_{t_1}^{t_2} |G_t(t,s)| ds \\ &\leq \frac{(t_2 - t_1)}{2} D(x,y). \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we obtain easily

$$D(T_1 x, T_2 y) \leq \left( \gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} \right) D(x,y).$$

Consequently, in view of hypothesis (d), the contractive condition (5) is satisfied with

$$q = \gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} < 1.$$

Therefore, Corollary 2.16 applied to $T_1$ and $T_2,$ which have common fixed point $x^* \in \mathcal{C},$ that is, $x^*$ is a common solution of (3.9). □

### 3.3. Second-order differential equation

Now, we consider the boundary value problem for second order differential equation

$$\begin{cases} x''(t) = -f(t, x(t)), & t \in I, \\ x(0) = x(1) = 0, \end{cases} \tag{3.14}$$

where $I = [0, 1]$ and $f : I \times \mathbb{R} \to \mathbb{R}$. is a continuous function.

It is known, and easy to check, that the problem (3.14) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t, s) f(s, x(s)) ds, \quad \text{for} \quad t \in I, \tag{3.15}$$

where $G$ is the Green's function define by

$$G(t, s) = \begin{cases} t(1 - s) & \text{if} \quad 0 \leq t \leq s \leq 1 \\ s(1 - t) & \text{if} \quad 0 \leq s \leq t \leq 1. \end{cases}$$

That is, if $x \in C^2(I, \mathbb{R})$, then $x$ is a solution of problem (3.14) iff $x$ is a solution of the integral Eq. (3.15).

Let $X = C(I)$ be the space of all continuous functions defined on $I$. Consider the metric-like $\sigma$ on $X$ define by

$$\sigma(x, y) = \|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty \text{ for all } x, y \in X,$$

where $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$ for all $u \in X$.

Note that $\sigma$ is also a partial metric on $X$ and since

$$d_\sigma(x, y) := 2\sigma(x, y) - \sigma(x, x) - \sigma(y, y) = 2\|x - y\|_\infty.$$

By Lemma 2.20, hence $(X, \sigma)$ is complete since the metric space $(X, \|\cdot\|)$ is complete.

**Theorem 3.6.** ref. [12] Suppose the following conditions:

**i.** there exist continuous functions $p : I \to \mathbb{R}^+$ such that

$$|f(s, a) - f(s, b)| \leq 8p(s)|a - b|$$

for all $s \in I$ and $a, b \in \mathbb{R}$;

**ii.** there exist continuous functions $q : I \to \mathbb{R}^+$ such that

$$|f(s, a)| \leq 8q(s)|a|$$

for all $s \in I$ and $a \in \mathbb{R}$;

**iii.** $\max_{s \in I} p(s) = \alpha\lambda_1 < \frac{1}{49}$, which is $0 \leq \alpha < \frac{1}{7}$;

**iv.** $\max_{s \in I} q(s) = \alpha\lambda_2 < \frac{1}{49}$ which is $0 \leq \alpha < \frac{1}{7}$.

Then problem (3.14) has a unique solution $u \in X = C(I, \mathbb{R})$.

*Proof.* Define the mapping $T: X \to X$ by

$$Tx(t) = \int_0^1 G(t,s)f(s,x(s))ds,$$

for all $x \in X$ and $t \in T$. Then the problem (3.14) is equivalent to finding a fixed point $u$ of $T$ in $X$. Let $x, y \in X$, we obtain

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left|\int_0^1 G(t,s)f(s,x(s))ds - \int_0^1 G(t,s)f(s,y(s))ds\right| \\ &\leq \int_0^1 G(t,s)|f(s,x(s)) - f(s,y(s)|ds \\ &\leq 8\int_0^1 G(t,s)p(s)|x(s) - y(s)|ds \\ &\leq 8\alpha\lambda_1\|x-y\|_\infty \int_0^1 G(t,s)ds \\ &= \alpha\lambda_1\|x-y\|_\infty. \end{aligned}$$

In the above equality, we used that for each $t \in I$, we have $\int_0^1 G(t,s)ds = \frac{t}{2}(1-t)$ and so $\sup_{t\in I}\int_0^1 G(t,s)ds = \frac{1}{8}$. Therefore,

$$\|Tx - Ty\|_\infty \leq \alpha\lambda_1\|x-y\|_\infty. \tag{3.16}$$

Moreover, we have

$$\begin{aligned} Tx(t) &= \left|\int_0^1 G(t,s)f(s,x(s))ds\right| \\ &\leq 8\int_0^1 G(t,s)q(s)|x(s)|ds \\ &\leq 8\alpha\lambda_2\|x\|_\infty. \end{aligned}$$

Hence,

$$\|Tx\|_\infty \leq \alpha\lambda_2\|x\|_\infty. \tag{3.17}$$

Similar method, we obtain

$$\|Ty\|_\infty \leq \alpha\lambda_2\|y\|_\infty. \tag{3.18}$$

Let $e^{-\tau} = \lambda_1 + 2\lambda_2 < 1$ where $\tau > 0$. Form (3.16), (3.17) and (3.18), we obtain

$$\begin{aligned} \sigma(Tx,Ty) &= |Tx - Ty|_\infty + |Tx|_\infty + |Ty|_\infty \\ &\leq \alpha\lambda_1|x-y|_\infty + \alpha\lambda_2|x|_\infty + \alpha\lambda_2|y|_\infty \\ &\leq (\lambda_1 + 2\lambda_2)[(\alpha)(|Tx-Ty|_\infty + |Tx|_\infty + |Ty|_\infty)] \\ &= (e^{-\tau})\alpha\sigma(x,y). \end{aligned} \tag{3.19}$$

Let $\beta, \gamma, \eta, \delta > 0$ where $\beta < \frac{1}{7}, \gamma < \frac{1}{7}, \eta < \frac{1}{7}, \delta < \frac{1}{7}$. It following (3.19), we get

$$\sigma(Tx, Ty) \leq (e^{-\tau})\left[\alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \eta\sigma(x,Ty) + \delta\sigma(y,Tx)\right], \tag{3.20}$$

where $\alpha + \beta + \gamma + 2\eta + 2\delta < 1$. Taking the function $F : \mathbb{R}^+ \to \mathbb{R}$ in (3.20), where $F(t) = \ln(t)$, which is $F \in \mathcal{F}$, we get

$$\tau + F(\sigma(Tx,Ty)) \leq F\big(\alpha\sigma(x,y) + \beta\sigma(x,Tx) + \gamma\sigma(y,Ty) + \eta\sigma(x,Ty) + \delta\sigma(y,Tx)\big).$$

Therefore all hypothesis of Theorem (2.25) are satisfied, and so $T$ has a unique fixed point $u \in X$, that is, the problem (3.14) has a unique solution $u \in C^2(I)$. □

### 3.4. Partial differential equation

Consider the Laplace operator is a second order differential operator in the n-dimensional Euclidean space, defined as the divergence $(\nabla\cdot)$ of the gradient $(\nabla f)$. Thus if $f$ is a twice-differentiable real-valued function, then the Laplacian of $f$ is defined by

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f \tag{3.21}$$

where the latter notations derive from formally writing $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$. Equivalently, the Laplacian of $f$ the sum of all the unmixed

$$\Delta f = \sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i^2}. \tag{3.22}$$

As a second-order differential operator, the Laplace operator maps $C^k$ functions to $C^{k-2}$ functions for $k \geq 2$. the expression (3.21)(or equivalently(3.22)) defines an operator $\Delta : C^{(k)}(\mathbb{R}^n) \to C^{(k-2)}(\mathbb{R}^n)$ or more generator $\Delta : C^{(k)}(\Omega) \to C^{(k-2)}(\Omega)$ for any open set $\Omega$ Consider semilinear elliptic equation. Look for a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ that solves

$$-\Delta u = f(u) \ \text{ in } \ \Omega \tag{3.23}$$

$$u = u_0 \quad \text{on} \quad \partial\Omega \tag{3.24}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a typically nonlinear function. Equivalently look for a fixed point of $Tu{:=}(-\Delta u_0)^{-1}(f(u))$.

**Theorem 3.7.** ref. [5] Let $f \in C(\mathbb{R})$ and $\sup_{x\in\mathbb{R}}|f(x)| = a < \infty$. then (3.23) has a weak solution $u \in H_0^1(\Omega)$, i.e.

$$\int_\Omega \nabla u \cdot \nabla \Phi dx = \int_\Omega f(u)\Phi dx, \qquad \forall \Phi \in C_0^\infty(\Omega).$$

*Proof.* Our strategy is to apply Schauder's Fixed Point Theorem to the map

$$T : L^2(\Omega) \to L^2(\Omega)$$
$$u \mapsto (-\Delta)^{-1}(f(u)),$$

where $T$ is continuous. Lemma (2.9) show that $u \to f(u)$ is continuous form $L^2(\Omega)$ into itself. Corollary (2.10) shows that $(-\Delta)^{-1}$ is continuous form $L^2(\Omega)$ into $H_0^1(\Omega)$, which is continuously embedded in $L^2(\Omega)$. Find a closed, non-empty bounded convex set such that $T : M \to M$. Given $u \in L^2(\Omega)$, $Tu$ satisfies

$$\int_\Omega \nabla Tu \cdot \nabla Tudx = \int_\Omega f(u)Tudx \le a|\Omega|\|Tu\|_{L^2(\Omega)} \tag{3.25}$$

Cauchy-Schwarz. $T$ here fore, using Ponincare's inequality

$$\|Tu\|^2_{L^2(\Omega)} \le C(\Omega)\|Tu\|^2_{L^2(\Omega)} \le a|\Omega|\|Tu\|^2_{L^2(\Omega)}.$$

Thus if we set $R = a|\Omega|C(\Omega)$ and choose $M = \left\{u : \quad \|u\|^2_{L^2(\Omega)} \le R\right\}$. We have established that $T : M \to M$, $T$ is compact. Using Poincare's inequality on the right-hand-side in (3.25), we obtain. $\|\nabla Tu\|^2_{L^2(\Omega)} \le R\|\nabla Tu\|_{L^2(\Omega)}$. Thus $T(M) \subset \left\{u : \quad \|u\|_{H^1(\Omega)} \le R\right\}$, and since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, $T$ is compact. □

### 3.5. A non-homogeneous linear parabolic partial differential equation

We consider the following initial value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t)), & -\infty < x < \infty, 0 < t \le T, \\ u(x,0) = \varphi(x) \ge 0, & -\infty < x < \infty, \end{cases} \tag{3.26}$$

where $H$ is continuous and $\varphi$ assume to be continuously differentiable such that $\varphi$ and $\varphi'$ are bounded.

By a *solution* of the problem (3.26), we mean a function $u \equiv u(x,t)$ defined on $\mathbb{R} \times I$, where $I$:=$[0,T]$, satisfying the following conditions:

**i.** $u, u_t, u_x, u_{xx} \in C(\mathbb{R} \times I)$. $\{C(\mathbb{R} \times I)$ denote the space of all continuous real valued functions$\}$,

**ii.** $u$ and $u_x$ are bounded in $\mathbb{R} \times I$,

**iii.** $u_t(x,t) = u_{xx}(x,t) + H(x,t,u(x,t),u_x(x,t))$ for all $(x,t) \in \mathbb{R} \times I$,

**iv.** $u(x,0) = \varphi(x)$ for all $x \in \mathbb{R}$.

It is important to note that the differential equation problem (3.26) is equivalent to the following integral equation problem

$$u(x,t) = \int_{-\infty}^{\infty} k(x-\xi,t)\varphi(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} k(x-\xi,t-\tau)H(\xi,\tau,u(\xi,\tau),u_x(\xi,\tau))d\xi d\tau \tag{3.27}$$

for all $x \in \mathbb{R}$ and $0 < t \leq T$, where

$$k(x,t) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

The problem (3.26) admits a solution if and only if the corresponding problem (3.27) has a solution.

Let

$$\Omega := \{u(x,t); u, u_x \in C(\mathbb{R} \times I) \text{ and } \|u\| < \infty\},$$

where

$$\|u\| := \sup_{x \in \mathbb{R},\, t \in I} |u(x,t)| + \sup_{x \in \mathbb{R},\, t \in I} |u_x(x,t)|.$$

Obviously, the function $\omega : \mathbb{R}^+ \times \Omega \times \Omega \to \mathbb{R}_+$ given by

$$\omega_\lambda(u,v) := \frac{1}{\lambda} \|u - v\| = \frac{1}{\lambda} d(u,v)$$

is a metric modular on $\Omega$. Clearly, the set $\Omega_\omega$ is a complete modular metric space independent of generators.

**Theorem 3.8.** ref. [17] Consider the problem (3.26) and assume the following:

**i.** for $c > 0$ with $|s| < c$ and $|p| < c$, the function $F(x,t,s,p)$ is uniformly Hölder continuous in $x$ and $t$ for each compact subset of $\mathbb{R} \times I$,

**ii.** there exists a constant $c_H \leq \left(T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)^{-1} \leq q$, where $q \in (0,1)$ such that

$$0 \leq \frac{1}{\lambda}\left[H(x,t,s_2,p_2) - H(x,t,s_1,p_1)\right]$$

$$\leq c_H \left[\frac{s_2 - s_1 + p_2 - p_1}{\lambda}\right]$$

for all $(s_1,p_1), (s_2,p_2) \in \mathbb{R} \times \mathbb{R}$ with $s_1 \leq s_2$ and $p_1 \leq p_2$,

**iii.** $H$ is bounded for bounded $s$ and $p$.

Then the problem (3.26) admits a solution.

*Proof.* It is well known that $u \in \Omega_\omega$ is a solution (3.26) iff $u \in \Omega_\omega$ is a solution integral Eq. (3.27).

Consider the graph $G$ with $V(G) = D = \Omega_\omega$ and $E(G) = \{(u, v) \in D \times D : u(x, t) \leq v(x, t)$ and $u_x(x, t) \leq v_x(x, t)$ at each $(x, t) \in \mathbb{R} \times I\}$. Clearly $E(G)$ is partial ordered and $(D, E(G))$ satisfy property (A).

Also, define a mapping $\Lambda : \Omega_\omega \to \Omega_\omega$ by

$$(\Lambda u)(x, t) := \int_{-\infty}^{\infty} k(x - \xi, t)\varphi(\xi) d\xi + \int_0^t \int_{-\infty}^{\infty} k(x - \xi, t - \tau) H(\xi, \tau, u(\xi, \tau), u_x(\xi, \tau)) d\xi d\tau$$

for all $(x, t) \in \mathbb{R} \times I$. Then, finding solution of problem (3.27) is equivalent to the ensuring the existence of fixed point of $\Lambda$.

Since $(u, v) \in E(G)$, $(u_x, v_x) \in E(G)$ and hence $(\Lambda u, \Lambda v) \in E(G)$, $(\Lambda u_x, \Lambda v_x) \in E(G)$.

Thus, from the definition of $\Lambda$ and by (ii) we have

$$\begin{aligned} &\frac{1}{\lambda}|(\Lambda v)(x, t) - (\Lambda u)(x, t)| \\ &\leq \frac{1}{\lambda} \int_0^t \int_{-\infty}^{\infty} k(x - \xi, t - \tau) |H(\xi, \tau, v(\xi, \tau), v_x(\xi, \tau)) - H(\xi, \tau, u(\xi, \tau), u_x(\xi, \tau))| d\xi d\tau \\ &\leq \int_0^t \int_{-\infty}^{\infty} k(x - \xi, t - \tau) c_H \left[ \frac{1}{\lambda} |(v(\xi, \tau) - u(\xi, \tau) + v_x(\xi, \tau) - u_x(\xi, \tau))| \right] d\xi d\tau \\ &\leq c_H \omega_\lambda(u, v) T. \end{aligned} \tag{3.28}$$

Similarly, we have

$$\begin{aligned} \frac{1}{\lambda}|(\Lambda v)_x(x, t) - (\Lambda u)_x(x, t)| &\leq c_H \omega_\lambda(u, v) \int_0^t \int_{-\infty}^{\infty} |k_x(x - \xi, t - \tau)| d\xi d\tau \\ &\leq 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}} c_H \omega_\lambda(u, v). \end{aligned} \tag{3.29}$$

Therefore, from (3.28) and (3.29) we have

$$\omega_\lambda(\Lambda u, \Lambda v) \leq \left( T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}} \right) c_H \omega_\lambda(u, v)$$

i.e.

$$\omega_\lambda(\Lambda u, \Lambda v) \leq q \omega_\lambda(u, v), \quad q \in (0, 1)$$

i.e.

$$d(\Lambda u, \Lambda v) \leq e^{-\tau} d(u, v), \tau > 0.$$

Now, by passing to logarithms, we can write this as

$$\ln(d(\Lambda u, \Lambda v)) \leq \ln(e^{-\tau} d(u, v))$$

$$\tau + \ln(d(\Lambda u, \Lambda v)) \leq \ln(d(u, v)).$$

Now, from example 2.22 (i) and taking $T = \Lambda$ and $R = \mathcal{I}$ (Identity map), we deduce that the operator T satisfies all the hypothesis of theorem 2.32.

Therefore, as an application of theorem 2.32 we conclude the existence of $u^* \in \Omega_\omega$ such that $u^* = \Lambda u^*$ and so $u^*$ is a solution of the problem 3.26.

### 3.6. Fractional differential equation

Before we will discuss the source of fractional differential equation.

*Cauchy's formula for repeated integration.* Let $f$ be a continuous function on the real line. Then the $n_{th}$ repeated integral of $f$ based at $a$,

$$f^{(-n)}(x) = \int_a^x \int_a^{\sigma_1} \int_a^{\sigma_2} \dots \int_a^{\sigma_{n-1}} f(\sigma_n) d\sigma_n \dots d\sigma_3 d\sigma_2 d\sigma_1$$

is given by single integration

$$f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt.$$

A proof is given by mathematical induction. Since $f$ is continuous, the base case follows from the fundamental theorem of calculus.

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

where

$$f^{-1}(a) = \int_a^a f(t) dt = 0.$$

Now, suppose this is true for n, and let us prove it for $n+1$.

Firstly, using the Leibniz integral rule. Then applying the induction hypothesis

$$\begin{aligned} f^{(-n+1)}(x) &= \int_a^x \int_a^{\sigma_1} \int_a^{\sigma_2} \dots \int_a^{\sigma_n} f(\sigma_{n+1}) d\sigma_n \dots d\sigma_3 d\sigma_2 d\sigma_1 \\ &= \int_a^x \frac{1}{(n-1)!} \int_a^{\sigma_1} (\sigma_1 - t)^{n-1} f(t) dt d\sigma_1 \\ &= \int_a^x \frac{d}{d\sigma_1} \left[ \frac{1}{n!} \int_a^{\sigma_1} (\sigma_1 - t)^n f(t) dt \right] d\sigma_1 \\ &= \frac{1}{n!} \int_a^x (x-t)^n f(t) dt. \end{aligned}$$

This completes the proof. In fractional calculus, this formula can be used to construct a notion of differintegral, allowing one to differentiate or integrate a fractional number of time.

Integrating a fractional number of time with this formula is straightforward, one can use fractional $n$ by interpreting $(n-1)!$ as $\Gamma(n)$, that is the Riemann-Liouville integral which is defined by

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}dt.$$

This also makes sense if $a = -\infty$, with suitable restriction on $f$. The fundamental relation hold

$$\frac{d}{dx}I^{\alpha+1}f(x) = I^{\alpha}f(x)$$
$$I^{\alpha}\left(I^{\beta}f\right) = I^{\alpha+\beta}f(x)$$

the latter of which is semigroup properties. These properties make possible not only the definition of fractional differentiation by taking enough derivative of $I^{\alpha}f$. One can define fractional-order derivative of as well by

$$\frac{d^{\alpha}}{dx^{\alpha}}f = \frac{d^{[\alpha]}}{dx^{[\alpha]}}I^{[\alpha]-\alpha}f$$

where $[\cdot]$ denote the ceilling function. One also obtains a differintegral interpolation between differential and integration by defining

$$D_x^{\alpha}f(x) = \begin{cases} \dfrac{d^{[\alpha]}}{dx^{[\alpha]}}I^{[\alpha]-\alpha}f(x) & \text{if } \alpha > 0 \\ f(x) & \text{if } \alpha = 0 \\ I^{-\alpha}f(x) & \text{if } \alpha < 0. \end{cases}$$

An alternative fractional derivative was introduced by Caputo in 1967, and produce a derivative that has different properties it produces zero from constant function and more importantly the initial value terms of the Laplace Transform are expressed by means of the value of that function and of its derivative of integer order rather than the derivative of fractional order as in the Riemann-Liouville derivative. The Caputo fractional derivative with base point $x$ is then

$${}^{c}D_x^{\alpha}f(x) = I^{[\alpha]-\alpha}\frac{d^{[\alpha]}}{dx^{[\alpha]}}f(x).$$

**Lemma 3.9.** ref. [24] Let $u : [0, \infty] \to X$ be continuous function such that $u \in C([0, \tau], X)$ for all $\tau > 0$. Then $u$ is a global solution of

$${}^{c}D_t^{\alpha}u(t) = Bu(t); t > 0 \tag{3.30}$$

$$u(0) = u_0 \in X \tag{3.31}$$

if and only if $u$ the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s)ds, \, t \geq 0.$$

*Proof.* ($\Rightarrow$) Let $\tau > 0$. Since $u$ is a global solution of (3.30), then $u \in C([0,\tau], X)$, ${}^cD_t^\alpha u \in C([0,\tau], X)$ andt

$${}^cD_t^\alpha u(t) = Bu(t), \quad t \in (0,\tau].$$

Thus, by applying $I_t^\alpha$ in both sides of the equality (since ${}^cD_t^\alpha u \in L^1(0,\tau; X)$) we obtain

$$u(t) = u(0) + I_t^\alpha Bu(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s)ds, \, t \geq 0.$$

Since $\tau > 0$ was an arbitrary choice, $u$ satisfies the integral equation for all $t \geq 0$, as we wish.

($\Leftarrow$) On the other hand, choose $\tau > 0$ (but arbitrary). By hypothesis, $u \in C([0,\tau], X)$, and satisfies the integral equation,

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu(s)ds, \quad t \in [0,\tau].$$

Observing also $u(0) = u_0$ and rewriting the equality above, we obtain

$$u(t) = u(0) + I_t^\alpha Bu(s), \quad t \in [0,\tau].$$

Since $Bu(s) \in C([0,\tau], X)$, we conclude, by ${}^cD_t^\alpha I_t^\alpha f(t) = f(t)$ of the fractional integral and derivative property that we can apply ${}^cD_t^\alpha$ in both sides of the integral equation, obtaining

$${}^cD_t^\alpha u(t) = Bu(t), \quad t \in [0,\tau]$$

what lead us to verify that ${}^cD_t^\alpha u \in C([0,\tau], X)$. Since $\tau > 0$ was an arbitrary choice, we conclude that the function $u$ is a global solution of (3.30). □

**Theorem 3.10.** ref. [24] Let $\alpha \in (0,1)$, $B \in L(X)$ and $u_0 \in X$ then the problem (3.30).

have a unique global solution.

*Proof.* Choose $\tau > 0$. then consider $K_\tau = u \in C([0,\tau], X); u(0) = u_0$ and operator.

$T : K_\tau \to K_\tau$ given by

$$T(u(t)) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t Bu(t)dt.$$

We will show that a power (with respect to be composition) of this operator is a contraction and therefore by Banach's Fixed Point Theorem, $T$ have a unique fixed point in $K_\tau$ to this end, observe that for any $u, v \in K_\tau$

$$\begin{aligned}
\|T(u(t)) - T(v(t))\| &= \|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}B(u(s)-v(s))ds\| \\
&\leq \|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}\|B\|_{L(X)}\|u(s)-v(s))\|ds \\
&\leq \frac{\|B\|_{L(X)}}{\Gamma(\alpha)}\|u(s)-v(s))\|\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{t^\alpha\|B\|_{L(X)}}{\alpha\Gamma(\alpha)}\|u(s)-v(s))\| \\
&\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\|u(s)-v(s))\| \\
&\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\sup_{0\leqslant s\leqslant \tau}\|u(s)-v(s))\|.
\end{aligned}$$

By iterating this relation, we find that

$$\begin{aligned}
\|T^2(u(t)) - T^2(v(t))\| &\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\sup_{0\leqslant s\leqslant \tau}\|Tu(s)-Tv(s))\| \\
&\leq \frac{t^2\alpha}{\Gamma^2(\alpha+1)}\|B\|^2_{L(X)}\sup_{0\leqslant s\leqslant \tau}\|u(s)-v(s))\| \\
\|T^3(u(t)) - T^3(v(t))\| &\leq \frac{t^\alpha}{\Gamma(\alpha+1)}\|B\|_{L(X)}\sup_{0\leqslant s\leqslant \tau}\|T^2u(s)-T^2v(s))\| \\
&\leq \frac{t^3\alpha}{\Gamma^3(\alpha+1)}\|B\|^3_{L(X)}\sup_{0\leqslant s\leqslant \tau}\|u(s)-v(s))\| \\
&\leq \cdots \\
\|T^n(u(t)) - T^n(v(t))\| &\leq \frac{t^n\alpha}{\Gamma^n(\alpha+1)}\|B\|^n_{L(X)}\sup_{0\leqslant s\leqslant \tau}\|u(s)-v(s))\|
\end{aligned}$$

and for an sufficiently large $n$,the constant in question is less than 1, i.e., there exists a fixed point $u\in K_\tau$. Observe now that $\tau > 0$ was an arbitrary choice, so we conclude that the fixed point $u\in C([0,\tau],X)$ for all $\tau > 0$ and Lemma (3.9), we obtain the existence and uniqueness of a global solution to the problem (3.30). □

**Corollary 3.11.** ref. [24] Consider the same hypothesis of theorem (3.10).

**i.** Let $\{U_n(t)\}|_{n=0}^{\infty}$ be a sequence of continuous functions $U_n : [0,\infty) \to X$ given by $U_0(t) = u_0$, $U_n = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}BU_{n-1}(s)ds$, $n\in\{1,2,\ldots\}$.

Then there exists a continuous function $U : [0,\infty) \to X$, such that for any $\tau > 0$, we conclude that $U_n \to U$ in $C([0,\tau],X)$. Moreover, $U(t)$ is the unique global solution of (3.30).

**ii.** It holds that

$$U(t) = \sum_{k=0}^{\infty}\frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)}.$$

*Proof.* (i) It follows directly from proof of Theorem (3.10).

(ii) It is trivial that $U_0(t) = u_0$. So we compute, using the gamma function properties, that

$$U_1(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Bu_0(s)ds = u_0 + \frac{t^\alpha Bu_0}{\alpha\Gamma(\alpha)} = u_0 + \frac{t^\alpha Bu_0}{\Gamma(\alpha+1)}.$$

By a simple induction process, we conclude that

$$U_n(t) = \sum_{k=0}^{n} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)}$$

and therefore

$$U(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{(t^\alpha B)^k u_0}{\Gamma(\alpha k+1)} := E_\alpha(t^\alpha B)u_0.$$

□

From the above works, we can see a fact, although the fractional boundary value problems have been studied, to the best of our knowledge, there have been a few works using the lower and upper solution method. However, only positive solution are useful for many application, motivated by the above works, we study the existence and uniqueness of positive solution of the following integral boundary value problem.

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2 \tag{3.32}$$

$$u(0) = 0, \quad u(1) = \int_0^1 u(s)ds, \tag{3.33}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function and $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative.

We need the following lemmas that will be used to prove our main results.

**Lemma 3.12.** ref. [25] Let $\alpha > 0$ and $u \in C(0,1) \cap L(0,1)$. Then fractional differential equation

$$D_{0+}^{\alpha} u(t) = 0$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{3.34}$$

$C_i \in \mathbb{R}, \quad i = 1, 2, \cdots, N, \quad N = [\alpha] + 1$ as unique solution.

**Lemma 3.13.** ref. [25] Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2} - \cdots - C_N t^{\alpha-N} \tag{3.35}$$

for some $C_i \in \mathbb{R}, i = 1, 2, \cdots, N, \quad N = [\alpha] + 1$.

In the following, we present the Green function of fractional differential equation with integral boundary value condition.

**Theorem 3.14.** ref. [26] Let $1 < \alpha < 2$, Assume $y(t) \in C[0, 1]$, then the following equation

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1 \tag{3.36}$$

$$u(0) = 0, \quad u(1) = \int_0^1 u(s)ds, \tag{3.37}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) y(s) ds \tag{3.38}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1}(\alpha - 1 + s) - [t-s]^{\alpha-1}(\alpha-1)}{(\alpha-1)\Gamma(\alpha)} & \text{if } 0 \le s \le t \le 1 \\ \dfrac{[t(1-s)]^{\alpha-1}(\alpha - 1 + s)}{(\alpha-1)\Gamma(\alpha)} & \text{if } 0 \le t \le s \le 1. \end{cases}$$

*Proof.* We may apply Lemma (3.13) to reduce Eq. (3.36) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$$

for some $C_1, C_2 \in \mathbb{R}$. Therefore, the general solution of (3.36) is

$$u(t) = -\int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \tag{3.39}$$

By $u(0) = 0$, we can get $C_2 = 0$. In addition, $u(1) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1$, it follows

$$C_1 = \int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_0^1 u(s) ds. \tag{3.40}$$

Take (3.40) into (3.39), we have

$$u(t) = -\int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + t^{\alpha-1} \int_0^1 u(s) ds. \tag{3.41}$$

Let $\int_0^1 u(s)ds = A$,by (3.41), we can get

$$\begin{aligned}\int_0^1 u(t)dt &= -\int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)dsdt + \int_0^1 t^{\alpha-1}\int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)dsdt + A\int_0^1 t^{\alpha-1}dt\\ &= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}y(s)ds + \frac{A}{\alpha}\\ &= \int_0^1 \frac{s(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}y(s)ds + \frac{A}{\alpha}.\end{aligned}$$

So,

$$A = \frac{\alpha}{\alpha-1}\int_0^1 \frac{s(1-s)^{\alpha-1}}{\alpha\Gamma(\alpha)}y(s)ds = \int_0^1 \frac{s(1-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)}y(s)ds.$$

Combine with (3.41), we have

$$\begin{aligned}u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + t^{\alpha-1}\int_0^1 \frac{s(1-s)^{\alpha-1}}{(\alpha-1)\Gamma()\alpha}y(s)ds\\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \int_0^1 \frac{\left[t(1-s)^{\alpha-1}(\alpha-1+s)\right]}{(\alpha-1)\Gamma(\alpha)}y(s)ds\\ &= \int_0^1 \frac{\left[t(1-s)^{\alpha-1}(\alpha-1+s)-(t-s)^{\alpha-1}(\alpha-1)\right]}{(\alpha-1)\Gamma(\alpha)}y(s)ds + \int_t^1 \frac{[t(1-s)^{\alpha-1}(\alpha-1+s)}{(\alpha-1)\Gamma(\alpha)}y(s)ds\\ &= \int_0^1 G(t,s)y(s)ds.\end{aligned}$$

This complete the proof.

**Remark 3.15.** Obviously, the Green function $G(t,s)$ satisfies the following properties:

**i.** $G(t,s) > 0, \quad t,s \in (0,1);$

**ii.** $G(t,s) \leq \frac{2}{(\alpha-1)\Gamma(\alpha)}; \quad 0 \leq t,s \leq 1.$

**Theorem 3.16.** ref. [26] Assume that function $f$ satisfies

$$|f(t,u) - f(t,v)| \leq a(t)|u-v| \tag{3.42}$$

where $t \in [0,1], \ u,v \in [0,\infty), \ a:[0,1] \to [0,\infty)$ is a continuous function. If

$$\int_0^1 s^{\alpha-1}(\alpha-1+s)a(s)ds < (\alpha-1)\Gamma(\alpha) \tag{3.43}$$

then the Eq. (3.32) has a unique positive solution.

*Proof.* If $T^n$ is a contraction operator for $n$ sufficiently large, then the Eq. (3.32) has a unique positive solution.

In fact, by the definition of Green function $G(t,s)$, for $u, v \in P$, we have the estimate

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \int_0^1 G(t,s)|f(s,u(s)) - f(s,v(s))|ds \\
&\leq \int_0^1 G(t,s)a(s)|u(s) - v(s)|ds \\
&\leq \int_0^1 \frac{[t(1-s)]^{\alpha-1}(\alpha - 1 + s)}{(\alpha-1)\Gamma(\alpha)} a(s)\|u - v\|ds \\
&\leq \frac{\|u - v\|t^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha - 1 + s)a(s)ds.
\end{aligned}$$

Denote $K = \int_0^1 (1-s)^{\alpha-1}(\alpha - 1 + s)a(s)ds$, then

$$|Tu(t) - Tv(t)| \leq \frac{Kt^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \|u - v\|.$$

Similarly,

$$\begin{aligned}
|T^2u(t) - T^2v(t)| &= \int_0^1 G(t,s)|f(s,Tu(s)) - f(s,Tv(s))|ds \\
&\leq \int_0^1 G(t,s)a(s)|Tu(s) - Tv(s)|ds \\
&\leq \int_0^1 G(t,s)a(s)\frac{Ks^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)}\|u - v\|ds \\
&\leq \int_0^1 \frac{K[t(1-s)]^{\alpha-1}(\alpha - 1 + s)}{(\alpha-1)^2\Gamma^2(\alpha)} a(s)s^{\alpha-1}\|u - v\|ds \\
&\leq \frac{K\|u - v\|t^{\alpha-1}}{(\alpha-1)^2\Gamma^2(\alpha)} \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}(\alpha - 1 + s)a(s)ds \\
&= \frac{KHt^{\alpha-1}}{(\alpha-1)^2\Gamma^2(\alpha)}\|u - v\|
\end{aligned}$$

where $H = \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}(\alpha - 1 + s)a(s)ds$. By mathematical induction, it follows

$$|T^n u(t) - T^n v(t)| \leq \frac{KH^{n-1}t^{\alpha-1}}{(\alpha-1)^n\Gamma^n(\alpha)}\|u - v\|$$

by (3.43), for $n$ large enough, we have

$$\frac{KH^{n-1}t^{\alpha-1}}{(\alpha-1)^n\Gamma^n(\alpha)} = \frac{K}{(\alpha-1)\Gamma(\alpha)}\left(\frac{H}{(\alpha-1)\Gamma(\alpha)}\right)^{n-1} < 1.$$

Hence, it holds

$$\|T^n u - T^n v\| < \|u - v\|,$$

which implies $T^n$ is a contraction operator for $n$ sufficiently large, then the Eq. (3.32) has a unique positive solution. □

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Chapter 2

# Existence Theory of Differential Equations of Arbitrary Order

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Additional information is available at the end of the chapter

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**Abstract**

The aims of this chapter are devoted to investigate a system of fractional-order differential equations (FDEs) with multipoint boundary conditions. Necessary and sufficient conditions are investigated for at most one solution to the proposed problem. Also, results for the existence of at least one or two positive solutions are developed by using a fixed-point theorem of concave-type operator for the considered problem. Further, we extend the conditions for more than two solutions and established some adequate conditions for multiplicity results to the proposed problem. Also, a result devoted to Hyers-Ulam stability is discussed. Suitable examples are provided to verify the established results.

**Keywords:** fractional differential equations, coupled system, boundary condition, concave operator

**Mathematics subject classification**: 26A33, 34A08, 35B40

## 1. Introduction

Arbitrary-order differential equations are the excellent tools in the description of many phenomena and process in different fields of science, technology, and engineering (see [1, 2]). Therefore, considerable attention has been paid to the subject of differential equations of arbitrary order (see [3–5] and the references therein). The area devoted to the existence of positive solutions to fractional differential equations and their system especially coupled systems was greatly studied by many authors (for details see [6–9]). In all these articles, the concerned results were obtained by using classical fixed point theorems like Banach contraction principle, Leray-Schauder fixed point theorem, and fixed point theorems of cone type. The aforesaid area has been very well explored for both ordinary- and arbitrary-order differential equations. Existence and uniqueness results for nonlinear and linear, classical, as well as arbitrary-order differential equations have been investigated in many papers (see few of them as [10–13]).

Another warm area of research in the theory of fractional-order differential equations (FDEs) is devoted to the multiplicity of solutions. Plenty of research articles are available on this topic in literature. In [14], the author studied the given boundary value problem (BVP) for existence of multiple solutions:

$$\begin{cases} \mathscr{D}^{\theta_1}p(t) + \mathcal{H}(t, p(t)) = 0, & t \in \mathbf{I}, \quad \theta_1 \in (1, 2], \\ p(t)|_{t=0} = p(t)\big|_{t=1} = 0. \end{cases}$$

where $\mathscr{D}$ is the Riemann-Liouville derivative of non-integer order and $\mathbf{I} = [0, 1]$. In same line, Kaufmann and Mboumi [15] studied the given boundary value problem of fractional differential equations for multiplicity of positive solutions:

$$\begin{cases} \mathscr{D}^{\theta_1}p(t) + \phi(t)\mathcal{H}(t, p(t)) = 0, & t \in \mathbf{I}, \quad \theta_1 \in (1, 2], \\ p(t)\big|_{t=0} = p^{'}(t)\big|_{t=1} = 0, \end{cases}$$

where $\mathscr{D}$ is the Riemann-Liouville derivative and $\phi \in C(\mathbf{I}, \mathbf{R}), \ \mathcal{H} \in C(\mathbf{I} \times \mathbf{R}, \mathbf{R})$.

In the last few decades, the theory devoted to the multiplicity of solutions is very well extended to coupled systems of nonlinear FDEs, and we refer to few papers in [16–18]. Wang et al. [19] established some conditions under which the given system of three point BVP

$$\begin{cases} \mathscr{D}^{\theta_1}p(t) = \mathcal{H}_1(t, q(t)); & t \in \mathbf{I}, \\ \mathscr{D}^{\theta_2}q(t) = \mathcal{H}_2(t, p(t)); & t \in \mathbf{I}, \\ p(t)_{t=0} = 0, \ p(t)_{t=1} = \mu p(t)\big|_{t=\xi}, \ q(t)\big|_{t=0} = 0, \ q(t)_{t=1} = \nu q(t)\big|_{t=\xi}, \end{cases}$$

has a solution, where $\theta_1, \ \theta_2 \in (1, 2]$ and $\mu, \nu \in \mathbf{I}, \ \xi \in (0, 1), \ \mathcal{H}_i : [0, 1] \times \mathbf{R} \to \mathbf{R}$ for $i = 1, 2$ are nonlinear functions.

In the last few decades, another important aspect devoted to stability analysis of FDEs with initial/boundary conditions has been given much attention. This is because stability is very important from the numerical and optimization point of view. Various forms of stabilities were studied for the aforesaid FDEs including exponential, Mittag-Leffler, and Lyapunov stability. Recently, Hyers-Ulam stability has given more attention. This concept was initially introduced by Ulam and then by Hyers (for details see [20–22]). Now, many articles have been written on this concept (see [23–27]). So far, the aforementioned stability has not yet well studied for multipoint BVPs of FDEs. Motivated by the aforesaid discussion, we propose the following coupled system of four-point BVP provided as

$$\begin{cases} \mathscr{D}^{\theta_1}p(t) = \mathcal{H}_1(t, p(t), q(t)); & t \in \mathbf{I}; \quad \theta_1 \in (m-1, m], \\ \mathscr{D}^{\theta_2}q(t) = \mathcal{H}_2(t, p(t), q(t)); & t \in \mathbf{I}; \quad \theta_2 \in (m-1, m], \\ p^{(j)}(t)_{t=0} = q^{(j)}(t)\big|_{t=0} = 0, \ p(t)|_{t=1} = p(t)\big|_{t=\eta}, \ q(t)|_{t=1} = q(t)\big|_{t=\xi}. \end{cases} \tag{1}$$

where $j = 0, 1, 2, \cdots m-2,\ \ m \geq 3,\ \ \mathbf{I} = [0,1],\ \ \eta, \xi \in (0,1),\ \ \mathcal{H}_1, \mathcal{H}_2 : [0,1] \times \{0\} \cup \mathbf{R}^+ \times \{0\} \cup \mathbf{R}^+ \to \{0\} \cup \mathbf{R}^+$ are continuous functions, and $\mathscr{D}^{\theta_1},\ \ \mathscr{D}^{\theta_2}$ stand for Riemann-Liouville fractional derivative of order $\theta_1, \theta_2$ in sequel. We obtain necessary and sufficient conditions for the existence of solution to system (1) by using another type of fixed point result based on a concave-type operator with increasing or decreasing property. The idea then extends to form some conditions which ensure multiplicity of solutions to the considered problem. Also, we discuss some results about the Hyers-Ulam stability for the considered problem. Further by providing examples, we illustrate the established results.

## 2. Preliminaries

In the current section, we review few fundamental lemmas and results found in [2, 4, 6, 28, 29].

**Definition 2.1.** *Arbitrary-order integral of function* $\psi : (0, \infty) \to \mathbf{R}$ *is recalled as*

$$\mathcal{I}^{\theta_1} \psi(t) = \frac{1}{\Gamma(\theta_1)} \int_0^t (t-s)^{\theta_1 - 1} \psi(s) ds,$$

*where* $\theta_1 > 0$ *is a real number and also the integral is pointwise defined on* $\mathbf{R}^+$

**Definition 2.2.** *Arbitrary-order derivative in Riemann-Liouville sense for a function* $\psi \in ((0, \infty), \mathbf{R})$ *is given by*

$$\mathscr{D}^{\theta_1} \psi(t) = \left(\frac{d}{dt}\right)^m \int_0^t \frac{(t-s)^{m-\theta_1-1}}{\Gamma(m-\theta_1)} \psi(s) ds, \theta_1 > 0, \text{where } m = [\theta_1] + 1.$$

**Lemma 2.3.** [16] *Let* $\theta_1 > 0,$ *then for arbitrary* $C_j \in \mathbf{R},\ \ j = 1, 2, \ldots, m,\ \ m = [\theta_1] + 1,$ *and the solution of*

$$\mathscr{D}^{\theta_1} \psi(t) = f(t)$$

is provided by

$$\psi(t) = \mathcal{I}^{\theta_1} f(t) + C_1 t^{\theta_1 - 1} + C_2 t^{\theta_1 - 2} + \ldots + C_m t^{\theta_1 - m}.$$

**Definition 2.4.** [17, 28] *Consider a Banach space* $\mathbf{E}$ *with a closed set* $\mathbf{C} \subset \mathbf{E}$*. Then,* $\mathbf{C}$ *is said to be partially ordered if* $p \preceq q$ *such that* $q - p \in \mathbf{C}$*. Further,* $\mathbf{C}$ *is said to be a cone if it holds the given conditions:*

1. $p \in \mathbf{C}$ *and for a real constant* $\kappa \geq 0$ *the relation* $\kappa p \in \mathbf{C}$ *holds.*
2. $p$ *and* $-p \in \mathbf{C}$ *yield that* $0 \in \mathbf{C},$ *where* $0$ *is zero element of Banach space* $\mathbf{E}$

**Definition 2.5.** [17, 28] *A closed and convex set* $\mathbf{C}$ *of* $\mathbf{E}$ *is said to be a normal cone if it obeys the given properties:*

1. *For* $0 \preceq p \preceq q \in \mathbf{E}$, *there exists* $\beta > 0$, *such that* $\|p\|_{\mathbf{E}} \leq \beta \|q\|_{\mathbf{E}}$*;*
2. $p \sim q$, *for all* $p, q \in \mathbf{E}$ *yields that there exist constants* $a, b > 0$ *such that* $ap \preceq q \preceq bq$.

**Remark 2.6.** *As* $\sim$ *is an equivalence relation, therefore defines a set* $\mathbf{C}_f = \{p \in \mathbf{E} : p \sim f\}$ *for* $f \in \mathbf{C}$. *Obviously, one can derive that* $\mathbf{C}_f \subset \mathbf{C}$ *for* $f \succ 0$.

**Definition 2.7.** *The operator* $\mathcal{S} : \mathbf{C} \to \mathbf{C}$ *is said to be* $\lambda$ *concave for every* $\theta, \lambda \in (0,1), p \in \mathbf{C}$, *if and only if* $\mathcal{S}(\lambda p) \succeq \theta^{\lambda} \mathcal{S}p$.

**Definition 2.8.** *The operator* $\mathcal{S} : \mathbf{C} \to \mathbf{C}$ *is said to be to be increasing if* $p, q \in \mathbf{C}, p \preceq q$ *gives that* $\mathcal{S}p \preceq \mathcal{S}q$.

**Lemma 2.9.** [17, 28] *Assume that* $\mathcal{S} : \mathbf{C} \to \mathbf{C}$ *is increasing* $\lambda-$*concave operator for a normal cone* $\mathbf{C}$ *produced by Banach space* $\mathbf{E}$, *such that there exists* $p \succ 0$ *with* $\mathcal{S}f \in \mathbf{C}_f$. *Then,* $\mathcal{S}$ *has a unique fixed point* $p \in \mathbf{C}_f$

**Theorem 2.10.** [30] *Let* $\mathbf{E}$ *be a Banach space with* $\mathbf{C} \subseteq \mathcal{B}$, *which is closed and convex. Let* $\mathcal{E}$ *be a relatively open subset of* $\mathbf{C}$ *with* $0 \in \mathcal{E}$ *and* $\mathcal{S} : \overline{\mathcal{E}} \to \mathbf{C}$ *be a continuous and compact operator. Then.*

1. *The operator* $\mathcal{S}$ *has a fixed point in* $\overline{\mathcal{E}}$,
2. *There exist* $w \in \partial\mathscr{E}$ *and* $\lambda \in (0,1)$ *with* $w = \lambda \mathcal{S} w$.

**Lemma 2.11.** [30] *For a Banach space* $\mathbf{E}$ *together with a cone* $\mathbf{C}$, *there exist two relatively open subsets* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *of* $\mathbf{E}$ *such that* $0 \in \mathbf{A}_1 \subset \overline{\mathbf{A}}_1 \subset \mathbf{A}_2$. *Moreover, for a completely continuous operator* $\mathcal{S} : \mathbf{C} \cap (\overline{\mathbf{A}}_2 \setminus \mathbf{A}_1) \to \mathbf{C}$, *one of the given conditions holds:*

1. $\|\mathcal{S}p\| \leq \|p\|$ *for all* $p \in \mathbf{C} \cap \partial \mathbf{A}_1$*;* $\|\mathcal{S}p\| \geq \|p\|$, *for all* $p \in \mathbf{C} \cap \partial \mathbf{A}_2$*;*
2. $\|\mathcal{S}p\| \geq \|p\|$ *for all* $p \in \mathbf{C} \cap \partial \mathbf{A}_1$*;* $\|\mathcal{S}p\| \leq \|p\|$, *for all* $p \in \mathbf{C} \cap \partial \mathbf{A}_2$

*Then,* $\mathcal{S}$ *has at least one fixed point in* $\mathbf{C} \cap (\overline{\mathbf{A}}_2 \setminus \mathbf{A}_1)$.

## 3. Main results

**Theorem 3.1.** *Let* $\varphi \in C([0,1], \mathbf{R})$, $\eta \in (0,1)$ *and* $\lambda_1 = 1 - \eta^{\theta_1 - 1} < 1$, *and then the unique solution to BVP of linear FDE*

$$\begin{cases} \mathscr{D}^{\theta_1} p(t) = \varphi(t), t \in \mathbf{I}, \quad \theta_1 \in (m-1, m], \\ p^{(j)}(t)_{t=0} = 0, p(t)|_{t=1} = p(t)|_{t=\eta}, \quad j = 0, 1, 2, \cdots m-2, \quad m \geq 3, \end{cases} \tag{2}$$

is given by

$$p(t) = \int_0^1 \mathbf{G}(t,s) \varphi(s) ds, \tag{3}$$

*where* $\mathbf{G}(t,s)$ *is the Green's function defined by*

$$\mathbf{G}(t,s)=\frac{1}{\Gamma(\theta_1)}\begin{cases}\frac{1}{\lambda_1}\left[-[t(1-s)]^{\theta_1-1}+[t(\eta-s)]^{\theta_1-1}\right]+(t-s)^{\theta_1-1}, & 0\le s\le t\le\eta\le 1,\\ \frac{1}{\lambda_1}\left[-[t(1-s)]^{\theta_1-1}+[t(\eta-s)]^{\theta_1-1}\right], & 0\le t\le s\le\eta\le 1,\\ -\frac{1}{\lambda_1}[t(1-s)]^{\theta_1-1}+(t-s)^{\theta_1-1}, & 0\le\eta\le s\le t\le 1,\\ -\frac{1}{\lambda_1}[t(1-s)]^{\theta_1-1}, & 0\le\eta\le t\le s\le 1.\end{cases} \tag{4}$$

*Proof.* In view of Lemma 2.3, we may write Eq. (2) as

$$p(t)=\mathcal{I}^{\theta_1}\varphi(t)+C_1t^{\theta_1-1}+C_2t^{\theta_1-2}+\ldots+C_mt^{\theta_1-m}. \tag{5}$$

In view of conditions $p^{(j)}(t)_{t=0}=0,\ \ j=0,1,\ldots m-2,\ \ m\ge 3,$, Eq. (5) suffers from singularity; therefore, we have $C_2=C_3=\ldots=C_n=0$. Hence, Eq. (5) becomes

$$p(t)=\mathcal{I}^{\theta_1}\varphi(t)+C_1t^{\theta_1-1}. \tag{6}$$

Applying boundary condition $p(t)|_{t=1}=p(t)|_{t=\eta}$ and $d=1-\eta_1^{\theta}$ in Eq. (6), one has

$$\begin{aligned}p(t)&=\mathcal{I}^{\theta_1}\varphi(t)+\frac{t^{\theta_1-1}}{\lambda_1}\left[\mathcal{I}^{\theta_1}\varphi(\eta)-\mathcal{I}^{\theta_1}\varphi(1)\right]\\ p(t)&=\int_0^1\mathbf{G}(t,s)\varphi(s)ds.\end{aligned} \tag{7}$$

where $\mathbf{G}(t,s)$ is Green's function given in Eq. (4).

In view of Theorem 3.1 and using $\lambda_1=1-\eta^{\theta_1-1},\ \ \lambda_2=1-\xi^{\theta_2-1}$, the corresponding coupled system of integral equations to the proposed system (1) is given as

$$\begin{cases}p(t)=\int_0^1\mathbf{G}_1(t,s)\mathcal{H}_1(s,p(s),q(s))ds,\\ q(t)=\int_0^1\mathbf{G}_2(t,s)\mathcal{H}_2(s,p(s),q(s))ds,\end{cases} \tag{8}$$

where $\mathbf{G}_1(t,s)$, $\mathbf{G}_2(t,s)$ are Green's functions, which can be similarly computed like in Theorem 3.1. Further, they are continuous on $\mathbf{I}\times\mathbf{I}$ and satisfy the following properties:

**i.** $\max_{t\in\mathbf{I}}|\mathbf{G}_1(t,s)|\le\frac{(\lambda_1+1)(1-s)^{\theta_1-1}}{\lambda_1}=\mathbf{G}_1(1,s),\ \ \text{for all}\ \ s\in\mathbf{I},$

$\max_{t\in\mathbf{I}}|\mathbf{G}_1(t,s)|\le\frac{(\lambda_2+1)(1-s)^{\theta_2-1}}{\lambda_2}=\mathbf{G}_2(1,s),\ \ \text{for all}\ \ s\in\mathbf{I};$

**ii.** $\min_{t\in[\theta,1-\theta]}\mathbf{G}_1(t,s)\ge\frac{\gamma_1(s)}{2}\mathbf{G}(1,s)$ for every $\theta\ \ s\in(0,1)$;

$\min_{t\in[\theta,1-\theta]}\mathbf{G}_2(t,s)\ge\frac{\gamma_2(s)}{2}\mathbf{G}(1,s)$ for every $\theta\ \ s\in(0,1)$;

Further, taking that $\gamma=\inf\{\gamma_1=\theta^{\theta_1-1},\gamma_2=\theta^{\theta_2-1}\}$.

Let us define a Banach space by $\mathbf{E} = \{p(t)|p \in C(\mathbf{I})\}$ endowed with a norm $\|p\|_{\mathbf{E}} = \max_{t\in\mathbf{I}}|p(t)|$. Further, in the norm for the product space, we define it as $\|(p,q)\|_{\mathbf{E}\times\mathbf{E}} = \|p\|_{\mathbf{E}} + \|q\|_{\mathbf{E}}$. Clearly, $(\mathbf{E}\times\mathbf{E}, \|\cdot\|_{\mathbf{E}\times\mathbf{E}})$ is a Banach space. Onward, we define the cone $\mathbf{C} \subset \mathbf{E}\times\mathbf{E}$ by

$$\mathbf{C} = \left\{(p,q) \in \mathbf{E}\times\mathbf{E} : \min_{t\in\mathbf{I}}[p(t) + q(t)] \geq \gamma\|(p,q)\|_{\mathbf{E}\times\mathbf{E}}\right\}.$$

Consider an operator $\mathcal{S} : \mathbf{E}\times\mathbf{E} \to \mathbf{E}\times\mathbf{E}$ defined by

$$\begin{aligned}\mathcal{S}(p,q)(t) &= \left(\int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,p(s),q(s))ds,\ \int_0^1 \mathbf{G}_2(t,s)\mathcal{H}_2(s,p(s),q(s))ds\right).\\ &= (\mathcal{S}_1 p(t), \mathcal{S}_2 q(t)).\end{aligned} \tag{9}$$

It is to be noted that the fixed points of the operator $\mathcal{S}$ correspond with the solution of the system (1) under consideration.

**Theorem 3.2.** *Under the continuity of $\mathcal{H}_1, \mathcal{H}_2 : \mathbf{I}\times\mathbf{R}^+\cup\{0\}\times\mathbf{R}^+\cup\{0\} \to \mathbf{R}^+\{0\}$, the operator $\mathcal{S}$ satisfies that $\mathcal{S}(\mathbf{C}) \subset \mathbf{C}$ and $\mathcal{S} : \mathbf{C} \to \mathbf{C}$ is completely continuous.*

*Proof.* To derive $\mathcal{S}(\mathbf{C}) \subset \mathbf{C}$, let $(p,q) \in \mathbf{C}$, and then we have

$$\mathcal{S}_1(p(t),q(t)) = \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,p(s),q(s))ds \geq \gamma_1 \int_0^1 \mathbf{G}_1(1,s)\mathcal{H}_1(s,p(s)),q(s)ds. \tag{10}$$

Also, we get

$$\mathcal{S}_1(p(t),q(t)) = \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,p(s),q(s))ds \leq \int_0^1 \mathbf{G}_1(1,s)\mathcal{H}_1(s,p(s)),q(s)ds. \tag{11}$$

Thus, from Eqs. (10) and (11), we have

$$\mathcal{S}_1(p(t),q(t)) \geq \gamma\|\mathcal{S}_1(p,q)\|_{\mathbf{E}}, \quad \text{for every} \quad t \in \mathbf{I}.$$

Similarly, we can obtain

$$\mathcal{S}_2(p(t),q(t)) \geq \gamma\|\mathcal{S}_2(p,q)\|_{\mathbf{E}}, \quad \text{for every} \quad t \in \mathbf{I}.$$

$$\text{Thus} \quad \mathcal{S}_1(p(t),q(t)) + \mathcal{S}_2(p(t),q(t)) \geq \gamma\|(p,q)\|_{\mathbf{E}\times\mathbf{E}}, \quad \text{for all} \quad t \in \mathbf{I},$$

$$\min_{t\in\mathbf{I}}[\mathcal{S}_1(p(t),q(t)) + \mathcal{S}_2(p(t),q(t))] \geq \gamma\|(p,q)\|_{\mathbf{E}\times\mathbf{E}}.$$

Hence, we have $\mathcal{S}(p,q) \in \mathbf{C} \Rightarrow \mathcal{S}(\mathbf{C}) \subset \mathbf{C}$.

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Let us consider

$$\max_{t\in\mathbf{I}} |\mathcal{H}_1(t,p(t),q(t))| \le \mathcal{M}_1, \quad \max_{t\in\mathbf{I}} |\mathcal{H}_2(t,p(t),q(t))| \le \mathcal{M}_2.$$

Then, we consider $t_1 < t_2 \in \mathbf{I}$, such that

$$\begin{aligned}
|\mathcal{S}_1(p,q)(t_2) - \mathcal{S}_1(p,q)(t_1)| &= \left|\int_0^1 (\mathbf{G}(t_2,s) - \mathbf{G}_1(t_1,s))\mathcal{H}_1(s,p(s),q(s))ds\right| \\
&\le \frac{\mathcal{M}_1}{\Gamma(\theta_1)}\left[\frac{\left(t_2^{\theta_1-1} - t_1^{\theta_1-1}\right)}{\lambda_1}\left(\int_0^\eta (\eta-s)^{\theta_1-1}ds - \int_0^1 (1-s)^{\theta_1-1}ds\right)\right] \\
&+ \frac{\mathcal{M}_1}{\Gamma(\theta_1)}\left[\int_0^{t_2} (t_2-s)^{\theta_1-1}ds - \int_0^{t_1} (t_1-s)^{\theta_1-1}ds\right] \\
&\le \frac{\mathcal{M}_1}{\lambda_1\Gamma(\theta_1+1)}\left[\left(t_2^{\theta_1-1} - t_1^{\theta_1-1}\right)\left(\eta^{\theta_1} - \lambda_1\right) + \lambda_1\left(t_2^{\theta_1} - t_1^{\theta_1}\right)\right].
\end{aligned} \tag{12}$$

By the same fashion, we obtain for $\mathcal{S}_2$ as

$$|\mathcal{S}_2(p,q)(t_2) - \mathcal{S}_2(p,q)(t_1)| \le \frac{\mathcal{M}_2}{\lambda_2\Gamma(\theta_2+1)}\left[\left(t_2^{\theta_2-1} - t_1^{\theta_2-1}\right)\left(\xi^{\theta_2} - \lambda_2\right) + \lambda_2\left(t_2^{\theta_2} - t_1^{\theta_2}\right)\right]. \tag{13}$$

The right hand sides of Eqs. (12) and (13) are approaching to zero at $t_1 \to t_2$. Thus, the operator $\mathcal{S}$ is equi-continuous. Therefore, thanks to the Arzelá-Ascoli theorem, we receive that $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2) : \mathbf{C} \to \mathbf{C}$ is completely continuous.

**Theorem 3.3.** *Due to continuity of* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *on* $\mathbf{I} \times \mathbf{R}^+ \cup\{0\} \times \mathbf{R}^+\cup\{0\} \to \mathbf{R}^+$, *there exist* $\varphi_j, \psi_j, \sigma_j (j = 1,2) : (0,1) \to \mathbf{R}^+\cup\{0\}$ *for* $t \in (0,1)$, $p$, $q \ge 0$ *such that*

$$\begin{aligned}
|\mathcal{H}_1(t,p(t),q(t))| &\le \varphi_1(t) + \psi_1(t)|p(t)| + \sigma_1(t)|q(t)|; \\
|\mathcal{H}_2(t,p(t),q(t))| &\le \varphi_2(t) + \psi_2(t)|p(t)| + \sigma_2(t)|q(t)|,
\end{aligned}$$

along with the following conditions:

**i.** $\Delta_1 = \int_0^1 \mathbf{G}_1(1,s)\varphi_1(s)ds < \infty, \ \Lambda_1 = \int_0^1 \mathbf{G}_1(1,s)\left[\psi_1(s) + \sigma_1(s)\right]ds < 1;$

**ii.** $\Delta_2 = \int_0^1 \mathbf{G}_2(1,s)\varphi_2(s)ds < \infty, \ \Lambda_2 = \int_0^1 \mathbf{G}_2(1,s)\left[\psi_2(s) + \sigma_2(s)\right]ds < 1$

*are satisfied. Then, the system* (1) *has at least one solution* $(p,q)$ *which lies in*

$$\mathcal{E} = \left\{(p,q) \in \mathbf{C} : \|(p,q)\|_{\mathbf{E}\times\mathbf{E}} < \min\left(\frac{2\Delta_1}{1-2\Lambda_1}, \frac{2\Delta_2}{1-2\Lambda_2}\right)\right\}.$$

*Proof.* Let $\mathcal{E} = \{(p,q) \in \mathbf{C} : \|(p,q)\|_{\mathbf{E}\times\mathbf{E}} < r\}$ with $\min\left(\frac{2\Delta_1}{1-2\Lambda_1}, \frac{2\Delta_2}{1-2\Lambda_2}\right) < r$.

Define the operator $\mathcal{S} : \overline{\mathcal{E}} \to \mathbf{C}$ as in Eq. (9).

Let $(p,q) \in \mathcal{E}$ that is $\|(p,q)\|_{\mathbf{E}\times\mathbf{E}} < r$. Then, we have

$$\begin{aligned} |\mathcal{S}_1(p,q)(t)| &= \max_{t\in\mathbf{I}} \left| \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,p(s),q(s))ds \right| \\ &\le \left( \int_0^1 \mathbf{G}_1(1,s)\varphi_1(s)ds + \int_0^1 \mathbf{G}_1(1,s)\psi_1(s)|p(s)|ds + \int_0^1 \mathbf{G}_1(1,s)\sigma_1(s)|q(s)|ds \right) \\ &\le \int_0^1 \mathbf{G}_1(1,s)\varphi_1(s)ds + r\left[\int_0^1 \mathbf{G}_1(1,s)\left[\psi_1(s)+\sigma_1(s)\right]ds\right] = \Delta_1 + r\Lambda_1 \le \frac{r}{2}. \end{aligned} \tag{14}$$

Thus, from Eq. (14), we have

$$\|\mathcal{S}_1(p,q)\|_{\mathbf{E}} \le \frac{r}{2}. \tag{15}$$

Similarly, one can derive that

$$\|\mathcal{S}_2(p,q)\|_{\mathbf{E}} \le \frac{r}{2}. \tag{16}$$

Thus, from Eqs. (15) and (16), we get

$$\|\mathcal{S}(p,q)\|_{\mathbf{E}\times\mathbf{E}} \le r. \tag{17}$$

Therefore, $\mathcal{S}(p,q) \subseteq \overline{\mathcal{E}}$. Hence, by Theorem 3.2 the operator $\mathcal{S} : \overline{\mathcal{E}} \to \mathcal{E}$ is completely continuous.

Consider the eigenvalue problem:

$$(p,q) = \rho\mathcal{S}(p,q), \quad \text{with } \rho \in (0,1). \tag{18}$$

Under the assumption that $(p,q)$ is a solution of Eq. (18) for $\rho \in (0,1)$, we have

$$\begin{aligned} |p(t)| &\le \rho \max_{t\in\mathbf{I}} \int_0^1 \mathbf{G}_1(t,s)|\mathcal{H}_1(s,p(s),q(s))ds| \\ &\le \rho\left[\int_0^1 \mathbf{G}_1(1,s)\varphi_1(s)ds + \int_0^1 \mathbf{G}_1(1,s)(\psi_1(s)|p(s)| + \sigma_1(s)|q(s)|)ds\right] \\ &\le \rho(\Delta_1 + r\Lambda_1) \end{aligned}$$

which implies that $\|p\|_{\mathbf{E}} < \frac{r}{2}$.

Similarly, we can obtain that $\|q\|_{\mathbf{E}} < \frac{r}{2}$, so $\|(p,q)\|_{\mathbf{E}\times\mathbf{E}} < r$, which implies that $(p,q)$ does not belong to $\partial\mathcal{E}$ for all $\rho \in (0,1)$. Therefore, due to Theorem 2.10, $\mathcal{S}$ has a fixed point in $\overline{\mathcal{E}}$

Assume that the given hypothesis holds:

$(H_1)$ The nonlinear functions $\mathcal{H}_1$ and $\mathcal{H}_2$ are continuous on $\mathbf{I} \times \mathbf{R}^+ \cup \{0\} \times \mathbf{R}^+ \cup \{0\} \rightarrow \mathbf{R}^+ \cup \{0\}$

$(H_2)$ For all $t \in \mathbf{I}$, we have

$$\mathcal{H}_1(t,p,q) = \neq 0, \;\; \mathcal{H}_2(t,p,q) \neq 0, \;\; \text{at} \;\; (p,q) = (0,0)$$

and

$$\mathcal{H}_1(t,p,q) \neq 1, \;\; \mathcal{H}_2(t,1,1) \neq 1; \;\; \text{at} \;\; (p,q) = (1,1);$$

$(H_3)$ For all $t \in \mathbf{I}$ such that

$$0 \leq p \leq p_1, \;\; 0 \leq q \leq q_1 \Rightarrow \mathcal{H}_1(t,p,q) \leq \mathcal{H}_1\big(t,p_1,q_1\big), \;\; \mathcal{H}_2(t,p,q) \leq \mathcal{H}_1\big(t,p_1,q_1\big);$$

$(H_4)$ For $p,q \geq 0$, there exist real numbers $0 < \lambda, \mu < 1$, such that for each $t \in \mathbf{I}, \tau \in (0,1)$, we have

$$\mathcal{H}_1(t,\tau p,\tau q) \geq \tau^{\lambda}\mathcal{H}_1(t,p,q), \;\; \mathcal{H}_2(t,\tau p,\tau q) \geq \tau^{\mu}\mathcal{H}_2(t,p,q).$$

**Theorem 3.4.** *Under the assumptions $(H_1) - (H_4)$, the BVP* (1) *has a unique solution in $\mathbf{C}_f$ where $f(t) = \big(t^{\theta_1 - 1}, t^{\theta_2 - 1}\big)$.*

*Proof.* Let $\max\{\lambda,\mu\} = \kappa$ and $(p,q) \in \mathbf{C}$. For each $t \in \mathbf{I}$, using $(H_4)$, we have

$$\mathcal{S}_1(\tau p,\tau q)(t) = \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,\tau p(s),\tau q(s))ds$$

$$\geq \tau^{\lambda}\int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,p(s),q(s))ds = \tau^{\lambda}\mathcal{S}_1(p,q)(t) \geq \tau^{\kappa}\mathcal{S}_1(p,q)(t),$$

Analogously, we also get

$$\mathcal{S}_2(\tau p,\tau q)(t) \geq \tau^{\kappa}\mathcal{S}_2(p,q)(t).$$

In view of partial order $\succeq$ on $\mathbf{E} \times \mathbf{E}$ induced by the cone $\mathbf{C}$, we get $\mathcal{S}(\tau p,\tau q) \succeq \tau^{\kappa}\mathcal{S}(p(t),q(t))$, $\tau \in (0,1)$, $(p,q) \in \mathbf{C}$. Which yields that $\mathcal{S}$ is $\tau-$ concave and nondecreasing operator with respect to the partial order by using hypothesis $(H_4)$. Hence, taking $f \in \mathbf{C}$ for each $t \in \mathbf{I}$ defined by

$$f(t) = \big(t^{\theta_1 - 1}, t^{\theta_2 - 1}\big) = \big(f_1(t), f_2(t)\big).$$

Suppose that

$$\mathbf{w}_1 = \max\left\{\frac{1}{\Gamma(\theta_1)}\int_0^1 \mathcal{H}_1(s,1,1)ds, \frac{1}{\Gamma(\theta_2)}\int_0^1 \mathcal{H}_2(s,1,1)ds\right\}$$

and

$$\mathbf{w}_2 = \max\left\{\frac{1}{\Gamma(\theta_1)}\int_0^1 \mathbf{L}(s)\mathcal{H}_1(s,0,0)ds, \frac{1}{\Gamma(\theta_2)}\int_0^1 \mathbf{K}(s)\mathcal{H}_2(s,0,0)ds\right\}.$$

Also, from Green's functions, we can obtain that

$$\mathbf{L}(s) = (1-s)^{\theta_1-1}\left(\frac{1+\lambda_1}{\lambda_1}\right), \ \mathbf{K}(s) = (1-s)^{\theta_2-1}\left(\frac{1+\lambda_2}{\lambda_2}\right). \tag{19}$$

Due to nondecreasing property of $\mathcal{H}_1, \mathcal{H}_2$ in view of $(H_3)$, we get $\mu > 0$, $\nu > 0$. Therefore, applying (19) together with $(H_4)$, one has

$$\mathcal{S}_1 h(t) = \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1\big(s, f_1(s), f_2(s)\big)ds$$

$$= \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1\big(s, s^{\theta_1-1}, s^{\theta_2-1}\big)ds \leq \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,1,1)ds$$

$$\leq \left(\frac{1}{\Gamma(\theta_1)}\int_0^1 (1-s)^{\theta_1-1}\mathcal{H}_1(s,1,1)ds\right) t^{\theta_1-1} \leq \mu f_1(t).$$

Similarly, we can get

$$\mathcal{S}_2 f(t) \leq \mu f_2(t).$$

Then, we obtain

$$\mathcal{S}f \preceq \mu f. \tag{20}$$

Like the aforesaid process, applying Eq. (19) together with $(H_4)$, for each $t \in \mathbf{I}$, one has

$$\mathcal{S}_1 f(t) = \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1\big(s, s^{\theta_1-1}, s^{\theta_2-1}\big)ds \geq \int_0^1 \mathbf{G}_1(t,s)\mathcal{H}_1(s,0,0)ds$$

$$\geq \left(\frac{1}{\Gamma\theta_1}\int_0^1 \mathbf{L}(s)\mathcal{H}_1(s,0,0)ds\right) t^{\theta_1-1} \geq \nu h_1(t),$$

With same fashion, we can obtain

$$\mathcal{S}_2 f(t) \geq \nu f_2(t).$$

Thus, we have

$$\mathcal{S}f(t) \succeq \nu f. \tag{21}$$

From Eqs. (20) and (21), we produce

$$\nu f \preceq \mathcal{S}f \preceq \mu f,$$

which implies that $\mathcal{S}f \in \mathbf{C}_f$. So, thanks to Lemma 2.9, we see that the operator $\mathcal{S}$ is concave; hence, it has at most one fixed point $(p, q) \in \mathbf{C}_f$ which is the corresponding solution of BVPs (1).

Now, we define the following:

$(C_1)$ $\mathcal{H}_j (j = 1, 2) : \mathbf{I} \times \mathbf{R}^+ \cup \{0\} \times \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ is uniformly bounded and continuous on $\mathbf{I}$ with respect to $t$.

$(C_2)$ Green's functions $\mathbf{G}_1(1, s), \mathbf{G}_2(1, s)$ satisfy

$$0 < \int_0^1 \mathbf{G}_1(1, s) ds < \infty, \quad 0 < \int_0^1 \mathbf{G}_2(1, s) ds < \infty;$$

$(C_3)$ Let these limits hold:

$$\mathcal{H}_1^{\varrho} = \lim_{p+q \to \varrho} \max_{t \in \mathbf{I}} \frac{\mathcal{H}_1(t, p, q)}{p + q}, \quad \mathcal{H}_2^{\varrho} = \lim_{p+g \to \varrho} \max_{t \in \mathbf{I}} \frac{\mathcal{H}_2(t, p, q)}{p + q},$$

$$\mathcal{H}_{1,\varrho} = \lim_{p+q \to \varrho} \inf_{t \in \mathbf{I}} \frac{\mathcal{H}_1(t, p, q)}{p + q}, \quad \mathcal{H}_{2,\varrho} = \lim_{p+q \to \varrho} \inf_{t \in \mathbf{I}} \frac{\mathcal{H}_2(t, p, q)}{p + q}, \quad \text{where } \varrho \in \{0, \infty\}.$$

$$\delta_1 = \max_{t \in \mathbf{I}} \int_0^1 \mathbf{G}_1(t, s) ds, \quad \delta_2 = \max_{t \in \mathbf{I}} \int_0^1 \mathbf{G}_2(t, s) ds.$$

**Theorem 3.5.** *Assume that the conditions $(C_1) - (C_3)$ together with given assumptions are satisfied:*

$(H_5)$ $\mathcal{H}_{1,0} \left( \gamma_1^2 \int_\theta^{1-\theta} \mathbf{G}_1(1, s) ds \right) > 1, \mathcal{H}_{1,\infty} \left( \gamma_1^2 \int_\theta^{1-\theta} \mathbf{G}_1(1, s) ds \right) > 1$ and

$$\mathcal{H}_{2,0} \left( \gamma_2^2 \int_\theta^{1-\theta} \mathbf{G}_2(1, s) ds \right) > 1, \mathcal{H}_{2,\infty} \left( \gamma_2^2 \int_\theta^{1-\theta} \mathbf{G}_2(1, s) ds \right) > 1.$$

*Moreover, $\mathcal{H}_{1,0} = \mathcal{H}_{2,0} = \mathcal{H}_{1,\infty} = \mathcal{H}_{2,\infty} = \infty$ also hold:*

$(H_6)$ *There exists constant $\alpha > 0$ such that*

$$\max_{t\in\mathbf{I},\,(p,q)\in\partial\mathbf{C}_\alpha} \mathcal{H}_1(t,p,q) < \frac{\alpha}{2\delta_1}$$

and

$$\max_{t\in\mathbf{I},\,(p,q)\in\partial\mathbf{C}_\alpha} \mathcal{H}_2(t,p,q) < \frac{\alpha}{2\delta_2}.$$

*Then, the system* (1) *of BVPs has at least two positive solutions* $(p,q)$, $(\overline{p},\overline{q})$ *which obeying*

$$0 < \|(p,q)\|_{\mathbf{E}\times\mathbf{E}} < \alpha < \|(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}}. \tag{22}$$

*Proof.* Assume that $(H_5)$ holds, and consider $\epsilon,\ \alpha,\ \lambda$ such that $0<\epsilon<\alpha<\lambda$. Further we define a set by

$$\Omega_r = \{(u,v)\in\mathbf{E}\times\mathbf{E} : \|(u,v)\|_{\mathbf{E}\times\mathbf{E}} < r\},\ \text{ where }\ r\in\{\epsilon,\alpha,\lambda\}.$$

Now, if

$$\mathcal{H}_{1,0}\left(\gamma_1^2\int_\theta^{1-\theta}\mathbf{G}_1(1,s)ds\right) > 1 \text{ and } \mathcal{H}_{2,0}\left(\gamma_2^2\int_\theta^{1-\theta}\mathbf{G}_2(1,s)ds\right) > 1.$$

Then, obviously, we can obtain that

$$\|\mathcal{S}(p,q)\|_{\mathbf{E}\times\mathbf{E}} \geq \|(p,q)\|_{\mathbf{E}\times\mathbf{E}},\ \text{ for }\ (p,q)\in\mathbf{C}\cap\partial\Omega_\varepsilon. \tag{23}$$

Now, if $\mathcal{H}_{1,\infty}\left(\gamma_1^2\int_\theta^{1-\theta}\mathbf{G}_1(1,s)ds\right) > 1$ and $\mathcal{H}_{2,\infty}\left(\gamma_2^2\int_\theta^{1-\theta}\mathbf{G}_2(1,s)ds\right) > 1.$

Then, like the proof of Eq. (23), we have

$$\|\mathcal{S}(p,q)\|_{\mathbf{E}\times\mathbf{E}} \geq \|(p,q)\|_{\mathbf{E}\times\mathbf{E}},\ \text{ for }\ (p,q)\in\mathbf{C}\cap\partial\Omega_\lambda. \tag{24}$$

Also, from $(H_5)$ and $(p,q)\in\mathbf{C}\cap\partial\Omega_\alpha$, we get

$$\begin{aligned}|\mathcal{S}_1(p,q)(t)| &= \left|\int_0^1 \mathbf{G}_1(t,s)\mathcal{H}(s,u(s),v(s))ds\right| \\ &\leq \int_0^1 \mathbf{G}_1(1,s)|\mathcal{H}_1(s,p(s),q(s))|ds.\end{aligned}$$

From which we have

$$\|\mathcal{S}_1(p,q)\|_{\mathbf{E}\times\mathbf{E}} < \frac{\alpha}{2\varrho_1}\int_0^1 \mathbf{G}_1(1,s)ds = \frac{\alpha}{2}.$$

Similarly, we have $\|\mathcal{S}_1(p,q)\|_{\mathbf{E}\times\mathbf{E}} < \frac{\alpha}{2}$ as $(p,q)\in\mathbf{C}\cap\partial\Omega_\alpha$. Hence, we have

$$\|\mathcal{S}(p,q)\|_{\mathbf{E}\times\mathbf{E}} < \|(p,q)\|_{\mathbf{E}\times\mathbf{E}}, \text{ for } (p,q)\in\mathbf{C}\cap\partial\Omega_\alpha. \tag{25}$$

Now, applying Lemma 2.11 to Eqs. (23) and (25) yields that $\mathcal{S}$ has a fixed point $(p,q)\in\mathbf{C}\cap\left(\overline{\Omega}_\alpha\setminus\mathbf{C}_\varepsilon\right)$ and a fixed point in $(\overline{p},\overline{q})\in\mathbf{C}\cap\left(\overline{\Omega}_\lambda\setminus\Omega_\alpha\right)$. Hence, we conclude that the system of BVPs (1) has at least two positive solutions $(p,q)$, $(\overline{p},\overline{q})$ such that $\|(p,q)\|_{\mathbf{E}\times\mathbf{E}}\neq\alpha$ and $\|(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}}\neq\alpha$. Thus, relation (22) holds.

**Theorem 3.6.** *Consider that* $(C_1)-(C_3)$ *together with the following hypothesis are satisfied:*

$(H_7)$ $\delta_1\mathcal{H}_{1,0}<1$, $\delta_1\mathcal{H}_{1,\infty}<1$; $\delta_2\mathcal{H}_{1,0}<1$, *and* $\delta_2\mathcal{H}_{2,\infty}<1$;

$(H_8)$ *There exist* $\rho>0$ *such that*

$$\max_{t\in\mathbf{I},\,(p,q)\in\partial\mathbf{C}_\alpha}\gamma_1^2\mathcal{H}_1(t,p,q)>\frac{\alpha}{2}\left(\int_\theta^{1-\theta}\mathbf{G}_1(1,s)ds\right)^{-1},$$
$$\max_{t\in\mathbf{I},\,(p,q)\in\partial\mathbf{C}_\alpha}\gamma_2^2\mathcal{H}_2(t,p,q)>\frac{\alpha}{2}\left(\int_\theta^{1-\theta}\mathbf{G}_2(1,s)ds\right)^{-1}$$

such that

$$0<\|(p,q)\|_{\mathbf{E}\times\mathbf{E}}<\alpha<\|(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}}.$$

*Then, the proposed coupled system of BVPs* (1) *has at least two positive solutions*.

*Proof.* Proof is like the proof of Theorem 3.4.

Analogously, we deduce from Theorem 3.5 and 3.6 the following results for multiplicity of solutions to the system (1) of BVPs.

**Theorem 3.7.** *Under the conditions* $(C_1)-(C_3)$, *there exist* $2k$ *positive numbers* $\mathbf{a}_j,\widehat{\mathbf{a}}_j$, $j=1,2\ldots k$ *with* $\mathbf{a}_1<\gamma_1\widehat{\mathbf{a}}_1<\widehat{\mathbf{a}}_1<\mathbf{a}_2<\gamma_1\widehat{\mathbf{a}}_2<\widehat{\mathbf{a}}_2\ldots\mathbf{a}_k<\gamma_1\widehat{\mathbf{a}}_k<\widehat{\mathbf{a}}_k$ *and* $\mathbf{a}_1<\gamma_2\widehat{\mathbf{a}}_1<\widehat{\mathbf{a}}_1<\mathbf{a}_2<\gamma_2\widehat{\mathbf{a}}_2<\widehat{\mathbf{a}}_2\ldots\mathbf{a}_k<\gamma_2\widehat{\mathbf{a}}_k<\widehat{\mathbf{a}}_k$ *such that.*

$(H_9)$ $\mathcal{H}_1(t,p(t),q(t))\left(\gamma_1\int_0^1\mathbf{G}_1(1,s)ds\right)\geq\mathbf{a}_j$, *for* $(t,p,q)\in\mathbf{I}\times\left[\gamma_1\mathbf{a}_j,\mathbf{a}_j\right]\times\left[\gamma_2\mathbf{a}_j,\mathbf{a}_j\right]$, *and*

$$\mathcal{H}_1(t,p(t),q(t))\delta_1\leq\widehat{\mathbf{a}}_i, \textit{for } (t,p,q)\in\mathbf{I}\times\left[\gamma_1\widehat{\mathbf{a}}_j,\widehat{\mathbf{a}}_j\right]\times\left[\gamma_2\mathbf{a}_j,\mathbf{a}_i\right], j=1,2\ldots k;$$

$(H_{10})$ $\mathcal{H}_2(t,p(t),q(t))\left(\gamma_2\int_0^1\mathbf{G}_2(1,s)ds\right)\geq\mathbf{a}_j$, *for* $(t,p,q)\in\mathbf{I}\times\left[\gamma_1\mathbf{a}_j,\mathbf{a}_j\right]\times\left[\gamma_2\mathbf{a}_j,\mathbf{a}_j\right]$, *and*

$$\mathcal{H}_1(t, p(t), q(t))\delta_2 \leq \widehat{\mathbf{a}}_j, \textit{for } \ (t, p, q) \in \mathbf{I} \times [\gamma_1 \mathbf{a}_j, \mathbf{a}_j] \times \left[\gamma_2 \widehat{\mathbf{a}}_j, \widehat{\mathbf{a}}_j\right], \ \ j = 1, 2 \ldots k.$$

*Then, system* (1) *of BVPs has at least k solutions* $\left(p_j, q_j\right)$*, satisfying*

$$\mathbf{a}_j \leq \| \left(p_j, q_j\right) \|_{\mathbf{E}\times\mathbf{E}} \leq \widehat{\mathbf{a}}_j, \ \ j = 1, 2 \ldots k.$$

*Further, if assumptions* $(C_1) - (C_3)$ *hold such that there exist* $2k$ *positive numbers* $\mathbf{b}_j, \widehat{\mathbf{b}}_j, \ \ j = 1, 2 \ldots k,$ *with*

$$\mathbf{b}_1 < \widehat{\mathbf{b}}_1 < \mathbf{b}_2 < \widehat{\mathbf{b}}_2 \ldots < \mathbf{b}_k < \widehat{\mathbf{b}}_k,$$

together with following hypothesis hold:

$(H_{11})$ $\mathcal{H}_1(t, p, q)$ *and* $\mathcal{H}_2(t, p, q)$ *are nondecreasing on* $\left[0, \widehat{\mathbf{b}}_k\right]$ *for all* $t \in \mathbf{I}$*;*

$$(H_{11}) \mathcal{H}_1(t, p(t), q(t)) \left(\gamma_1 \int_\theta^{1-\theta} \mathbf{G}_1(1, s) ds\right) \geq \mathbf{b}_j, \mathcal{H}_1(t, p(t), q(t))\delta_1 \leq \widehat{\mathbf{b}}_j, \ \ j = 1, 2 \ldots k, \tag{26}$$

$$\mathcal{H}_2(t, u(t), v(t)) \left(\gamma_2 \int_\theta^{1-\theta} \mathbf{G}_2(1, s) ds\right) \geq \mathbf{b}_j, \mathcal{H}_2(t, p(t), q(t))\delta_2 \leq \widehat{\mathbf{b}}_j, \ \ j = 1, 2 \ldots k.$$

*Then, system* (1) *of BVPs has at least k solutions* $\left(p_j, q_j\right)$*, satisfying*

$$\mathbf{b}_j \leq \| \left(p_j, q_j\right) \|_{\mathbf{E}\times\mathbf{E}} \leq \widehat{\mathbf{b}}_j, \ \ j = 1, 2 \ldots k.$$

## 4. Hyers-Ulam stability

**Definition 4.1.** *(*[31, *Definition 2]) Consider a Banach space* $\mathbf{E} \times \mathbf{E}$ *such that* $\mathcal{S}_1, \mathcal{S}_2 : \mathbf{E} \times \mathbf{E} \to \mathbf{E} \times \mathbf{E}$ *be the two operators. Then, the operator system provided by*

$$\begin{cases} p(t) = \mathcal{S}_1(p, q)(t), \\ q(t) = \mathcal{S}_2(p, q)(t) \end{cases} \tag{27}$$

*is called Hyers-Ulam stability if we can find* $\mathscr{C}_i (i = 1, 2, 3, 4) > 0,$ *such that for each* $\rho_i (i = 1, 2) > 0$ *and for each solution* $(p^*, q^*) \in \mathbf{E} \times \mathbf{E}$ *of the inequalities given by*

$$\begin{cases} \|p^* - \mathcal{H}_1(p^*, q^*)\|_{\mathbf{E}\times\mathbf{E}} \leq \rho_1, \\ \|q^* - \mathcal{H}_2(p^*, q^*)\|_{\mathbf{E}\times\mathbf{E}} \leq \rho_2, \end{cases} \tag{28}$$

*there exist a solution* $(\overline{p}, \overline{q}) \in \mathbf{E} \times \mathbf{E}$ *of system* (26) *which satisfy*

$$\begin{cases} \|p^* - \overline{p}\|_{\mathbf{E}\times\mathbf{E}} \leq \mathcal{C}_1\rho_1 + \mathcal{C}_2\rho_2, \\ \|q^* - \overline{q}\|_{\mathbf{E}\times\mathbf{E}} \leq \mathcal{C}_3\rho_1 + \mathcal{C}_4\rho_2. \end{cases} \tag{29}$$

**Definition 4.2.** *If* $\lambda_i$, *for* $i = 1, 2, \cdots, n$ *be the (real or complex) eigenvalues of a matrix* $\mathbf{M} \in \mathcal{C}^{n\times n}$, *then the spectral radius* $\rho(\mathbf{M})$ *is defined by*

$$\rho(\mathbf{M}) = \max\{|\lambda_i|, \ \ \textit{for} \ \ i = 1, 2, \cdots, n\}.$$

*Further, the matrix will converge to zero if* $\rho(\mathbf{M}) < 1$..

**Theorem 4.3.** *(*[31, *Theorem 4]) Consider a Banach space* $\mathbf{E} \times \mathbf{E}$ *with* $\mathcal{S}_1, \mathcal{S}_2 : \mathbf{E} \times \mathbf{E} \to \mathbf{E} \times \mathbf{E}$ *be the two operators such that*

$$\begin{cases} \|\mathcal{S}_1(p, q) - \mathcal{S}_1(p^*, q^*)\|_{\mathbf{E}\times\mathbf{E}} \leq \mathcal{C}_1\|p - p^*\|_{\mathbf{E}\times\mathbf{E}} + \mathcal{C}_2\|q - q^*\|_{\mathbf{E}\times\mathbf{E}}, \\ \|\mathcal{S}_2(p, q) - \mathcal{S}_2(p^*, q^*)\|_{\mathbf{E}\times\mathbf{E}} \leq \mathcal{C}_3\|p - p^*\|_{\mathbf{E}\times\mathbf{E}} + \mathcal{C}_4\|q - q^*\|_{\mathbf{E}\times\mathbf{E}}, \\ \textit{for all} \ \ (p, q), (p^*, q^*) \in \mathbf{E} \times \mathbf{E}, \end{cases} \tag{30}$$

and if the matrix

$$\mathbf{M} = \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{C}_3 & \mathcal{C}_4 \end{bmatrix}$$

*converges to zero (*[31, *Theorem 1]), then the fixed points corresponding to operatorial system* (26) *are Hyers-Ulam stable.*

For the stability results, the following should be hold:

$(H_{13})$ Under the continuity of $\mathcal{H}_i, i = 1, 2$, there exist $a_i, b_i \in c(0, 1), \ \ i = 1, 2$ and $(p, q), (\overline{p}, \overline{q})$ such that

$$|\mathcal{H}_i(t, p, q) - \mathcal{H}_i(t, \overline{p}, \overline{q})| \leq a_i(t)|p - \overline{p}| + b_i(t)|q - \overline{q}|, \ \ i = 1, 2.$$

In this section, we study Hyers-Ulam stability for the solutions of our proposed system. Thanks to Definition 4.1 and Theorem 4.3, the respective results are received.

**Theorem 4.4.** *Suppose that the assumptions* $(H_{13})$ *along with condition that matrix*

$$\mathbf{M} = \begin{bmatrix} \int_0^1 \mathbf{G}_1(1, s)a_1(s)ds & \int_0^1 \mathbf{G}_1(1, s)b_1(s)ds \\ \int_0^1 \mathbf{G}_2(1, s)a_2(s)ds & \int_0^1 \mathbf{G}_2(1, s)b_2(s)ds \end{bmatrix}$$

*is converging to zero. Then, the solutions of* (1) *are Hyers-Ulam stable.*

*Proof.* In view of Theorem 4.3, we have

$$\|\mathcal{S}_1(p,q)-\mathcal{S}_1(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}} \leq \int_0^1 \mathbf{G}_1(1,s)a_1(s)\|p-\overline{p}\|_{\mathbf{E}\times\mathbf{E}}ds + \int_0^1 \mathbf{G}_1(1,s)b_1(s)\|q-\overline{q}\|_{\mathbf{E}\times\mathbf{E}}ds$$
$$\|\mathcal{S}_2(p,q)-\mathcal{S}_2(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}} \leq \int_0^1 \mathbf{G}_2(1,s)a_2(s)\|p-\overline{p}\|_{\mathbf{E}\times\mathbf{E}}ds + \int_0^1 \mathbf{G}_2(1,s)b_2(s)\|q-\overline{q}\|_{\mathbf{E}\times\mathbf{E}}ds.$$

From which we get

$$\|\mathcal{S}_1(p,q)-\mathcal{S}_1(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}} \leq \left[\int_0^1 \mathbf{G}_1(1,s)a_1(s)ds]\right]\|p-\overline{p}\|_{\mathbf{E}\times\mathbf{E}} + \left[\int_0^1 \mathbf{G}_1(1,s)b_1(s)ds\right]\|q-\overline{q}\|_{\mathbf{E}\times\mathbf{E}}$$
$$\|\mathcal{S}_2(p,q)-\mathcal{S}_2(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}} \leq \left[\int_0^1 \mathbf{G}_2(1,s)a_2(s)ds\right]\|p-\overline{p}\|_{\mathbf{E}\times\mathbf{E}} + \left[\int_0^1 \mathbf{G}_2(1,s)b_2(s)ds\right]\|q-\overline{q}\|_{\mathbf{E}\times\mathbf{E}}. \tag{31}$$

Hence, we get

$$\|\mathcal{S}(p,q)-\mathcal{S}(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}} \leq \mathbf{M}\|(p,q)-(\overline{p},\overline{q})\|_{\mathbf{E}\times\mathbf{E}}, \tag{32}$$

where $\mathbf{M}=\begin{bmatrix}\int_0^1 \mathbf{G}_1(1,s)a_1(s)ds & \int_0^1 \mathbf{G}_1(1,s)b_1(s)ds\\ \int_0^1 \mathbf{G}_2(1,s)a_2(s)ds & \int_0^1 \mathbf{G}_2(1,s)b_2(s)ds\end{bmatrix}$. Hence, we received the required results.

## 5. Illustrative examples

**Example 5.1.** *Consider the given system of BVPs*

$$\begin{cases}\mathscr{D}^{\frac{7}{2}}p(t)+\left(1-t^2\right)+[p(t)q(t)]^{\frac{1}{3}}=0, \;\; \mathscr{D}^{\frac{11}{3}}q(t)+1+t+[p(t)q(t)]^{\frac{1}{4}}=0, \;\; t\in(0,1),\\ p(t)=p'(t)=p''(t)=q(t)=q'(t)=q''(t)=0, \;\; at \;\; t=0,\\ p(1)=p\left(\dfrac{1}{4}\right), \;\; q(1)=q\left(\dfrac{1}{3}\right).\end{cases} \tag{33}$$

*Clearly,* $\mathcal{H}_1(t,p,q)\neq 0$, $\mathcal{H}_2(t,p,q)\neq 0$, *at* $(p,q)=(0,0)$, *and* $\mathcal{H}_1(t,p,q)\neq 0$, $\mathcal{H}_2(t,p,q)\neq 0$, *at* $(p,q)=(1,1)$. *Simple computation yields that* $\mathcal{H}_1,\mathcal{H}_2$ *are nondecreasing for every* $t\in(0,1)$. *Also, for* $\tau$, $t\in(0,1)$, *and* $p,q\geq 0$, *one has* $\max\left\{\frac{1}{4},\frac{1}{3}\right\}=\frac{1}{3}$,

$$\mathcal{H}_1(t,\tau p,\tau q)\geq\tau^{\frac{1}{3}}\mathcal{H}_1(t,p,q), \qquad \mathcal{H}_2(t,\tau p,\tau q)\geq\tau^{\frac{1}{3}}\mathcal{H}_2(t,p,q).$$

*Thus, all the conditions of Theorem 3.4 are fulfilled, so the system* (32) *of BVPs has unique positive solution in* $\mathbf{B}_f$ *where* $f(t)=\left(t^{\frac{5}{2}},t^{\frac{9}{2}}\right)$.

**Example 5.2.** *Consider the following system of BVPs:*

$$\begin{cases}\mathscr{D}^{\frac{9}{2}}p(t)+(1+t)^2+[p(t)+q(t)]^3=0, \;\; \mathscr{D}^{\frac{9}{2}}q(t)+1+t+[p(t)+q(t)]^2=0, \;\; t\in(0,1),\\ p^{(j)}(t)=q^{(j)}(t)=0, \;\; j=0,1,2,3, \;\; at \;\; t=0,\\ p(1)=p\left(\dfrac{1}{2}\right), \;\; q(1)=q\left(\dfrac{1}{2}\right).\end{cases} \tag{34}$$

*It is obvious that* $\mathcal{H}_1(t,p,q)\neq 0$, $\mathcal{H}_2(t,p,q)\neq 0$, *at* $(p,q)=(0,0)$, *and* $\mathcal{H}_1(t,p,q)\neq 0$, $\mathcal{H}_2(t,p,q)\neq 0$, *at* $(p,q)=(1,1)$. *Also, an easy computation yields that* $\mathcal{H}_1,\mathcal{H}_2$ *are nondecreasing for each* $t\in(0,1)$. *Moreover, for* $\tau$, $t\in(0,1)$, *and* $p,q\geq 0$, *we see that* $\max\{3,2\}=3$,

$$\mathcal{H}_1(t,\tau p,\tau q)\geq\tau^3\mathcal{H}_1(t,p,q),\qquad \mathcal{H}_2(t,\tau p,\tau q)\geq\tau^3\mathcal{H}_2(t,p,q).$$

*Thus, all the assumption of Theorem 3.4 is fulfilled, so the coupled system* (33) *has a unique positive solution in* $\mathbf{B}_f$ *where* $f(t)=\left(t^{\frac{3}{4}},t^{\frac{4}{3}}\right)$.

**Example 5.3.** *Consider the following system of BVPs:*

$$\begin{cases}\mathscr{D}^{\frac{7}{2}}p(t)=\dfrac{t}{40}+\dfrac{t}{20}\cos|p(t)|+\dfrac{t^2}{20}\sin|q(t)|, & t\in(0,1),\\ \mathscr{D}^{\frac{7}{2}}q(t)=\dfrac{t^2}{50}+\dfrac{t^2}{60}\sin|p(t)|+\dfrac{t}{60}\cos|q(t)|, & t\in(0,1),\\ p^{(j)}(t)=q^{(j)}(t)=0,\ \ j=0,1,2,\ \ at\ \ t=0,\\ p(1)=p\left(\dfrac{1}{2}\right),\ \ q(1)=q\left(\dfrac{1}{2}\right).\end{cases} \tag{35}$$

*From system* (33), *we see that*

$$|\mathcal{H}_1(t,p,q)|\leq\frac{t}{40}+\frac{t}{20}\cos|p(t)|+\frac{t^2}{20}\sin|q(t)|$$

and

$$|\mathcal{H}_2(t,p,q)|\leq\frac{t^2}{50}+\frac{t^2}{60}\sin|p(t)|+\frac{t}{60}\cos|q(t)|.$$

*where* $\varphi_1(t)=\frac{t}{40}$, $\varphi_2(t)=\frac{t^2}{50}$, $\psi_1(t)=\frac{t}{20}$, $\psi_2(t)=\frac{t^2}{60}$, $\sigma_1(t)=\frac{t^2}{20}$, $\sigma_2(t)=\frac{t}{60}$. Also, $\eta=\xi=\frac{1}{2}$, $\lambda_1=\lambda_2=0.17677$. *Thus, by computation, we have*

$$\mathbf{G}_j(1,s)=6.65710\,\frac{(1-s)^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)},\ \ for\ \ j=1,2.$$

*Upon computation, we get*

$$\Delta_1=\int_0^1\mathbf{G}_1(1,s)\varphi_1(s)ds=0.003577<\infty,\ \ \Delta_2=\int_0^1\mathbf{G}_2(1,s)\varphi_2(s)ds=0.000924<\infty.$$

*Similarly, we can also compute.*

$\Lambda_1=\int_0^1\mathbf{G}_1(1,s)[\psi_1(s)+\sigma_1(s)]ds=0.03092853<1$, $\Lambda_2=\int_0^1\mathbf{G}_2(1,s)[\psi_2(s)+\sigma_2(s)]ds=0.00289<1$. *Further, we see that* $\max\{0.007626,\ 0.00185\}=0.007626$. *So, all the conditions of Theorem 3.3 are satisfied. So, the BVP* (34) *has at least one solution and the solution lies in*

$$\mathcal{E}=\{(p,q)\in\mathbf{C}:\|(p,q)\|_{\mathbf{E}\times\mathbf{E}}<0.007626\}.$$

**Example 5.4.** *Taking the following system of BVPs*

$$\begin{cases} \mathscr{D}^{\frac{11}{2}}p(t) + \dfrac{[p(t)+q(t)]^2+1}{(15+t^2)\delta_1} = 0, \;\; \mathscr{D}^{\frac{11}{2}}q(t) + \dfrac{[p(t)+q(t)]^2+t}{(15+t^2)\delta_2} = 0, \;\; t \in (0,1), \\ p^{(j)}(t) = q^{(j)}(t) = 0, \;\; j = 0,1,2,3,4, \;\; at \;\; t = 0, \\ p(1) = p\left(\dfrac{1}{4}\right), \;\; q(1) = q\left(\dfrac{1}{4}\right). \end{cases} \tag{36}$$

*It is simple to check that* $\mathcal{H}_{1,0} = \mathcal{H}_{2,0} = \mathcal{H}_{1,\infty} = \mathcal{H}_{2,\infty} = \infty$. *Also, for any* $(t,p,q) \in \mathbf{I} \times \mathbf{I} \times \mathbf{I}$, *we see that*

$$\begin{aligned} \mathcal{H}_1(t,p,q) &\leq \frac{1}{3\delta_1} \\ \mathcal{H}_2(t,p,q) &\leq \frac{1}{3\delta_2}. \end{aligned}$$

*Thus, all the assumptions of Theorem 3.5 are satisfied with taking* $\alpha = 1$, *so the coupled system* (35) *has two solutions satisfying* $0 < \|(p,q)\|_{\mathbf{E}\times\mathbf{E}} < 1 < \|(p,q)\|_{\mathbf{E}\times\mathbf{E}}$.

**Example 5.5.** *Consider the following coupled systems of boundary value problems:*

$$\begin{cases} \mathscr{D}^{\frac{5}{2}}p(t) + \Gamma\left(\dfrac{5}{2}\right)\left[\dfrac{tp(t)}{16} + \dfrac{t^2q(t)}{32}\right] = 0, t \in (0,1), \\ \mathscr{D}^{\frac{5}{2}}q(t) + \Gamma\left(\dfrac{5}{2}\right)\left[\dfrac{9t^2|\cos(p(t))|}{16\sqrt{\pi}} + \dfrac{9t|\cos(q(t))|}{32\sqrt{\pi}}\right] = 0, t \in (0,1), \\ p^{(j)}(t) = q^{(j)}(t) = 0, \;\; j = 0,1,2, \;\; at \;\; t = 0, \\ p(1) = p\left(\dfrac{1}{2}\right), \;\; q(1) = q\left(\dfrac{1}{2}\right). \end{cases} \tag{37}$$

*Here,* $a_1(t) = \Gamma\left(\frac{5}{2}\right)\frac{t}{16}$, $b_1(t) = \Gamma\left(\frac{5}{2}\right)\frac{t^2}{32}$, $a_2(t) = \Gamma\left(\frac{5}{2}\right)\frac{9t^2}{16\sqrt{\pi}}$, $b_2(t) = \Gamma\left(\frac{5}{2}\right)\frac{9t}{32\sqrt{\pi}}$. *Moreover*

$$\mathbf{M} = \begin{bmatrix} \int_0^1 \mathbf{G}_1(1,s)a_1(s)ds & \int_0^1 \mathbf{G}_1(1,s)b_1(s)ds \\ \int_0^1 \mathbf{G}_2(1,s)a_2(s)ds & \int_0^1 \mathbf{G}_2(1,s)b_2(s)ds \end{bmatrix} = \begin{bmatrix} 0.0460 & 0.0007 \\ 0.0068 & 0.0058 \end{bmatrix}.$$

*Here,* $\rho(\mathbf{M}) = 4.61 \times 10^{-2} < 1$. *Therefore, matrix* $\mathbf{M}$ *converges to zero, and hence the solutions of* (36) *are Hyers-Ulam stable by using Theorem 4.4.*

## 6. Conclusion

We have developed a comprehensive theory on existence of solutions and its Hyers-Ulam stability for system of multipoint BVP of FDEs. The concerned theory has been enriched by providing suitable examples.

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## Conflict of interest

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Chapter 3

# An Extension of Massera's Theorem for *N*-Dimensional Stochastic Differential Equations

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Additional information is available at the end of the chapter

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**Abstract**

In this chapter, we consider a periodic SDE in the dimension $n \geq 2$, and we study the existence of periodic solutions for this type of equations using the Massera principle. On the other hand, we prove an analogous result of the Massera's theorem for the SDE considered.

**Keywords:** stochastic differential equations, periodic solution, Markov process, Massera theorem

## 1. Introduction

The theory of stochastic differential equations is given for the first time by Itô [7] in 1942. This theory is based on the concept of stochastic integrals, a new notion of integral generalizing the Lebesgue–Stieltjes one.

The stochastic differential equations (SDE) are applied for the first time in the problems of Kolmogorov of determining of Markov processes [8]. This type of equations was, from the first work of Itô, the subject of several investigations; the most recent include the generalization of known results for EDO, such as the existence of periodic and almost periodic solutions. It has, among others, the work of Bezandry and Diagana [1, 2], Dorogovtsev [4], Vârsan [12], Da Prato [3], and Morozan and his collaborators [10, 11].

The existence of periodic solutions for differential equations has received a particular interest. We quote the famous results of Massera [9]. In its approach, Massera was the first to establish a relation between the existence of bounded solutions and that of a periodic solution for a nonlinear ODE.

In this work, we will prove an extension of Massera's theorem for the following:

nonlinear SDE in dimension $n \geq 2$

$$dx = a(t,x)dt + b(t,x)dW_t$$

## 2. Preliminaries

Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be the complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions

- $\{F_t\}_{t\geq 0}$ is an increasing family of sub algebras containing negligible sets of $F$ and is continuous at right.

$$F_\infty = \sigma\{\cup_{t\geq 0} F_t\}.$$

Let a Brownian motion $W(t)$, adapted to $\{F_t, t\geq 0\}$, i.e., $W(0) = 0, \forall t \geq 0, W(t)$ is $F_t-$measurable. We consider the SDE

$$\begin{cases} dx = a(t,x)dt + b(t,x)dW_t \\ x(t_0) = z. \end{cases} \tag{1}$$

in $(\Omega, F, \{F_t\}_{t\geq 0}, P)$.

The functions $a(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $b(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable. We suppose that $F_t$ is the completion of $\sigma\{W_r, t_0 \leq r \leq t\}$ for all $t \geq t_0$, and the initial condition $z$ is independent of $W_t$, for $t \geq t_0$ and $E|z|^p < \infty$.

Suppose that the functions $a(t,x)$ and $b(t,x)$ satisfy the global Lipschitz and the linear growth conditions

$$\exists k > 0, \forall t \in \mathbb{R}_+, \forall x, y \in \mathbb{R}^n : \|a(t,x) - a(t,y)\| + \|b(t,x) - b(t,y)\| \leq k\|x - y\|$$

and

$$\|a(t,x)\|^p + \|b(t,x)\|^p \leq k^p(1 + \|x\|^p)$$

We know that if $a$ and $b$ satisfy these conditions, then the system (1) admits a single global solution.

We note by $B$ the space of random $F_t-$measurable functions $x(t)$ for all $t$, satisfying the relation

$$\sup_{t\geq 0} E|x(t)|^2,$$

we consider in $B$ the norm

$$\|x\|_B = \sup_{t\geq 0}\left(E|x|^2\right)^{\frac{1}{2}}$$

$(B, \|.\|_B)$ is the Banach space.

### 2.1. Markov property

The following result proves that the solution of the SDE (1) is a Markov process.

**Theorem 1**. ([5], Th. 2, p. 466) *Assume that $a(t,x)$ and $b(t,x)$ satisfy the hypothesis of the theorem ([5], Th. 1, p. 461) and that $X^{(t,x)}(s)$ is a process such that for $s \in t, \infty)$ for all $t > t_0$ is a solution of SDE*

$$X^{(t,x)}(s) = x + \int_t^s a\left(u, X^{(t,x)}(u)\right)du + \int_t^s b\left(u, X^{(t,x)}(u)\right)dW_u \tag{2}$$

*Then the process $X_t$, solution of SDE (1), is a Markovian process with a transition function*

$$p(t,x;s,A) = P\left(X^{(t,x)}(s) \in A\right).$$

Let $p(s,x;t,A)$ be a transition function; we construct a Markov process with an initial arbitrary distribution. In a particular case, for $t > s$, we associate with the function $p(s,x;t,A)$ a family $X^{(s,z)}(t,\omega)$ of a Markov process such that the processes $X^{(s,z)}(t,\omega)$ exist with initial point $z$ in $s$, i.e.,

$$P\left(X^{(s,z)}(t,\omega) = z\right) = 1 \tag{3}$$

### 2.2. Notions of periodicity and boundedness

**Définition 1.** *A stochastic process $X(t,\omega)$ is said to be periodic with period $T$ $(T > 0)$ if its finite dimensional distributions are periodic with periodic $T$, i.e., for all $m \geq 0$, and $t_1, t_2, \ldots t_m \in \mathbb{R}^+$ the joint distributions of the stochastic processes $X_{t_1+kT}(\omega), X_{t_2+kT}(\omega), \ldots X_{t_m+kT}(\omega)$ are independent of $k$ $(k \in \mathbb{Z})$.*

**Remark 1.** *If $X(t,\omega)$ is $T-$periodic, then $m(t) = EX(t)$, $v(t) = VarX(t)$ are $T-$periodic, in this case, this process is said to be $T-$periodic in the wide sense.*

**Définition 2.** *The function $p(s,x;t,A) = P(X_t \in A/X_s)$ for $0 \leq s \leq t$, is said to be periodic if $p(s,x;t+s,A)$ is periodic in $s$.*

**Définition 3.** *The Markov families* $X^{(t_0,z)}(\omega)$ *are said to be* $p-$*uniformly bounded* $(p>2)$, *if* $\forall\alpha>0, \exists\theta(\alpha)>0, \forall t\geq t_0$:

$$\|z\|_{B,p}\leq\alpha\Rightarrow\left\|X^{(t_0,z)}(\omega)\right\|_{B,p}\leq\theta(\alpha)$$

We denote $X^{(t_0,z)}(\omega)$ as the family of all Markov process for $t_0\in\mathbb{R}^+$and $z$ in $L^p$.

**Remark 2.** *It is easy to see that all* $L_p-$*borné Markov processes* $X_t$, $\left(i.e\exists M>0, \forall t\geq t_0 : \|X_t\|^p_{B,p}\leq M,\right)$ *is* $p-$*uniformly bounded.*

**Lemme 1.** ([6], Theorem 3.2 and Remark 3.1, pp. 66–67) *A necessary and sufficient condition for the existence of a Markov* $T-$*periodic* $X^{(t_0,z)}(\omega)$ *with a given* $T-$*periodic transition function* $p(s,x;t,A)$, *is that for some* $t_0$, $z$, $X^{(t_0,z)}(\omega)$ *are uniformly stochastically continuous and*

$$\lim_{R\to\infty}\lim_{L\to\infty}\inf\frac{1}{L}\int_{t_0}^{t_0+L}p\big(t_0,z;t,\overline{U}_{R,p}\big)dt=0 \tag{4}$$

*if the transition function* $p(s,X_s;t,A)$ *satisfies the following not very restrictive assumption*

$$\alpha(R)=\sup_{z\in U_{\beta(R),p}}0<t_0, t-t_0<Tp\big(t_0,z;t,\overline{U}_{R,p}\big)\to_{R\to\infty}0 \tag{5}$$

*for some function* $\beta(R)$ *which tends to infinity as* $R\to\infty$.

In Eq. (4), we have $R\in\mathbb{R}^*_+$:

$$U_{R,p}=\{x\in\mathbb{R}^n : |x|^p<R\}$$

$$\overline{U}_{R,p}=\{x\in\mathbb{R}^n : |x|^p\geq R\}$$

The conditions of Lemma 1 are of little use for stochastic differential equations, since the properties of transition functions of such processes are usually not expressible in terms of the coefficients of the equation. So, in the following, we will give some new useful sufficient conditions in terms of uniform boundedness and point dissipativity of systems.

**Lemme 2.** *If Markov families* $X^{(t_0,z)}(\omega)$ *with* $T-$*periodic transition functions are uniformly bounded uniformly stochastically continuous, then there is a* $T-$*periodic Markov process.*

*Proof.* By using a Markov inequality [13], we have

$$\begin{aligned}p\big(t_0,z;t,\overline{U}_{R,p}\big)&=\frac{1}{RP(X_{t_0}=z)}E\left|X^{(t_0,z)}(\omega)\right|^p\\&\leq\frac{1}{RP(z)}\left\|X^{(t_0,z)}(\omega)\right\|^p_{B,p}\end{aligned}$$

Then, for $\alpha>0, \exists\theta(\alpha)>0,$ such that for all $t\geq t_0$

$$\|z\|_{B,p} \leq \alpha \Rightarrow \left\|X^{(t_0,z)}(\omega)\right\|_{B,p} \leq \theta(\alpha)$$

we get

$$p\left(t_0, z; t, \overline{U}_{R,p}\right) \leq \frac{1}{RP(z)} \theta^p(\alpha)$$

Then

$$0 \leq \lim_{R\to\infty} \lim_{L\to\infty} \inf \frac{1}{L} \int_{t_0}^{t_0+L} p\left(t_0, z; t, \overline{U}_{R,p}\right) dt \leq \lim_{R\to\infty} \frac{1}{RP(z)} \theta^p(\alpha) \left( \lim_{L\to\infty} \inf \frac{1}{L} \int_{t_0}^{t_0+L} dt \right)$$
$$= \lim_{R\to\infty} \frac{\theta^p(\alpha)}{RP(z)} = 0,$$

that is, Eq. (4). From Lemma 1, we have a $T-$periodic Markov process.

## 3. Main result

Let the SDE

$$\begin{cases} dx = a(t,x)dt + b(t,x)dW_t \\ x_{t_0} = z, \quad E|z|^p < \infty \end{cases} \tag{6}$$

We assume that this SDE satisfies the conditions as in Section 2 after Eq. (1).

Suppose that

$H_1$) the functions $a(t,x)$ and $b(t,x)$ are $T-$periodic in $t$.

$H_2$) the functions $a(t,x)$ and $b(t,x)$ satisfy the condition

$$\|a(t,x)\|^p + \|b(t,x)\|^p \leq \phi(\|x\|^p), p > 2 \tag{7}$$

where $\phi$ is a concave non-decreasing function.

**Lemme 3.** ([13], Lemme 3.4) *Assume that $a(t,x)$and $b(t,x)$ verify*

$$E(\|a(t,x)\|^p) + E(\|b(t,x)\|^p) \leq \eta, p > 2$$

then, the solutions of periodic SDE (6) are uniformly stochastically continuous.

We prove the Massera's theorem for the SDE in dimension $n \geq 2$.

**Theorem 2.** *Under $(H_1), (H_2)$, if the solutions of the SDE (6) are $L_p-$bounded, then there is a $T-$periodic Markov process.*

*Proof.* We note by $X^{(t_0,z)}(t,\omega)$ an $L_p$-bounded solution of SDE (6), from Theorem 1, this solution is unique a Markov process that is $F_t-$measurable. Suppose that $p(t_0,z;t,A)$ is a transition function of Markov process $X^{(t_0,z)}(t,\omega)$, under $(H_1)$ and since $p(t_0,z;t,A)$ depend of $a(t,x), b(t,x)$ then this function is $T-$periodic in $t$. In the other hand, $\phi$ is concave non-decreasing function, we get

$$E\phi(|x|^p) \leq \phi(E|x|^p)$$

From the $L_p-$boundedness of $X^{(t_0,z)}(t,\omega)$, then under $(H_2)$: $\exists\eta > 0$ such that

$$E\left\|a\left(t, X^{(t_0,z)}(t,\omega)\right)\right\|^p + E\left\|b\left(t, X^{(t_0,z)}(t,\omega)\right)\right\|^p < \eta$$

for $p > 2$. By Lemma 3, we have $X^{(t_0,z)}(t,\omega)$ is $p-$uniformly bounded and $p-$uniformly stochastically continuous, this gives, the conditions of Lemma 2 are verified, finally, we can conclude the existence of the $T-$periodic Markov process. □

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Chapter 4

# Phase Portraits of Cubic Dynamic Systems in a Poincare Circle

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Additional information is available at the end of the chapter

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**Abstract**

In the proposed chapter, we are going to outline the results of a study on an arithmetical plane of a broad family of dynamic systems having polynomial right parts. Let these polynomials be of cubic and square reciprocal forms. The task of our investigation is to find out all the different (in the topological sense) phase portraits in a Poincare circle and indicate the coefficient criteria of their appearance. To achieve this goal, we use the Poincare method of central and orthogonal consecutive displays (or mappings). As a result of this thorough investigation, we have constructed more than 250 topologically different phase portraits in total. Every portrait we present using a special table called a descriptive phase portrait. Each line of such a special table corresponds to one invariant cell of the phase portrait and describes its boundaries, as well as a source of its phase flow and a sink of it.

**Keywords:** dynamic systems, phase portraits, phase flows, Poincare sphere, Poincare circle, singular points, separatrices, trajectories

## 1. Introduction

A dynamic system appears to be a mathematical model of some process or phenomenon, in which fluctuations and other so-called statistical events are not taken into consideration. It can be characterized by its initial state and a law according to which the system goes into a different state. A *phase space* of a dynamic system is the totality of all admissible states of this system.

It is necessary to distinguish dynamic systems with the discrete time and with the continuous time. For dynamic systems with the discrete time (they are called *cascades*), a system's behavior is described with a sequence of its states. For dynamic systems with continuous time (which are called *flows*), a state of the system is defined for each moment of time on a real or an imaginary axis. Cascades and flows are the main subject of study in symbolic and topological dynamics.

Dynamic systems, both with discrete and continuous time, can be usually described by an autonomous system of differential equations, defined in a certain domain and satisfying in it the conditions of the Cauchy theorem of existence and uniqueness of solutions of the differential equations.

Singular points of differential equations correspond to equilibrium positions of dynamic systems, and periodical solutions of differential equations correspond to closed phase curves of dynamic systems.

The main task of the theory of dynamic systems is a study of curves, defined by differential equations. This process includes splitting of a phase space into trajectories and studying their limit behavior—finding and classifying the equilibrium positions, and revealing the attracting and repulsive manifolds (i.e., attractors and repellers; sinks and sources). The most important notions of the theory of dynamic systems are the notion of stability of equilibrium states, which means the ability of a system under considerably small changes of initial data to remain near an equilibrium state (or on a given manifold) for an arbitrary long period of time, as well as the notion of roughness of a system (i.e., the saving of a system's properties under small changes of a model itself). A rough dynamic system is a system that preserves its qualitative character of motion under small changes of parameters.

The research methods proposed in this chapter are new and effective; they can also be used for the study of applied dynamic systems of the second order with polynomial right parts.

According to Jules H. Poincare, a normal autonomous second-order differential system with polynomial right parts, in principle, allows its full qualitative investigation on an extended arithmetical plane $\overline{R}^2_{x,y}$ [1]. Inspired by the great Poincare's works, mathematicians of the next generations, including contemporary researchers, have studied some of such systems, for example, quadratic dynamic systems [2], ones containing nonzero linear terms, homogeneous cubic systems, and dynamic systems with nonlinear homogeneous terms of the odd degrees (3, 5, 7) [3], which have a center or a focus in a singular point O (0, 0) [4], as well as other particular kinds of systems.

We consider in the present chapter a family of dynamic systems on a real plane *x, y*.

$$\frac{dx}{dt} = X(x,y), \frac{dy}{dt} = Y\,(x,y) \tag{1}$$

such that $X\,(x,\,y)$, $Y\,(x,\,y)$ are reciprocal forms of $x$ and $y$, $X$ is a cubic, $Y$ a square form, and $X\,(0,1) > 0$, $Y\,(0,\,1) > 0$. Our objective is to depict in a Poincare circle all kinds (different in the topological sense) of possible for systems phase portraits for Eq. (1), and also to indicate the criteria of every portrait realization close to coefficient ones. With this aim, we apply Poincare's method of consecutive mappings: first, the central mapping of a plane $x$, $y$ (from a center (0, 0, 1) of a sphere $\sum$), augmented with a line at infinity (i.e., $\overline{R}^2_{x,y}$ plane) on a sphere $\sum$: $X^2 + Y^2 + Z^2 = 1$ with identified diametrically opposite points, and second, the orthogonal mapping of a lower enclosed semi-sphere of a sphere $\sum$ to a circle $\overline{\Omega}$: $x^2 + y^2 \leq 1$ with identified diametrically opposite points of its boundary $\Gamma$. We will now describe this process in more detail.

The circle $\overline{\Omega}$ and the sphere $\sum$ in this process are called the Poincare circle and the Poincare sphere, respectively [1].

## 2. Basic definitions and notation

$\varphi(t, p)$, $p = (x, y)$—a fixed point: = a solution (a motion) of Eq. (1) – system with initial data$(0, p)$.

$L_p$:$\varphi = \varphi(t, p)$, $t \in I_{\max}$, – a trajectory of motion $\varphi(t, p)$.

$L_p^{+(-)}$:= + $(-)$ – a semi-trajectory of a trajectory $L_p$.

$O$-curve of a system := the system's semi-trajectory $L^s{}_p$($p \neq O$, $s \in \{+, -\}$) adjoining to a point $O$ under a condition such that $st \to +\infty$.

$O^{+(-)}$- curve of a system: = the system's $O$-curve $L_p^{+(-)}$.

$O_{+(-)}$-curve of a system: = the system's $O$-curve adjoining to a point $O$ from a domain $x > 0$ ($x < 0$).

$TO$-curve of a system: = the system's $O$-curve, which, being supplemented by a point $O$, touches some ray in it.

A nodal bundle of $NO$-curves of a system := an open continuous family of the system's $TO$-curves $L^s{}_p$, where $s \in \{+, -\}$ is a fixed index, $p \in \wedge$, $\wedge$ a simple open arc, $L^s{}_p \cap \wedge = \{p\}$.

A saddle bundle of $SO$-curves of a system, a separatrix of the point $O$:= a fixed $TO$-curve, which is not included in some bundle of $NO$-curves of a system.

$E$, $H$, $P$-$O$-sectors of a system: an elliptical, a hyperbolic, a parabolic sector.

A topological type (T-type) of a singular point $O$ of a system:= a word $A_O$ consisting of letters $N$, $S$ (a word $B_O$ consisting of letters $E$, $H$, $P$), which describes a circular order of bundles $N$, $S$ of its $O$-curves (of its $O$-sectors $E$, $H$, $P$) when traversing the point $O$ in the " + "-direction, i. e., counterclockwise, starting with some of them.

$$P(u) := X(1, u) \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3,$$

$$Q(u) := Y(1, u) \equiv a + bu + cu^2.$$

**Note 1.** For every Eq. (1) system:

1) T-type of a singular point $O$ in its form $B_O$ is easy to construct using its T-type in the form $A_O$, and going backward (we need to determine both forms, see Corollary 1);

2) Real roots of a polynomial $P(u)$ (polynomial $Q(u)$) are in fact angular coefficients of isoclines of infinity (isoclines of a zero));

3) When we write out the real roots of the system's polynomials $P(u)$, $Q(u)$, separately or all together, we always number the roots of each one of them in an ascending order.

## 3. Topological type (T-type) of a singular point $O(0, 0)$

In order to find all $O$-curves and to split their totality into the bundles $N$, $S$, let us use the method of exceptional directions of a system in the point $O$ [1]. According to this method, the equation of exceptional directions for the point $O$ of the Eq. (1) system has the form.

$$xY(x, y) \equiv x\left(ax^2 + bxy + cy^2\right) = 0.$$

For this, the following cases are possible:

**1.** When $d \equiv b^2 - 4ac > 0$, this equation defines simple straight lines $x = 0$ and.

$$y = q_i x, \quad i = 1, 2, \ q_1 < q_2$$

**2.** When $d = 0$, this equation defines the straight line $x = 0$ and the double straight line.

$$y = qx, q = -\frac{b}{2c}$$

**3.** When $d < 0$, the equation defines only the straight line $x = 0$.

Theorem 1 is true for the aforementioned cases [5].

**Theorem 1.** Words $A_O$ and $B_O$, which define a topological type (T-type) of a singular point $O$ (0, 0) of the Eq. (1) system*:*

1) in the case of $d > 0$, depending on signs of values $P(q_i) = 1, 2,$ have forms, indicated in a **Table 1**;

| $r$ | $P(q_1)$ | $P(q_2)$ | $A_O$ | $B_O$ |
|---|---|---|---|---|
| 1, 4 | + | + | $S_0S^1_+N^2_+S^0N^1_-S^2_- = S_0S^1_+NS^2_-$ | $PH^2$ |
| 2 | – | – | $S_0N^1_+S^2_+S^0S^1_-N^2_- = NS^2_+S^0S^1_+$ | $PH^2$ |
| 3, 6 | – | + | $S_0N^1_+N^2_+S^0S^1_-S^2_-$ | $PEPH^3$ |
| 5 | + | – | $S_0S^1_+S^2_+S^0N^1_-N^2_-$ | $H^3PEP$ |

**Table 1.** T-type of a singular point when $d > 0$ $(r = \overline{1, 6})$.

| $q$ | $P(q)$ | $A_O$ | $B_O$ |
|---|---|---|---|
| + | + | $S_0 S_+ N_+ S^0$ | $H^2P$ |
| – | – | $S_0 N_+ S_+ S^0$ | $PH^2$ |
| + | – | $S_0 S^0 S_- N_-$ | $H^2P$ |
| – | + | $S_0 S^0 N_- S_-$ | $PH^2$ |
| 0 | + | $S_0 S_+ N S_-$ | $H^2P$ |
| 0 | – | $N S_+ S^0 S_-$ | $PH^2$ |

**Table 2.** T-type of the singular point $O(0, 0)$ when $d = 0$.

2) in the case of $d = 0$ depending on signs of values $q$ and $P(q)$, they have forms, indicated in a **Table 2**,

3) in the case of $d < 0$ they have forms: $A_O = S_0 S^0$, $B_O = HH$ (**Table 1**).

**Note 2**. Let us clarify the meaning of the new symbols introduced in Theorem 1.

$S_0$ ($S^0$) means a bundle $S$, adjoining to point $O(0,0)$ from the domain $x > 0$ along a semi-axis $x = 0, y < 0$, when $t \to +\infty$ (along a semi-axis $x = 0,\ \ y > 0,$ when $t \to -\infty$).

The lower sign index " + " or "–" on every bundle $N$ or $S$, different from $S_0$ and $S^0$, indicates whether the bundle consists of $O_+$-curves or of $O_-$-curves. Upper index 1 or 2 on every such a bundle indicates whether its $O$-curves are adjoining to point $O$ along a straight line $y = q_1 x$ or along a straight line $y = q_2 x$.

In **Table 2**, row 5, 6, a bundle $N$ does not have a lower sign index because it contains both $O_+$ -curves and $O_-$ -curves simultaneously.

**Corollary 1.** From Theorem 1, it follows, that Eq. (1) systems do not have limit cycles on the $R^2_{x;y}$ plane.

Indeed, such a cycle could surround a singular point $O$ (0,0) of an Eq. (1) system, and then the Poincare index of this singular point must be equal to 1 [1]. However, Bendixon's formula for the index of an isolated singular point of a smooth dynamic system is as follows:

$$I(O) = 1 + \frac{e - h}{2}$$

where $e(h)$ is the number of elliptical (hyperbolic) $O$-sectors of the system. This formula and our Theorem 1 give: for the singular point $O$ (0, 0) of every Eq. (1) system, Poincare index $I(O) = 0$.

**Corollary 2.** For the singular point $O$ (0, 0) of an Eq. (1) system, 11 different topological types (T-types) are possible, and from the analysis of these 11 T-types we can conclude:

for every Eq. (1) system, the singular point $O(0, 0)$ has not more than four separatrices (actually 2, 3, or 4 ones).

## 4. Infinitely remote singular points (IR points)

Now it is time to discuss the behavior of trajectories of the Eq. (1) systems in a neighborhood of infinity. For the investigation of this question we use the method of Poincare consecutive transformations, or mappings [1].

The first Poincare transformation

$$x = \frac{1}{z}, \quad y = \frac{u}{z} \quad \left(u = \frac{y}{x}, \quad z = \frac{1}{x}\right).$$

unambiguously maps a phase plane $R^2_{x,y}$ of the Eq. (1) system onto a Poincare sphere $\sum$: $x^2 + y^2 + z^2 = 1$ (where $z = -Z$ [1]) with the diametrically opposite points identified, which is considered without its equator $E$, and an infinitely remote straight line of a plane $\overline{R^2_{x,y}}$. The first Poincare transformation maps onto the equator $E$ of the sphere $\sum$; the diametrically opposite points are also considered to be identified.

The Eq. (1) system in this mapping transforms into a system, which in the Poincare coordinates $u, z$ after a time change $dt = -z^2 d\tau$ looks like the following:

$$\frac{du}{d\tau} = P(u)u - Q(u)z, \qquad \frac{dz}{d\tau} = P(u)z,$$

where $P(u) :\equiv X(1, u)$ and $Q(u) :\equiv Y(1, u)$ are reciprocal polynomials.

This new system is determined on the whole sphere $\sum$, including its equator, and on the whole $(u, z)$ − plane $\alpha^*$, which is tangent to a sphere $\sum$ at point C = (1, 0, 0). We shall study this system, namely on a plane $\overline{R^2_{u,z}}$, and project the received results onto a closed circle $\overline{\Omega}$, sequentially mapping, first, a plane $R^2_{u,z}$ onto the sphere $\sum$ from its center, and second, its lower semi-sphere $\overline{H}$ onto the Poincare circle $\overline{\Omega}$, i. e., onto a closed unit circle of a plane $R^2_{x,y}$ through the orthogonal mapping.

For our new system, the axis $z = 0$ is invariant (consists of this system's trajectories). On this axis, lie its singular points $O_i(u_i, 0)$, $i = \overline{0, m}$, where $u_i$, $i = \overline{1, m}$ are all real roots of the polynomial $P(u)$, and $u_0 = 0$; at the same time, there may exist $i_0 \in \{1, \ldots, m\}$: $u_{i_0} = 0$. Let us call such points IR points of the first kind for the Eq. (1) system.

The second Poincare transformation

$$x = \frac{v}{z}, \quad y = \frac{1}{z} \left(v = \frac{x}{y}, \quad z = \frac{1}{y}\right)$$

also unambiguously maps a phase plane $R^2_{x,y}$ onto a Poincare sphere $\sum$ with the diametrically opposite points identified, considered without its equator. Every Eq. (1) system transforms into a system, which in the coordinates $\tau, v, z$ looks like the following:

$$\frac{dv}{d\tau} = -X(v,1) + Y(v,1)vz, \qquad \frac{dz}{d\tau} = Y(v,1)z^2.$$

This last system is determined on the whole sphere $\sum$, and on the whole $(v,z)$ − plane $\widehat{\alpha}$, which is tangent to a sphere $\sum$ at point $D = (0,1,0)$ [1]. A set $z = 0$ is invariant for this last system. On this set, lie its singular points$(v_0, 0)$, where $v_0$ is any real root of the polynomial $X(v,1) \equiv p_3 + p_2 v + p_1 v^2 + p_0 v^3$. It would be natural to call such points IR points of the second kind for Eq. (1) systems, but each of these points, for which $v_0 \neq 0$, obviously coincides with one of the IR-points of the first kind, namely with the point $\left(\frac{1}{v_0}, 0\right)$,

while $v_0 = 0$ is not a root of the polynomial $X(x, 1)$, because $X(0, 1) = p_3 \neq 0$ for the Eq. (1) system. Consequently, the following corollary is correct.

**Corollary 3.** The infinitely remote singular points of any Eq. (1) system are only IR-points of the first kind.

With the orthogonal projection of a closed lower semi-sphere $\overline{H}$ of a Poincare sphere $\sum$ onto a plane *x, y*, its open part *H* one-to-one maps onto an open Poincare circle Ω, while its boundary *E* (an equator of the Poincare sphere $\sum$) maps onto the boundary of the Poincare circle $\Gamma = \partial\Omega$, which implies the following. 1) Trajectories of any Eq.(– (including its singular point *O* (0, 0)) are displayed in a circle Ω, filling it.

2) Such a system's infinitely remote trajectories (including IR points) are displayed on the boundary *Γ* of a circle Ω, filling it.

Following Poincare, we call the first trajectories of the Eq. (1) system in Ω, and the second, we call trajectories of the Eq. (1) system on *Γ*.

As it follows from the aforementioned conclusions, to each IR point $O_i(u_i, 0)$, of the Eq. (1) system, $i \in \{1, \dots, m\}$, correspond two diametrically opposite points situated on the *Γ* circle.

$$O_i^{\pm}(u_i, 0) : O_i^{+}\left(O_i^{-}\right) \in \Gamma^{+(-)} := \Gamma\big|_{x>0\ (x<0)}.$$

$\forall i \in \{1, \dots, m\}$ for the point $O_i^{+}\left(O_i^{-}\right)$, we shall introduce the following notation.

1. Let a $O_i^{+}\left(O_i^{-}\right)$—curve be a semi-trajectory of the Eq. (1) system in Ω, starting in an ordinary point $p \in \Omega$ and adjacent to a point $O_i^{+(-)}$.
2. A notation for bundles *N*, *S*, adjacent to a point $O_i^{+}\left(O_i^{-}\right)$ from the circle Ω, similar to the notation introduced for the point *O* (0, 0).

**3.** A notation of a word $A_i^+(A_i^-)$ consisting of letters $N,S$, which fixes an order of bundles of $O_i^+(O_i^-)$-curves at a semi-circumvention of the point $O_i^+(O_i^-)$ in the circle $\Omega$ in the direction of increasing $u$.

We shall describe a T-type of a point $O_i^+(O_i^-)$with a word $A_i^+(A_i^-)$, and a T-type of a point $O_i$ with words $A_i^{\pm}$.

T-types of IR points $O_0^{\pm}(0,0)$ of Eq. (1) systems are described in the following theorem.

**Theorem 2.** Let a number $u = 0$ be the multiplicity $k \in \{0, \dots, 3\}$ of the root of a polynomial $P(u)$ of the Eq. (1) system. Then, words $A_0^{\pm}$, which determine the topological types (T-types) of IR points $O_0^{\pm}(0,0)$ of this system, depending on the value of $k$ and a sign of a number $ap_k$ (where $a$ and $p_k$ are coefficients of the system), have the forms as shown in **Table 3** [5].

**Corollary 4**. IR points $O_0^{\pm}$of any Eq.(1)—system do not have separatrices.

T-types of IR points $O_i(u_i, 0) \neq O_o(0,0)$, $i = \overline{1,m}$, of Eq. (1) systems are described in the following theorem.

**Theorem 3**. Let a real number $u_i(\neq 0)$ be a multiplicity $k_i \in \{1,2,3\}$ of the root of a polynomial $P(u)$ of an Eq. (1) system. Then for this system, a value $g_i = P^{(ki)}(u_i)Q(u_i) \neq 0$ and words $A_i^{\pm}$, which determine topological types (T-types) of IR points $O_i^{\pm}(u_i, 0)$ of this system, depending on the value of $k_i$ and signs of numbers $u_i$ and $g_i$, have forms as shown in **Table 4** [5].

**Corollary 5**. As can be seen from Theorems 2 and 3, for the IR points of Eq. (1) systems, only a finite number (13) of different T-types are possible. The investigation of these T-types shows that IR-points of each Eq. (1) system have only $m$ separatrices: one separatrice for every singular point $O_i(u_i, 0)$, $i = \overline{1,m}$.

**Note 3**. In **Tables 3** and **4**, the lower sign index " + " or "–" on every bundle $N$ or $S$, indicates whether the bundle adjusts to the point $O_i^+$ (or to the point $O_i^-$) from the side $u > u_i$ or from the side $u < u_i$ of the isocline $u = u_i$.

In **Table 3**, row 1, a bundle $N$ does not have a lower sign index because as the detailed study of this case shows, it contains $O_i^+$-curves ($O_i^-$-curves) in every domain $|u|> 0$ [5].

| $k$ | $ap_k$ | $A_0^+$ | $A_0^-$ |
|---|---|---|---|
| 0 | 0 | $N$ | $N$ |
| 0, 2 | + (−) | $N_+(N_-)$ | $N_-(N_+)$ |
| 1, 3 | + (−) | $N_-N_+(\varnothing)$ | $\varnothing(N_-N_+)$ |

**Table 3.** T-types of IR points $O_0^{\pm}(0,0)$.

| $u_i$ | $k_i$ | $g_i$ | $A_i^+$ | $A_i^-$ |
|---|---|---|---|---|
| +(−) | 1, 3 | + | $N_+(N_-)$ | $S_-(S_+)$ |
| +(−) | 1, 3 | − | $S_-(S_+)$ | $N_+(N_-)$ |
| +(−) | 2 | + | $S_-N_+(\varnothing)$ | $\varnothing(N_-S_+)$ |
| +(−) | 2 | − | $\varnothing(N_-S_+)$ | $S_-N_+(\varnothing)$ |

**Table 4.** T-types of IR points $O_i^{\pm}(u_i, 0)$, $i \in \{1, \ldots, m\}$.

## 5. Systems containing 3 and 2 multipliers in their right parts

In this section, we present a solution to the main assigned problem for those Eq. (1) systems whose decompositions of forms *X (x, y), Y (x, y)* into real forms of lower degrees contain 3 and 2 multipliers, respectively:

$$\mathrm{X}(x,y) = p_3(y-u_1x)(y-u_2x)(y-u_3x), \mathrm{Y}(x,y) = \mathrm{c}(y-q_1x)(y-q_2x) \tag{2}$$

where $p_3 > 0$, $c > 0$, $u_1 < u_2 < u_3$, $q_1 < q_2$, $u_i \neq q_j$ for each *i* and *j*.

The solution process contains the follows steps.

### 5.1. Basic concepts and notation

The following notations are introduced for the arbitrary system under consideration in the Section 5.

*P(u), Q(u)* – the system's polynomials *P, Q:*

$$\mathrm{P(u)}:= X(1,u) \equiv p_3(u-u_1)(u-u_2)(u-u_3), \ \ Q(u):= Y(1,u) \equiv c(u-q_1)(u-q_2)$$

*RSP* (*RSQ*) – an ascending sequence of all real roots of then system's polynomial *P(u) (Q(u)), RSPQ* – an ascending sequence of all real roots of both the system's polynomials *P(u), Q(u).*

### 5.2. The double change (DC) transformation

Let us call a double change of variables in this dynamic system: *(t, y)* → *(−t, −y).* The double change transformation transforms the system under consideration into another such system, for which numberings and signs of roots of polynomials *P(u), Q(u),* as well as the direction of motion upon trajectories with the increasing of *t* are reversed. Let us agree to call a pair of different Eq. (2) systems mutually inversed in relation to the DC transformation, if this transformation appears to convert one into another, and call them independent of a DC transformation in the opposite case.

Clearly, 10 different types of *RSPQ* are possible for an arbitrary Eq. (2) system, as $C_5^2 = \frac{5!}{3!2!} = 10$.

As we can conclude using the DC transformation of Eq. (2) systems, six of the *RSPQ*s appear to be independent in pairs. Similarly, each of the remaining four systems has the mutually inversed one among the first six Eq. (2)-systems.

Let us assign a specific number $r \in \{1, \ldots, 10\}$ to each one of the different *RSPQ*s of the Eq. (2) system in such a manner that *RSPQr* = $\overline{1,6}$ are independent in pairs, while *RSPQ* sequences with numbers $r = \overline{7,10}$ are mutually inversed to *RSPQ*`s which have numbers $r = \overline{1,4}$.

It is time to introduce the important notion of a family number *r* of Eq. (2) systems.

An *r family* of Eq. (2) systems $:=$ the totality of systems (belonging to Eq. (2) family) having the *RSPQ* number *r*.

Now following a single plan, we consistently investigate the families of Eq. (2)systems that have numbers $r = \overline{1,6}$. For families having numbers $r = \overline{7,10,}$ we obtain data through the DC-transformation of families, $r = \overline{1,4}$.

A plan of the investigation of each selected Eq. (2) family contains the follows items.

1. We determine a list of singular points of systems of the fixed family in a Poincare circle $\overline{\Omega}$. They appear to be a point $O\ (0,\ 0) \in \Omega$ and points $O_i^{\pm}$ $(u_i,\ 0)$ $\in \Gamma$, $i = \overline{0,3}$, $u_0 = 0$. For every point in the list, we use the notions of a saddle (S) and node (N) bundles adjacent to this point's semi-trajectories, of a separatrix of the singular point, and of a topodynamical type of the singular point (TD type).

2. Further, we split the family under consideration to subfamilies with numbers $s = \overline{1,7}$. For every subfamily, we reveal topodynamical types of singular points and separatrices of them.

3. We investigate the separatrices' behavior for all singular points of systems belonging to the chosen subfamily $\forall$ s $\in$ {1, …, 7}. Very important are the following questions: a question of a uniqueness of a continuation of every given separatrix from a small neighborhood of a singular point to all the lengths of this separatrix, as well as a question about a mutual arrangement of all separatrices in a Poincare circle Ω. We answer these questions for all families of systems under consideration.

4. As a result of all previous studies, we depict phase portraits of dynamic systems of a given family and outline the criteria of every portrait appearance [5, 6].

From this section, we can conclude the following:

Systems of the family number $r = 1$ have 25 different types of phase portraits.

Systems of families number 2 and 3: there are 9 types of phase portraits per family.

Systems of families 4 and 5: there exist 7 types of phase portraits per family.

Systems belonging to the family number $r = 6$ show 36 different types of phase portraits.

Hence, we have obtained 93 different types in total for the systems described in this section—a lot of possible types at first glance. However, it is important to keep this in mind: every given family includes an uncountable number of differential systems.

## 6. Two classes of systems containing various combinations of two different multipliers in both right parts: an A-class

In Sections 6 and 7, the problem has been solved for an Eq. (3) family. The Eq. (3) family of Eq. (1) systems is as follows—the family consists of a totality of all Eq. (1) systems; for each of them, decompositions of forms *X (x, y), Y (x, y)* into real multipliers of the lowest degrees contain two multipliers each:

$$X(x,y) = p(y-u_1x)^{k_1}(y-u_2x)^{k_2}, \mathrm{Y}(x,y) = \mathrm{q}(y-q_1x)(y-q_2x) \tag{3}$$

where $p, \ q, \ u_1, u_2, q_1, q_2 \in R, p>0, \ q>0, \ u_1<u_2, \ q_1<q_2, \ u_i \neq q_j$ for each $i,j \ \in \{1,2\}$, $k_1, k_2 \in N, \ k_1+k_2 = 3$.

It is natural to distinguish two classes of Eq. (3) systems. The A class contains systems with $k_1 = 1, \ k_2 = 2;$ and the B class contains systems with $k_1 = 2, k_2 = 1$.

In this section, we give a full solution of the assigned task for systems belonging to the A class of the Eq. (3) family, i.e.,

$$\frac{dx}{dt} = p(y-u_1x)(y-u_2x)^2, \ \frac{dy}{dt} = \mathrm{q}(y-q_1x)(y-q_2x) \tag{4}$$

The process of forming the solution contains steps similar to the ones described in Section 4 of this chapter.

For an arbitrary Eq.(4)– system, we introduce the following concepts.

Let *P(u), Q(u)* be the system's polynomials *P, Q:*

$$\mathrm{P(u)}:= X(1,u) \equiv p(u-u_1)(u-u_2)^2, \quad Q(u):= Y(1,u) \equiv q(u-q_1)(u-q_2),$$

and *RSP* (*RSQ*) be an ascending sequence of all the real roots of the system's polynomial, while *P(u) (Q(u)),RSPQ* is an ascending sequence of all the real roots of both system's polynomials *P(u)* and *Q(u).* There exist 6 different possible variants of *RSPQ* as $C_4^2 = \frac{4!}{2!2!} = 6$. Let us number them from 1 to 6 in some order.

Now let us put into use an important notion:

An *r-family* of Eq.(4) – systems is the totality of Eq. (4) dynamic systems with the *RSPQ* number *r* from the list of six allowable variants.

**A consistent research of families of Eq.** (4) **dynamic systems**.

The steps of research of every fixed family belonging to Eq. (4) dynamic systems are as follows.

1. For all singular points of a given dynamic system that belongs to the family under consideration, let us introduce notions of S (saddle) and N (node) bundles of semi-trajectories, which are adjacent to a chosen singular point; also let us introduce a notion for its separatrix and a notion for its topodynamical type (TD-type).

2. Now the considered family must be divided into subfamilies numbered $s \in \{1, \ldots, 5\}$.Then it is necessary to determine the TD-types of singular points of systems belonging to the obtained subfamilies, and separatrices of singular points $\forall s = \overline{1,5}$.
3. For all five subfamilies, we investigate the separatrices` of singular points behavior and find an answer to a question concerning a uniqueness of a global continuation of every chosen separatrix from a tiny neighborhood of a singular point to all the lengths of this separatrix in the Poincare circle Ω, as well as an answer to a question of all separatrices` mutual arrangement in Ω.

The mutual arrangement of all separatrices in the Poincare circle is invariable when, for a given *s*, a global continuation of every separatrix of each singular point of the subfamily number *s* is unique. Consequently, all systems of a chosen subfamily number *s* have, in a Poincare circle, one common type of phase portrait.

But in a different situation, when, for a fixed number *s*, systems of such subfamily have, for example, *m* separatrices with global continuations that are not unique, this subfamily is divided into *m* additional subfamilies (so as to say *subsubfamilies*) of the next order.

As we could understand conducting their further study, for each of *subsubfamilies*, the global continuation of every separatrix is unique, and the mutual arrangement of separatrices in the Poincare circle Ω is invariable.

As a result, the topological type of phase portrait of all systems belonging to this *subsubfamily* in the $\overline{\Omega}$ circle is common for the chosen *subsubfamily*.

**4**. We depict phase portraits in $\overline{\Omega}$ for the systems of Eq.(4) families, $r = \overline{1,6}$, in the two possible forms (the table and the graphic ones), and indicate for each portrait close to coefficient criteria of its realization.

A conclusion for the Section 6 of our chapter is:

1. Eq. (4)–systems belonging to the number 1 family have in the Poincare circle $\overline{\Omega}$, 13 different topological types of phase portraits.
2. Eq.(4)– systems of the family number 2 have 7 types.
3. Family number 3 have 10 types.
4. Family numbers 4, 5, and 6 have 5 different types of phase portraits per number.

This means that in total, all large families of Eq.(4) dynamic systems of the A class may have 45 different topological types of phase portraits in a Poincare circle.

## 7. Systems with 2 different multipliers in both right parts, belonging to a B class

In this section, the full solution of our task for Eq. (3) systems of the B class is given:

$$\frac{dx}{dt} = p(y - u_1 x)^2 (y - u_2 x),\ \frac{dy}{dt} = \mathrm{q}(y - q_1 x)(y - q_2 x). \tag{5}$$

For an arbitrary Eq.(5)– system, *P(u), Q(u)* are the system's polynomials *P, Q.*

$$\mathrm{P(u)} :\ = X(1, u) \equiv p(u - u_1)^2 (u - u_2), \quad Q(u) :\ = Y(1, u) \equiv q(u - q_1)(u - q_2),$$

*RSPQ* shows 6 different variants, because $C_4^2 = 6$.

We can thus conclude that all Eq. (5) family of systems is split into 52 different subfamilies, and all systems of each chosen subfamily show in a circle $\overline{\Omega}$, one common type of a phase portrait belonging to this particular subfamily. We have constructed all 52 topologically different phase portraits.

## 8. Systems containing 3 and 1 different multipliers in right parts

In this section, we solve the problem for an Eq. (6) family, i.e., for a family of Eq. (1) systems

$$\frac{dx}{dt} = p_3(y - u_1 x)(y - u_2 x)(y - u_3 x),\ \frac{dy}{dt} = c(y - q_1 x)^2 \tag{6}$$

$$p_3 \ > 0,\ \ c > 0,\ \ u_1 < u_2 < u_3, q(\in R) \neq u_i, i = \overline{1, 3}.$$

The solution process includes the follows steps. Let us break the Eq. (6) family into subfamilies numbered $r = \overline{1, 4}$.

Each of these is a totality of systems with an *RSPQ* number *r,* where *r* is the system's number in the list of possible *RSPQ*s.

1. $u_1, u_2, u_3, q,$
2. $u_1, u_2, q, u_3,$
3. $u_1, q, u_2, u_3,$
4. $q, u_1, u_2, u_3.$

Applying to the Eq. (6) system, a double change of variables (DC): *(t, y)* $\rightarrow$ *(−t,-y),* we reveal that it transforms families of these systems having the numbers *r* = 1, 2, 3, 4, into their families with numbers *r* = 4, 3, 2, 1 respectively, and backward. We emphasize: this fact means that families of Eq. (6) systems having numbers 1 and 2 are not connected with the DC transformation, and that families having numbers 3 and 4 are not related to each other; at the same time, family number 3 is mutually inversed by the DC transformation to the family number 2, and family number 4 is mutually inversed to the family number 1 correspondingly. This conclusion follows from the consideration of their *RSPQ* sequences [5, 6].

1. We study alternately the families of systems, *r* = 1,2, following the common program of Eq. (1) systems study [5], i.e.:

1. We fix $r \in \{1, 2\}$, then we break the chosen family into subfamilies numbered $s$ [5, 6], $s = \overline{1, 9,}$ and find the topodynamical types (TD-types) of singular points of these systems.

2. We construct for the systems of a fixed subfamily $\forall s = \overline{1, 9,}$ the so-called off-road map (ORM) [5–7]. The ORM helps us to find an $\alpha(\omega)-$ limit set of every $\alpha(\omega)-$ separatrix. It also lets us describe the mutual arrangement of all separatrices in the Poincare circle $\Omega$.

3. We depict all possible topologically different phase portraits for Eq. (6) systems.

**2.** We investigate consistently families of Eq. (6) systems, $r = 3, 4$, using the DC transformation of the results obtained for families, $r = 2, 1$. Then, we depict all types of existing phase portraits for the families 3 and 4.

Then, we conclude the following.

For families of Eq. (6) systems with numbers 1, 2, 3, and 4, there exist

$$15 + 11 + 11 + 15 = 52$$

different topological types of phase portraits in a Poincare circle $\overline{\Omega}$.

## 9. Systems containing 2 and 1 different multipliers in right parts

In this section, we give the full solution of the problem for Eq. (7) systems, i.e., for the Eq. (1) systems of the kind

$$\dot{x} = p_0x^3 + p_1x^2y + p_2xy^2 + p_3y^3 \equiv p_3(y - u_1x)^2(y - u_2x) \tag{7}$$

$$\dot{y} = x^2 + bxy + cy^2 \equiv c(y - qx)^2,$$

where $p_3 > 0, \ \ c > 0, \ \ u_1 < u_2, q \ (\in R) \neq u_{1,2}$.

The process of study of these systems is quite similar to that previously described for other families of Eq. (1) systems. For an arbitrary Eq. (7) system, *P(u), Q(u)* are the system's polynomials *P, Q:*

$$P(u) := X(1, u) \equiv p_3(u - u_1)^2(u - u_2), \qquad Q(u) := Y(1, u) \equiv c(u - q)^2,$$

and there exists 3 different variants for their *RSPQ*s.

A conclusion from our research for this particular type of systems is the following.

We`ve revealed, that for every possible family of Eq. (7) systems, 7 different topological types of their phase portraits are being implemented. This means that for all three existing families of such systems, $r = \overline{1, 3,}$ the number of different phase portraits is 21 [8, 9].

## 10. Conclusions

The presented work is devoted to the original study.

The main task of the work was to depict and describe all the different, in the topological meaning, phase portraits in a Poincare circle, possible for the dynamical differential systems belonging to a broad family of Eq. (1) systems, and to its numerical subfamilies. The authors have constructed all such phase portraits in two ways—in a descriptive (table) and in a graphic form. Each table contains 5–6 rows. Every row describes one invariant cell of the phase portrait in detail—it describes its boundary, source, and sink of its phase flow. The table was the descriptive phase portrait.

The second objective of this work was to develop, outline, and successfully apply some new effective methods of investigation [8–10].

This was a theoretical work, but due to aforementioned new methods, the chapter may be useful for applied studies of dynamic systems of the second order with polynomial right parts. The authors hope that this work may be interesting and useful for researchers and for both students and postgraduates.

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Chapter 5

# Differential Equations Arising from the 3-Variable Hermite Polynomials and Computation of Their Zeros

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Additional information is available at the end of the chapter

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**Abstract**

In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. Finally, we investigate the zeros of the 3-variable Hermite polynomials by using computer.

**Keywords:** differential equations, heat equation, Hermite polynomials, the 3-variable Hermite polynomials, generating functions, complex zeros

## 1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers see [1–15]. The special polynomials of two variables provided new means of analysis for the solution of a wide class of differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

In [1], the Hermite polynomials are given by the exponential generating function

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}.$$

We can also have the generating function by using Cauchy's integral formula to write the Hermite polynomials as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \frac{n!}{2\pi i} \oint_C \frac{e^{2tx-t^2}}{t^{n+1}} dt$$

with the contour encircling the origin. It follows that the Hermite polynomials also satisfy the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

Further, the two variables Hermite Kampé de Fériet polynomials $H_n(x, y)$ defined by the generating function (see [3])

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = e^{xt+yt^2} \tag{1}$$

are the solution of heat equation

$$\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y), \quad H_n(x, 0) = x^n.$$

We note that

$$H_n(2x, -1) = H_n(x).$$

The 3-variable Hermite polynomials $H_n(x, y, z)$ are introduced [4].

$$H_n(x, y, z) = n! \sum_{k=0}^{[\frac{n}{3}]} \frac{z^k H_{n-3k}(x, y)}{k!(n-3k)!}.$$

The differential equation and he generating function for $H_n(x, y, z)$ are given by

$$\left(3z \frac{\partial^3}{\partial x^3} + 2y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n\right) H_n(x, y, z) = 0$$

and

$$e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}, \tag{2}$$

respectively.

By (2), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} H_n(x_1 + x_2, y, z) \frac{t^n}{n!} &= e^{(x_1+x_2)t+yt^2+zt^3} \\
&= \sum_{n=0}^{\infty} x_2^n \frac{t^n}{n!} \sum_{n=0}^{\infty} H_n(x_1, y, z) \frac{t^n}{n!} \qquad (3) \\
&= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y, z) x_2^{n-l} \right) \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides of (3), we have the following theorem.

**Theorem 1**. For any positive integer $n$, we have

$$H_n(x_1 + x_2, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y, z) x_2^{n-l}.$$

Applying Eq. (2), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(x, y, z_1 + z_2) \frac{t^n}{n!} &= e^{xt+yt^2+(z_1+z_2)t^3} \\ &= \sum_{k=0}^{\infty} z_2^n \frac{t^{3k}}{k!} \sum_{l=0}^{\infty} H_l(x, y, z_1) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\left[\frac{n}{3}\right]} \frac{H_{n-3k}(x, y, z_1) z_2^k n!}{k!(n-3k)!} . \right) \frac{t^n}{n!}.\end{aligned}$$

On equating the coefficients of the like power of $t$ in the above, we obtain the following theorem.

**Theorem 2**. For any positive integer $n$, we have

$$H_n(x, y, z_1 + z_2) = n! \sum_{k=0}^{\left[\frac{n}{3}\right]} \frac{H_{n-3k}(x, y, z_1) z_2^k}{k!(n-3k)!}.$$

Also, the 3-variable Hermite polynomials $H_n(x, y, z)$ satisfy the following relations

$$\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z),$$

and

$$\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z).$$

The following elementary properties of the 3-variable Hermite polynomials $H_n(x, y, z)$ are readily derived form (2). We, therefore, choose to omit the details involved.

**Theorem 3**. For any positive integer $n$, we have

**1** $H_n(2x, -1, 0) = H_n(x)$.

**2** $H_n(x, y_1 + y_2, z) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{H_{n-2k}(x, y_1, z) y_2^k}{k!(n-2k)!}$.

**3** $H_n(x, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_l(x) H_{n-l}(-x, y+1, z)$.

**Theorem 4**. For any positive integer $n$, we have

**1** $H_n(x_1+x_2, y_1+y_2, z) = \sum_{l=0}^{n}\binom{n}{l} H_l(x_1, y_1, z) H_{n-l}(x_2, y_2)$.

**2** $H_n(x_1+x_2, y_1+y_2, z_1+z_2) = \sum_{l=0}^{n}\binom{n}{l} H_l(x_1, y_1, z) H_{n-l}(x_2, y_2, z_2)$.

The 3-variable Hermite polynomials can be determined explicitly. A few of them are

$$
\begin{aligned}
H_0(x,y,z) &= 1,\\
H_1(x,y,z) &= x,\\
H_2(x,y,z) &= x^2+2y,\\
H_3(x,y,z) &= x^3+6xy+6z,\\
H_4(x,y,z) &= x^4+12x^2y+12y^2+24xz,\\
H_5(x,y,z) &= x^5+20x^3y+60xy^2+60x^2z+120yz,\\
H_6(x,y,z) &= x^6+30x^4y+180x^2y^2+120y^3+120x^3z+720xyz+360z^2,\\
H_7(x,y,z) &= x^7+42x^5y+420x^3y^2+840xy^3+210x^4z+2520x^2yz+2520y^2z+2520xz^2,\\
H_8(x,y,z) &= x^8+56x^6y+840x^4y^2+3360x^2y^3+1680y^4+336x^5z+6720x^3yz\\
&\quad +20160xy^2z+10080x^2z^2+20160yz^2.\\
H_9(x,y,z) &= x^9+72x^7y+1512x^5y^2+10080x^3y^3+15120xy^4+504x^6z+15120x^4yz\\
&\quad +90720x^2y^2z+60480y^3z+30240x^3z^2+181440xyz^2+60480z^3,\\
H_{10}(x,y,z) &= x^{10}+90x^8y+2520x^6y^2+25200x^4y^3+75600x^2y^4+30240y^5+720x^7z\\
&\quad +30240x^5yz+302400x^3y^2z+604800xy^3z+75600x^4z^2\\
&\quad +907200x^2yz^2+907200y^2z^2+604800xz^3.
\end{aligned}
$$

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials (see [7, 8, 12, 16–19]). In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. In addition, we investigate the zeros of the 3-variable Hermite polynomials using numerical methods. Using computer, a realistic study for the zeros of the 3-variable Hermite polynomials is very interesting. Finally, we observe an interesting phenomenon of 'scattering' of the zeros of the 3-variable Hermite polynomials.

## 2. Differential equations associated with the 3-variable Hermite polynomials

In this section, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials.

Let

$$F = F(t,x,y,z) = e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x,y,z)\frac{t^n}{n!}, \quad x, y, z, t \in \mathbb{C}. \tag{4}$$

Then, by (4), we have

$$F^{(1)} = \frac{\partial}{\partial t}F(t,x,y,z) = \frac{\partial}{\partial t}\left(e^{xt+yt^2+zt^3}\right) = e^{xt+yt^2+zt^3}\left(x+2yt+3zt^2\right) = \left(x+2yt+3zt^2\right)F(t,x,y,z), \tag{5}$$

$$F^{(2)} = \frac{\partial}{\partial t}F^{(1)}(t,x,y,z) = (2y+6zt)F(t,x,y,z) + \left(x+2yt+3zt^2\right)F^{(1)}(t,x,y,z) = \left(\left(x^2+2y\right)+(6z+4xy)t+\left(4y^2+6xz\right)t^2+(12yz)t^3+\left(9z^2\right)t^4\right)F(t,x,y,z). \tag{6}$$

Continuing this process, we can guess that

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \sum_{i=0}^{2N} a_i(N,x,y,z)t^i F(t,x,y,z), \ (N=0,1,2,\ldots). \tag{7}$$

Differentiating (7) with respect to $t$, we have

$$\begin{aligned}F^{(N+1)} = \frac{\partial F^{(N)}}{\partial t} &= \sum_{i=0}^{2N} a_i(N,x,y,z)it^{i-1}F(t,x,y,z) + \sum_{i=0}^{2N} a_i(N,x,y,z)t^iF^{(1)}(t,x,y,z)\\
&= \sum_{i=0}^{2N} a_i(N,x,y,z)it^{i-1}F(t,x,y,z) + \sum_{i=0}^{2N} a_i(N,x,y,z)t^i\left(x+2yt+3zt^2\right)F(t,x,y,z)\\
&= \sum_{i=0}^{2N} ia_i(N,x,y,z)t^{i-1}F(t,x,y,z) + \sum_{i=0}^{2N} xa_i(N,x,y,z)t^iF(t,x,y,z)\\
&\quad + \sum_{i=0}^{2N} 2ya_i(N,x,y,z)t^{i+1}F(t,x,y,z) + \sum_{i=0}^{2N} 3za_i(N,x,y,z)t^{i+2}F(t,x,y,z)\\
&= \sum_{i=0}^{2N-1} (i+1)a_{i+1}(N,x,y,z)t^iF(t,x,y,z) + \sum_{i=0}^{2N} xa_i(N,x,y,z)t^iF(t,x,y,z)\\
&\quad + \sum_{i=1}^{2N+1} 2ya_{i-1}(N,x,y,z)t^iF(t,x,y,z) + \sum_{i=2}^{2N+2} 3za_{i-2}(N,x,y,z)t^iF(t,x,y,z)\end{aligned}$$

Hence we have

$$\begin{aligned}F^{(N+1)} = &\sum_{i=0}^{2N-1} (i+1)a_{i+1}(N,x,y,z)t^iF(t,x,y,z)\\
&+ \sum_{i=0}^{2N} xa_i(N,x,y,z)t^iF(t,x,y,z)\\
&+ \sum_{i=1}^{2N+1} 2ya_{i-1}(N,x,y,z)t^iF(t,x,y,z)\\
&+ \sum_{i=2}^{2N+2} 3za_{i-2}(N,x,y,z)t^iF(t,x,y,z).\end{aligned} \tag{8}$$

Now replacing $N$ by $N+1$ in (7), we find

$$F^{(N+1)} = \sum_{i=0}^{2N+2} a_i(N+1,x,y,z)t^i F(t,x,y,z). \tag{9}$$

Comparing the coefficients on both sides of (8) and (9), we obtain

$$\begin{aligned}
&a_0(N+1,x,y,z) = a_1(N,x,y,z) + xa_0(N,x,y,z),\\
&a_1(N+1,x,y,z) = 2a_2(N,x,y,z) + xa_1(N,x,y,z) + 2ya_0(N,x,y,z),\\
&a_{2N}(N+1,x,y,z) = xa_{2N}(N,x,y,z) + 2ya_{2N-1}(N,x,y,z) + 3za_{2N-2}(N,x,y,z),\\
&a_{2N+1}(N+1,x,y,z) = 2ya_{2N}(N,x,y,z) + 3za_{2N-1}(N,x,y,z),\\
&a_{2N+2}(N+1,x,y,z) = 3za_{2N}(N,x,y,z),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
a_i(N+1,x,y,z) = {}& (i+1)a_{i+1}(N,x,y,z) + xa_i(N,x,y,z)\\
&+ 2ya_{i-1}(N,x,y,z) + 3za_{i-2}(N,x,y,z), \ (2 \le i \le 2N-1).
\end{aligned} \tag{11}$$

In addition, by (7), we have

$$F(t,x,y,z) = F^{(0)}(t,x,y,z) = a_0(0,x,y,z)F(t,x,y,z), \tag{12}$$

which gives

$$a_0(0,x,y,z) = 1. \tag{13}$$

It is not difficult to show that

$$\begin{aligned}
&xF(t,x,y) + 2ytF(t,x,y,z) + 3zt^2F(t,x,y,z)\\
&= F^{(1)}(t,x,y,z)\\
&= \sum_{i=0}^{2} a_i(1,x,y,z)F(t,x,y,z)\\
&= \left(a_0(1,x,y,z) + a_1(1,x,y,z)t + a_2(1,x,y,z)t^2\right)F(t,x,y,z).
\end{aligned} \tag{14}$$

Thus, by (14), we also find

$$a_0(1,x,y,z) = x, \quad a_1(1,x,y,z) = 2y, \quad a_2(1,x,y,z) = 3z. \tag{15}$$

From (10), we note that

$$\begin{aligned}
&a_0(N+1,x,y,z) = a_1(N,x,y,z) + xa_0(N,x,y,z),\\
&a_0(N,x,y,z) = a_1(N-1,x,y,z) + xa_0(N-1,x,y,z), \ldots\\
&a_0(N+1,x,y,z) = \sum_{i=0}^{N} x^i a_1(N-i,x,y,z) + x^{N+1},
\end{aligned} \tag{16}$$

and

$$\begin{aligned} &a_{2N+2}(N+1,x,y,z) = 3za_{2N}(N,x,y,z),\\ &a_{2N}(N,x,y,z) = 3za_{2N-2}(N-1,x,y,z), \ldots\\ &a_{2N+2}(N+1,x,y,z) = (3z)^{N+1}. \end{aligned} \tag{17}$$

Note that, here the matrix $a_i(j,x,y)_{0\le i\le 2N+2,\,0\le j\le N+1}$ is given by

$$\begin{pmatrix} 1 & x & 2y+x^2 & \cdot & \cdots & \cdot \\ 0 & 2y & 4xy+6z & \cdot & \cdots & \cdot \\ 0 & 3z & 6xz+4y^2 & \cdot & \cdots & \cdot \\ 0 & 0 & 12yz & \cdot & \cdots & \cdot \\ 0 & 0 & (3z)^2 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & (3z)^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & (3z)^{N+1} \end{pmatrix}$$

Therefore, we obtain the following theorem.

**Theorem 5.** For $N = 0, 1, 2, \ldots,$ the differential equation

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \left(\sum_{i=0}^{N} a_i(N,x,y,z)t^i\right)F(t,x,y,z)$$

has a solution

$$F = F(t,x,y,z) = e^{xt+yt^2+zt^3},$$

where

$$\begin{aligned} &a_0(N+1,x,y,z) = \sum_{i=0}^{N} x^i a_1(N-i,x,y,z) + x^{N+1},\\ &a_1(N+1,x,y,z) = 2a_2(N,x,y,z) + xa_1(N,x,y,z) + 2ya_0(N,x,y,z),\\ &a_{2N}(N+1,x,y,z) = xa_{2N}(N,x,y,z) + 2ya_{2N-1}(N,x,y,z) + 3za_{2N-2}(N,x,y,z),\\ &a_{2N+1}(N+1,x,y,z) = 2ya_{2N}(N,x,y,z) + 3za_{2N-1}(N,x,y,z),\\ &a_{2N+2}(N+1,x,y,z) = (3z)^{N+1}, \end{aligned}$$

and

$$a_i(N+1,x,y,z) = (i+1)a_{i+1}(N,x,y,z) + xa_i(N,x,y,z) \\ + 2ya_{i-1}(N,x,y,z) + 3za_{i-2}(N,x,y,z), (2 \le i \le 2N-1).$$

From (4), we note that

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \sum_{k=0}^{\infty} H_{k+N}(x,y,z)\frac{t^k}{k!}. \tag{18}$$

By (4) and (18), we get

$$\begin{aligned} e^{-nt}\left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) &= \left(\sum_{m=0}^{\infty}(-n)^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}H_{m+N}(x,y,z)\frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}(-n)^{m-k}H_{N+k}(x,y,z)\right)\frac{t^m}{m!}. \end{aligned} \tag{19}$$

By the Leibniz rule and the inverse relation, we have

$$\begin{aligned} e^{-nt}\left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) &= \sum_{k=0}^{N}\binom{N}{k}n^{N-k}\left(\frac{\partial}{\partial t}\right)^k\left(e^{-nt}F(t,x,y,z)\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{N}\binom{N}{k}n^{N-k}H_{m+k}(x-n,y,z)\right)\frac{t^m}{m!}. \end{aligned} \tag{20}$$

Hence, by (19) and (20), and comparing the coefficients of $\frac{t^m}{m!}$ gives the following theorem.

**Theorem 6.** Let $m, n, N$ be nonnegative integers. Then

$$\sum_{k=0}^{m}\binom{m}{k}(-n)^{m-k}H_{N+k}(x,y,z) = \sum_{k=0}^{N}\binom{N}{k}n^{N-k}H_{m+k}(x-n,y,z). \tag{21}$$

If we take $m = 0$ in (21), then we have the following corollary.

**Corollary 7.** For $N = 0, 1, 2, \ldots,$ we have

$$H_N(x,y,z) = \sum_{k=0}^{N}\binom{N}{k}n^{N-k}H_k(x-n,y,z).$$

For $N = 0, 1, 2, \ldots,$ the differential equation

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t,x,y,z) = \left(\sum_{i=0}^{N}a_i(N,x,y,z)t^i\right)F(t,x,y,z)$$

has a solution

$$F = F(t, x, y, z) = e^{xt+yt^2+zt^3}.$$

Here is a plot of the surface for this solution. In **Figure 1(left)**, we choose $-2 \le z \le 2$, $-1 \le t \le 1$, $x = 2$, and $y = -4$. In **Figure 1(right)**, we choose $-5 \le x \le 5$, $-1 \le t \le 1$, $y = -3$, and $z = -1$.

## 3. Distribution of zeros of the 3-variable Hermite polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$. By using computer, the 3-variable Hermite polynomials $H_n(x, y, z)$ can be determined explicitly. We display the shapes of the 3-variable Hermite polynomials $H_n(x, y, z)$ and investigate the zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$. We investigate the beautiful zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ by using a computer. We plot the zeros of the $H_n(x, y, z)$ for $n = 20$, $y = 1, -1, 1 + i, -1 - i$, $z = 3, -3, 3 + i, -3 - i$ and $x \in \mathbb{C}$ (**Figure 2**). In **Figure 2(top-left)**, we choose $n = 20$, $y = 1$, and $z = 3$. In **Figure 2(top-right)**, we choose $n = 20$, $y = -1$, and $z = -3$. In **Figure 2(bottom-left)**, we choose $n = 20$, $y = 1 + i$, and $z = 3 + i$. In **Figure 2(bottom-right)**, we choose $n = 20$, $y = -1 - i$, and $z = -3 - i$.

In **Figure 3(top-left)**, we choose $n = 20$, $x = 1$, and $y = 1$. In **Figure 3(top-right)**, we choose $n = 20$, $x = -1$, and $y = -1$. In **Figure 3(bottom-left)**, we choose $n = 20$, $x = 1 + i$, and $y = 1 + i$. In **Figure 3(bottom-right)**, we choose $n = 20$, $x = -1 - i$, and $y = -1 - i$.

Stacks of zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \le n \le 20$ from a 3-D structure are presented (**Figure 3**). In **Figure 4(top-left)**, we choose $n = 20$, $y = 1$, and $z = 3$. In **Figure 4 (top-right)**, we choose $n = 20$, $y = -1$, and $z = -3$. In **Figure 4(bottom-left)**, we choose $n = 20$, $y = 1 + i$, and $z = 3 + i$. In **Figure 4(bottom-right)**, we choose $n = 20$, $y = -1 - i$, and $z = -3 - i$.

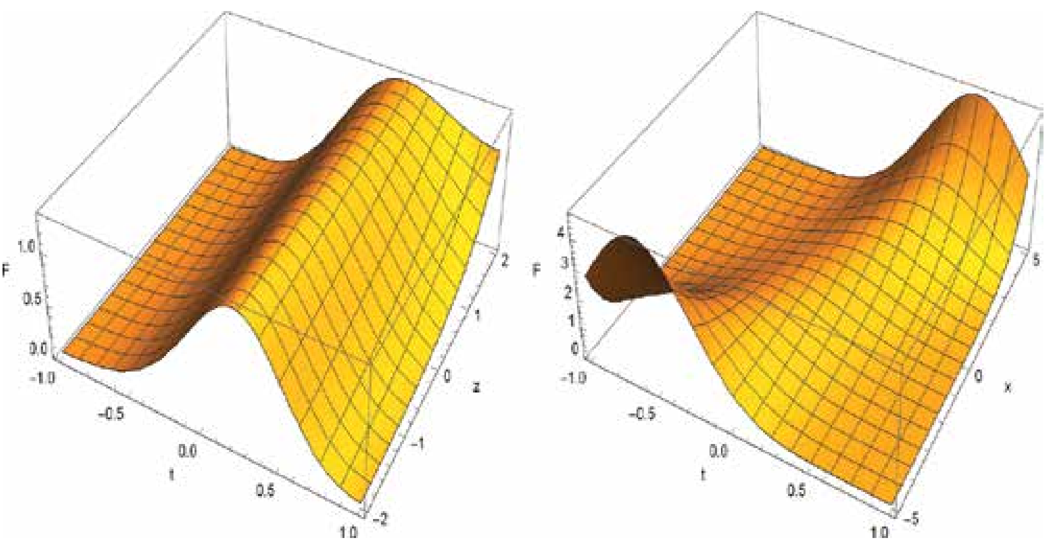


**Figure 1.** The surface for the solution $F(t, x, y, z)$.

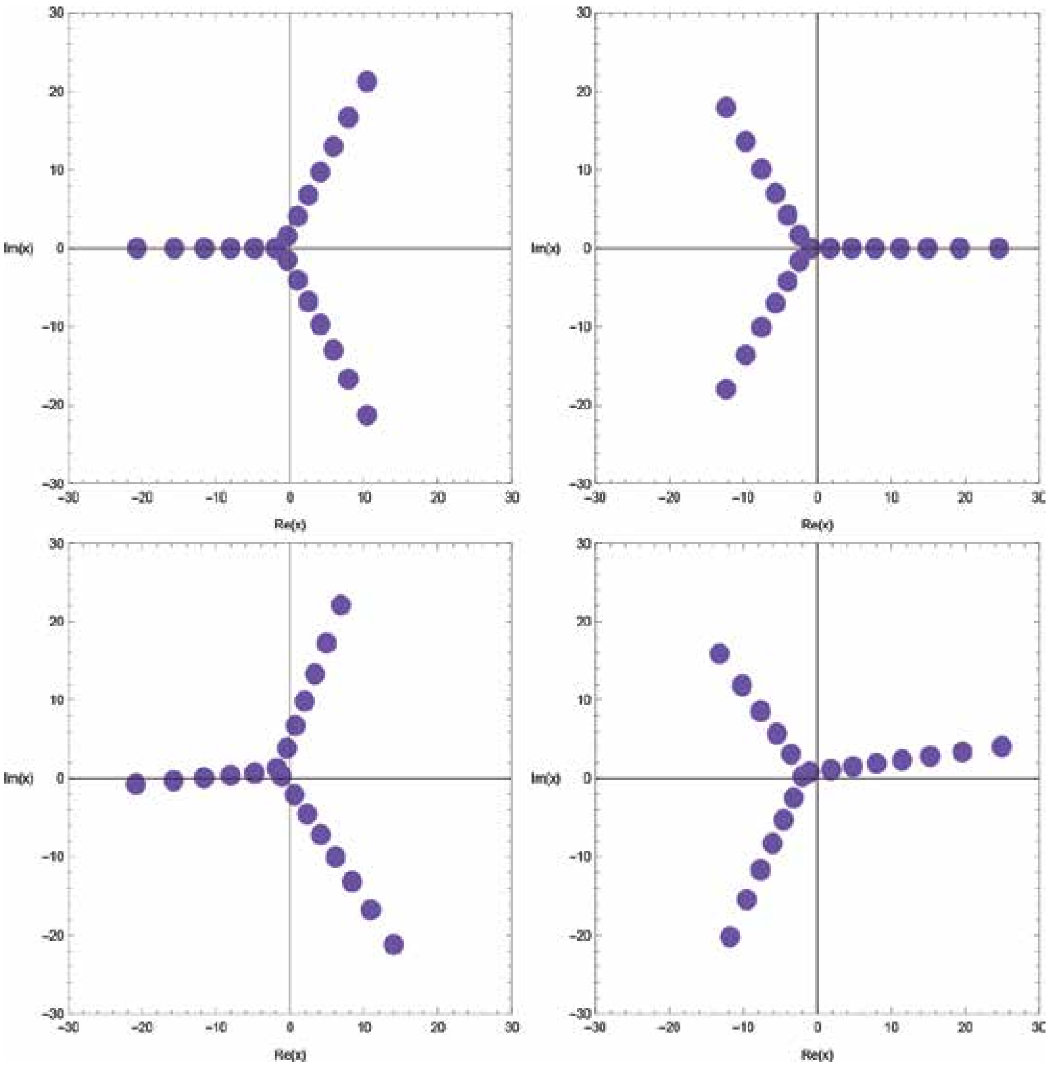


**Figure 2.** Zeros of $H_n(x, y, z)$.

Our numerical results for approximate solutions of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ are displayed (**Tables 1–3**).

The plot of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \le n \le 20$ structure are presented (**Figure 5**).

In **Figure 5(left)**, we choose $y = 1$ and $z = 3$. In **Figure 5(right)**, we choose $y = -1$ and $z = -3$.

Stacks of zeros of $H_n(x, -2, 4)$ for $1 \le n \le 40$, forming a 3D structure are presented (**Figure 6**). In **Figure 6(top-left)**, we plot stacks of zeros of $H_n(x, -2, 4)$ for $1 \le n \le 20$. In **Figure 6(top-right)**, we draw $x$ and $y$ axes but no $z$ axis in three dimensions. In **Figure 6(bottom-left)**, we draw $y$

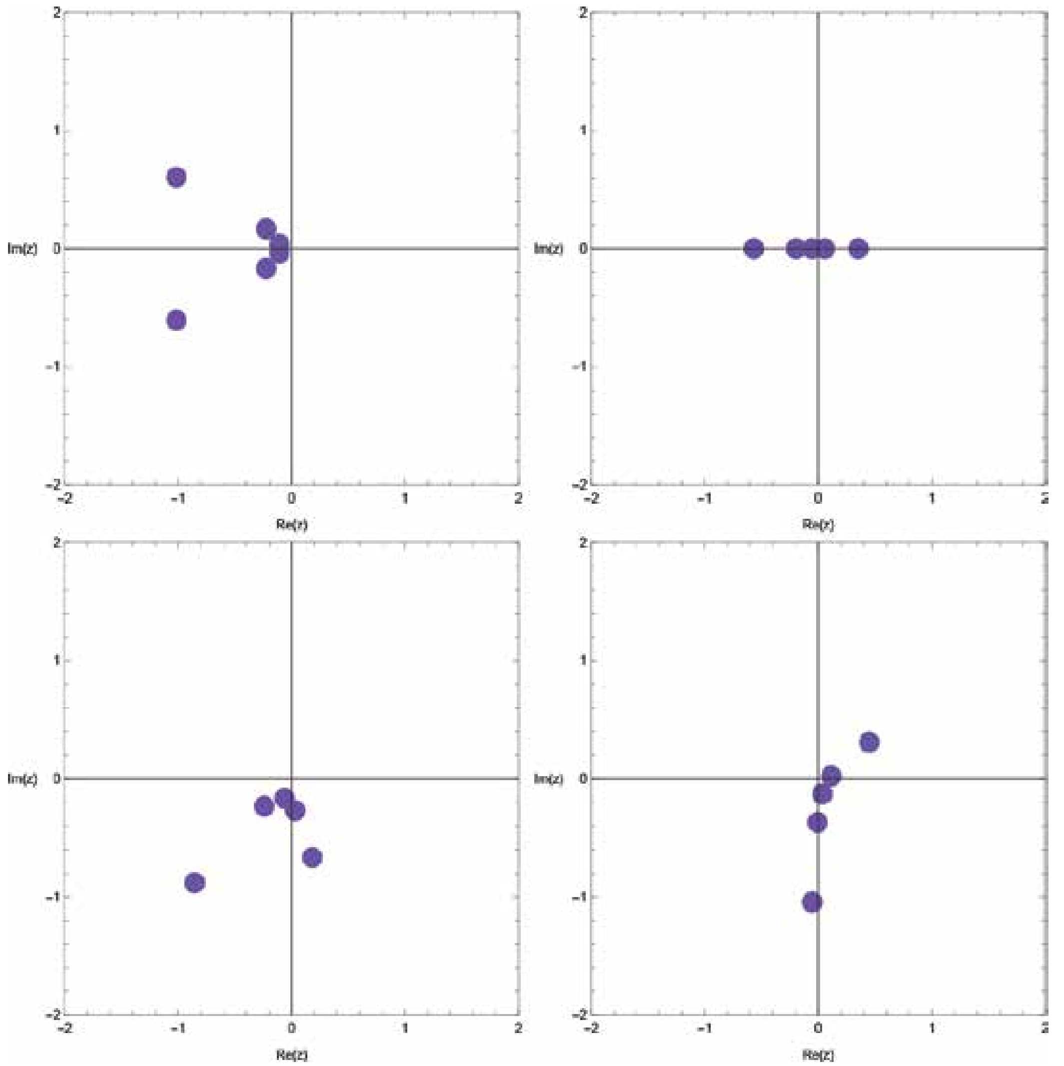


**Figure 3.** Zeros of $H_n(x, y, z)$.

and $z$ axes but no $x$ axis in three dimensions. In **Figure 6(bottom-right)**, we draw $x$ and $z$ axes but no $y$ axis in three dimensions.

It is expected that $H_n(x, y, z), x \in \mathbb{C}, y, z \in \mathbb{R}$, has $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions (see **Figures 2–7**). We observe a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $y, z \in \mathbb{R}$. We also hope to verify a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $y, z \in \mathbb{R}$ (**Tables 1** and **2**). Next, we calculated an approximate solution satisfying $H_n(x, y, z) = 0, x \in \mathbb{C}$. The results are given in **Tables 3** and **4**.

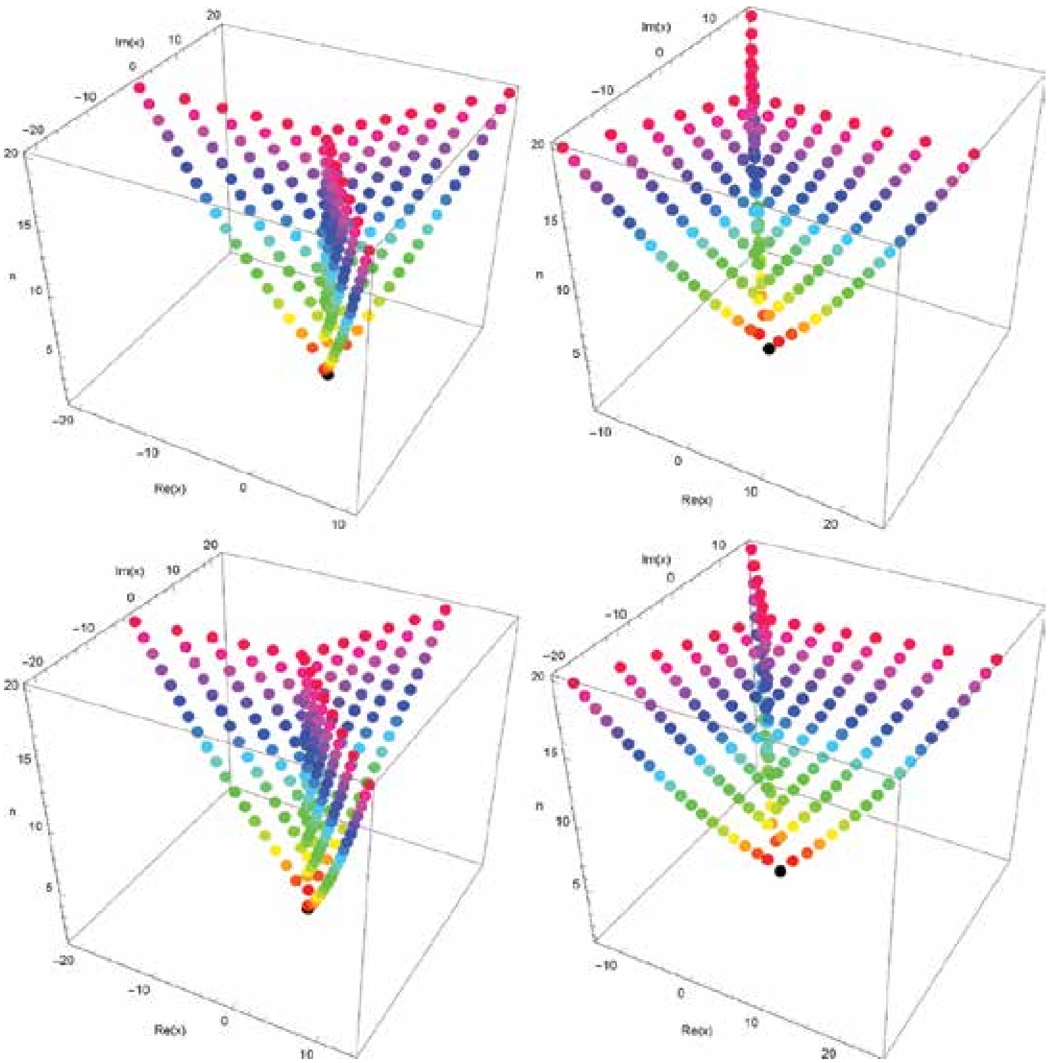


**Figure 4.** Stacks of zeros of $H_n(x, y, z), 1 \le n \le 20$.

The plot of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \le n \le 20$ structure are presented (**Figure 7**).

In **Figure 7(left)**, we choose $x = 1$ and $y = 2$. In **Figure 7(right)**, we choose $x = -1$ and $y = -2$.

Finally, we consider the more general problems. How many zeros does $H_n(x, y, z)$ have? We are not able to decide if $H_n(x, y, z) = 0$ has $n$ distinct solutions. We would also like to know the number of complex zeros $C_{H_n(x,y,z)}$ of $H_n(x, y, z), \mathrm{Im}(x) \neq 0$. Since $n$ is the degree of the polynomial $H_n(x, y, z)$, the number of real zeros $R_{H_n(x,y,z)}$ lying on the real line $\mathrm{Im}(x) = 0$ is then $R_{H_n(x,y,z)} = n - C_{H_n(x,y,z)}$, where $C_{H_n(x,y,z)}$ denotes complex zeros. See **Tables 1** and **2** for

http://dx.doi.org/10.5772/intechopen.74355

| **Degree $n$** | **Real zeros** | **Complex zeros** |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 0 | 2 |
| 3 | 1 | 2 |
| 4 | 2 | 2 |
| 5 | 1 | 4 |
| 6 | 2 | 4 |
| 7 | 3 | 4 |
| 8 | 2 | 6 |
| 9 | 3 | 6 |
| 10 | 4 | 6 |
| 11 | 3 | 8 |
| 12 | 4 | 8 |
| 13 | 3 | 10 |
| 14 | 4 | 10 |

**Table 1.** Numbers of real and complex zeros of $H_n(x, 1, 3)$.

| **Degree $n$** | **Real zeros** | **Complex zeros** |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 2 | 0 |
| 3 | 1 | 2 |
| 4 | 2 | 2 |
| 5 | 3 | 2 |
| 6 | 2 | 4 |
| 7 | 3 | 4 |
| 8 | 4 | 4 |
| 9 | 3 | 6 |
| 10 | 4 | 6 |
| 11 | 5 | 6 |
| 12 | 6 | 6 |
| 13 | 5 | 8 |
| 14 | 6 | 8 |

**Table 2.** Numbers of real and complex zeros of $H_n(x, -1, -3)$.

| Degree $n$ | $x$ |
|---|---|
| 1 | 0 |
| 2 | — |
| 3 | $-1.8845$ |
| 4 | $3.1286, \quad -0.17159$ |
| 5 | $-4.5385$ |
| 6 | $-5.8490, \quad -1.3476$ |
| 7 | $-7.1098, \quad -2.1887, \quad -0.36350$ |
| 8 | $-8.3241, \quad -3.4645$ |
| 9 | $-9.4984, \quad -4.6021, \quad -1.1118$ |
| 10 | $-10.637, \quad -5.7212, \quad -1.5785, \quad -0.61919$ |
| 11 | $-11.745, \quad -6.8105, \quad -2.8680$ |
| 12 | $-12.824, \quad -7.8743, \quad -3.8894, \quad -0.99513$ |

**Table 3.** Approximate solutions of $H_n(x,1,3)=0, x \in \mathrm{R}$.

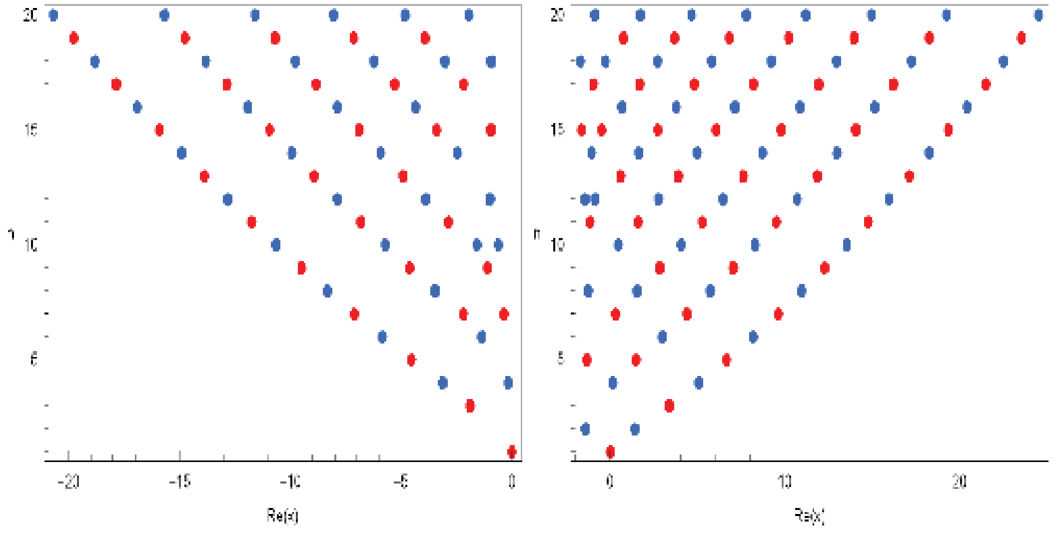


**Figure 5.** Real zeros of $H_n(x,y,z), 1 \leq n \leq 20$.

tabulated values of $R_{H_n(x,y,z)}$ and $C_{H_n(x,y,z)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the 3-variable Hermite polynomials $H_n(x,y,z)$ which appear in mathematics and physics. The reader may refer to [2, 11, 13, 20] for the details.

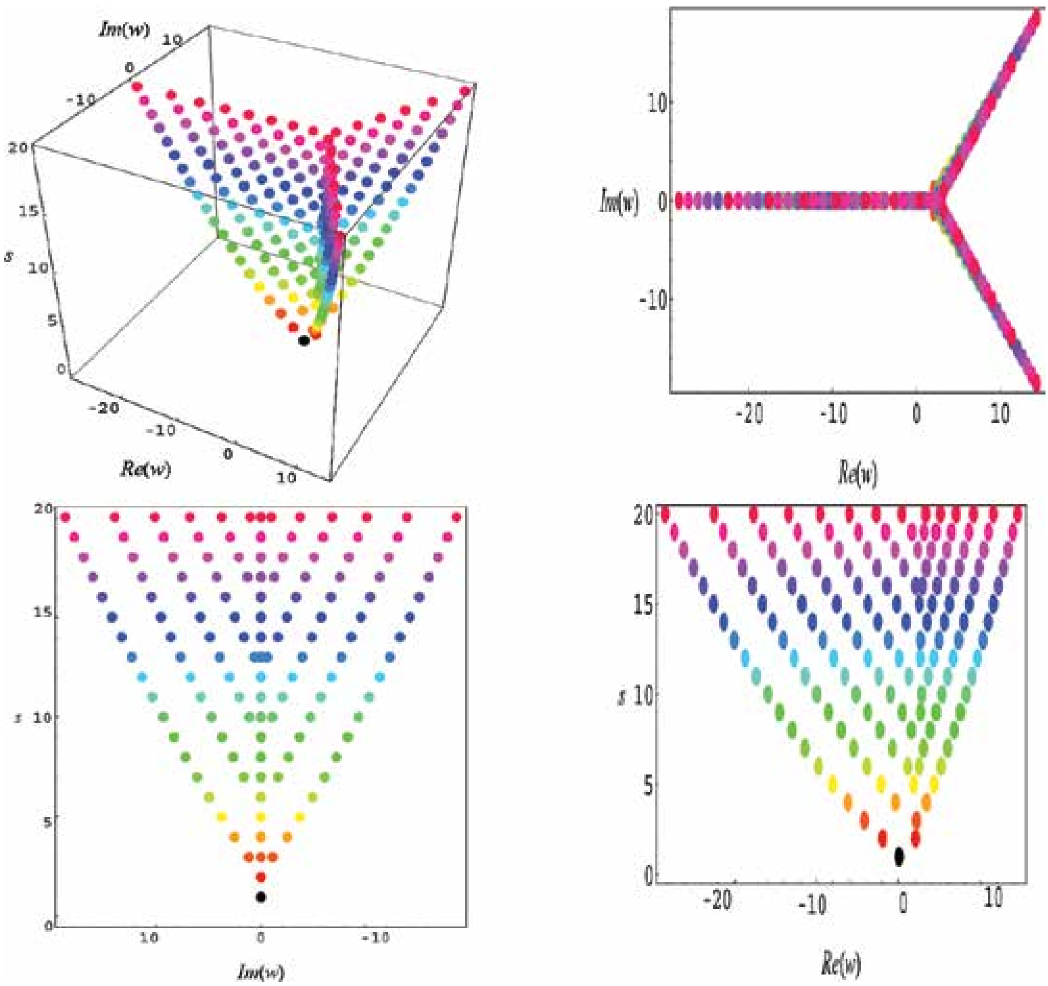


**Figure 6.** Stacks of zeros of $H_n(x, -2, 4)$ *for* $1 \le n \le 20$.

| degree $n$ | $x$ |
|---|---|
| 1 | 0 |
| 2 | −1.4142, 1.4142 |
| 3 | 3.3681 |
| 4 | 0.16229, 5.0723 |
| 5 | −1.3404, 1.4745, 6.6661 |
| 6 | 2.9754, 8.1678 |
| 7 | 0.31213, 4.3783, 9.5946 |

| degree $n$ | $x$ |
|---|---|
| 8 | −1.2604, 1.5304, 5.7274, 10.959 |
| 9 | 2.8224, 7.0271, 12.270 |
| 10 | 0.44594, 4.0615, 8.2834, 13.535 |
| 11 | −1.1740, 1.5825, 5.2667, 9.5013, 14.760 |
| 12 | −1.4659, −0.87728, 2.7469, 6.4398, 10.685, 15.949 |

**Table 4.** Approximate solutions of $H_n(x,-1,-3)=0, x\in\mathrm{R}$.

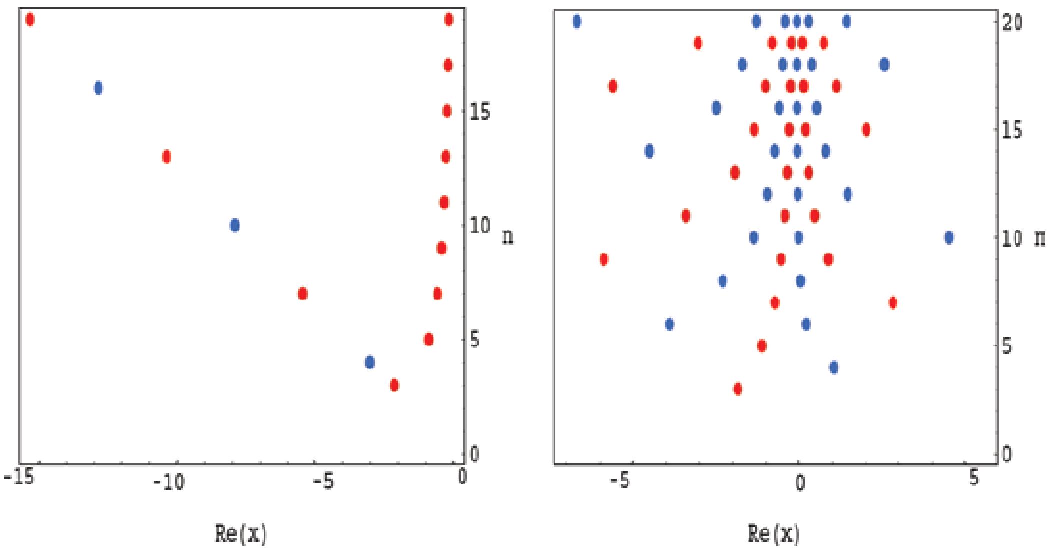


**Figure 7.** Real zeros of $H_n(x,y,z), 1\le n\le 20$.

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Chapter 6

# Reproducing Kernel Functions

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Additional information is available at the end of the chapter

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**Abstract**

In this chapter, we obtain some reproducing kernel spaces. We obtain reproducing kernel functions in these spaces. These reproducing kernel functions are very important for solving ordinary and partial differential equations.

**Keywords:** reproducing kernel functions, reproducing kernel spaces, ordinary and partial differential equations

## 1. Introduction

Reproducing kernel spaces are special Hilbert spaces. These spaces satisfy the reproducing property. There is an important relation between the order of the problems and the reproducing kernel spaces.

## 2. Reproducing kernel spaces

In this section, we define some useful reproducing kernel functions [1–23].

*Definition 2.1* (reproducing kernel). Let E be a nonempty set. A function $K : E \times E \to \mathbb{C}$ is called a reproducing kernel of the Hilbert space $H$ if and only if

**a.** $K(\cdot, t) \in H$ for all $t \in E$,

**b.** $\langle \varphi, K(\cdot, t) \rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

The last condition is called the reproducing property as the value of the function $\varphi$ at the point $t$ is reproduced by the inner product of $\varphi$ with $K(\cdot, t)$.

Then, we need some notation that we use in the development of this chapter. Next, we define several spaces with inner product over those spaces. Thus, the space defined as

$$W_2^3[0,1] = \left\{ v | v, v', v'' : [0,1] \to \mathbb{R} \text{ are absolutely continuous}, v^{(3)} \in L^2[0,1] \right\} \tag{1}$$

is a Hilbert space. The inner product and the norm in $W_2^3[0,1]$ are defined by

$$\begin{gathered} \langle v, g \rangle_{W_2^3} = \sum_{i=0}^{2} v^{(i)}(0) g^{(i)}(0) + \int_0^1 v^{(3)}(x) g^{(3)}(x) \mathrm{d}x, \quad v, g \in W_2^3[0,1], \\ \|v\|_{W_2^3} = \sqrt{\langle v, v \rangle_{W_2^3}}, \quad v \in W_2^3[0,1], \end{gathered} \tag{2}$$

respectively. Thus, the space $W_2^3[0,1]$ is a reproducing kernel space, that is, for each fixed $y \in [0,1]$ and any $v \in W_2^3[0,1]$, there exists a function $R_y$ such that

$$v(y) = \langle v(x), R_y(x) \rangle_{W_2^3}, \tag{3}$$

and similarly, we define the space

$$T_2^3[0,1] = \left\{ \begin{array}{c} v | v, v', v'' : [0,1] \to \mathbb{R} \text{ are absolutely continuous}, \\ \\ v'' \in L^2[0,1], v(0) = 0, v'(0) = 0 \end{array} \right\} \tag{4}$$

The inner product and the norm in $T_2^3[0,1]$ are defined by

$$\begin{gathered} \langle v, g \rangle_{T_2^3} = \sum_{i=0}^{2} v^{(i)}(0) g^{(i)}(0) + \int_0^1 v'''(t) g'''(t) \mathrm{d}t, \quad v, g \in T_2^3[0,1], \\ \|v\|_{T_2^3} = \sqrt{\langle v, v \rangle_{T_2^3}}, \quad v \in T_2^3[0,1], \end{gathered} \tag{5}$$

respectively. The space $T_2^3[0,1]$ is a reproducing kernel Hilbert space, and its reproducing kernel function $r_s$ is given by [1] as

$$r_s = \begin{cases} \dfrac{1}{4} s^2 t^2 + \dfrac{1}{12} s^2 t^3 - \dfrac{1}{24} s t^4 + \dfrac{1}{120} t^5, & t \leq s, \\ \dfrac{1}{4} s^2 t^2 + \dfrac{1}{12} s^3 t^2 - \dfrac{1}{24} t s^4 + \dfrac{1}{120} s^5, & t > s, \end{cases} \tag{6}$$

and the space

$$G_2^1[0,1] = \{ v | v : [0,1] \to \mathbb{R} \text{ is absolutely continuous}, v'(x) \in L^2[0,1] \}, \tag{7}$$

is a Hilbert space, where the inner product and the norm in $G_2^1[0,1]$ are defined by

$$\begin{aligned}\langle v, g\rangle_{G_2^1} &= v^{(i)}(0)g^{(i)}(0) + \int_0^1 v'(x)g'(x)\mathrm{d}x, \quad v, g \in G_2^1[0,1],\\ \|v\|_{G_2^1} &= \sqrt{\langle v, v\rangle_{G_2^1}}, \quad v \in G_2^1[0,1],\end{aligned} \tag{8}$$

respectively. The space $G_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function $Q_y$ is given by [1] as

$$Q_y = \begin{cases} 1+x, & x \leqslant y \\ 1+y, & x > y. \end{cases} \tag{9}$$

**Theorem 1.1.** *The space $W_2^3[0,1]$ is a complete reproducing kernel space whose reproducing kernel $R_y$ is given by*

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{6} c_i(y)x^{i-1}, & x \leq y, \\ \displaystyle\sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases} \tag{10}$$

where

$$\begin{aligned} &c_1(y) = 1, \quad c_2(y) = y, \quad c_3(y) = \frac{y^2}{4}, \quad c_4(y) = \frac{y^2}{12}, \quad c_5(y) = -\frac{1}{24y}, \quad c_6(y) = \frac{1}{120},\\ &d_1(y) = 1 + \frac{y^5}{120}, \quad d_2(y) = \frac{-y^4}{24} + y, \quad d_3(y) = \frac{y^2}{4} + \frac{y^3}{12}, \quad d_4(y) = d_5(y) = d_6(y) = 0. \end{aligned}$$

*Proof.* Since

$$\left\langle v, R_y \right\rangle_{W_2^3} = \sum_{i=0}^{2} v^{(i)}(0)R_y^{(i)}(0) + \int_0^1 v^{(3)}(x)R_y^{(3)}(x)\mathrm{d}x, \quad \left(v, R_y \in W_2^3[0,1]\right. \tag{11}$$

through iterative integrations by parts for (11), we have

$$\begin{aligned}\left\langle v(x), R_y(x) \right\rangle_{W_2^4} &= \sum_{i=0}^{2} v^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(2-i)}R_y^{(5-i)}(0)\right]\\ &+\sum_{i=0}^{2}(-1)^{(2-i)}v^{(i)}(1)R_y^{(5-i)}(1) + \int_0^1 v(x)R_y^{(6)}(x)\mathrm{d}x.\end{aligned} \tag{12}$$

Note, the property of the reproducing kernel as

$$\left\langle v(x), R_y(x) \right\rangle_{W_2^3} = v(y). \tag{13}$$

If

$$
\begin{aligned}
R_y(0) - R_y^{(5)}(0) &= 0,\\
R_y'(0) + R_y^{(4)}(0) &= 0,\\
R_y''(0) - R_y'''(0) &= 0,\\
R_y^{(3)}(1) &= 0,\\
R_y^{(4)}(1) &= 0,\\
R_y^{(5)}(1) &= 0,
\end{aligned}
\tag{14}
$$

Then by (11), we obtain

$$R_y^{(6)}(x) = \delta(x-y), \tag{15}$$

when $x \neq y$,

$$R_y^{(6)}(x) = 0, \tag{16}$$

therefore,

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y)x^{i-1}, & x \le y, \\ \sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases} \tag{17}$$

Since

$$R_y^{(6)}(x) = \delta(x-y), \tag{18}$$

we have

$$
\begin{aligned}
\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0,1,2,3,4,\\
\partial^5 R_{y^+}(y) - \partial^5 R_{y^-}(y) = -1.
\end{aligned}
\tag{19}
$$

From (14) and (19), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1,2,\ldots,6)$ can be obtained. Thus, $R_y$ is given by

$$R_y = \begin{cases} 1 + yx + \frac{1}{4}y^2x^2 + \frac{1}{12}y^2x^3 - \frac{1}{24}yx^4 + \frac{1}{120}x^5, & x \le y \\ 1 + yx + \frac{1}{4}y^2x^2 + \frac{1}{12}y^3x^2 - \frac{1}{24}xy^4 + \frac{1}{120}y^5, & x > y. \end{cases} \tag{20}$$

Now, we note that the space given in [1] as

$$W(\Omega)=\left\{\begin{array}{c} v(x,t)|\dfrac{\partial^4 v}{\partial x^2\partial t^2},\text{ is completely continuous in }\ \Omega=[0,1]\times[0,1],\\ \\ \dfrac{\partial^6 v}{\partial x^3\partial t^3}\in L^2(\Omega), v(x,0)=0, \dfrac{\partial v(x,0)}{\partial t}=0\end{array}\right\} \tag{21}$$

is a binary reproducing kernel Hilbert space. The inner product and the norm in $W(\Omega)$ are defined by

$$\begin{aligned}\langle v(x,t),g(x,t)\rangle_W &= \sum_{i=0}^{2}\int_0^1\left[\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}v(0,t)\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}g(0,t)\right]\mathrm{d}t\\ &+\sum_{j=0}^{2}\left\langle\frac{\partial^j}{\partial t^j}v(x,0),\frac{\partial^j}{\partial t^j}g(x,0)\right\rangle_{W_2^3}\\ &+\int_0^1\int_0^1\left[\frac{\partial^3}{\partial x^3}\frac{\partial^3}{\partial t^3}v(x,t)\frac{\partial^3}{\partial x^3}\frac{\partial^3}{\partial t^3}g(x,t)\right]\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{22}$$

$$\|v\|_w=\sqrt{\langle v,v\rangle_W},\quad v\in W(\Omega),$$

respectively.

**Theorem 1.2.** *The $W(\Omega)$ is a reproducing kernel space, and its reproducing kernel function is*

$$K_{(y,s)}=R_y r_s \tag{23}$$

*such that for any $v\in W(\Omega)$,*

$$\begin{aligned} v(y,s)&=\langle v(x,t),K_{(y,s)}(x,t)\rangle_{W'}\\ K_{(y,s)}(x,t)&=K_{(x,t)}(y,s).\end{aligned} \tag{24}$$

*Similarly, the space*

$$\widehat{W}(\Omega)=\left\{v(x,t)|v(x,t)\ \text{ is completely continuous in }\ \Omega=[0,1]\times[0,1],\frac{\partial^2 v}{\partial x\partial t}\in L^2(\Omega)\right\} \tag{25}$$

*is a binary reproducing kernel Hilbert space. The inner product and the norm in $\widehat{W}(\Omega)$ are defined by* [1] *as*

$$\langle v(x,t), g(x,t)\rangle_{\widehat{W}} = \int_0^1 \left[\frac{\partial}{\partial t} v(0,t) \frac{\partial}{\partial t} g(0,t)\right] \mathrm{d}t + \langle v(x,0), g(x,0)\rangle_{W_2^1}$$

$$+ \int_0^1 \int_0^1 \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t} v(x,t) \frac{\partial}{\partial x}\frac{\partial}{\partial t} g(x,t)\right] \mathrm{d}x\mathrm{d}t, \tag{26}$$

$$\|v\|_{\widehat{W}} = \sqrt{\langle v, v\rangle_{\widehat{W}}}, \quad v \in \widehat{W}(\Omega),$$

respectively. $\widehat{W}(\Omega)$ is a reproducing kernel space, and its reproducing kernel function $G_{(y,s)}$ is

$$G_{(y,s)} = Q_y Q_s. \tag{27}$$

**Definition 1.3.**

$$W_2^3[0,1] = \left\{ \begin{array}{c} u(x)|u(x), u'(x), u''(x), \text{are absolutely continuous in } [0,1] \\ \\ u^{(3)}(x) \in L^2[0,1], x \in [0,1], u(0) = 0, u(1) = 0. \end{array} \right\}$$

The inner product and the norm in $W_2^3[0,1]$ are defined, respectively, by

$$\langle u(x), g(x)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0) g^{(i)}(0) + \int_0^1 u^{(3)}(x) g^{(3)}(x) \mathrm{d}x, u(x), g(x) \in W_2^3[0,1]$$

and

$$\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}, \quad u \in W_2^3[0,1].$$

The space $W_2^3[0,1]$ is a reproducing kernel space, that is, for each fixed $y \in [0,1]$ and any $u(x) \in W_2^3[0,1]$, there exists a function $R_y(x)$ such that

$$u(y) = \langle u(x), R_y(x)\rangle_{W_2^3}.$$

**Definition 1.4.**

$$W_2^1[0,1] = \left\{ \begin{array}{c} u(x)|u(x), \text{is absolutely continuous in } [0,1] \\ \\ u'(x) \in L^2[0,1], x \in [0,1], \end{array} \right\}$$

The inner product and the norm in $W_2^1[0,1]$ are defined, respectively, by

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$$\langle u(x), g(x)\rangle_{W_2^1} = u(0)g(0) + \int_0^1 u'(x)g'(x)\mathrm{d}x, u(x), g(x) \in W_2^1[0,1], \tag{28}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, \quad u \in W_2^1[0,1]. \tag{29}$$

The space $W_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function $T_x(y)$ is given by

$$T_x(y) = \begin{cases} 1+x, & x \leq y, \\ 1+y, & x > y. \end{cases} \tag{30}$$

**Theorem 1.5.** *The space $W_2^3[0,1]$ is a complete reproducing kernel space, and its reproducing kernel function $R_y(x)$ can be denoted by*

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y)x^{i-1}, & x \leq y, \\ \sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases}$$

where

$$\begin{aligned}
c_1(y) &= 0,\\
c_2(y) &= \frac{5}{516}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y,\\
c_3(y) &= \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 - \frac{5}{312}y^3 - \frac{5}{26}y,\\
c_4(y) &= \frac{5}{1872}y^4 - \frac{1}{1872}y^5 + \frac{7}{104}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y,\\
c_5(y) &= -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 - \frac{1}{104}y,\\
c_6(y) &= \frac{1}{120} + \frac{1}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 - \frac{1}{1872}y^3 - \frac{1}{156}y,\\
d_1(y) &= \frac{1}{120}y^5,\\
d_2(y) &= -\frac{1}{104}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y,\\
d_3(y) &= \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 + \frac{7}{104}y^3 - \frac{5}{26}y,\\
d_4(y) &= \frac{5}{1872}y^4 - \frac{1}{1872}y^5 - \frac{5}{312}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y,\\
d_5(y) &= -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 + \frac{5}{156}y,\\
d_6(y) &= -\frac{1}{156}y + \frac{1}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 - \frac{1}{1872}y^3.
\end{aligned}$$

*Proof.* We have

$$\begin{aligned}\left\langle u(x), R_y(x)\right\rangle_{W_2^3} = & \sum_{i=0}^{2} u^{(i)}(0)R_y^{(i)}(0) \\ & + \int_0^1 u^{(3)}(x)R_y^{(3)}(x)\mathrm{d}x.\end{aligned} \tag{31}$$

Through several integrations by parts for (31), we have

$$\begin{aligned}\left\langle u(x), R_y(x)\right\rangle_{W_2^6} = & \sum_{i=0}^{2} u^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(2-i)}R_y^{(5-i)}(0)\right] \\ & + \sum_{i=0}^{2} (-1)^{(2-i)} u^{(i)}(1)R_y^{(5-i)}(1) \\ & - \int_0^1 u(x)R_y^{(6)}(x)\mathrm{d}x.\end{aligned} \tag{32}$$

Note that property of the reproducing kernel

$$\left\langle u(x), R_y(x)\right\rangle_{W_2^3} = u(y),$$

If

$$\begin{cases} R_y''(0) - R_y^{(3)}(0) = 0, \\ R_y'(0) + R_y^{(4)}(0) = 0, \\ R_y^{(3)}(1) = 0, \\ R_y^{(4)}(1) = 0, \end{cases} \tag{33}$$

then by (31), we have the following equation:

$$-R_y^{(6)}(x) = \delta(x-y),$$

$$\text{when } x \neq y,$$

$$R_y^{(6)}(x) = 0,$$

therefore,

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{6} c_i(y)x^{i-1}, & x \leq y, \\ \displaystyle\sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases}$$

Since

$$-R_y^{(6)}(x) = \delta(x-y),$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0,1,2,3,4, \tag{34}$$

and

$$\partial^5 R_{y^+}(y) - \partial^5 R_{y^-}(y) = -1. \tag{35}$$

Since $R_y(x) \in W_2^3[0,1]$, it follows that

$$R_y(0) = 0, R_y(1) = 0, \tag{36}$$

From (33)–(36), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1,2,\dots,6)$ can be obtained. Thus $R_y(x)$ is given by

$$R_y(x) = \begin{cases} \frac{5}{516}xy^4 - \frac{1}{156}xy^5 - \frac{5}{26}xy^2 - \frac{5}{78}xy^3 + \frac{3}{13}xy + \frac{5}{624}x^2y^4 - \frac{1}{624}x^2y^5 + \frac{21}{104}x^2y^2 \\ -\frac{5}{312}x^2y^3 - \frac{5}{26}x^2y + \frac{5}{1872}x^3y^4 - \frac{1}{1872}x^3y^5 + \frac{7}{104}x^3y^2 - \frac{5}{936}x^3y^3 - \frac{5}{78}x^3y \\ -\frac{5}{3744}x^4y^4 + \frac{1}{3744}x^4y^5 + \frac{5}{624}x^4y^2 + \frac{5}{1872}x^4y^3 - \frac{1}{104}x^4y - \frac{1}{156}x^5y + \frac{1}{3744}x^5y^4 \\ -\frac{1}{18720}x^5y^5 - \frac{1}{624}x^5y^2 - \frac{1}{1872}x^5y^3, \quad x \le y \\ \frac{5}{516}yx^4 - \frac{1}{156}yx^5 - \frac{5}{26}yx^2 - \frac{5}{78}yx^3 + \frac{3}{13}xy + \frac{5}{624}y^2x^4 - \frac{1}{624}y^2x^5 + \frac{21}{104}x^2y^2 \\ -\frac{5}{312}y^2x^3 - \frac{5}{26}y^2x + \frac{5}{1872}y^3x^4 - \frac{1}{1872}y^3x^5 + \frac{7}{104}y^3x^2 - \frac{5}{936}x^3y^3 - \frac{5}{78}y^3x \\ -\frac{5}{3744}x^4y^4 + \frac{1}{3744}y^4x^5 + \frac{5}{624}y^4x^2 + \frac{5}{1872}y^4x^3 - \frac{1}{104}y^4x - \frac{1}{156}y^5x + \frac{1}{3744}y^5x^4 \\ -\frac{1}{18720}x^5y^5 - \frac{1}{624}y^5x^2 - \frac{1}{1872}y^5x^3, \quad x > y \end{cases}$$

$$W_2^4[0,1] = \left\{ \begin{array}{c} v(x) | v(x), v'(x), v''(x), v'''(x) \\ \text{are absolutely continuous in } [0,1], \\ v^{(4)}(x) \in L^2[0,1], x \in [0,1] \end{array} \right\} \tag{37}$$

The inner product and the norm in $W_2^4[0,1]$ are defined, respectively, by

$$\langle v(x), g(x)\rangle_{W_2^4} = \sum_{i=0}^{3} v^{(i)}(0)g^{(i)}(0) + \int_0^1 v^{(4)}(x)g^{(4)}(x)\mathrm{d}x, \quad v(x), g(x) \in W_2^4[0,1], \\ \|v\|_{W_2^4} = \sqrt{\langle v, v\rangle_{W_2^4}}, \quad v \in W_2^4[0,1]. \tag{38}$$

The space $W_2^4[0,1]$ is a reproducing kernel space, that is, for each fixed.

$y \in [0,1]$ and any $v(x) \in W_2^4[0,1]$, there exists a function $R_y(x)$ such that

$$v(y) = \langle v(x), R_y(x)\rangle_{W_2^4} \tag{39}$$

Similarly, we define the space

$$W_2^2[0,T] = \left\{ \begin{array}{c} v(t)|v(t), v'(t) \\ \text{are absolutely continuous in } [0,T], \\ v''(t) \in L^2[0,T], t \in [0,T], v(0) = 0 \end{array} \right\} \tag{40}$$

The inner product and the norm in $W_2^2[0,T]$ are defined, respectively, by

$$\langle v(t), g(t)\rangle_{W_2^2} = \sum_{i=0}^{1} v^{(i)}(0)g^{(i)}(0) + \int_0^T v''(t)g''(t)\mathrm{d}t, \quad v(t), g(t) \in W_2^2[0,T], \\ \|v\|_{W_1} = \sqrt{\langle v, v\rangle_{W_2^2}}, \quad v \in W_2^2[0,T]. \tag{41}$$

Thus, the space $W_2^2[0,T]$ is also a reproducing kernel space, and its reproducing kernel function $r_s(t)$ can be given by

$$r_s(t) = \begin{cases} st + \frac{s}{2}t^2 - \frac{1}{6}t^3, & t \le s, \\ st + \frac{t}{2}s^2 - \frac{1}{6}s^3, & t > s, \end{cases} \tag{42}$$

and the space

$$W_2^2[0,1] = \left\{ \begin{array}{c} v(x)|v(x), v'(x) \\ \text{are absolutely continuous in } [0,1], \\ v''(x) \in L^2[0,1], x \in [0,1] \end{array} \right\} \tag{43}$$

where the inner product and the norm in $W_2^2[0,1]$ are defined, respectively, by

$$\begin{gathered}\langle v(t), g(t)\rangle_{W_2^2} = \sum_{i=0}^{1} v^{(i)}(0)g^{(i)}(0) + \int_0^T v''(t)g''(t)\mathrm{d}t, \quad v(t), g(t) \in W_2^2[0,1], \\ \|v\|_{W_2} = \sqrt{\langle v, v\rangle_{W_2^2}}, \quad v \in W_2^2[0,1].\end{gathered} \tag{44}$$

The space $W_2^2[0,1]$ is a reproducing kernel space, and its reproducing kernel function $Q_y(x)$ is given by

$$Q_y(x) = \begin{cases} 1 + xy + \dfrac{y}{2}x^2 - \dfrac{1}{6}x^3, & x \le y, \\ 1 + xy + \dfrac{x}{2}y^2 - \dfrac{1}{6}y^3, & x > y. \end{cases} \tag{45}$$

Similarly, the space $W_2^1[0,T]$ is defined by

$$W_2^1[0,T] = \left\{ \begin{array}{c} v(t)|v(t) \text{ is absolutely continuous in } [0,T], \\ \\ v(t) \in L^2[0,T], t \in [0,T] \end{array} \right\} \tag{46}$$

The inner product and the norm in $W_2^1[0,T]$ are defined, respectively, by

$$\begin{gathered}\langle v(t), g(t)\rangle_{W_2^1} = v(0)g(0) + \textstyle\int_0^T v'(t)g'(t)\mathrm{d}t, \quad v(t), g(t) \in W_2^1[0,T], \\ \|v\|_{W_2^1} = \sqrt{\langle v, v\rangle_{W_2^1}}, \quad v \in W_2^1[0,T].\end{gathered} \tag{47}$$

The space $W_2^1[0,T]$ is a reproducing kernel space, and its reproducing kernel function $q_s(t)$ is given by

$$q_s(t) = \begin{cases} 1 + t, & t \le s, \\ 1 + s, & t > s. \end{cases} \tag{48}$$

Further, we define the space $W(\Omega)$ as

$$W(\Omega) = \left\{ \begin{array}{c} v(x,t)| \dfrac{\partial^4 v}{\partial x^3 \partial t}, \text{ is completely continuous,} \\ \\ in\Omega = [0,1] \times [0,T], \\ \\ \dfrac{\partial^6 v}{\partial x^4 \partial t^2} \in L^2(\Omega), v(x,0) = 0 \end{array} \right\} \tag{49}$$

and the inner product and the norm in $W(\Omega)$ are defined, respectively, by

$$
\begin{aligned}
\langle v(x,t), g(x,t)\rangle_W = & \sum_{i=0}^{3}\int_0^T \left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i} v(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i} g(0,t)\right]\mathrm{d}t \\
& + \sum_{j=0}^{1}\left\langle \frac{\partial^j}{\partial t^j} v(x,0), \frac{\partial^j}{\partial t^j} g(x,0)\right\rangle_{W_2^4} \\
& + \int_0^T\int_0^1 \left[\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial t^2} v(x,t)\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial t^2} g(x,t)\right]\mathrm{d}x\mathrm{d}t, \\
\|v\|_W = & \sqrt{\langle v, v\rangle_W}, \quad v \in W(\Omega).
\end{aligned} \tag{50}
$$

Now, we have the following theorem:

**Theorem 1.6.** *The space $W_2^4[0,1]$ is a complete reproducing kernel space, and its reproducing kernel function $R_y(x)$ can be denoted by*

$$
R_y(x) = \begin{cases} \sum_{i=1}^{8} c_i(y)x^{i-1}, & x \le y, \\ \sum_{i=1}^{8} d_i(y)x^{i-1}, & x > y, \end{cases} \tag{51}
$$

where

$$
\begin{aligned}
& c_1(y) = 1, \quad c_2(y) = y, \quad c_3(y) = \frac{1}{4}y^2, \\
& c_4(y) = \frac{1}{36}y^3, \quad c_5(y) = \frac{1}{144}y^3, \quad c_6(y) = -\frac{1}{240}y^2, \\
& c_7(y) = \frac{1}{720}y, \quad c_8(y) = -\frac{1}{5040}, \\
& d_1(y) = 1 - \frac{1}{5040}y^7, \quad d_2(y) = y + \frac{1}{720}y^6, \\
& d_3(y) = \frac{1}{4}y^2 - \frac{1}{240}y^5, \quad d_4(y) = \frac{1}{36}y^3 + \frac{1}{144}y^4, \\
& d_5(y) = 0, \quad d_6(y) = 0, \quad d_7(y) = 0, \quad d_8(y) = 0.
\end{aligned} \tag{52}
$$

*Proof.* Since

$$
\begin{aligned}
\left\langle v(x), R_y(x)\right\rangle_{W_2^4} = & \sum_{i=0}^{3} v^{(i)}(0)R_y^{(i)}(0) + \int_0^1 v^{(4)}(x)R_y^{(4)}(x)\mathrm{d}x, \\
& \left(v(x), R_y(x) \in W_2^4[0,1]\right)
\end{aligned} \tag{53}
$$

through iterative integrations by parts for (53), we have

$$\begin{aligned}\left\langle v(x), R_y(x)\right\rangle_{W_2^4} = & \sum_{i=0}^{3} v^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(3-i)} R_y^{(7-i)}(0)\right] \\ & + \sum_{i=0}^{3} (-1)^{(3-i)} v^{(i)}(1) R_y^{(7-i)}(1) \\ & + \int_0^1 v(x) R_y^{(8)}(x) \mathrm{d}x.\end{aligned} \tag{54}$$

Note that property of the reproducing kernel

$$\left\langle v(x), R_y(x)\right\rangle_{W_2^4} = v(y). \tag{55}$$

If

$$\begin{aligned} R_y(0) + R_y^{(7)}(0) = 0, \\ R_y'(0) - R_y^{(6)}(0) = 0, \\ R_y''(0) + R_y^{(5)}(0) = 0, \\ R_y'''(0) - R_y^{(4)}(0) = 0, \\ R_y^{(4)}(1) = 0, \\ R_y^{(5)}(1) = 0, \\ R_y^{(6)}(1) = 0, \\ R_y^{(7)}(1) = 0, \end{aligned} \tag{56}$$

then by (54), we obtain the following equation:

$$R_y^{(8)}(x) = \delta(x - y), \tag{57}$$

when $x \neq y$,

$$R_y^{(8)}(x) = 0; \tag{58}$$

therefore,

$$R_y(x) = \begin{cases} \sum_{i=1}^{8} c_i(y) x^{i-1}, & x \leq y, \\ \sum_{i=1}^{8} d_i(y) x^{i-1}, & x > y. \end{cases} \tag{59}$$

Since

$$R_y^{(8)}(x) = \delta(x-y), \tag{60}$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0,1,2,3,4,5,6, \tag{61}$$

$$\partial^7 R_{y^+}(y) - \partial^7 R_{y^-}(y) = 1. \tag{62}$$

From (56)–(62), the unknown coefficients $c_i(y)$ ve $d_i(y)(i = 1, 2, \ldots, 8)$ can be obtained. Thus, $R_y(x)$ is given by

$$R_y(x) = \begin{cases} 1 + yx + \frac{1}{4}y^2x^2 + \frac{1}{36}y^3x^3 + \frac{1}{144}y^3x^4 \\ -\frac{1}{240}y^2x^5 + \frac{1}{720}yx^6 - \frac{1}{5040}x^7, \quad x \leq y, \\ 1 + xy + \frac{1}{4}x^2y^2 + \frac{1}{36}x^3y^3 + \frac{1}{144}x^3y^4 \\ -\frac{1}{240}x^2y^5 + \frac{1}{720}xy^6 - \frac{1}{5040}y^7, \quad x > y. \end{cases} \tag{63}$$

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Section 2

# Applications of Differential Equations

Chapter 7

# Local Discontinuous Galerkin Method for Nonlinear Ginzburg-Landau Equation

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Additional information is available at the end of the chapter

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**Abstract**

The Ginzburg-Landau equation has been applied widely in many fields. It describes the amplitude evolution of instability waves in a large variety of dissipative systems in fluid mechanics, which are close to criticality. In this chapter, we develop a local discontinuous Galerkin method to solve the nonlinear Ginzburg-Landau equation. The nonlinear Ginzburg-Landau problem has been expressed as a system of low-order differential equations. Moreover, we prove stability and optimal order of convergence $O(h^{N+1})$ for Ginzburg-Landau equation where $h$ and $N$ are the space step size and polynomial degree, respectively. The numerical experiments confirm the theoretical results of the method.

**Keywords:** Ginzburg-Landau equation, discontinuous Galerkin method, stability, error estimates

## 1. Introduction

The Ginzburg-Landau equation has arisen as a suitable model in physics community, which describes a vast variety of phenomena from nonlinear waves to second-order phase transitions, from superconductivity, superfluidity, and Bose-Einstein condensation to liquid crystals and strings in field theory [1]. The Taylor-Couette flow, Bénard convection [1] and plane Poiseuille flow [2] are such examples where the Ginzburg-Landau equation is derived as a wave envelop or amplitude equation governing wave-packet solutions. In this chapter, we develop a nodal discontinuous Galerkin method to solve the nonlinear Ginzburg-Landau equation

$$\frac{\partial u}{\partial t} - (\nu + i\eta)\Delta u + (\kappa + i\zeta)|u|^2 u - \gamma u = 0, \tag{1}$$

and periodic boundary conditions and $\eta, \zeta, \gamma$ are real constants, $\nu, \kappa > 0$. Notice that the assumption of periodic boundary conditions is for simplicity only and is not essential: the method as well as the analysis can be easily adapted for nonperiodic boundary conditions.

The various kinds of numerical methods can be found for simulating solutions of the nonlinear Ginzburg-Landau problems [3–11]. The local discontinuous Galerkin (LDG) method is famous for high accuracy properties and extreme flexibility [12–20]. To the best of our knowledge, however, the LDG method, which is an important approach to solve partial differential equations, has not been considered for the nonlinear Ginzburg-Landau equation. Compared with finite difference methods, it has the advantage of greatly facilitating the handling of complicated geometries and elements of various shapes and types as well as the treatment of boundary conditions. The higher order of convergence can be achieved without many iterations.

The outline of this chapter is as follows. In Section 2, we derive the discontinuous Galerkin formulation for the nonlinear Ginzburg-Landau equation. In Section 3, we prove a theoretical result of $L^2$ stability for the nonlinear case as well as an error estimate for the linear case. Section 4 presents some numerical examples to illustrate the efficiency of the scheme. A few concluding remarks are given in Section 5.

## 2. LDG scheme for Ginzburg-Landau equation

In order to construct the LDG method, we rewrite the second derivative as first-order derivatives to recover the equation to a low-order system. However, for the first-order system, central fluxes are used. We introduce variables $r, s$ and set

$$r = \frac{\partial}{\partial x} s, \quad s = \frac{\partial}{\partial x} u, \tag{2}$$

then, the Ginzburg-Landau problem can be rewritten as

$$\begin{aligned} &\frac{\partial u}{\partial t} - (\nu + i\eta) r + (\kappa + i\zeta)|u|^2 u - \gamma u = 0, \\ &r = \frac{\partial}{\partial x} s, \quad s = \frac{\partial}{\partial x} u. \end{aligned} \tag{3}$$

We consider problem posed on the physical domain $\Omega$ with boundary $\partial\Omega$ and assume that a nonoverlapping element $D^k$ such that

$$\Omega = \bigcup_{k=1}^{K} D^k. \tag{4}$$

Now we introduce the broken Sobolev space for any real number $r$

$$H^r(\Omega) = \{v \in L^2(\Omega) : \forall k = 1, 2, \ldots.K, v|_{D^k} \in H^r(D^k)\}. \tag{5}$$

We define the local inner product and $L^2(D^k)$ norm

$$(u, v)_{D^k} = \int_{D^k} uvdx, \quad \|u\|^2_{D^k} = (u, u)_{D^k}, \tag{6}$$

as well as the global broken inner product and norm

$$(u, v)_\Omega = \sum_{k=1}^{K} (u, v)_{D^k}, \quad \|u\|^2_{L^2(\Omega)} = \sum_{k=1}^{K} (u, u)_{D^k}. \tag{7}$$

We define the jumps along a normal, $\hat{n}$, as

$$[u] = \hat{n}^- u^- + \hat{n}^+ u^+. \tag{8}$$

The numerical traces $(u,s)$ are defined on interelement faces as the central fluxes

$$u^* = \{u\} = \frac{u^+ + u^-}{2}, \quad s^* = \{s\} = \frac{s^+ + s^-}{2}. \tag{9}$$

Let us discretize the computational domain $\Omega$ into $K$ nonoverlapping elements, $D^k = \left[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}\right]$, $\Delta x_k = x_{k+\frac{1}{2}} - x_{k-\frac{1}{2}}$ and $k = 1, \ldots, K$. We assume $u_h, r_h, s_h \in V_k^N$ be the approximation of $u, r, s$ respectively, where the approximation space is defined as

$$V_k^N = \left\{v : v_k \in \mathbb{P}^N(D^k), \forall D^k \in \Omega\right\}, \tag{10}$$

where $\mathbb{P}^N(D^k)$ denotes the set of polynomials of degree up to $N$ defined on the element $D^k$. We define local discontinuous Galerkin scheme as follows: find $u_h, r_h, s_h \in V_k^N$, such that for all test functions $\vartheta, \phi, \varphi \in V_k^N$,

$$\begin{aligned}
&\left(\tfrac{\partial u_h}{\partial t}, \vartheta\right)_{D^k} - (\nu + i\eta)(r_h, \vartheta)_{D^k} + (\kappa + i\zeta)\left(|u_h|^2 u_h, \vartheta\right)_{D^k} - \gamma(u_h, \vartheta)_{D^k} = 0,\\
&(r_h, \phi)_{D^k} = \left(\tfrac{\partial}{\partial x} s_h, \phi\right)_{D^k},\\
&(s_h, \varphi)_{D^k} = \left(\tfrac{\partial}{\partial x} u_h, \varphi\right)_{D^k}.
\end{aligned} \tag{11}$$

Applying integration by parts to (11), and replacing the fluxes at the interfaces by the corresponding numerical fluxes, we obtain

$$\begin{aligned}
&\left(\tfrac{\partial u_h}{\partial t}, \vartheta\right)_{D^k} - (\nu + i\eta)(r_h, \vartheta)_{D^k} + (\kappa + i\zeta)\left(|u_h|^2 u_h, \vartheta\right)_{D^k} - \gamma(u_h, \vartheta)_{D^k} = 0,\\
&(r_h, \phi)_{D^k} = -(s_h, \phi_x)_{D^k} + \left(s_h^* \phi^-\right)_{k+\frac{1}{2}} - \left(s_h^* \phi^+\right)_{k-\frac{1}{2}},\\
&(s_h, \varphi)_{D^k} = -(u_h, \varphi_x)_{D^k} + \left(u_h^* \varphi^-\right)_{k+\frac{1}{2}} - \left(u_h^* \varphi^+\right)_{k-\frac{1}{2}},
\end{aligned} \tag{12}$$

we can rewrite (12) as

$$
\begin{aligned}
&\left(\tfrac{\partial u_h}{\partial t},\vartheta\right)_{D^k}-(\nu+i\eta)(r_h,\vartheta)_{D^k}+(\kappa+i\zeta)\left(|u_h|^2u_h,\vartheta\right)_{D^k}-\gamma(u_h,\vartheta)_{D^k}=0,\\
&(r_h,\phi)_{D^k}=-\left(s_h,\phi_x\right)_{D^k}+\left(\hat{n}.s_h^*,\phi\right)_{\partial D^k},\\
&(s_h,\varphi)_{D^k}=-\left(u_h,\varphi_x\right)_{D^k}+\left(\hat{n}.u_h^*,\varphi\right)_{\partial D^k}.
\end{aligned}
\tag{13}
$$

where $\hat{n}$ is simply a scalar and takes the value of +1 and $-1$ at the right and the left interface, respectively.

## 3. Stability and error estimates

In this section, we discuss stability and accuracy of the proposed scheme, for the Ginzburg-Landau problem.

### 3.1. Stability analysis

In order to carry out the analysis of the LDG scheme, we have the following results.

**Theorem 3.1.** ($L^2$ *stability). The solution given by the LDG method defined by* (13) *satisfies*

$$\|u_h(x,T)\|_\Omega \le e^{-2\gamma T}\|u_0(x)\|_\Omega$$

*for any* $T>0$.

**Proof.** Set $(\vartheta,\phi,\varphi)=(u_h,\nu u_h,\nu s_h)$ in (13) and consider the integration by parts formula $\left(u,\frac{\partial r}{\partial x}\right)_{D^k}+\left(r,\frac{\partial u}{\partial x}\right)_{D^k}=[ur]_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}$, we get

$$
\begin{aligned}
&\left((u_h)_t,u_h\right)_{D^k}+(s_h,s_h)_{D^k}\\
&=-\nu(r_h,u_h)_{D^k}+(\nu+i\eta)(r_h,u_h)_{D^k}-(\kappa+i\zeta)\left(|u_h|^2u_h,u_h\right)_{D^k}\\
&+\gamma(u_h,u_h)_{D^k}+\nu\left(\hat{n}.s_h^*,u_h\right)_{\partial D^k}+\nu\left(\hat{n}.u_h^*,s_h\right)_{\partial D^k}-\nu(\hat{n}.s_h,u_h)_{\partial D^k}.
\end{aligned}
\tag{14}
$$

Taking the real part of the resulting equation, we obtain

$$
\begin{aligned}
\left((u_h)_t,u_h\right)_{D^k}+(s_h,s_h)_{D^k}=&-\kappa\left(|u_h|^2u_h,u_h\right)_{D^k}+\gamma(u_h,u_h)_{D^k}\\
&+\nu\left(\hat{n}.s_h^*,u_h\right)_{\partial D^k}+\nu\left(\hat{n}.u_h^*,s_h\right)_{\partial D^k}-\nu(\hat{n}.s_h,u_h)_{\partial D^k}.
\end{aligned}
\tag{15}
$$

Removing the positive term $\kappa\left(|u_h|^2u_h,u_h\right)_{D^k}$, we obtain

$$\left((u_h)_t,u_h\right)_{D^k}+(s_h,s_h)_{D^k}\le\gamma\|u_h\|^2_{L^2\left(D^k\right)}+\nu\left(\hat{n}.s_h^*,u_h\right)_{\partial D^k}+\nu\left(\hat{n}.u_h^*,s_h\right)_{\partial D^k}-\nu(\hat{n}.s_h,u_h)_{\partial D^k}. \tag{16}$$

Summing over all elements (16), we easily obtain

$$\left((u_h)_t, u_h\right)_{L^2(\Omega)} + (s_h, s_h)_{L^2(\Omega)} \le \gamma \|u_h\|_\Omega^2. \tag{17}$$

Employing Gronwall's inequality, we obtain

$$\|u_h(x,T)\|_\Omega \le e^{-2\gamma T}\|u_0(x)\|_\Omega. \quad \square$$

### 3.2. Error estimates

We consider the linear Ginzburg-Landau equation

$$\frac{\partial u}{\partial t} - (\nu + i\eta)\Delta u + (\kappa + i\zeta)u - \gamma u = 0. \tag{18}$$

It is easy to verify that the exact solution of the above (18) satisfies

$$\begin{aligned}
&(u_t, \vartheta)_{D^k} - (\nu + i\eta)(r, \vartheta)_{D^k} + (\kappa + i\zeta)(u, \vartheta)_{D^k} - \gamma(u, \vartheta)_{D^k} = 0,\\
&(r, \phi)_{D^k} = -\left(s, \phi_x\right)_{D^k} + \left(\hat{n}.s^*, \phi\right)_{\partial D^k},\\
&(s, \varphi)_{D^k} = -\left(u, \varphi_x\right)_{D^k} + (\hat{n}.u^*, \varphi)_{\partial D^k}.
\end{aligned} \tag{19}$$

Subtracting (19) from the linear Ginzburg-Landau Eq. (13), we have the following error equation

$$\begin{aligned}
&\left((u-u_h)_t, \vartheta\right)_{D^k} + \left(s - s_h, \phi_x\right)_{D^k} + \left(u - u_h, \varphi_x\right)_{D^k} + (\kappa + i\zeta)(u - u_h, \vartheta)_{D^k}\\
&-\gamma(u - u_h, \vartheta)_{D^k} + \left(r - r_h, \phi\right)_{D^k} + (s - s_h, \varphi)_{D^k} - \left(\hat{n}.(s - s_h)^*, \phi\right)_{\partial D^k}\\
&-(\nu + i\eta)(r - r_h, \vartheta)_{D^k} - (\hat{n}.(u - u_h)^*, \varphi)_{\partial D^k} = 0.
\end{aligned} \tag{20}$$

For the error estimate, we define special projections $\mathcal{P}^-$ and $\mathcal{P}^+$ into $V_h^k$. For all the elements, $D^k$, $k = 1, 2, \ldots, K$ are defined to satisfy

$$\begin{aligned}
&(\mathcal{P}^+u - u, v)_{D^k} = 0, \quad \forall v \in \mathbb{P}_N^k\left(D^k\right), \quad \mathcal{P}^+u\left(x_{k-\frac{1}{2}}\right) = u\left(x_{k-\frac{1}{2}}\right),\\
&(\mathcal{P}^-u - u, v)_{D^k} = 0, \quad \forall v \in \mathbb{P}_N^{k-1}\left(D^k\right), \quad \mathcal{P}^-u\left(x_{k+\frac{1}{2}}\right) = u\left(x_{k+\frac{1}{2}}\right).
\end{aligned} \tag{21}$$

Denoting

$$\begin{aligned}
&\pi = \mathcal{P}^-u - u_h, \quad \pi^e = \mathcal{P}^-u - u, \quad \varepsilon = \mathcal{P}^+r - r_h, \quad \varepsilon^e = \mathcal{P}^+r - r,\\
&\tau = \mathcal{P}^+s - s_h, \quad \tau^e = \mathcal{P}^+s - s.
\end{aligned} \tag{22}$$

For the abovementioned special projections, we have, by the standard approximation theory [21], that

$$\begin{aligned}
&\|\mathcal{P}^+u(.) - u(.)\|_{L^2(\Omega_h)} \le Ch^{N+1},\\
&\|\mathcal{P}^-u(.) - u(.)\|_{L^2(\Omega_h)} \le Ch^{N+1},
\end{aligned} \tag{23}$$

where here and below $C$ is a positive constant (which may have a different value in each occurrence) depending solely on u and its derivatives but not of $h$.

**Theorem 3.2.** *Let u be the exact solution of the problem* (18)*, and let* $u_h$ *be the numerical solution of the semi-discrete LDG scheme* (13)*. Then for small enough h, we have the following error estimates:*

$$\|u(.,t) - u_h(.,t)\|_{L^2(\Omega_h)} \leq Ch^{N+1}, \tag{24}$$

where the constant $C$ is dependent upon $T$ and some norms of the solutions.

**Proof**. From the Galerkin orthogonality (20), we get

$$\begin{aligned}
&\left((\pi-\pi^e)_t,\vartheta\right)_{D^k}+\left(\tau-\tau^e,\phi_x\right)_{D^k}+\left(\pi-\pi^e,\varphi_x\right)_{D^k}+(\kappa+i\zeta)(\pi-\pi^e,\vartheta)_{D^k}-\gamma(\pi-\pi^e,\vartheta)_{D^k}\\
&+\left(\varepsilon-\varepsilon^e,\phi\right)_{D^k}+(\tau-\tau^e,\varphi)_{D^k}+\left(\phi-\phi^e,\beta\right)_{D^k}-\left(\hat{n}.(\tau-\tau^e)^*,\phi\right)_{\partial D^k}-(\nu+i\eta)\\
&\times(\varepsilon-\varepsilon^e,\vartheta)_{D^k}-(\hat{n}.(\pi-\pi^e)^*,\varphi)_{\partial D^k}=0.
\end{aligned} \tag{25}$$

Taking the real part of the resulting equation, we obtain

$$\begin{aligned}
&\left((\pi-\pi^e)_t,\vartheta\right)_{D^k}+\left(\tau-\tau^e,\phi_x\right)_{D^k}+\left(\pi-\pi^e,\varphi_x\right)_{D^k}+\kappa(\pi-\pi^e,\vartheta)_{D^k}\\
&-\gamma(\pi-\pi^e,\vartheta)_{D^k}+\left(\varepsilon-\varepsilon^e,\phi\right)_{D^k}+(\tau-\tau^e,\varphi)_{D^k}-\left(\hat{n}.(\tau-\tau^e)^*,\phi\right)_{\partial D^k}\\
&-\nu(\varepsilon-\varepsilon^e,\vartheta)_{D^k}-(\hat{n}.(\pi-\pi^e)^*,\varphi)_{\partial D^k}=0.
\end{aligned} \tag{26}$$

We take the test functions

$$\vartheta=\pi, \quad \phi=\nu\pi, \quad \varphi=\nu\tau, \tag{27}$$

we obtain

$$\begin{aligned}
&\left((\pi-\pi^e)_t,\pi\right)_{D^k}+\nu(\tau-\tau^e,\pi_x)_{D^k}+\nu(\pi-\pi^e,\tau_x)_{D^k}\\
&+\kappa(\pi-\pi^e,\pi)_{D^k}-\gamma(\pi-\pi^e,\pi)_{D^k}+\nu(\varepsilon-\varepsilon^e,\pi)_{D^k}\\
&+\nu(\tau-\tau^e,\tau)_{D^k}-\nu(\hat{n}.(\tau-\tau^e)^*,\pi)_{\partial D^k}-\nu(\varepsilon-\varepsilon^e,\pi)_{D^k}-\nu(\hat{n}.(\pi-\pi^e)^*,\tau)_{\partial D^k}=0.
\end{aligned} \tag{28}$$

Summing over $k$, simplify by integration by parts and (9), we get

$$\begin{aligned}
&(\pi_t,\pi)_\Omega+\nu(\tau,\tau)_\Omega=\nu(\tau^e,\pi_x)_\Omega+\nu(\pi^e,\tau_x)_\Omega+(\pi^e_t,\pi)_\Omega-\gamma(\pi^e,\pi)_\Omega+\kappa(\pi^e,\pi)_\Omega\\
&+\nu(\tau^e,\tau)_\Omega+\gamma(\pi,\pi)_\Omega-\kappa(\pi,\pi)_\Omega-\nu\sum_{k=1}^{K}(\hat{n}.(\pi^e)^*,\tau)_{\partial D^k}-\nu\sum_{k=1}^{K}(\hat{n}.(\tau^e)^*,\pi)_{\partial D^k},
\end{aligned} \tag{29}$$

we can rewrite (29) as

$$(\pi_t,\pi)_\Omega+\nu(\tau,\tau)_\Omega=I+II+III, \tag{30}$$

where

$$I=\nu(\tau^e,\pi_x)_\Omega+\nu(\pi^e,\tau_x)_\Omega, \tag{31}$$

$$\begin{aligned}
II=&\left(\pi^e_t,\pi\right)_\Omega-\gamma(\pi^e,\pi)_\Omega+\kappa(\pi^e,\pi)_\Omega+\nu(\tau^e,\tau)_\Omega\\
&-\nu\sum_{k=1}^{K}(\hat{n}.(\pi^e)^*,\tau)_{\partial D^k}-\nu\sum_{k=1}^{K}(\hat{n}.(\tau^e)^*,\pi)_{\partial D^k},
\end{aligned} \tag{32}$$

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$$III = \gamma(\pi,\pi)_\Omega - \kappa(\pi,\pi)_\Omega. \tag{33}$$

Using the definitions of the projections $\mathcal{P}, \mathcal{S}$ (21) in (31), we get

$$I = 0. \tag{34}$$

From the approximation results (23) and Young's inequality in (32), we obtain

$$II \le c_1 \|\pi\|^2_{L^2(\Omega)} + c_2\|\tau\|^2_{L^2(\Omega)} + Ch^{2N+2}. \tag{35}$$

and

$$III \le c_1 \|\pi\|^2_{L^2(\Omega)}. \tag{36}$$

Combining (34), (35), (36) and (30), we obtain

$$(\pi_t,\pi)_\Omega + \nu(\tau,\tau)_\Omega \le c_1\|\pi\|^2_{L^2(\Omega)} + c_2\|\tau\|^2_{L^2(\Omega)} + Ch^{2N+2}, \tag{37}$$

provided $c_2$ is sufficiently small such that $c_2 \le \nu$, we obtain that

$$(\pi_t,\pi)_\Omega \le c_1\|\pi\|^2_{L^2(\Omega)} + Ch^{2N+2}. \tag{38}$$

From the Gronwall's lemma and standard approximation theory, the desired result follows. □.

## 4. Numerical examples

In this section, we present several numerical examples to illustrate the previous theoretical results. We use the high-order Runge-Kutta time discretizations [22], when the polynomials are of degree $N$, a higher order accurate Runge-Kutta (RK) method must be used in order to guarantee that the scheme is stable. In this chapter, we use a fourth-order non-total variation diminishing (TVD) Runge-Kutta scheme [23]. Numerical experiments demonstrate its numerical stability

$$\frac{\partial \mathbf{u}_h}{\partial t} = \mathcal{F}(\mathbf{u}_h, t), \tag{39}$$

where $\mathbf{u}_h$ is the vector of unknowns, we can use the standard fourth-order four-stage explicit RK method (ERK)

$$\begin{aligned}
\mathbf{k}^1 &= \mathcal{F}(\mathbf{u}_h^n, t^n),\\
\mathbf{k}^2 &= \mathcal{F}\left(\mathbf{u}_h^n + \frac{1}{2}\Delta t\mathbf{k}^1, t^n + \frac{1}{2}\Delta t\right),\\
\mathbf{k}^3 &= \mathcal{F}\left(\mathbf{u}_h^n + \frac{1}{2}\Delta t\mathbf{k}^2, t^n + \frac{1}{2}\Delta t\right),\\
\mathbf{k}^4 &= \mathcal{F}(\mathbf{u}_h^n + \Delta t\mathbf{k}^3, t^n + \Delta t),\\
\mathbf{u}_h^{n+1} &= \mathbf{u}_h^n + \frac{1}{6}(\mathbf{k}^1 + 2\mathbf{k}^2 + 2\mathbf{k}^3 + \mathbf{k}^4),
\end{aligned} \tag{40}$$

to advance from $\mathbf{u}_h^n$ to $\mathbf{u}_h^{n+1}$, separated by the time step, $\Delta t$. In our examples, the condition $\Delta t \leq C \Delta x_{min}^{\alpha}$ $(0 < C < 1)$ is used to ensure stability.

**Example 4.1** *We consider the following linear Ginzburg-Landau equation*

$$\frac{\partial u}{\partial t} - (\nu + i\eta)\Delta u + (\kappa + i\zeta)u = 0, \quad x \in [-20, 20], \quad t \in (0, 0.5], \quad u(x, 0) = u_0(x), \tag{41}$$

with

$$\eta = \frac{1}{2}, \kappa = -\frac{\nu\left(3\sqrt{1+4\nu^2} - 1\right)}{2(2+9\nu^2)}, \zeta = -1, \gamma = 0. \tag{42}$$

The exact solution $u(x, t) = a(x)e^{id\ln(a(x)) - i\omega t}$ where

$$a(x) = F sech(x), F = \sqrt{\frac{d\sqrt{1+4\nu^2}}{-2\kappa}}, d = \frac{\sqrt{1+4\nu^2} - 1}{2\nu}, \omega = -\frac{d\left(1+4\nu^2\right)}{2\nu}. \tag{43}$$

The convergence rates and the numerical $L^2$ error are listed in **Figure 1** for several different values of $\nu$, confirming optimal $O\left(h^{N+1}\right)$ order of convergence across.

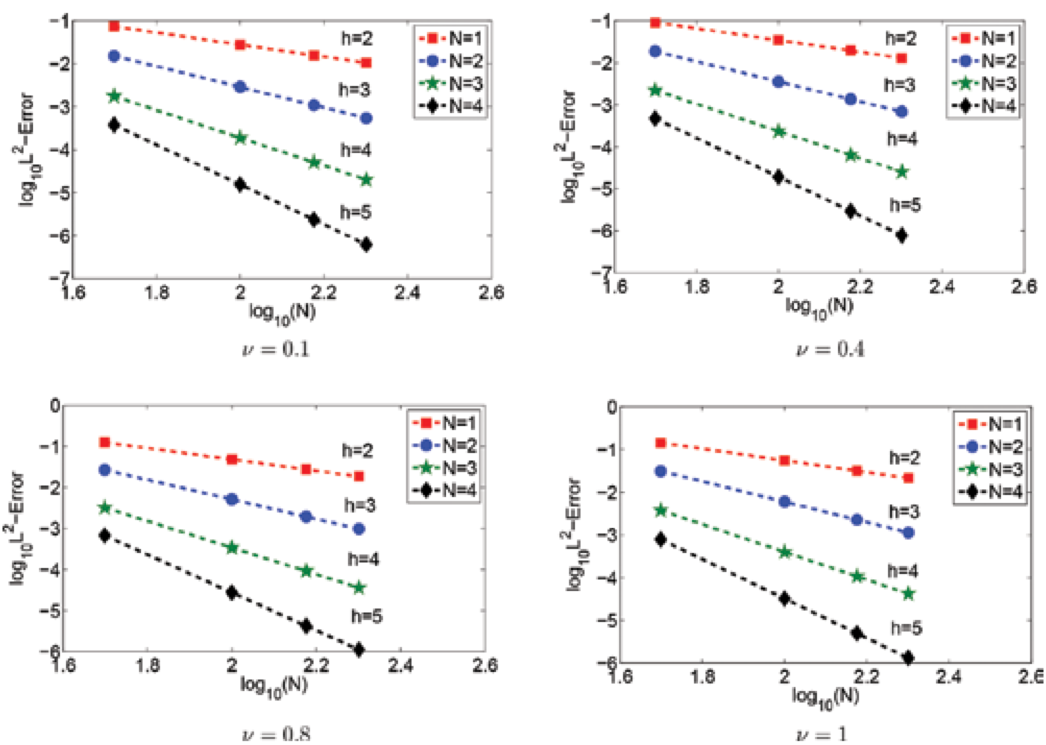


**Figure 1.** The rate of convergence for the solving the nonlinear Ginzburg-Landau equation in Example 4.2.

**Example 4.2** *We consider the nonlinear Ginzburg-Landau Eq.* (1) *with initial condition,*

$$u(x,0) = e^{-x^2}, \tag{44}$$

with parameters $\nu = 1, \kappa = 1, \eta = 1, \zeta = 2,\ x \in [-10, 10]$. We consider cases with $N = 2$ and $K = 40$ and solve the equation for several different values of $\gamma$. The numerical solution $u_h(x,t)$ for $\gamma = 2, 1, 0, -1, -2$ is shown in **Figures 2** and **3**. The parameter $\gamma$ will affect the wave shape. From these figures, it is obvious that the solution decays rapidly with time evolution especially for $\gamma < 0$ and the parameter $\gamma$ dramatically affects the wave shape.

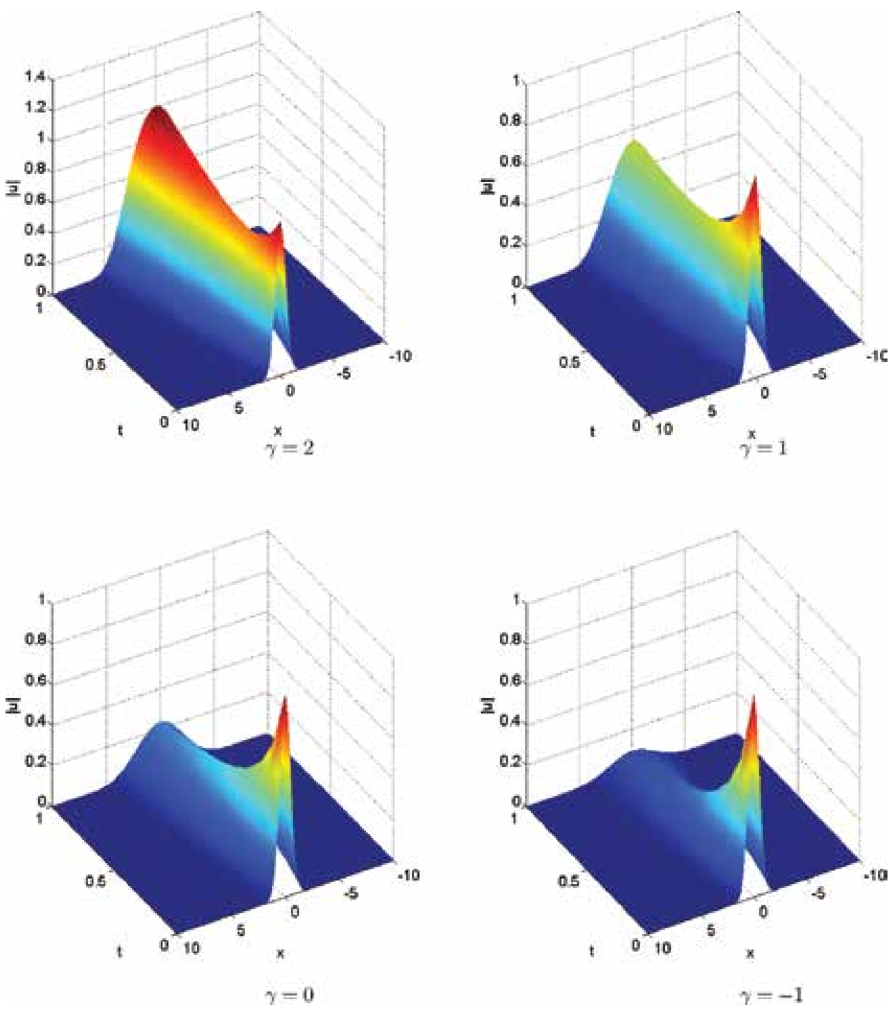


**Figure 2.** Numerical results for the nonlinear Ginzburg-Landau equation in Example 4.2.

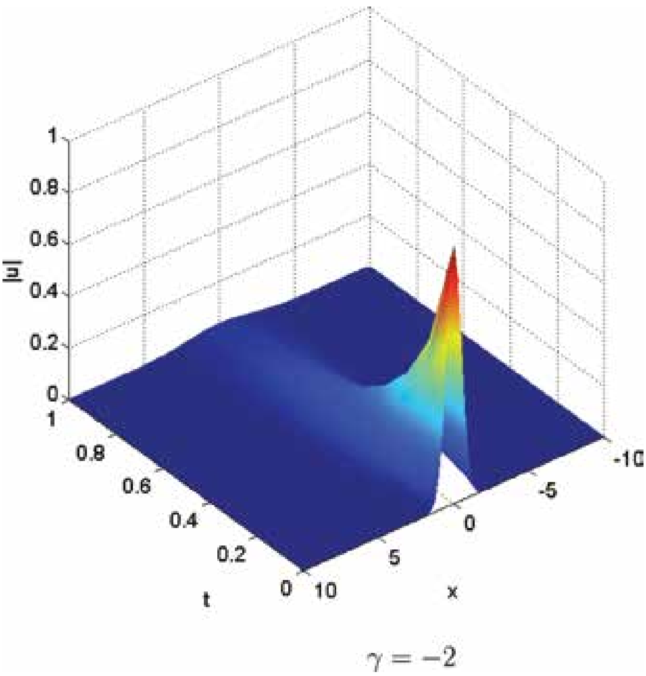


**Figure 3.** Numerical results for the nonlinear Ginzburg-Landau equation with $\gamma = -2$ in Example 4.2.

## 5. Conclusions

In this chapter, we developed and analyzed a local discontinuous Galerkin method for solving the nonlinear Ginzburg-Landau equation and have proven the stability of this method. Numerical experiments confirm that the optimal order of convergence is recovered. As a last example, the Ginzburg-Landau equation with initial condition is solved for different values of $\gamma$ and results show that the parameter $\gamma$ dramatically affects the wave shape. In addition, the solution decays rapidly with time evolution especially for $\gamma<0$.

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Chapter 8

# General Functions Method in Transport Boundary Value Problems of Elasticity Theory

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Additional information is available at the end of the chapter

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**Abstract**

The Lame system describing the dynamics of an isotropic elastic medium affected by a steady transport load moving at subsonic, transonic, and supersonic speed is considered. Its fundamental and generalized solutions in a moving frame of reference tied to the transport load are analyzed. Shock waves arising in the medium at supersonic speeds are studied. Conditions on the jump in the stress, displacement rate, and energy across the shock front are obtained using distribution theory. Transport boundary value problem for an elastic medium bounded by a cylindrical surface of arbitrary cross section and subjected to transport loads is considered in the subsonic and supersonic case with regard to shock waves. To solve problems, the generalized functions method is developed. In the space of generalized functions, generalized solutions are constructed and their regular integral presentations are obtained. Singular boundary equations solving the boundary value problems are presented.

**Keywords:** elastic medium, transport load, subsonic, transonic, supersonic speed, shock waves, boundary value problem, generalized functions method, generalized solutions, singular boundary equations

## 1. Introduction

A widespread source of wave generation in continuous media is transport loading, i.e., moving loads whose form does not change over time. The velocity of a transport load has a large effect on the type of differential equations describing the dynamics of the medium. The equations depend parametrically on the Mach numbers, i.e., on the ratio of the speed of motion to the propagation speeds of perturbations in the medium (sound speeds). It is well known [1] that, in an isotropic elastic medium, there are two sound speeds ($c_1$,$c_2$), which determine the velocities of dilatational and shear waves propagation. This has a large effect on the type of equations and leads to systems of elliptic, hyperbolic, or mixed equations. For transport problems, typical factors are shock effects generated by supersonic loading. At shock fronts, the stresses, displacement rates, and energy density are discontinuous. A convenient research method for such problems is provided by the theory of generalized functions (distributions), which makes it possible to significantly expand the class of processes amenable to study by using singular generalized functions in the simulation of observed phenomena. In this chapter, methods of this theory are used to solve boundary value problems using motion equations of the theory of elasticity in cylindrical domains under the action of transport loads, moving at supersonic and supersonic speeds.

## 2. Motion equation of elastic medium

We consider an isotropic elastic medium with Lame's parameters $\lambda,\mu$, and a density $\rho$. Let us denote $x = x_j e_j$, $e_j$ as the unit vectors of Cartesian coordinate system in the space $R^3$; displacements vector $u(\mathrm{x},t) = u_j e_j$; stress tensors $\sigma_{ij}$ deformation tensor $\varepsilon_{ij}$. These tensors are connected by Hook's law [1]:

$$\varepsilon_{ij} = 0,5\left(u_{i,j} + u_{j,i}\right), \quad i,j,k = 1,2,3. \tag{1}$$

$$\sigma_{ij} = C_{ij}^{kl}\varepsilon_{kl} = C_{ij}^{kl}u_{k,l} \tag{2}$$

The elastic constant tensor has the symmetry properties.

$$C_{ij}^{kl} = C_{ji}^{kl} = C_{ij}^{lk} = C_{kl}^{ij}.$$

In the case of an isotropic medium, it is equal to

$$C_{ij}^{kl} = \lambda\delta_i^j\delta_l^k + \mu(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l),$$

and Hook's law has the form

$$\sigma_{ij} = \lambda\,\mathrm{div}\,u\,\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right)$$

Here $\delta_{ij} = \delta_i^j$ is the Kronecker symbol. Everywhere, there are tensor convolutions over of the same name indexes from 1 to 3, $u_{i,j} \triangleq \frac{\partial u_i}{\partial x_j}$.

Motion equations for material continuum

$$\frac{\partial \sigma_{ij}}{\partial x_j} + G_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \quad i,j = 1,2,3 \tag{3}$$

for elastic medium by using Eqs. (1) and (2) have the form:

$$L_i^j(\partial_{\mathbf{x}}, \partial_t)\, u_j + G_i = 0, \tag{4}$$

Here $L$ is the matrix Lame's operator:

$$L_i^j(\partial_{\mathbf{x}}, \partial_t) = (c_1^2 - c_2^2)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \delta_i^j\left(c_2^2\Delta - \frac{\partial^2}{\partial t^2}\right)$$

$c_1 = \sqrt{(\lambda + 2\mu)/\rho},\ c_2 = \sqrt{\mu/\rho}$ are the velocities of dilatational and shear waves ($c_1 > c_2$), $G(\mathrm{x},t)$ is the mass force, $\Delta$ is the Laplace operator.

The system shown in Eq. (4) was fairly well studied by Petrashen [2]. Since the elastic potential of the medium is positive definite, this system is strictly hyperbolic. Such systems can have solutions with discontinuous derivatives. The discontinuity surface $F$ in $R^4 = R^3 \times t(-\infty < t < \infty)$ coincides with a characteristic surface of the system. It corresponds to a wave front $F_t$ moving in $R^3$ at the velocity $V$:

$$V = -\nu_t/\|\nu\|_3, \qquad \|\nu\|_3 = \sqrt{\sum_{k=1}^{3}\nu_k^2}. \tag{5}$$

We note that $\nu(x,t) = (\nu_1,\nu_2,\nu_3,\nu_t)$ is a normal vector to $F$ in $R^4$, satisfying the characteristic equation

$$\det\left\{\left(c_1^2 - c_2^2\right)\nu_i\nu_j + \delta_{ij}\left(c_2^2\|\nu\|_3^2 - \nu_t^2\right)\right\} = (c_1^2\|\nu\|_3^2 - \nu_t^2)(c_2^2\|\nu\|_3^2 - \nu_t^2)^2 = 0. \tag{6}$$

This equation has the roots:

$$\nu_t = \pm c_j\|\nu\|_3,\ \ j = 1, 2. \tag{7}$$

From Eqs. (5) and (7), we get that $F_t$ moves in $R^3$ at the sound velocity $V = c_1$ or $V = c_2$.

We introduce a wave vector $m = (m_1, m_2, m_3)$. It is a unit normal vector to $F_t$ in $R^3$ for fixed $t$ in the direction of wave propagation. By virtue of Eq. (7),

$$m_j = \frac{\nu_j}{\|\nu\|_3} = -V\nu_j/\nu_t \tag{8}$$

Let $\nu_t = \nu_4$. The requirement that the displacements be continuous across the wave front, i.e.,

$$[u(x,t)]_{F_t} = 0 \tag{9}$$

which is associated with the preservation of the continuity of the medium, leads to kinematic consistency conditions for solutions at the wave front:

$$\left[m_j\frac{\partial u_i}{\partial t} + V\frac{\partial u_i}{\partial x_j}\right]_{F_t} = 0, \quad i, j = 1, 2, 3. \tag{10}$$

(the continuity of the tangent derivatives on $F_t$). Additionally, Eq. (4) implies dynamical consistency conditions for solutions at the wave front, which are equivalent to the momentum conservation law in its neighborhood:

$$[\sigma_{ij}]\, m_j = -\rho V \left[\frac{\partial u_i}{\partial t}\right]_{F_t}, \quad i,j = 1,2,3 \tag{11}$$

*Definition*. A wave is called a *shock wave* if the jump in the stresses across the wave front is finite: $e_i m_j [\sigma_{ij}]_{Ft} = \not{0}$. If $m_j[\sigma_{ij}]_{Ft} = 0$, then this is a *weak shock* wave. If $m_j[\sigma_{ij}]_{Ft} = \infty$, then this is a *strong shock* wave.

Velocity suffers a jump discontinuity across a shock front. At fronts of weak shock waves, the velocities are continuous, but the second derivatives of solutions are not. Strong shock waves (in the sense of the aforementioned definition) do not occur in actual media, since, at large stress jumps, the medium is destroyed and ceases to be elastic. However, strong shock waves in elastic media play an important theoretical role in the construction of solutions, specifically, fundamental solutions of Eq. (4).

## 3. Lame transport equations and Mach numbers

Suppose that the force affecting the medium moves at a constant velocity $c$ along the $X_3$ axis (for convenience, in its negative direction) and, in a moving coordinate system $x' = (x_1, x_2, z = x_3 + ct)$ it does not depend on $t$:

$$G(x,z) = G_j(x_1, x_2, x_3 + ct)\, e_j \tag{12}$$

Transport solutions are solutions of Eq. (4) with the same structure:

$$\mathrm{u} = \mathrm{u}(x_1, x_2, x_3 + ct) = \mathrm{u}(\mathrm{x}, \mathrm{z}) \tag{13}$$

The speed of transport loads is called *subsonic* if $c < c_2$,*transonic* if $c_2 < c < c_1$, and *supersonic* if $c > c_1$. A speed is called the *first* or *second sound* speed if $c = c_j$, $j = 1, 2$, respectively.

In the new variables, the equations of motion are brought to the form

$$L_j^i\left(\frac{\partial}{\partial x'}\right) u_i = \left\{\left(M_1^{-2} - M_2^{-2}\right)\frac{\partial^2}{\partial x_i' \partial' x_j} + \left(M_2^{-2}\Delta - \frac{\partial^2}{\partial x_3'^2}\right)\delta_j^i\right\} u_i + g_j = 0 \tag{14}$$

Here $g_j = \left(\rho c^2\right)^{-1} G_j; \quad M_j = c/c_j$ are Mach numbers: ($M_1 < M_2$).

As $M_j < 1$($j = 1, 2$) the load is subsonic and the system of equations is elliptic. If the load is supersonic, i.e., $M_j > l$, $j = 1, 2$, then the system becomes hyperbolic. In the case of transonic speeds, i.e., $M_1 < 1$ and $M_2 > 1$, the equations are hyperbolic-elliptic. In the case of sound speeds, the equations are parabolic-elliptic if $M_2 = 1$ and parabolic-hyperbolic if $M_1 = 1$. We will show this later when considering fundamental solutions of Eq. (14).

Since the original system is hyperbolic, Eq. (14) can also have discontinuous solutions. Let $F$ be a discontinuity surface in the space of variables $x'$ such that it is stationary in this space and moves at one of the sound velocities $V = c_1, c_2$ in the space of $(x_1, x_2, x_3)$. It follows from Eq. (7) that $V = cn_3$, where $n = (n_1, n_2, n_3)$ is the unit normal to $F$ in $R^3$. Therefore, since $c = c_j/n_3$ and $|n_3| \leq 1$, such surfaces can arise only at supersonic speeds: $c \geq c_j$.

It follows from Eqs. (9) to (11) and Eq. (13) that the kinematic and dynamical consistency conditions for solutions at discontinuities in the mobile coordinate system have the form:

$$[u(x,z)]_F = 0 \quad \Rightarrow \qquad [n_z u_{i,j} - n_j u_{i,z}]_F = 0; \tag{15}$$

$$[\sigma_{ij}] n_j = -\rho c_k c [u_{i,z}]_F; \quad n_z = -c_k/c, \quad for\ c \geq c_k; \tag{16}$$

$n = \{n_1, n_2, n_z = n_3\}$ is a *wave vector*, $k = 1$ for shock dilatational waves, $k = 2$ for shock shear waves. Here and hereafter, the derivative with respect to $x_j$ is denoted by the index $j$ after a comma in the function notation or by the variable itself.

*Definition.* If $c > c_2$, the solution of the system in Eq. (14) is called *classical* if it is continuous and twice differentiable everywhere, except for, possibly, wave fronts. The number of fronts is finite at any fixed $t$ and the conditions on the gaps, Eqs. (15) and (16), are satisfied on the wave fronts.

At first, we construct the solutions of the transport Lame equation using methods of generalized functions theory.

## 4. Shock waves as generalized solutions of transport Lame equations: conditions on wave front

Consider Eq. (14) and its solutions on the space of generalized vector functions $D'_3(R^3)$ with components being generalized functions from $D'(R^3)$ (see [3]). Obviously, if $u$ is a solution of Eq. (14) that is twice differentiable, then it is also a generalized solution of Eq. (14). If a vector function $u$ satisfies Eq. (14) in the classical sense almost everywhere, except for some surfaces, on which its derivatives are discontinuous, then, generally speaking, $u$ is not a generalized solution of Eq. (14).

Let $u(x,z)$ be a shock wave ($x = (x_1, x_2)$), i.e., a classical solution of the Lame transport equations, Eq. (14), that satisfies conditions Eqs. (15) and (16) at the front $F$. Let $\hat{u}(x,z)$ denote the corresponding regular generalized function.

**Theorem 4.1.** *The shock wave $\hat{u}(x,z)$ is a generalized solution of the Lame equation in $D'_3(R^3)$.*

**Proof.** Using the rules for differentiating generalized functions with derivatives having jump discontinuities across some surfaces (see [3]), for the equations of motion in $D'_3(R^3)$, we obtain

$$\frac{\partial\hat{\sigma}_{ij}}{\partial x'_j} - \rho c^2 \frac{\partial^2 \hat{u}_i}{\partial x_z^2} + G_i = \left[ \sigma_{ij}h_j - \rho c^2 \ h_z \frac{\partial u_i}{\partial z} \right]_F \delta_F + \\ + \frac{\partial}{\partial x'_j} \left\{ \left[ \lambda \ u_k h_k \delta_{ij} + \mu (u_i h_j + u_j h_i \right]_F \delta_F \right\} - \frac{\partial}{\partial z} \left\{ [u_i \ ]_F h_z \delta_F \right\}, \tag{17}$$

Here, the right-hand side involves singular generalized functions, namely, single layers $\delta_F\ (x,z)$ and double layers on $F$. By virtue of conditions Eqs. (15) and (16), the densities of these layers are equal to zero, so the right-hand side of Eq. (17) vanishes; i.e., the shock wave satisfies the same equations, Eq. (14), but in the generalized sense.

As a result, we obtain a simple formal method for deriving conditions at jumps in solutions and their derivatives across the shock fronts in hyperbolic equations. Namely, these equations are written in the space of generalized functions and the densities of the singular functions corresponding to single, double, etc., layers are set to zero.

Define as follows the *kinetic energy density*

$$\mathrm{K} = 0.5\rho \|u,_t\|^2 = 0.5\rho c^2 \|u,_z\|^2 \tag{18}$$

and *elastic potential*

$$W = 0.5\sigma_{ij}u_{i,j} = 0.5\sigma_{ij}\varepsilon_{ij} \tag{19}$$

Consider the following functions: the *energy density* $E = K + W$ of elastic deformations and the *Lagrangian* $\Lambda = K - W$.

**Theorem 4.2.** *If G is continuous, then the Lagrangian $\Lambda$ is continuous at the shock waves fronts.*

$([\Lambda]_F = 0)$ *and the jump in the energy density satisfies the relation*

$$h_z[E]_F = \left[ (\sigma_{ij}h_j) u_{i,z} \right]_F, \tag{20}$$

*First formula is equivalent to the equality:*

$$[E]_{F_{c_k}} = -\frac{c_k}{c} h_j^k \left[ \ \sigma_{ij}u_{i,z} \right]_{F_{c_k}}, \quad k = 1, 2$$

where $c_k$ is the sound velocity corresponding to front $F$, $h_j^k$ is the components of the wave vector to $F$.

The last formula may be easy to get if we write the equation for $E$ in $D'_3(R^3)$ in the form

$$\hat{E},_z = E,_z + [E]_F h_z \delta_F = (\sigma_{ij}u_{i,z}),_j + \rho(G, u,_z) + \left[ \sigma_{ij}u_{i,z} \right]_F \delta_F h_j + \rho(G, [u]_F) h_z \delta_F \quad \Rightarrow$$

$$[E]_F h_z = \left[ \sigma_{ij}u_{i,z} \right]_F h_j$$

as $[u]_F = 0$. For the gaps of these functions, the theorem has been proved on the basis of classic methods (see [4, 5]). For full proof of this theorem, see [6].

## 5. Fundamental Green's tensors and generalized solutions of transport Lame equations

The matrix of fundamental solutions $\widehat{U}(x, z)$ satisfies Eq. (14) with a delta function in the mass force:

$$L_i^j\left(\frac{\partial}{\partial \mathbf{x}'}\right)\hat{U}_j^k + \delta(\mathbf{x}')\delta_i^k = 0, \qquad i, j = 1, 2, 3 \tag{21}$$

This matrix is called *Green's tensor* for the transport Lame equations if it satisfies the decay conditions at infinity

$$\hat{U}_i^k \to 0, \quad \partial_j \hat{U}_i^k \to 0, \quad x' \to \infty, \quad i,j,k = 1,2,3. \tag{22}$$

For a fixed $k$, its components describe the displacements of the elastic medium under a concentrated force moving at the velocity $c$ along the axis $Z = X_3$ and acting in the $X_k$ direction.

Green's tensor can be obtained by taking the Fourier transform of Eq. (17) and solving the corresponding system of linear algebraic equations for the Fourier transforms $\overline{U}(\xi_1, \xi_2, \xi_3)$. It is reduced to the form (see [4]).

$$\bar{U}_i^j = \frac{M_2^2\delta_i^j}{(\|\xi\|^2 - M_2^2\xi_3^2)} + \frac{\xi_i\xi_j}{\xi_3^2}\left(\frac{1}{\|\xi\|^2 - M_2^2\xi_3^2} - \frac{1}{\|\xi\|^2 - M_1^2\xi_3^2},\right) \tag{23}$$

It can be seen that $\widehat{U}(x, z)$ has no classical inverse Fourier transform since it has non-integrable singularities in its denominators. This is associated with the fact that the matrix of fundamental solutions is defined, generally speaking, up to solutions of the homogeneous system of equations. The functions

$$\bar{f}_{km} = \xi^{-m}\left(\|\xi\|^2 - M_m^2\xi_3^2\right)^{-1}, \quad m = 0, 1, 2$$

are of crucial importance in the construction of the original Green's tensor. It is easy to see that $\bar{f}_{0m}$ is the Fourier transform of the fundamental solution to the equation

$$\frac{\partial^2 \hat{f}_{0k}}{\partial x_1^2} + \frac{\partial^2 \hat{f}_{0k}}{\partial x_2^2} + (1 - M_k^2)\frac{\partial^2 \hat{f}_{0k}}{\partial z^2} + \delta(x)\delta(z) = 0 \tag{24}$$

This equation is similar to the elliptic Laplace equation at subsonic speeds if $M_k < 1$ and to the wave equation at supersonic speeds if $M_k > 1$. At the sound speed ($M_k = 1$), the variable $z$

disappears from the equation and the equation becomes parabolic, since the space dimension is higher by one, which determines the type of Eq. (14), as noted earlier, since the solutions contain waves of two types. Green's tensor for the Lame transport equation was constructed by Alekseyeva [4] by applying fundamental solutions of the Laplace and wave equations and regularization functions $\bar{f}_{km}$, which depends on the speed of transport load. Green's tensor has the regular form:

$$U_i^j(x,z) = c_2^{-2}\delta_i^j f_{02}(\|x\|,z) + c^{-2}\left(f_{21',ij}(\|x\|,z) - f_{22',ij}(\|x\|,z)\right), \tag{25}$$

where the type of basic function depends on velocity $c$.

*In subsonic case* ($M_k < 1$):

$$4\pi f_{oj}(r,z) = \frac{1}{\sqrt{z^2 + m_j^2 r^2}}, \quad 4\pi f_{1j} = \operatorname{sgn}\ |z|\ \ln\left(\frac{|z| + \sqrt{z^2 + m_j^2 r^2}}{m_j r}\right),$$

$$4\pi f_{2j} = |z|\ \ln\left(\frac{|z| + V_j}{m_j r}\right) - V_j + m_j\|x\|,$$

*In sonic case* ($M_k = 1$):

$$f_{ok}(|x|,z) = -0.5\ \delta(z)|x|, \quad f_{1k} = 0.5\ \theta(z)|x|, \quad f_{2k} = 0.5\ z\theta(z)|x|.$$

*In supersonic case* ($M_k > 1$):

$$f_{oj}(r,z) = \frac{\theta(z - m_j r)}{2\pi\sqrt{z^2 - m_j^2 r^2}}, \quad f_{1j} = \frac{\theta(z - m_j r)}{2\pi}\ \ln\left(\frac{z + V_j^-}{m_j r}\right), \quad f_{2j} = \frac{\theta(z - m_j r)}{2\pi}\left(z\ln\left(\frac{z + V_j^-}{m_j r}\right) - V_j^-\right),$$

Here and hereafter, we use the following notation: $\theta(z)$ is the Heaviside step function,

$$m_k = \sqrt{1 - M_k^2}, \quad V_k = \sqrt{z^2 + m_k^2 r^2}, \quad V_k^- = \sqrt{z^2 - m_k^2 r^2}, \quad r = \sqrt{x_1^2 + x_2^2} = \|x\|,$$

The dilatational and shear components of $\widehat{U}(x,z)$ are easy to write out

$$\begin{aligned} &U_i^j(x,z) = U_{i1}^j(x,z) + U_{i2}^j(x,z) \\ &U_{i1}^j = c^{-2} f_{21',ij}(\|x\|,z), \quad U_{i2}^j(x,z) = c_2^{-2}\delta_i^j f_{02}(\|x\|,z) - f_{22',ij}(\|x\|,z) \end{aligned} \tag{26}$$

In the supersonic case, the support of the functions is the cone $z > m_k\|x\|$. This determines a radiation condition as physical considerations imply that there are no displacements of the elastic medium outside this cone since the perturbations have a finite propagation velocity, which cannot be higher than the corresponding sound velocity for a particular type of deformation. At the fronts of shock waves ($z = m_k\|x\|$), Green's tensor grows to infinity.

If the following convolution exists,

$$\widehat{u}_i = \widehat{U}_i^j * G_j(x,z)/\rho c^2 \tag{27}$$

it is easy to prove that it is the generalized solution of the transport Lame equations, Eq. (14).

If mass forces are regular, then Eq. (28) has an integral presentation:

$$u_i(x,z) = \int_{D^-} U_i^j(x-y, z-\tau) g_j(y,\tau) dy_1 dy_2 d\tau = u_i(x,z) \tag{28}$$

If mass forces are concentrated on surface $D$ and described by singular generalized functions of the type of single layers $g = g_j(y,\tau) e_j \delta_D(y,\tau)$, then

$$\widehat{u}_i(x,z) = \int_D \Big( U_i^j(x-y, z-\tau) g_j(y,\tau) dD(y,\tau) = u_i(x,z) \tag{29}$$

Moreover, by the Du Bois-Reymond lemma [3], these solutions are classical. For other types of singular mass forces, to calculate Eq. (28), we use the definition of convolution of a generalized function [3].

It is easy to see from Eqs. (23) to (25) that the solution is represented as a composition of fundamental solutions distributed over the support of the function *f*(*x*,*z*); their intensities are determined by its value.

In Alexeyeva and Kayshibayeva's paper [5], there are some numerical examples of calculation of the dynamic of elastic medium at subsonic, transonic, and supersonic speed of transport loads moving along the strip in an elastic medium.

## 6. Subsonic Green's tensor, fundamental stress tensors, and their properties

In the subsonic case from Eq. (25), we obtain the components of Green's tensor in the form:

$$\hat{U}_1^1 = \frac{1}{4\pi c^2}\left(\frac{1}{V_2} + \frac{z^2 x_1^2 W_{12}}{r^4 M_2^2} - \frac{x_2^2 V_{12}}{r^4 M_2^2}\right),$$
$$\hat{U}_2^2 = \frac{1}{4\pi c^2}\left(\frac{1}{V_2} + \frac{z^2 x_2^2 W_{12}}{r^4 M_2^2} - \frac{x_1^2 V_{12}}{r^4 M_2^2}\right),$$
$$\hat{U}_1^2 = \hat{U}_2^1 = \frac{x_1 x_2}{4\pi c^2 r^4}\left(z^2 W_{12} + V_{12}\right),\ \hat{U}_3^3 = \frac{1}{4\pi c^2}\left(\frac{1}{V_1} - \frac{m_2^2}{V_2}\right),$$
$$\hat{U}_1^3 = \hat{U}_3^1 = -\frac{x_1 z}{4\pi c^2 r^2} W_{12},\quad \hat{U}_2^3 = \hat{U}_3^2 = -\frac{x_2 z W_{12}}{4\pi c^2 r^2},$$
$$V_{12} = V_1 - V_2,\quad V_i = \sqrt{z^2 + m_i^2 r^2},\quad m_i^2 = 1 - M_i^2,\quad W_{12} = V_1^{-1} - V_2^{-1},\ r = \sqrt{x_1^2 + x_2^2}$$

They are regular functions. Since by $x' \to 0$ [6]:

$$V_{12} \sim \frac{r^2(m_1^2 - m_2^2)}{2\,|z|}, \quad , W_{12} \sim \frac{r^2(m_2^2 - m_1^2)}{2\,|z|}, \quad \frac{z^2 x_1^2}{r^4} W_{12} - \frac{x_2^2}{r^4} V_{12} \sim \frac{(m_2^2 - m_1^2)}{2\,|z|} \tag{30}$$

these components are bounded for $(x, z) \neq (0, 0, 0)$. At the point $(x, z) = (0, 0, 0)$, they have a weak singularity of order $R^{-1}$, $R = \sqrt{z^2 + r^2}$. It has a similar asymptotic at infinity. Accordingly, $R^{-2}$ is the order of the tensor derivatives asymptotic and the behavior of at ∞.

Tensor $\widehat{U}$ generates next fundamental stress tensors if we use Hook's law (Eq. (2)):

$$\begin{gathered}\Sigma^i_{jk}(x, z) = \lambda\, U^i_{l,\,l}\,\delta_{jk} + \mu\left(U^i_{j,\,k} + U^i_{k,\,j}\right), \quad \Gamma^i_j(x, z, n) = \Sigma^i_{jk}(x, z) n_k \\ \widehat{T}^i_j(x, z, n) = -\left(\rho c^2\right)^{-1} \Gamma^j_i(x, z, n)\end{gathered} \tag{31}$$

Then the elastic constant tensor is presented in the form

$$\widehat{T}^j_i(x, z, n) = \tilde{C}^{jl}_{km} \widehat{U}^k_{i,\,m} n_l, \quad \tilde{C}^{jl}_{km} = C^{jl}_{km} / \left(\rho c^2\right) \tag{32}$$

Tensor $\Gamma^i_j(x, z, n)$ describes the stresses at the plate with normal $n$ in a point $x'$ = ($x$,$z$). Tensor $\widehat{T}$ have some remarkable properties.

**Theorem 6.1.** *Fundamental stress tensor $\widehat{T}$ is the generalized solution of the transport Lame equation with singular mass forces of the multipole type:*

$$\rho c^2 L^j_i(\partial_{x'}) \widehat{T}^k_j + K^i_k(\partial_{x'}, n) \delta\left(x'\right) = 0 \tag{33}$$

where

$$K^l_i(\partial_{x'}, n) = \lambda n_i \partial_l + \mu m_j \left(\delta^l_i \partial_j + \delta^l_j \partial_i\right).$$

*For any closed Lyapunov's surface D, bounding a domain $D^- \subset R^3$*

$$\delta^j_i H^-_D(x, z) = V.P. \int_D \left(T^j_i(x - y, \tau - z, n(y, \tau)) + U^j_{i,\,z}(x - y, \tau - z) n_z(y, \tau)\right) dS(y, \tau) \tag{34}$$

*where $H^-_D(x, z)$ is the characteristic function of $D^-$, which is equal to 0.5 at D; $n(y, \tau)$ is a unit normal vector to D. The integrals are regular for $(x, z) \notin D$ and are taken in value principle sense* for $(x, z) \in D$.

These formulas have been proved by Alexeyeva [6]. The formula in Eq. (35) can be referred to as a dynamic analog of the well-known Gauss formula for a double-layer potential of the fundamental solution of Laplace's equation ([3]: 403). It plays a fundamental role in the solution of transport boundary value problems (BVP).

## 7. Statement of subsonic transport boundary value problems. Uniqueness of solution

Let $D^-$ be an elastic medium bounded by a cylindrical surface $D$ with generator parallel to the axis $X_3$; let $S^-$ be the cross-section of the cylindrical domain; let $S$ be its boundary, and let $n$ be the unit outward normal of $D$. Obviously, $n = n(x)$ and $n_3 = 0$. We assume that $G$ is an integrable vector function and $\exists\varepsilon > 0$ such that

$$\|G(x,z)\| \leq O\left(\|x'\|^{-(3+\varepsilon)}\right) \, for \|x'\| \to \infty, \quad x' \in D^- + D. \tag{35}$$

There is the subsonic transport load $P(x,z)$ moving along the boundary $D$ ($c < c_2$):

$$\sigma_{ij}(x,z)n_j(x) = P(x,z) = \rho c^2 p_i(x,z), \quad (x,z) \in D \tag{36}$$

We assume that $\exists\varepsilon_i > 0$:

$$\|u^D(x,z)\| \leq O(|z|^{-\varepsilon_1}) \quad \text{for } |z| \to \infty, \quad x \in S, \tag{37}$$

$$\|p(x,z)\| \leq O\left(|z|^{-1-\varepsilon 2}\right) \text{ for } |z| \to \infty, x \in S. \tag{38}$$

A vector function $u(x,z)$ satisfying the aforementioned conditions is referred to as a classical solution of the BVP. Let $C_{ab}{}^- = \{(x,z)$: $x \in D^-$, $a < z < b\}$. The two useful energetic equalities have been proved by Alexeyeva [6].

**Theorem 7.1.** *Classic solution of transport BVP satisfying to the equalities:*

$$\int\limits_{D_{ab}} (P,u)dD(x,z) - \int\limits_{D_{ab}^-} \left(W - 0.5\rho c^2\|u_{,z}\|^2 - (G,u)\right) \, dx_1dx_2dz +$$

$$+ \int\limits_{S^-} \left\{ \left(\rho c^2 u_{i,z} - \sigma_{i3}\right)\big|_{(x,a)} u_i(x,a) - \left(\rho c^2 u_{i,z} - \sigma_{i3}(x,b)\right)\big|_{(x,b)} u_i(x,b) \right\} dx_1dx_2dz = 0$$

$$\int\limits_{S^-} \left(W + 0,5\rho c^2\|u_{,z}\|^2 - \sigma_{i3}u_{i,z}\right)\Bigg|_z^{\pm\infty} dx_1dx_2 = \int\limits_{D_{z,\pm\infty}} (P,u_{i,z})dx_1dx_2dz + \int\limits_{D_{z,\pm\infty}^-} (G,u_{,z})dx_1dx_2dz$$

$$\int\limits_{D} (P(x,z),u(x,z))dD(x,z) = \int\limits_{D^-} \left(0.5\rho c^2\|u_{,z}\|^2 - W - (G,u)\right)dx_1dx_2dz$$

$$\int\limits_{S^-} (W + 0,5\rho c^2\|u_{,z}\|^2)dV(x) = \int\limits_{D} (P,u_{,z})dD(x,z) + \int\limits_{D^-} (G,u_{,z})dV(x,z) \tag{39}$$

$$D_{ab} = \{(x,z) : x \in D, \ \ a \leq z \leq b\}, D_{ab}^- = \{(x,z) : x \in D^-, \ \ a < z < b\}.$$

The following assertion is its corollary.

**Theorem 7.2.** *The solution of the subsonic transport boundary value problem is unique.*

**Proof.** Since the problem is linear, it suffices to prove the uniqueness of the zero solution. Let $u(x,z)$ satisfy the zero boundary conditions $P(x,z) = 0$ on $D$ and be a solution of the homogeneous Lame equations ((Eq. (14)) by $G(x,z) = 0$.

Then for $\forall z$

$$\int_{S^-} \left(W + \rho c^2 \|u,_z\|^2\right) dV(x) = 0 \tag{40}$$

It follows from the formula (Eq. (40)) of Theorem 7.1. The integrand is a positive quadratic form in $u_{i,j}$, since the elastic potential satisfies the relation $W \geq 0$ ([1]: 589, 591); moreover, $W = 0$ only for displacements of the medium treated as an absolutely rigid body. Therefore, Eq. (40) is true only if $u_{i,j} = 0$ for all $i, j$. This, together with the decay of solutions at infinity and the arbitrary choice of $z$, implies that $u = 0$.

The proof of the theorem is complete. It is valid both for the internal and external boundary value problem. The asymptotic requirements on $G$ and the boundary functions may be weakened.

## 8. General functions method: statement of subsonic transport BVP in $D_3'(R^3)$

Our aim is to construct the solution of BVP by using boundary integral equations (BIE) for $u(x,z)$. The construction of an analog of Green's formula for solutions of elliptic equations ([3]: 366), which permits one to determine the values of the desired function inside the domain on the basis of the boundary values of the function and its normal derivative, is the key point in the construction of BIE of boundary value problems. An analog of this formula for equations of the static theory of elasticity is referred to as the *Somigliana formula* [1]. It determines the function $u(x,z)$ in the domain $D^-$, if the boundary values of displacements $u_D(x,z)$ and stresses $p(x,z)$ are given. We construct a dynamic analog of that formula in the case of transport solutions. To this end, we use the method of generalized functions (GFM).

We introduce the regular generalized solution of BVP

$$\widehat{u}(x,z) = u(x,z)H_D^-(x) = u(x,z)H_S^-(x)1(z), \tag{41}$$

which defines it as a regular vector function on all space $R^3$. Here $H_D^-(x,z)$ is the characteristic function of the set $D$: $1(z) \equiv 1$, $H_S^-(x)$ is the characteristic function of $S^-$, which is equal to 0.5 at $S$: $\partial_j H_S{}^-(x) = -n_j(x)\delta_S(x),$ where $n_j(x)\delta_S(x)$ is a simple layer at $S$.

By using the properties of the differentiation of regular generalized functions with jumps on $D$, we obtain the equation for $\widehat{u}(x,z)$:

$$\rho c^2 L_i^j(\partial_x, \partial_z,)\widehat{u}_j(x,z) = \widehat{G}_i +$$

$$+(\rho c^2 n_3 u_{i,z} - P_i)\delta_D + (n_3 u_i \delta_D)_{,z} - \left(C_{ij}^{kl} u_k n_l \delta_D\right)_{,j} \tag{42}$$

$\widehat{G} = GH_D^-(x,z), \quad \delta_D(x,z) = \delta_S(x)1(z), \quad 1(z) \equiv 1$, is a simple layer on $D$. Since $n_3 = 0$ on $D$, it follows from the properties of the Green tensor that an analog of the Somigliana formula holds in the space of generalized functions:

$$\rho c^2 \widehat{u}_i = \widehat{U}_i^j * P_j \delta_D + \left(\left(\lambda u_k n_k \delta_l^j + \mu\left(n_j u_l + n_l u_j\right)\right)\delta_D {}^* \widehat{U}_i^l\right)_{,j} + \widehat{U}_i^j * G_j H_D^-,$$

which we write in a form more suitable for transformation as:

$$\widehat{u}_i = \widehat{U}_i^j * p_j \delta_D(x,z) + \widehat{U}_{i,m}^j {}^* \tilde{C}_{jm}^{kl} u_k n_l \delta_D(x,z) \tag{43}$$

If we write out this convolution in integral form with regard to the notation introduced here and Eqs. (1) and (2), then we obtain a formula, whose form coincides with the Somigliana formula for problems of elastostatics ([1]: 605):

$$u_i H_D^-(x,z) = \int_D \left(U_i^j(x,y,z,\tau)\, p_j(y,\tau) - T_i^j(x,y,z,\tau,n(y,\tau))u_j(y,\tau)\right) dD(y,\tau), \quad i,j = 1,2,3 \tag{44}$$

where we introduce the shift tensors:

$$U_i^j(x,y,z,\tau) = U_i^j(x-y, z-\tau), \qquad T_i^j(x,y,z,\tau,n) = T_i^j(x-y, z-\tau, n).$$

This formula permits one to determine displacements in the medium on the basis of known boundary values of displacements and stresses. But the integrals are regular only *for* $(x,z) \notin D$ and do not exist for $(x,z) \in D$.

## 9. Singular boundary integral equations of subsonic transport BVP

The following assertion provides a solution for the aforementioned boundary value problems.

**Theorem 9.1.** *If the solution u(x;z) of subsonic transport BVP satisfies the Holder condition on D; namely,*

$$\|u_j(x,z) - u_j(y,t)\| \leq C\|(x,z) - (y,t)\|^\beta, \ x \in S, \ y \in S,$$

then u(x;z) satisfies the singular boundary integral equation

$$\begin{aligned} 0,5u_i(x,z) &= \widehat{g}_j{}^* \widehat{U}_i^j(x,z) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau) dD(y,\tau) - \\ &- V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau)) u_j(y,\tau) dD(y,\tau) - i,j = 1,2,3 \end{aligned} \tag{45}$$

**Proof.** Let consider Eq. (45) for $(x,z) \in D^-$. Let $(x^*,z^*) \in D$, $x' \to (x^*,z^*)$. Then, using Theorem 6.1, we have

$$\begin{aligned} &\lim_{(x,z)\to(x^*,,z^*)} u_i(x,z) = u_i(x^*,z^*) = \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \lim_{(x,z)\to(x^*,z^*)} \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau) dD(y,\tau) - \\ &- \lim_{(x,z)\to(x^*,z^*)} \int_D T_i^j(x,y,z,\tau,n(y,\tau)) \big(u_j(y,\tau) - u_j(x^*,z^*)\big) dD(y,\tau) + \\ &+ u_j(x^*,z^*) \lim_{(x,z)\to(x^*,z^*)} \int_D T_i^j(x,y,z,\tau,n(y,\tau)) dD(y,\tau) = \\ &= \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau) dD(y,\tau) - \\ &- V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau)) \big(u_j(y,\tau) - u_j(x^*,z^*)\big) dD(y,\tau) + \\ &+ u_j(x^*,z^*) \lim_{(x,z)\to(x^*,z^*)} \left( \delta_i^j - \int_D U_{i,z}^j(x,y,z,\tau)) n_z(y) dD(y,\tau) \right) = \\ &= \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau) dD(y,\tau) - \\ &- V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau)) \big(u_j(y,\tau) - u_j(x^*,z^*)\big) dD(y,\tau) + \\ &+ u_j(x^*,z^*) \delta_i^j = \widehat{g}_j{}^* \widehat{U}_i^j(x^*,z^*) + \int_D U_i^j(x,y,z,\tau)\, p_j(y,\tau) dD(y,\tau) - \\ &- V.P. \int_D T_i^j(x,y,z,\tau,n(y,\tau)) u_j(y,\tau) dD(y,\tau) - 0,5u_i(x^*,z^*) + u_i(x^*,z^*). \end{aligned}$$

In the last relation, we have used the obvious properties: integrals with $U_i^j$ exist by virtue of the Holder property of $u$ on $D$ and weak singularity $U_i^j$ at $D$. Then if the surface integral exists, its value coincides with the principal value; the principal value of the integral containing the difference of integrated functions is equal to the difference of the principal values of integrals corresponding to each of these functions if they exist.

By transposing the last two terms to the left-hand side of the relation, we obtain the formula of the theorem for the boundary points. The proof of the theorem is complete.

This theorem gives us resolving system of integral equations for defining unknown values of boundary displacements.

Note also that the subsonic analog of the Somigliana formula was obtained for generalized functions. But since they are regular, from the Dubois-Reymond lemma ([3]: 97), the solution is classical. However, if the acting loads are described by singular generalized functions, which often takes place in physical problems, then one should use a representation of a generalized solution in the convolution form (Eq. (43)) with the evaluation of convolutions by the definition (see [3]: 133).

## 10. Supersonic green's tensor and its antiderivative with respect to *z*

From Eq. (25), we get the regular representation of $\widehat{U}_i^j$ in the supersonic case which has the form

$$\begin{gathered}
2\pi U_1^1 = \frac{\theta_2}{V_2} + \frac{z^2 x_1^2}{r^4 M_2^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right) - \frac{x_2^2}{r^4 M_2^2}\left(\theta_1 V_1 - \theta_2 V_2\right), \\
2\pi U_2^2 = \frac{\theta_2}{V_2} + \frac{z^2 x_2^2}{r^4 M_2^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right) - \frac{x_1^2}{r^4 M_2^2}\left(\theta_1 V_1 - \theta_2 V_2\right), \\
2\pi U_1^2 = \frac{x_1 x_2}{r^4}\left(z^2\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right) + \left(\theta_1 V_1 - \theta_2 V_2\right)\right), 2\pi U_3^3 = \left(\frac{\theta_1}{V_1} + \frac{\theta_2 m_2^2}{V_2}\right), \\
2\pi U_1^3 = -\frac{x_1 z}{r^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right), 2\pi U_2^3 = -\frac{x_2 z}{r^2}\left(\frac{\theta_1}{V_1} - \frac{\theta_2}{V_2}\right)
\end{gathered} \tag{46}$$

Here $\theta_j = \theta\left(z - m_j\|x\|\right)$, $V_j^{-} = \sqrt{z^2 - m_j^2\|x\|^2}$, $m_j = \sqrt{M_j^2 - 1}$. It satisfies the radiation conditions:

$$\operatorname*{supp}_z\ U(x,z) \in \{z > 0\}, U_i^k \to 0, U_{i'j}^k \to 0 \text{ by } x' \to \infty. \tag{47}$$

One can readily see that its components are zero outside the sonic cones:

$$K_l^{+} = \{(x,z) : z > m_l\|x\|\}, l = 1, 2.$$

On the surfaces of the cones, the components $U_1^3$ have singularities of the type $(z^2 - m^2{}_j r^2)^{-1/2}$.

For solution of supersonic problems, we introduce the tensor $\widehat{W}_j^i(x,z)$, which is the antiderivative of $\widehat{U}_j^i$ with respect to $z$:

$$\widehat{W}_j^i = \sum_{k=1}^{2} \widehat{W}_{jk}^i = \widehat{U}_j^i * \delta(x_1)\delta(x_2)\theta(z) = \widehat{U}_j^i \underset{z}{*} \theta(z), \quad \widehat{W}_{j',z}^i = \widehat{U}_j^i \tag{48}$$

They are also fundamental solutions of Eq. (14) for the mass forces of the corresponding $F_j \underset{z}{*} \theta(z)$. After calculation, we define its components as:

$$\begin{aligned} 2\pi W_1^1 &= \frac{z}{r^4}\left(x_1^2 - x_2^2\right)(\theta_1 V_1 - \theta_2 V_2) + 0,5 m_1^2 \theta_1 \ln\frac{z+V_1}{m_1 r} + \left(M_2^2 - 0,5 m_2^2\right)\theta_2 \ln\frac{z+V_2}{m_2 r} \\ 2\pi W_2^2 &= -\frac{z}{r^4}\left(x_1^2 - x_2^2\right)(\theta_1 V_1 - \theta_2 V_2) + 0,5 m_1^2 \theta_1 \ln\frac{z+V_1}{m_1 r} + \left(M_2^2 - 0,5 m_2^2\right)\theta_2 \ln\frac{z+V_2}{m_2 r} \end{aligned} \tag{49}$$

$$2\pi W_3^3 = \theta_1 \ln\frac{z+V_1}{m_1 r} + m_2^2 \theta_2 \ln\frac{z+V_2}{m_2 r}, \quad 2\pi W_2^3 = -x_2 r^{-2}(\theta_1 V_1 - \theta_2 V_2)$$

$$2\pi W_1^2 = z x_1 x_2 r^{-4}(\theta_1 V_1 - \theta_2 V_2), \quad 2\pi W_1^3 = -x_1 r^{-2}(\theta_1 V_1 - \theta_2 V_2)$$

Tensor $\widehat{W}_j^i$ has the same support as $\widehat{U}_j^i$ but as at the cone $K_j$

$$m_j \|x\| = z \Rightarrow V_j^-(x,z) = 0 \quad \Rightarrow \ln\frac{z+V_j}{m_j \|x\|} = 0 \tag{50}$$

it continues on fronts $K_j$. $W_j{}^i(x,z)$ has weak singularity by $x' = 0$ and weak logarithmic singularity on Z with respect to $\|x\|$ by $x = 0$. To single out these singularities, we decompose it into the terms:

$$\begin{aligned} W_j^i(x,z) = W_j^{is}(x,z) + W_j^{id}(x,z) &= \sum_{k=1}^{2} \theta_k(z - m_k r)\ W_{jk}^{is}(x) + W_j^{id}(x,z), \\ 2\pi c^2 W_{j1}^{is}(x) &= -\left(\delta_{i3}\delta_{j3} + 0,5 m_1^2 (1-\delta_{i3})\delta_{ij}\right)\ln m_1 r, \\ 2\pi c^2 W_{j2}^{is}(x) &= \left(\delta_{i3}\delta_{j3} + \delta_{ij}\left(0,5 m_1^2 (1-\delta_{i3}) - M_2^2\right)\right)\ln m_2 r \end{aligned} \tag{51}$$

The tensors $W_j^{is}$ of diagonal form are independent of $z$ inside the sonic cones $K_l (l = 1, 2)$ and have a logarithmic singularity with respect to $\|x\|$ on the $Z$-axis. Unlike the generating tensor $W_j^{is}$, $W_j^{id}$ has bounded jumps on the $K_l$. One can readily see that the tensor shifts

$$U_i^j(x,y,z) = \widehat{U}_i^j(x-y,z), W_i^j(x,y,z) = \widehat{W}_i^j(x-y,z)$$

have the following symmetry properties around the Z-axis:

$$U_i^j(x,y,z) = U_i^j(y,x,z) = U_j^i(x,y,z), \quad W_i^j(x,y,z) = W_i^j(y,x,z) = W_j^i(x,y,z), \ \ i,j = 1,2 \tag{52}$$

But for the components with indices $(i,j) = (1,3), (3,1), (2,3), (3,2)$

$$U_i^j(x,y,z) = -U_i^j(y,x,z), \quad W_i^j(x,y,z) = -W_i^j(y,x,z) \tag{53}$$

## 11. Fundamental supersonic antiderivative stress tensor $\widehat{H}$ and its properties

We introduce antiderivative stress tensor

$$\begin{aligned} \tilde{\Sigma}_{i3}^j &= \widehat{\Sigma}_j^i * \theta(z)\delta(x) \\ \widehat{H}_j^i &= \widehat{T}_j^i * \theta(z)\delta(x) = \widehat{T}_j^i \underset{z}{*} \theta(z), \widehat{H}_{j,z}^i = \widehat{T}_j^i \end{aligned} \tag{54}$$

This tensor can be obtained in a different way, by analogy with *T*, using Hooke's law, except that the Green tensor should be replaced with its antiderivative *W*. By using the presentation of the basic functions of Green's tensor construction (Eq. (25)) in the supersonic case, it can be presented in the following form:

$$\hat{H}_i^j(x,z,n) = \left(2M_1^2 - M_2^2\right) n_j f_{11,i} - M_2^2 \left(\delta_{ij} \frac{\partial f_{12}}{\partial n} + n_i f_{12,j}\right) - 2\frac{\partial}{\partial n}\left(f_{31,ij} - f_{32,ij}\right)\Big)$$

$$2\pi f_{1k,i}\left(\|x\|, z\right) = \frac{\theta_k}{V_k^-}\left(\delta_{i3} - \frac{z}{\|x\|} r_{,i}\right),$$

$$\begin{aligned} 2\pi f_{3k,ij}\left(\|x\|, z\right) = \left(\delta_{i3}\delta_{j3} + 0,5 m_k^2 \delta_{ij}\epsilon_{[ij]3}\right) \theta_k \ln \frac{z + V_k^-}{m_k \|x\|} - \\ - \frac{V_k^- \theta_k}{\|x\|}\left(\delta_{i3} r_{,j} + \delta_{j3} r_{,i} + \frac{z}{\|x\|}\left(r_{,i} r_{,j} - 0,5\delta_{ij}\epsilon_{[ij]3}\right)\right) \end{aligned}$$

Obviously, for $z < \tau$, all the introduced shifted tensors are zero. It has the following symmetry properties around the Z-axis:

$$H_i^j(x,y,z,m) = -H_i^j(y,x,z,m) = -H_i^j(x,y,z,-m)$$

except for $(i,j)$ = (1,3), (2,3), (3,1):

$$H_i^3(x,y,z) = H_i^3(y,x,z), \quad H_i^3(x,y,z) = H_i^3(y,x,z), \quad i = 1,2.$$

Components $H_j^i(x,z)$ have weak singularities on the fronts of the type $\left(z^2 - m^2{}_j\|x\|^2\right)^{-1/2}$, but more stronger singularity of the type of $\|x\|^{-1}$ on the axis Z. If we put Eq. (51) in Hook's law, then we can again single out two terms in $H_j^i(x,z)$:

$$H_j^i(x,z) = H_j^{is}(x,z) + H_j^{id}(x,z) = \sum_{k=1}^{2} \theta_k(z - m_k r)\left(H_{jk}^{is}(x) + H_{jk}^{id}(x,z)\right) \tag{55}$$

Since the tensors $H_{jk}^{is}(x)$ independent of $z$ inside the sonic cones $K_l$ ($l$ = 1,2), we conventionally say that they are *stationary*. Accordingly, the tensors $H_j^{id}(x,z)$ are said to be *dynamic*, because they depend essentially on $z$, although they are regular functions. The aforementioned symmetry properties hold for both stationary and dynamic terms in the tensors.

For this type of tensors, the next theorem was proved (see [7]).

**Theorem 11.1.** *The fundamental stress tensor H satisfies the relation*

$$\delta_i^j H_S^-(x)\theta(z) = \int_0^z d\tau \int_S H_i^j(y-x,\tau,n(y))\ dS(y) +$$
$$+ \int_S \left( \left(\rho c^2\right)^{-1} \tilde{\Sigma}_{i3}^j(x-y,z) - U_{i',z}^j(y-x,z) \right) dy_1 dy_2$$

*For $x \notin D$ all integrals are regular; for $x \in D$ the first integral is singular, calculated in value principle sense.*

This theorem enables us to obtain solvable singular boundary integral equations for a supersonic transport boundary value problem.

## 12. Statement of supersonic transport BVP: uniqueness of solutions

We suppose here that supersonic transport loads, moving at supersonic velocity $c > c_1$, are known on the boundary $D$:

$$P = \sigma_{ij} n_i e_j = \rho c^2 p_j(x,z) e_j \theta(z), x = (x_1, x_2) \in S, \qquad i,j = 1,2,3 \tag{56}$$

Functions $p_j(x,z)$ are integrable on $D_+$. We assume here $G$ = 0 and

$$u(x,z) = 0, \quad u_{i,z}(x,z) = 0, \qquad z \le 0, \quad x \in S^- \tag{57}$$

For $\|(x,z)\| \to \infty$

$$u_j \to 0, \exists \varepsilon > 0: \ \|\partial_j u\| < O\left(\|(x,z)\|^{1+\varepsilon}\right), \qquad j = 1,2,z \tag{58}$$

The jump conditions, Eqs. (15) and (16) are satisfied on the shock wave fronts.

**Theorem 12.1.** *The solution of the supersonic transport boundary value problem is unique.*

**Proof.** Suppose that there exist two solutions. Since the problem is linear, it follows that their difference $u(x,z)$ satisfies the zero boundary conditions, i.e., $P(x;z)$ = 0, and is a solution of the homogeneous equations of motion ($G$ = 0). We note, that Lemma 8.1 is also true in the supersonic case for shock waves as there is Theorem 3.2 for the gaps of energy on their fronts (see full proof). Then together with conditions given in Eq. (59) of decay of the solutions at infinity and the zero conditions for $z = 0$,

$$\int_{S^-} E(x,z)dx_1dx_2 = \int_{S^-} \sigma_{i3}u_{i,z}(x,z)dx_1dx_2 \to 0 \quad by \quad z \to \infty$$

The energy density $E$ is a positive definite quadratic form of $u_{i,j}$ by construction. Therefore, by virtue of the decay of the solution at infinity, the relation only holds if $u_{i,j} = 0$ for all $i$ and $j$. Hence, we obtain $u = 0$; i.e., the solutions coincide. The proof of the theorem is complete.

Theorem 12.1 holds for both exterior and interior boundary value problems.

## 13. Statement of supersonic BVP in $D_3'\left(R^3\right)$ and its generalized solution

To solve the problem, we also use the method of generalized functions. We introduce here the regular generalized function with support on $D_+^-$:

$$\widehat{u}_j(x,z) = u_j(x,z)H_S^-(x)\theta(z) \tag{59}$$

Also using the properties of differentiation of regular generalized functions with gaps at $D$, and taking into account the boundary conditions and the conditions on the fronts, we obtain the transport Lame equations (Eq. (14)) on the space of distributions with singular mass forces:

$$\widehat{g}_j = p_j\delta_S(x)\theta(z) + \left(\left(\lambda u_k n_k \delta_{ij} + \mu\left(u_i n_j + u_j n_i\right)\right)\delta_S(x)\theta(z)\right)_{,i} \tag{60}$$

By using the properties of convolutions with the Green tensor and the boundary conditions, we obtain the generalized solution of BVP in the form:

$$\rho c^2 \widehat{u}_k = \widehat{U}_k^j * P_j \delta_S(x)\theta(z) + \widehat{U}_{k,i}^j * \left(\lambda u_m n_m \delta_{ij} + \mu\left(u_i n_j + u_j n_i\right)\right)\delta_S(x)\theta(z) \tag{61}$$

By analog with the subsonic case, if we use fundamental stress tensor, then the right-hand side of Eq. (61) may be represented in the form of a surface integral over the boundary of the domain. In our notation, on the boundary, it acquires the form

$$u_i H_S^-(x)\theta(z) = \int_{D_+} \Big(U_i^j(x,y,z-\tau)\, p_j(y,\tau) - T_i^j(x,y,z-\tau, n(y,\tau))u_j(y,\tau)\Big)dD(y,\tau) \tag{62}$$

This formula is similar to the Somigliana formula in the static theory of elasticity ([1]: 146), but it is impossible to use this formula to determine the solution of the boundary value problem in the case of supersonic loads, because the second term contains strong non-integrable singularities of the tensor $T$ on the shock wave fronts of fundamental solutions; therefore, the integrals are divergent. To construct a regular integral representation of the formula, we must regularize it. For this, we use the tensor $H$.

## 14. Dynamic analog of the Somigliana formula in supersonic case

For regularization of Eq. (61), we put $W_{,z}$ instead of $U$ in the second term and use the property of differentiation of convolution:

$$\begin{aligned}
&\rho c^2\widehat{u}_k = \widehat{U}_k^j * P_j\delta_S(x)\theta(z) + \widehat{W}_{k',iz}^j{}^*\big(\lambda u_m n_m\delta_{ij} + \mu\big(u_i n_j + u_j n_i\big)\big)\delta_S(x)\theta(z) = \\
&= \widehat{U}_k^j * P_j\delta_S(x)\theta(z) + \overleftarrow{W}_{k',i}^j{}^*\big(\lambda u_{m,z} n_m\delta_{ij} + \mu\big(u_{i,z} n_j + u_{j,z} n_i\big)\big)\delta_S(x)\theta(z) + \\
&+\widehat{W}_{k',i}^j{}^*\big(\lambda u_m n_m\delta_{ij} + \mu\big(u_i n_j + u_j n_i\big)\big)\delta_S(x)\delta(z) = \\
&= \widehat{U}_k^j * P_j\delta_S(x)\theta(z) + \widehat{W}_{k',m}^j{}^* C_{jm}^{il} u_{i,z} n_l(x)\delta_S(x)\theta(z) + C_{jm}^{il}\widehat{W}_{k',m}^j \underset{x}{*} u_i(x.0)n_l(x)\delta_S(x)
\end{aligned} \tag{63}$$

From here on, we use Eq. (57) we get the formula which can be written in integral form.

**Theorem 14.1.** *The generalized solution of supersonic transport BVP can be presented in the form:*

$$\widehat{u}_k = \widehat{U}_k^j * p_j\delta_S(x)\theta(z) + \tilde{C}_{jm}^{il}\widehat{W}_{k',m}^j{}^* u_{i,z} n_l(x)\delta_S(x)\theta(z) \tag{64}$$

*which for $x \notin S$ has the next integral presentation*

$$\begin{aligned}
u_i H_S^-(x)\theta(z) = &\sum_{k=1}^{2}\int_S \theta(z - m_k r)dS(y)\int_0^{z-m_k r}\Big\{U_i^j(x-y, z-\tau)\, p_j(y,\tau) - \\
&-H_i^{jd}(x-y, z-\tau, n(y))u_{j,z}(y,\tau)\Big\}d\tau - \int_S H_i^{js}(x-y, z, n(y))u_j(y, z - m_k r)dS(y)
\end{aligned} \tag{65}$$

$$r = \|x - y\|$$

**Proof.** Formula (65) follows from Eq. (64) in virtue of Eqs. (61) and (32). Its integral form is

$$u_i H_S^-(x)\theta(z) = \int_0^z \Big\{U_i^j(x, y, z-\tau)\, p_j(y,\tau) - H_i^j(x, y, z-\tau, n(y))u_{j,z}(y,\tau)\Big\}d\tau$$

If we use Eq. (55) for $H_j^i(x,z)$ as the support of $H_j^{is}(x,z)$, $H_j^{id}(x,z)$, we get

$$\begin{aligned}
u_i H_S^-(x)\theta(z) = &\sum_{k=1}^{2}\int_S \theta(z - m_k r)dS(y)\int_0^{z-m_k r}\Big\{U_i^j(x-y, z-\tau)\, p_j(y,\tau) - \\
&- (H_{ik}^{jd}(x-y, z-\tau, n(y) + H_{ik}^{js}(x-y, n(y)))u_{j,z}(y,\tau)\Big\}d\tau
\end{aligned} \tag{66}$$

Note that

$$\int_0^{z-m_k r} H_{ik}^{js}(x-y,n(y))u_{j,\tau}(y,\tau)d\tau = H_{ik}^{js}(x-y,n(y)))(u_j(y,z-m_k r)-u_j(y,0) =$$
$$= H_{ik}^{js}(x-y,n(y)))u_j(y,z-m_k r)$$

In virtue of this equity, we get from Eq. (66) the last formula of the theorem.

All integrals exist; indeed, the integrands are integrable everywhere, including the fronts of fundamental solutions, because the kernels of the integrands have weak singularities on the fronts of the form $\left(z^2 - {m^2}_j\|x\|^2\right)^{-1/2}$ in virtue of the properties of kernels $U$ and $H$. The proof is completed.

This formula is a dynamic analog of Somigliana formula for supersonic loads. It defines the displacement in elastic medium by using boundary values of stresses and velocity of displacements of boundary surface.

This formula also preserves its form for $(x,z)\in D$ with regard to the definition of $H_S{}^{-}(x)\theta(z)$ on $D$.

## 15. Singular boundary integral equations of supersonic transport BVP

**Theorem15.1**. *If the classical solution of BVP satisfies the Holder's conditions at $D_+$, i.e., $\exists C>0$, $\beta>0$ that*

$$\left|u_j(x,z)-u_j(y,z)\right| < C\|x-y\|^{\beta}, \quad x,y\in S.$$

*then it satisfies the singular boundary integral equation at $D_+$*

$$0,5u_i(x,z) = \sum_{k=1}^{2}\int_{S_z^k(x')}\theta(z-m_k r)dS(y)\int_0^{z-m_k r}\Big\{U_i^j(x-y,z-\tau)\,p_j(y,\tau)-$$
$$-H_i^{jd}(x-y,z-\tau,n(y))u_{j,z}(y,\tau)\Big\}d\tau-$$
$$-V.P.\int_{S_z^k(x)} H_i^{js}(x-y,z,n(y))u_j(y,z-m_k r)dS(y),\, r=\|x-y\|$$

$$S_\tau^k(x') = \{(y,\tau): m_k r < z-\tau\}, \quad S_z^k(x) = \{(y): m_k r < z\}.$$

**Proof.** The desired assertion follows from Theorem 14.1 and Theorem 11.1 for tensor $H$ by analogy of the proof of Theorem 12.1 about singular boundary integral equations in the subsonic case. Full proofs of these theorems can be found in [7].

This theorem gives us a resolving system of integral equations for definition of unknown values of boundary displacements in the supersonic case.

Moreover, the Somigliana formula for displacements was obtained for generalized functions. But since they are regular, from the Dubois-Reymond lemma ([3]: 97), this solution is classical. However, if the acting loads are described by singular generalized functions, which often takes place in physical problems, then one should use a representation of a generalized solution in the convolution form (Eq. (65)) with the evaluation of convolutions by the definition (see [3]: 133).

## 16. Conclusion

The constructed singular boundary integral equations in the supersonic case are not classical equations because the solution inside a domain is determined by the boundary values of stresses and displacement rates rather than displacements themselves, unlike the Somigliana formula. In addition, the domain of integration over a boundary surface substantially depends on $z$, which is specific for hyperbolic equations. This complicates finding solutions of such problems by the successive approximation method. However, for the numerical discretization of singular boundary integral equations, the method of boundary elements makes it possible to use standard methods of computational mathematics for a computer implementation of the solution of such problems. The aforementioned boundary value problems model the dynamics of underground structures like transport tunnels and extended excavations subjected to the dynamic influence of moving vehicles and seismic loads. They permit one to study the dynamics of a rock mass in a neighborhood of underground structures depending on its physical-mechanical properties, the velocity of moving transport, specific features of the transport load, and the geometric properties of structures in technical computations of displacements and the stress-strain state of the mass away from the tunnel.

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Chapter 9

# Solution of Nonlinear Partial Differential Equations by New Laplace Variational Iteration Method

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Additional information is available at the end of the chapter

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**Abstract**

Nonlinear equations are of great importance to our contemporary world. Nonlinear phenomena have important applications in applied mathematics, physics, and issues related to engineering. Despite the importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics, there is still the daunting problem of finding new methods to discover new exact or approximate solutions. The purpose of this chapter is to impart a safe strategy for solving some linear and nonlinear partial differential equations in applied science and physics fields, by combining Laplace transform and the modified variational iteration method (VIM). This method is founded on the variational iteration method, Laplace transforms and convolution integral, such that, we put in an alternative Laplace correction functional and express the integral as a convolution. Some examples in physical engineering are provided to illustrate the simplicity and reliability of this method. The solutions of these examples are contingent only on the initial conditions.

**Keywords:** nonlinear partial differential equations, Laplace transform, modified variational iteration method

## 1. Introduction

In the recent years, many authors have devoted their attention to study solutions of nonlinear partial differential equations using various methods. Among these attempts are the Adomian decomposition method, homotopy perturbation method, variational iteration method (VIM) [1–5], Laplace variational iteration method [6–8], differential transform method and projected differential transform method.

Many analytical and numerical methods have been proposed to obtain solutions for nonlinear PDEs with fractional derivatives such as local fractional variational iteration method [9], local fractional Fourier method, Yang-Fourier transform and Yang-Laplace transform and other methods. Two Laplace variational iteration methods are currently suggested by Wu in [10–13]. In this chapter, we use the new method termed He's variational iteration method, and it is employed in a straightforward manner.

Also, the main aim of this chapter is to introduce an alternative Laplace correction functional and express the integral as a convolution. This approach can tackle functions with discontinuities as well as impulse functions effectively. The estimation of the VIM is to build an iteration method based on a correction functional that includes a generalized Lagrange multiplier. The value of the multiplier is chosen using variational theory so that each iteration improves the accuracy of the result.

In this chapter, we have applied the modified variational iteration method (VIM) and Laplace transform to solve convolution differential equations.

## 2. Combine Laplace transform and variational iteration method to solve convolution differential equations

In this section, we combine Laplace transform and modified variational iteration method to figure out a new case of differential equation called convolution differential equations; it is possible to obtain the exact solutions or better approximate solutions of these equivalences. This method is utilized for solving a convolution differential equation with given initial conditions. The results obtained by this method show the accuracy and efficiency of the method.

**Definition (2.1)**

Let $f(x), g(x)$ be integrable functions, then the convolution of $f(x), g(x)$ is defined as:

$$f(\mathrm{x})^{*}\mathrm{g}(\mathrm{x}) = \int_{0}^{x} f(\mathrm{x}-\mathrm{t})\mathrm{g}(\mathrm{t})\mathrm{dt}$$

and the Laplace transform is defined as:

$$\ell\,[f(\mathrm{x})] = \mathrm{F}(\mathrm{s}) = \int_{0}^{\infty} e^{-sx} f(\mathrm{x})\mathrm{dx}$$

where Re s > 0, where s is complex valued and $\ell$ is the Laplace operator.

Further, the Laplace transform of first and second derivatives are given by:

$$\begin{aligned}
&(\mathrm{i})\ell\left[\mathrm{f}'(\mathrm{x})\right] = s\ell[\mathrm{f}(\mathrm{x})] - \mathrm{f}(0)\\
&(\mathrm{ii})\ell\left[\mathrm{f}''(\mathrm{x})\right] = \mathrm{s}^2\ell[\mathrm{f}(\mathrm{x})] - s\mathrm{f}(0) - \mathrm{f}'(0)
\end{aligned}$$

More generally:

$$\ell\left[f^{(n)}(x)\right] = s^n\ell[f(x)] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

and the one-sided inverse Laplace transform is defined by:

$$\ell^{-1}[F(s)] = f(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s)e^{sx}ds$$

where the integration is within the regions of convergence. The region of convergence is half-plane $\alpha < \mathrm{Re}\{s\}$.

**Theorem (2.2) (Convolution Theorem)**

If

$$\ell[f(x)] = F(s), \quad \ell[g(x)] = G(s),$$

then:

$$\ell[f(x)^*g(x)] = \ell[f(x)g(x)] = f(s)g(s)$$

or equivalently,

$$\ell^{-1}[F(s)G(s)] = f(x)^*g(x)$$

Consider the differential equation,

$$L\left[y(x)\right] + R\left[y(x)\right] + N\left[y(x)\right] + N^*\left[y(x)\right] = 0 \tag{1}$$

With the initial conditions

$$y(0) = h(x), \quad y'(0) = k(x) \tag{2}$$

where $L$ is a linear second-order operator, $R$ is a linear first-order operator, $N$ is the nonlinear operator and $N^*[y(x)]$ is the nonlinear convolution term which is defined by:

$$N^*\left[y(x)\right] = f\left(y, y', y'', \dots\dots, y^{(n)}\right)^* g\left(y, y', y'', \dots., y^{(n)}\right)$$

According to the variational iteration method, we can construct a correction functional as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\xi)\left[Ly_n(\xi) + R\tilde{y}_n(\xi) + N\tilde{y}_n(\xi) + N^*\tilde{y}_n(\xi)\right]d\xi \tag{3}$$

$Ry_n(\xi)$, $N\tilde{y}_n(\xi)$ and $N^*\tilde{y}_n(\xi)$ are considered as restricted variations, that is,

$$\delta R\tilde{y}_n = 0, \delta N\tilde{y}_n = 0 \text{ and } \delta N^*\tilde{y}_n = 0, \quad \lambda = -1$$

Then, the variational iteration formula can be obtained as:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[Ly_n(\xi) + Ry_n(\xi) + Ny_n(\xi) + N^*\tilde{y}_n(\xi)\right] d\xi \tag{4}$$

Eq. (4) can be solved iteratively using $y_0(x)$ as the initial approximation.

Then, the solution is $y(x) = \lim_{n\to\infty} y_n(x)$.

Now, we assume that $L = \frac{d^2}{dx^2}$ in Eq. (1).

Take Laplace transform ($\ell$) of both sides of Eq. (1) to find:

$$\ell[Ly(x)] + \ell[Ry(x)] + \ell[Ny(x)] + \ell[N^*y(x)] = 0 \tag{5}$$

$$s^2\ell y - sy(0) - y'(0) = -\ell\{Ry(x) + Ny(x) + N^*y(x)\} = 0 \tag{6}$$

By using the initial conditions and taking the inverse Laplace transform, we have:

$$y(x) = p(x) - \ell^{-1}\left[\frac{1}{s^2}Ry(x) + Ny(x) + N^*y(x)\right] = 0 \tag{7}$$

where $p(x)$ represents the terms arising from the source term and the prescribed initial conditions.

Now, the first derivative of Eq. (7) is given by:

$$\frac{dy(x)}{dx} = \frac{dp(x)}{dx} - \frac{d}{dx}\ell^{-1}\left[\frac{1}{s^2}\ell\{Ry(x) + Ny(x) + N^*y(x)\}\right] = 0 \tag{8}$$

By the correction functional of the irrational method, we have:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left\{ (y_n(\xi))_\xi - \frac{d}{d\xi}p(\xi) - \frac{d}{d\xi}\ell^{-1}\left[\frac{1}{s^2}\ell\{Ry(\xi) + Ny(\xi) + N^*y(\xi)\}\right] \right\} d\xi$$

Then, the new correction functional (new modified VIM) is given by:

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left[\frac{1}{s^2}\ell\left\{Ry_n(x) + Ny_n(x) + N^*y_n(x)\right]\right., \quad n \geq 0 \tag{9}$$

Finally, we find the answer in the strain; if inverse Laplace transforms exist, Laplace transforms exist.

In particular, consider the nonlinear ordinary differential equations with convolution terms,

$$1 - y''(x)\text{-}2 + 2y'^{*}y''\text{-}y'^{*}\left(y''\right)^2 = 0, y(0) = y'(0) = 0 \tag{10}$$

Take Laplace transform of Eq. (10), and making use of initial conditions, we have:

$$s^2 \ell y(x) - \frac{2}{s} = \ell\left[y'^{*}\left(y''\right)^2 - 2y'^{*}y''\right]$$

The inverse Laplace transform of the above equation gives that:

$$y(x) = x^2 + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[y'^{*}\left(y''\right)^2 - 2y'^{*}y''\right]\right\}$$

By using the new modified (Eq. (9)), we have the new correction functional,

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[y'^{*}\left(y''\right)^2 - 2y'^{*}y''\right]\right\}$$

or

$$y_{n+1}(x) = y_n(x) + \ell^{-1}\left\{\frac{1}{s^2}\left[\ell\left(y'\right)^{*}\ell\left(y''\right)^2 - 2\ell\left(y'\right)^{*}\ell\left(y''\right)\right]\right\} \tag{11}$$

Then, we have:

$$y_0(x) = x^2$$

$$y_1(x) = x^2 + \ell^{-1}\left\{\frac{1}{s^2}\left[\ell(4)\ell(2x) - 2\ell(2x)\ell(2)\right]\right\} = x^2$$

$$y_2(x) = x^2, \quad y_3(x) = x^2, \quad \ldots\ldots\ldots\ldots, y_n(x) = x^2$$

This means that:

$$y_0(x) = y_1(x) = y_2(x) = \ldots\ldots. = y_n(x) = x^2$$

Then, the exact solution of Eq. (10) is $y(x) = x^2$.

$$2 - y' - \left(y'\right)^2 - 2x + y'^{*}\left(y''\right)^2 = 0, y(0) = 1 \tag{12}$$

Take Laplace transform of Eq. (12), and using the initial condition, we obtain:

$$s\ell y - 1 - \frac{2}{s^2} = \ell\left[\left(y'\right)^2 - y'^{*}\left(y''\right)^2\right]$$

Take the inverse Laplace transform to obtain

$$y(\mathrm{x}) = 1 + x^2 + \ell^{-1}\left\{\frac{1}{s}\ell\left[(\mathrm{y}')^2 - \mathrm{y}'^*(\mathrm{y}'')^2\right]\right\}$$

Using Eq. (9) to find the new correction functional in the form

$$y_{n+1}(\mathrm{x}) = y_n(\mathrm{x}) + \ell^{-1}\left\{\frac{1}{s}\ell\left[(\mathrm{y}')^2 - \mathrm{y}_n'^*(\mathrm{y}_n'')^2\right]\right\}$$

or

$$y_{n+1}(\mathrm{x}) = y_n(\mathrm{x}) + \ell^{-1}\left\{\frac{1}{s}\left[\ell\left[(\mathrm{y}')^2\right] - \ell\left[\mathrm{y}_n'\right]\ell\left[(\mathrm{y}_n'')^2\right]\right]\right\} \tag{13}$$

Then, we have:

$$y_0(\mathrm{x}) = 1 + \mathrm{x}^2$$

$$y_1(\mathrm{x}) = 1 + \mathrm{x}^2 + \ell^{-1}\frac{1}{s}\left\{\ell(4x^2) - \ell(2\mathrm{x})\ell(4)\right\} = 1 + x^2 + \ell^{-1}\frac{1}{s}\left\{\frac{8}{s^3} - \left(\frac{2}{s^2}\right)\left(\frac{4}{s}\right)\right\} = 1 + x^2$$

$$y_0(\mathrm{x}) = y_1(\mathrm{x}) = \mathrm{y}_2(\mathrm{x}) = \ldots\ldots = \mathrm{y}_n(\mathrm{x}) = 1 + \mathrm{x}^2$$

Then, the exact solution of Eq. (12) is:

$$y(\mathrm{x}) = 1 + x^2$$

## 3. Solution of nonlinear partial differential equations by the combined Laplace transform and the new modified variational iteration method

In this section, we present a reliable combined Laplace transform and the new modified variational iteration method to solve some nonlinear partial differential equations. The analytical results of these equations have been obtained in terms of convergent series with easily computable components. The nonlinear terms in these equations can be handled by using the new modified variational iteration method. This method is more efficient and easy to handle such nonlinear partial differential equations.

In this section, we combined Laplace transform and variational iteration method to solve the nonlinear partial differential equations.

To obtain the Laplace transform of partial derivative, we use integration by parts, and then, we have:

$$\ell\left(\frac{\partial f(\mathrm{x},\mathrm{t})}{\partial t}\right) = sF(\mathrm{x},\mathrm{s}) - \mathrm{f}(\mathrm{x},0), \tag{14}$$

$$\ell\left(\frac{\partial^2 f(\mathrm{x},\mathrm{t})}{\partial t^2}\right) = s^2F(\mathrm{x},\mathrm{s}) - s\mathrm{f}(\mathrm{x},0) - \frac{\partial f(\mathrm{x},0)}{\partial t}$$

$$\ell\left(\frac{\partial f(\mathrm{x},\mathrm{t})}{\partial t}\right) = \frac{d}{dx}[F(\mathrm{x},\mathrm{s})],$$

$$\ell\left(\frac{\partial^2 f(\mathrm{x},\mathrm{t})}{\partial t^2}\right) = \frac{d^2}{dx^2}[F(\mathrm{x},\mathrm{s})].$$

where $f(x,s)$ is the Laplace transform of $(x,t)$.

We can easily extend this result to the nth partial derivative by using mathematical induction.

To illustrate the basic concept of He's VIM, we consider the following general differential equations,

$$\ell[Lu(\mathrm{x},\mathrm{t})] + \ell[Nu(\mathrm{x},\mathrm{t})] = \ell[g(x,t)] \tag{15}$$

with the initial condition

$$u(x,0) = h(x) \tag{16}$$

where $L$ is a linear operator of the first-order, $N$ is a nonlinear operator and $g(x,t)$ is inhomogeneous term. According to variational iteration method, we can construct a correction functional as follows:

$$u_{n+1} = u_n + \int_0^t \lambda[Lu_n(x,s) + N\tilde{u}_n(\mathrm{x},\mathrm{s}) - \mathrm{g}(\mathrm{x},\mathrm{s})]ds \tag{17}$$

where $\lambda$ is a Lagrange multiplier ($\lambda = -1$), the subscripts $n$ denotes the $nth$ approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$.

Eq. (17) is called a correction functional.

The successive approximation $u_{n+1}$ of the solution $u$ will be readily obtained by using the determined Lagrange multiplier and any selective function $u_0$; consequently, the solution is given by:

$$u = \lim_{u\to\infty} u_n$$

In this section, we assume that $L$ is an operator of the first-order $\frac{\partial}{\partial t}$ in Eq. (15).

Taking Laplace transform on both sides of Eq. (15), we get:

$$\ell[Lu(\mathrm{x},\mathrm{t})] + \ell[Nu(\mathrm{x},\mathrm{t})] = \ell[g(x,t)] \tag{18}$$

Using the differentiation property of Laplace transform and initial condition (16), we have:

$$s\ell[u(\mathrm{x},\mathrm{t})] - h(\mathrm{x}) = \ell[g(\mathrm{x},\mathrm{t})] - \ell[Nu(\mathrm{x},\mathrm{t})] \tag{19}$$

Applying the inverse Laplace transform on both sides of Eq. (19), we find:

$$u(\mathrm{x},\mathrm{t}) = G(\mathrm{x},\mathrm{t}) - \ell^{-1}\left\{\frac{1}{s}Nu[\mathrm{x},\mathrm{t}]\right\}, \tag{20}$$

where $G(x, t)$ represents the terms arising from the source term and the prescribed initial condition.

Take the first partial derivative with respect to $t$ of Eq. (20) to obtain:

$$\frac{\partial}{\partial t}u(\mathrm{x},\mathrm{t}) - \frac{\partial}{\partial t}G(\mathrm{x},\mathrm{t}) + \frac{\partial}{\partial t}\ell^{-1}\left\{\frac{1}{s}\ell[Nu(\mathrm{x},\mathrm{t})]\right\} \tag{21}$$

By the correction functional of the variational iteration method

$$u_{n+1} = u_n - \int_0^t \left\{(u_n)_\xi(\mathrm{x},\xi) - \frac{\partial}{\partial \xi}G(\mathrm{x},\xi) + \frac{\partial}{\partial \xi}\ell^{-1}\left\{\frac{1}{\xi}\ell[Nu(\xi,\mathrm{t})]\right\}\right\}d\xi$$

or

$$u_{n+1} = G(\mathrm{x},\mathrm{t}) - \ell^{-1}\left\{\frac{1}{s}\ell[Nu_n(\mathrm{x},\mathrm{t})]\right\} \tag{22}$$

Eq. (22) is the new modified correction functional of Laplace transform and the variational iteration method, and the solution $u$ is given by:

$$u(\mathrm{x},\mathrm{t}) = \lim_{u\to\infty} u_n(\mathrm{x},\mathrm{t})$$

In this section, we solve some nonlinear partial differential equations by using the new modified variational iteration Laplace transform method; therefore, we have:

**Example (3.1)**

Consider the following nonlinear partial differential equation:

$$u_t + uu_x = 0 \quad , \qquad \mathrm{u}(\mathrm{x},0) = -\mathrm{x} \tag{23}$$

Taking Laplace transform of Eq. (23), subject to the initial condition, we have:

$$\ell[u(\mathrm{x},\mathrm{t})] = -\frac{x}{s} - \frac{1}{s}\ell[uu_x]$$

The inverse Laplace transform implies that:

$$u(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \ell^{-1}\left\{\frac{1}{s}\ell[uu_x]\right\}$$

By the new correction functional, we find:

$$u_{n+1}(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \ell^{-1}\left\{\frac{1}{s}\ell\left[u_n(u_n)_x\right]\right\}$$

Now, we apply the new modified variational iteration Laplace transform method:

$$u_0(\mathrm{x},\mathrm{t}) = -\mathrm{x}$$

$$u_1(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \ell^{-1}\left\{\frac{1}{s}\ell[x]\right\} = -x - \ell^{-1}\left[\frac{x}{s^2}\right] = -x - xt$$

$$u_2(\mathrm{x},\mathrm{t}) = -\mathrm{x} - \ell^{-1}\left[x\left(\frac{1}{s^2} + \frac{2}{s^3} + \frac{2}{s^4}\right)\right] = -x - xt - xt^2 - \frac{1}{3}xt^3$$

$$\begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix}$$

Therefore, we deduce the series solution to be:

$$u(\mathrm{x},\mathrm{t}) = -\mathrm{x}\left(1 + \mathrm{t} + \mathrm{t}^2 + \mathrm{t}^3 + \ldots\right) = \frac{x}{t-1},$$

which is the exact solution.

**Example (3.2)**

Consider the following nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} \quad , \qquad \mathrm{u}(\mathrm{x}.0) = \mathrm{x}^2 \tag{24}$$

Taking Laplace transform of Eq. (24), subject to the initial condition, we have:

$$\ell[u(\mathrm{x},\mathrm{t})] = \frac{x^2}{s} + \frac{1}{s}\ell\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$$

Take the inverse Laplace transform to find that:

$$u(\mathrm{x},\mathrm{t}) = \mathrm{x}^2 + \ell^{-1}\left\{\frac{1}{s}\ell\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]\right\}$$

The new correction functional is given as

$$u_{n+1}(\mathrm{x},\mathrm{t}) = \mathrm{x}^2 + \ell^{-1}\left\{\frac{1}{s}\ell\left[\left(\frac{\partial u_n}{\partial x}\right)^2 + u_n\frac{\partial^2 u_n}{\partial x^2}\right]\right\}$$

This is the new modified variational iteration Laplace transform method.

The solution in series form is given by:

$$u_0(\mathrm{x},\mathrm{t}) = \mathrm{x}^2$$

$$u_1(\mathrm{x},\mathrm{t}) = \mathrm{x}^2 + \ell^{-1}\left\{\frac{6x^2}{s^2}\right\} = x^2 + 6x^2 t$$

$$u_2(\mathrm{x},\mathrm{t}) = \mathrm{x}^2\left(1 + 6\mathrm{t} + 36\mathrm{t}^2 + 72\mathrm{t}^3\right)$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

The series solution is given by:

$$u(\mathrm{x},\mathrm{t}) = \mathrm{x}^2\left(1 + 6\mathrm{t} + 36\mathrm{t}^2 + 72\mathrm{t}^3 + \ldots\right) = \frac{x^2}{1 - 6t}$$

**Example (3.3)**

Consider the following nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} = 2u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} \quad , \qquad \mathrm{u}(\mathrm{x}.0) = \frac{x+1}{2} \tag{25}$$

Using the same method in the above examples to find the new correction functional in the form:

$$u_{n+1}(\mathrm{x},\mathrm{t}) = \frac{x+1}{2} + \ell^{-1}\left\{\frac{1}{s}\,\ell\left[2u_n\left(\frac{\partial u_n}{\partial x}\right)^2 + u^2{}_n\frac{\partial^2 u_n}{\partial x^2}\right]\right\}$$

Then, we have:

$$u_0(\mathrm{x},\mathrm{t}) = \frac{x+1}{2}$$

$$u_1(\mathrm{x},\mathrm{t}) = \frac{x+1}{2} + \ell^{-1}\left\{\frac{x+1}{4}\frac{1}{s^2}\right\} = \frac{x+1}{2}\left[1 + \frac{t}{2}\right]$$

$$u_2(\mathrm{x},\mathrm{t}) = \frac{x+1}{2}\left(1 + \frac{\mathrm{t}}{2} + \frac{3}{8}\mathrm{t}^2 + \frac{1}{8}\mathrm{t}^3 + \frac{1}{64}\mathrm{t}^4\right)$$

$$\begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

The series solution is given by:

$$u(\mathrm{x},\mathrm{t}) = \frac{x+1}{2}\left(1 + \frac{\mathrm{t}}{2} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\mathrm{t}^2 + \ldots\right) = \frac{x+1}{2}(1-t)^{-\frac{1}{2}},$$

which is the exact solution of Eq. (25).

**Example (3.4)**

Consider the following nonlinear partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial x}\right)^2 + u - u^2 = \mathrm{te}^{-x} \quad , \qquad \mathrm{u(x.0)} = 0, \frac{\partial u}{\partial t} = e^{-x} \tag{26}$$

Taking the Laplace transform of the Eq. (26), subject to the initial conditions, we have:

$$s^2 \ell[u(\mathrm{x,t})] - e^{-x} = \ell\left[te^{-x} + u^2 - \left(\frac{\partial u}{\partial x}\right)^2 - u\right]$$

Take the inverse Laplace transform to find that:

$$u(\mathrm{x,t}) = te^{-x} + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[te^{-x} + u^2 - \left(\frac{\partial u}{\partial x}\right)^2 - u\right]\right\}$$

The new correct functional is given as:

$$u_{n+1}(\mathrm{x,t}) = te^{-x} + \ell^{-1}\left\{\frac{1}{s^2}\ell\left[te^{-x} + {u_n}^2 - \left(\frac{\partial u_n}{\partial x}\right)^2 - u_n\right]\right\}$$

This is the new modified variational iteration Laplace transform method.

The solution in series form is given by:

$$\begin{aligned} u_0(\mathrm{x,t}) &= te^{-x} \\ u_1(\mathrm{x,t}) &= te^{-x} \\ u_2(\mathrm{x,t}) &= te^{-x} \\ &. \\ &. \end{aligned} \tag{27}$$

The series solution is given by:

$$u(\mathrm{x,t}) = te^{-x}$$

## 4. New Laplace Variational iteration method

To illustrate the idea of new Laplace variational iteration method, we consider the following general differential equations in physics.

$$L\left[u(x,t)\right] + N\left[u(x,t)\right] = h(x,t) \tag{28}$$

where $L$ is a linear partial differential operator given by $\frac{\partial^2}{\partial t^2}$, $N$ is nonlinear operator and $h(x,t)$ is a known analytical function. According to the variational iteration method, we can construct a correction functional for Eq. (28) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \overline{\lambda}(x,\varsigma)[Lu_n(x,\varsigma) + N\tilde{u}_n(x,\varsigma) - h(x,\varsigma)]d\varsigma, \quad n \geq 0, \tag{29}$$

where $\lambda$ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript $n$ denotes the nth approximation, $N\tilde{u}_n(x,\varsigma)$ is considered as a restricted variation, that is, $\delta N\tilde{u}_n(x,\varsigma) = 0$.

Also, we can find the Lagrange multipliers, by using integration by parts of Eq. (28), but in this chapter, the Lagrange multipliers are found to be of the form $\lambda = \overline{\lambda}(x,t-\varsigma)$, and in such a case, the integration is basically the single convolution with respect to *t*, and hence, Laplace transform is appropriate to use.

Take Laplace transform of Eq. (29); then the correction functional will be constructed in the form:

$$\begin{aligned} \ell[u_{n+1}(x,t)] &= \ell[u_n(x,t)] \\ &+\ell\left[\int_0^t \overline{\lambda}(x,\varsigma)[Lu_n(x,\varsigma) + N\tilde{u}_n(x,\varsigma) - h(x,\varsigma)]d\varsigma\right], n \geq 0, \end{aligned} \tag{30}$$

Therefore

$$\begin{aligned} &\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)] \\ &+\ell\left[\overline{\lambda}(x,t)*[Lu_n(x,t) + N\tilde{u}_n(x,t) - h(x,t)]\right] \\ &= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\ell[Lu_n(x,t) + N\tilde{u}_n(x,t) - h(x,t)] \end{aligned} \tag{31}$$

where * is a single convolution with respect to *t*.

To find the optimal value of $\overline{\lambda}(x,t-\varsigma)$, we first take the variation with respect to $u_n(x,t)$. Thus:

$$\begin{aligned} &\frac{\delta}{\delta u_n}\ell[u_{n+1}(x,t)] = \frac{\delta}{\delta u_n}\ell[u_n(x,t)] + \\ &\frac{\delta}{\delta u_n}\ell\left[\overline{\lambda}(x,t)\right]\ell[Lu_n(x,t) + N\tilde{u}_n(x,t) - h(x,t)] \end{aligned} \tag{32}$$

Then, Eq. (32) becomes

$$\ell\left[\delta u_{n+1}(x,t)\right] = \ell\left[\delta u_n(x,t)\right] + \delta\ell\left[\overline{\lambda}(x,t)\right]\ell\left[Lu_n(x,t)\right] \tag{33}$$

In this chapter, we assume that $L$ is a linear partial differential operator given by $\frac{\partial^2}{\partial t^2}$, then, Eq. (33) can be written in the form:

$$\ell\left[\delta u_{n+1}(x,t)\right]=\ell\left[\delta u_n(x,t)\right]+\ell\left[\bar{\lambda}(x,t)\right]\left[s^2\ell\delta u_n(x,t)\right] \tag{34}$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t)=0$. This means that the right hand side of Eq. (34) should be set to zero; then, we have the following condition:

$$\ell\left[\bar{\lambda}(x,t)\right]=\frac{-1}{s^2} \quad \Rightarrow \quad \bar{\lambda}(x,t)=-t \tag{35}$$

Then, we have the following iteration formula

$$\begin{aligned}&\ell[u_{n+1}(x,t)]=\ell[u_n(x,t)]\\&-\ell\left[\int_0^t (t-\varsigma)[Lu_n(x,\varsigma)+N\tilde{u}_n(x,\varsigma)-h(x,\varsigma)]d\varsigma\right], n\geq 0,\end{aligned} \tag{36}$$

## 5. Applications

In this section, we apply the Laplace variational iteration method to solve some linear and nonlinear partial differential equations in physics.

**Example (5.1)**

Consider the initial linear partial differential equation

$$u_{tt}(x,t)-u_{xx}(x,t)+u(x,t)=0 \quad , \quad u(x,0)=0 \quad , \quad \frac{\partial u(x,0)}{\partial t}=x \tag{37}$$

The Laplace variational iteration correction functional will be constructed in the following manner:

$$\begin{aligned}&\ell[u_{n+1}(x,t)]=\ell[u_n(x,t)]\\&+\ell\left[\int_0^t \bar{\lambda}(x,t-\varsigma)\left[(u_n)_{tt}(x,\varsigma)-(u_n)_{xx}(x,\varsigma)+u_n(x,\varsigma)\right]d\varsigma\right]\end{aligned} \tag{38}$$

or

$$\begin{aligned}&\ell[u_{n+1}(x,t)]=\ell[u_n(x,t)]\\&+\ell\left[\bar{\lambda}(x,t)*\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n(x,t)\right]\right]\\&=\ell[u_n(x,t)]+\ell\left[\bar{\lambda}(x,t)\right]\ell\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n(x,t)\right]\\&=\ell[u_n(x,t)]+\ell\left[\bar{\lambda}(x,t)\right]\begin{bmatrix}s^2\ell u_n(x,t)-su_n(x,0)-\dfrac{\partial u_n}{\partial t}(x,0)\\-\ell(u_n)_{xx}(x,t)+\ell u_n(x,t)\end{bmatrix}\end{aligned} \tag{39}$$

Taking the variation with respect to $u_n(x,t)$ of Eq. (39), we obtain:

$$\begin{aligned}\frac{\delta}{\delta u_n}\ell[u_{n+1}(x,t)] &= \frac{\delta}{\delta u_n}\ell[u_n(x,t)]\\ &+\frac{\delta}{\delta u_n}\ell\left[\overline{\lambda}(x,t)\right]\begin{bmatrix} s^2\ell u_n(x,t) - s u_n(x,0) - \dfrac{\partial u_n}{\partial t}(x,0) \\ -\ell(u_n)_{xx}(x,t) + \ell u_n(x,t)\end{bmatrix}\end{aligned} \tag{40}$$

Then, we have.

$$\begin{aligned}\ell\left[\delta u_{n+1}(x,t)\right] &= \ell\left[\delta u_n(x,t)\right] + \ell\left[\overline{\lambda}(x,t)\right]\left[s^2\ell\delta u_n(x,t) + \ell\delta u_n(x,t)\right]\\ &= \ell\left[\delta u_n(x,t)\right]\left\{1+\ell\left[\overline{\lambda}(x,t)\right]\left(s^2+1\right)\right\}\end{aligned}$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t)=0$. Hence, we have:

$$1+(s^2+1)\ell\overline{\lambda}(x,t)=0\ , \quad and \quad \overline{\lambda}(x,t)=\ell^{-1}\left[\frac{-1}{s^2+1}\right]=-\sin t \tag{41}$$

Substituting Eq. (41) into Eq. (38), we obtain:

$$\begin{aligned}\ell[u_{n+1}(x,t)] &= \ell[u_n(x,t)]\\ &-\ell\left[\int_0^t \sin(t-\varsigma)\left[(u_n)_{tt}(x,\varsigma) - (u_n)_{xx}(x,\varsigma) + u_n(x,\varsigma)\right]d\varsigma\right]\\ &= \ell[u_n(x,t)] - \ell[\sin t]\ell\left[(u_n)_{tt}(x,t) - (u_n)_{xx}(x,t) + u_n(x,t)\right]\end{aligned} \tag{42}$$

Let $u_0(x,t)=u(x,0)+t\dfrac{\partial u}{\partial t}(x,0)=xt$, then, from Eq. (42), we have:

$$\ell\left[u_1(x,t)\right]=\ell\left[xt\right]-\ell\left[\sin t\right]\ell\left[xt\right]=\frac{x}{s^2}-\frac{x}{s^2(s^2+1)}$$

The inverse Laplace transforms yields:

$$u_1(x,t)=x\ \sin t \tag{43}$$

Substituting Eq. (43) into Eq. (38), we obtain:

$$\ell\left[u_2(x,t)\right]=\ell\left[x\ \sin t\right]-\ell\left[\sin t\right]\ell\left[0\right] \qquad then \qquad u_2(x,t)=x\ \sin t$$

Then, the exact solution of Eq. (37) is:

$$u(x,t)=x\ \sin t \tag{44}$$

**Example (4.2)**

Consider the nonlinear partial differential equation:

$$u_{tt}(x,t)-u_{xx}(x,t)+u^2(x,t)=x^2t^2 \quad , \quad u(x,0)=0 \quad , \quad \frac{\partial u(x,0)}{\partial t}=x \tag{45}$$

The Laplace variational iteration correction functional will be constructed as follows:

$$\ell\left[u_{n+1}(x,t)\right]=\ell\left[u_n(x,t)\right]-\ell\left[\int_0^t(t-\varsigma)\begin{bmatrix}(u_n)_{tt}(x,\varsigma)-(u_n)_{xx}(x,\varsigma)\\+u_n^2(x,\varsigma)-x^2\varsigma^2\end{bmatrix}d\varsigma\right] \tag{46}$$

or

$$\begin{aligned}\ell[u_{n+1}(x,t)]&=\ell[u_n(x,t)]+\ell\left[\overline{\lambda}(x,t)*\begin{bmatrix}(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)\\+u_n^2(x,t)-x^2t^2\end{bmatrix}\right]\\&=\ell[u_n(x,t)]+\ell\left[\overline{\lambda}(x,t)\right]\ell\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n^2(x,t)-x^2t^2\right]\\&=\ell[u_n(x,t)]+\ell\left[\overline{\lambda}(x,t)\right]\begin{bmatrix}s^2\ell u_n(x,t)-su_n(x,0)-\dfrac{\partial u_n}{\partial t}(x,0)\\-\ell(u_n)_{xx}(x,t)+\ell u_n^2(x,t)-\ell\left(x^2t^2\right)\end{bmatrix}\end{aligned} \tag{47}$$

Taking the variation with respect to $u_n(x,t)$ of Eq. (47) and making the correction functional stationary we obtain:

This implies that:

$$\ell+s\ell\overline{\lambda}(x,t)=0 \quad , \quad and \quad \overline{\lambda}(x,t)=\ell^{-1}\left[\frac{-1}{s}\right]=-1 \tag{48}$$

Substituting Eq. (21) into Eq. (19), we obtain:

$$\ell\left[u_{n+1}(x,t)\right]=\ell\left[u_n(x,t)\right]-\ell\left[\int_0^t(t-\varsigma)\begin{bmatrix}(u_n)_{tt}(x,\varsigma)-(u_n)_{xx}(x,\varsigma)\\+u_n^2(x,\varsigma)-x^2\varsigma^2\end{bmatrix}d\varsigma\right] \tag{49}$$

or

$$\ell[u_{n+1}(x,t)]=\ell[u_n(x,t)]+\ell[-t]\ell\begin{bmatrix}(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)\\+u_n^2(x,t)-x^2t^2\end{bmatrix} \tag{50}$$

Let $u_0(x,t)=u(x,0)+t\dfrac{\partial u}{\partial t}(x,0)=xt$, then, from Eq. (50), we have:

$$\begin{aligned}&\ell\left[u_1(x,t)\right]=\ell\left[xt\right]+\ell\left[-t\right]\ell\left[0-0+x^2t^2-x^2t^2\right]\\&u_1(x,t)=xt\end{aligned}$$

Then, the exact solution of Eq. (45) is: $u(x,t)=xt$

Again, the exact solution is obtained by using only few steps of the iterative scheme.

**Example (4.3)**

Consider the physics nonlinear boundary value problem,

$$u_t - 6uu_x + u_{xxx} = 0 \quad , \quad u(x,0)=\frac{6}{x^2} \, , \quad x \neq 0$$

The Laplace variational iteration correction functional is

$$\begin{aligned} \ell[u_{n+1}(x,t)] &= \ell[u_n(x,t)] \\ &+\ell\left[\int_0^t \overline{\lambda}(x,t-\varsigma)\begin{bmatrix}(u_n)_t(x,\varsigma)-6u_n(x,\varsigma)(u_n)_x(x,\varsigma)\\ +(u_n)_{xxx}(x,\varsigma)\end{bmatrix}d\varsigma\right] \end{aligned} \tag{52}$$

or

$$\begin{aligned} \ell[u_{n+1}(x,t)] &= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)*\left[(u_n)_t(x,t)-6(u_n)(x,t)(u_n)_x(x,t)+(u_n)_{xxx}(x,t)\right]\right] \\ &= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\ell\left[(u_n)_t(x,t)-6(u_n)(x,t)(u_n)_x(x,t)+(u_n)_{xxx}(x,t)\right] \\ &= \ell[u_n(x,t)] + \ell\left[\overline{\lambda}(x,t)\right]\left[s\ell u_n(x,t)-u_n(x,0)-\ell\left[6(u_n)(x,t)(u_n)_x(x,t)-(u_n)_{xxx}(x,t)\right]\right] \end{aligned}$$

Taking the variation with respect to $u_n(x,t)$ of the last equation and making the correction functional stationary we obtain:

$$\begin{aligned} \ell\left[\delta u_{n+1}(x,t)\right] &= \ell\left[\delta u_n(x,t)\right]+\ell\left[\overline{\lambda}(x,t)\right]\left[s\ell\,\delta u_n(x,t)\right] \\ &= \ell\left[\delta u_n(x,t)\right]\left\{\ell+s\ell\left[\overline{\lambda}(x,t)\right]\right\} \end{aligned}$$

This implies that:

$$1+s\ \ell\overline{\lambda}(x,t)=0 \ , \ \ and \ \ \overline{\lambda}(x,t)=\ell^{-1}\left[\frac{-1}{s}\right]=-t \tag{53}$$

Substituting Eq. (53) into Eq. (52), we obtain:

$$\ell\left[u_{n+1}(x,t)\right]=\ell\left[u_n(x,t)\right]+\ell\left[\int_0^t(-1)\begin{bmatrix}(u_n)_t(x,\varsigma)-6(u_n)(x,\varsigma)(u_n)_x(x,\varsigma)\\ +(u_n)_{xxx}(x,\varsigma)\end{bmatrix}d\varsigma\right]$$

or

$$\ell\left[u_{n+1}(x,t)\right]=\ell\left[u_n\right]+\ell\left[-1\right]\ell\left[(u_n)_t-(u_n)(u_n)_x+(u_n)_{xxx}\right] \tag{54}$$

Let $u_0(x,t)=u(x,0)=\frac{6}{x^2}$, then, from Eq. (54), we have:

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$$\ell[u_1(x,t)] = \ell\left[\frac{6}{x^2}\right] + \ell[-1]\ell\left[\frac{288}{x^5}\right] = \frac{6}{x^2} - \frac{288}{x^5}t$$

$$u_2(x,t) = \frac{6}{x^2} - \frac{288}{x^5}t - \frac{6048}{x^8}t^2 \quad ,..........$$

Then, the exact solution of Eq. (51) is: $u(x,t) = \dfrac{6x\left(x^3 - 24t\right)}{\left(x^3 - 12t\right)^2}$

**Exercises**

Solve the following nonlinear partial differential equations by new Laplace variational iteration method:

$$1)\, u_t + uu_x = 1 - e^{-x}(\mathrm{t} + \mathrm{e}^{-x}), \qquad \mathrm{u}(\mathrm{x}, 0) = \mathrm{e}^{-x}$$
$$2)\, u_t + uu_x = 2t + x + t^3 + xt^2, \quad \mathrm{u}(\mathrm{x}, 0) = 0$$
$$3)\, u_t + uu_x = 2x^2t + 2xt^2 + 2x^3t^4, \;\; \mathrm{u}(\mathrm{x}, 0) = 1$$
$$4)\, u_t + uu_x = 1 + t\cos x + \frac{1}{2}\sin 2x, \;\; \mathrm{u}(\mathrm{x}, 0) = \sin x$$
$$5)\, u_t + uu_x = 0, \qquad \mathrm{u}(\mathrm{x}, 0) = -x$$
$$6)\, u_t + uu_x - u = e^t, \quad \mathrm{u}(\mathrm{x}, 0) = 1 + x$$
$$7)\, u_{tt} - u_{xx} - u + u^2 = xt + x^2t^2, \qquad \mathrm{u}(\mathrm{x}, 0) = 1, \mathrm{u}_t(\mathrm{x}, 0) = x$$
$$8)\, u_{tt} - u_{xx} + u^2 = 1 + 2xt + x^2t^2, \qquad \mathrm{u}(\mathrm{x}, 0) = 1, \mathrm{u}_t(\mathrm{x}, 0) = x$$
$$9)\, u_{tt} - u_{xx} + u^2 = 6xt\left(\mathrm{x}^2 - \mathrm{t}^2\right) + x^6t^6, \;\; \mathrm{u}(\mathrm{x}, 0) = 0, \mathrm{u}_t(\mathrm{x}, 0) = 0$$
$$10)\, u_{tt} - u_{xx} + u^2 = \left(\mathrm{x}^2 + \mathrm{t}^2\right)^2, \qquad \mathrm{u}(\mathrm{x}, 0) = x^2, \mathrm{u}_t(\mathrm{x}, 0) = 0$$
$$11)\, u_{tt} - u_{xx} + u + u^2 = x^2\cos^2 t, \qquad \mathrm{u}(\mathrm{x}, 0) = x, \mathrm{u}_t(\mathrm{x}, 0) = 0$$
$$12)\, u_t + uu_x = 0, \quad \mathrm{u}(\mathrm{x}, 0) = x$$
$$13)\, u_t + uu_x = 0, \quad \mathrm{u}(\mathrm{x}, 0) = -x$$
$$14)\, u_t + uu_x = 0, \quad \mathrm{u}(\mathrm{x}, 0) = 2x$$
$$15)\, u_t + uu_x = u_{xx}, \quad \mathrm{u}(\mathrm{x}, 0) = -x$$
$$16)\, u_t + uu_x = u_{xx}, \quad \mathrm{u}(\mathrm{x}, 0) = 2x$$
$$17)\, u_t + uu_x = u_{xx}, \quad \mathrm{u}(\mathrm{x}, 0) = 4\tan 2x$$

## 6. Conclusions

The method of combining Laplace transforms and variational iteration method is proposed for the solution of linear and nonlinear partial differential equations. This method is applied in a direct way without employing linearization and is successfully implemented by using the initial conditions and convolution integral. But this method failed to solve the singular differential equations.

## Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this chapter.

## Answers

$$
\begin{array}{l}
1)\,\mathrm{u}(\mathrm{x},t)=t+e^{-x},\ \ 2)\mathrm{u}(\mathrm{x},t)=t^2+xt,\ 3)\ \mathrm{u}(\mathrm{x},t)=1+x^2t^2,\ 4)\,\mathrm{u}(\mathrm{x},t)=t+\ \sin x \\
5)\,\mathrm{u}(\mathrm{x},t)=\dfrac{x}{t-1},\ \ 6)\,\mathrm{u}(\mathrm{x},t)=x+e^t,\ \ 7)\,\mathrm{u}(\mathrm{x},t)=1+xt,\ \ 8)\,\mathrm{u}(\mathrm{x},t)=1+xt \\
9)\,\mathrm{u}(\mathrm{x},t)=x^3t^3,\ \ 10)\,\mathrm{u}(\mathrm{x},t)=t^2+x^2\ ,\ 11)\,\mathrm{u}(\mathrm{x},t)=x\mathrm{cost},\ 12)\,\mathrm{u}(\mathrm{x},t)=\dfrac{x}{1+t} \\
13)\,\mathrm{u}(\mathrm{x},t)=\dfrac{x}{t-1},\ 14)\,\mathrm{u}(\mathrm{x},t)=\dfrac{2x}{1+2t},\ \ 15)\,\mathrm{u}(\mathrm{x},t)=\dfrac{x}{t-1},\ \ 16)\,\mathrm{u}(\mathrm{x},t)=\dfrac{2x}{1+2t} \\
17)\,\mathrm{u}(\mathrm{x},t)=4\tan 2x
\end{array}
$$

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*Edited by Terry E. Moschandreou*

The editor has incorporated contributions from a diverse group of leading researchers in the field of differential equations. This book aims to provide an overview of the current knowledge in the field of differential equations. The main subject areas are divided into general theory and applications. These include fixed point approach to solution existence of differential equations, existence theory of differential equations of arbitrary order, topological methods in the theory of ordinary differential equations, impulsive fractional differential equations with finite delay and integral boundary conditions, an extension of Massera's theorem for n-dimensional stochastic differential equations, phase portraits of cubic dynamic systems in a Poincare circle, differential equations arising from the three-variable Hermite polynomials and computation of their zeros and reproducing kernel method for differential equations. Applications include local discontinuous Galerkin method for nonlinear Ginzburg-Landau equation, general function method in transport boundary value problems of theory of elasticity and solution of nonlinear partial differential equations by new Laplace variational iteration method.

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