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Meet the editor

Qiming Huang is a Ph.D. candidate in the Department of Statistics at Purdue University. He holds a B.S. in Mathematics from the Beijing Institute of Technology, M.S. in Statistics from Purdue University. His research interests include variable selection, dimension reduction, kernel methods, computational advertising and statistical application in psychometrics.

Contents

Preface

Game theory is a formal framework with mathematical tools to research on the complex interactions among interdependent rational players. The most well-known concept in game theory is the celebrated Nash equilibrium. Really, game theoretic approaches are multifarious, including among others cooperative and non-cooperative models, static and dynamic games, single-slot and repeated games, and finite- and infinite-horizon games. Game theory has led to revolutionary changes in economics and has found important applications in sociology, modern communication, biology engineering, and transportation. This book presents the introduction of game theory and supplies applications of game theory.

In the chapter "Introduction to game theory", an introduction to the concepts and history of game theory is presented, and the most common types of games are discussed in details.

The chapter "game application in cognitive radio networks" introduce an adaptive competitive second-price pay-to-bid sealed auction game as solution to the fairness problem of spectrum sharing among one primary user and a large number of secondary users in cognitive radio environment, and it is shown by numerical results the proposed mechanism could reach the maximum total profit for secondary with better fairness.

In the chapter "game theory in wireless ad-hoc opportunistic radios", a scenario based UMTS TDD opportunistic cellular system with an ad hoc behavior that operates over UMTS FDD licensed cellular network is considered, the ad hoc radio is modeled as a game and the unique Nash equilibrium for the game is applied in ad-hoc opportunistic radio.

The chapter "reliable aggregation routing for wireless sensor networks based on game theory" proposes a game-theoretic model of reliable data architecture in wireless sensor network, each selected group leaders uses game-theoretic model which tradeoffs between energy dissipation and data transmission delay to determine the degree of aggregation.

In the chapter "cooperative logistic games", the concepts, theory and application of the cooperative logistic games, which are focused mainly on transportation, inventory and supply chain games, are surveyed.

In the chapter "stochastic game theory approach to the robust synthetic gene network design", synthetic biological can increase efficiency of gene circuit design through registries of biological parts and standard datasheets. In synthetic gene networks, there is much uncertainly about what affects the behavior of biological circuitry and systems. The proposed robust minimax synthetic biology design method can predict the most robust value of genetic parameters from the perspective of stochastic game theory. The proposed synthetic genetic network not only can achieve the desired steady state but also can tolerate the worst-case effect due to these uncertain parameter variations and external noises on the host cell.

Game theory provides a powerful mathematical framework that can accommodate the preferences and requirements of various stakeholders in a given process as regards the outcome of the process. The chapters' content in this book will give an impetus to the application of game theory to the modeling and analysis of modern communication, biology engineering, and transportation, etc..

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1

Theory of Games: An Introduction

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1. Introduction

'Game Theory' is a mathematical concept, which deals with the formulation of the correct strategy that will enable an individual or entity (i.e., player), when confronted by a complex challenge, to succeed in addressing that challenge. It was developed based on the premise that for whatever circumstance, or for whatever 'game', there exists a strategy that will allow one player to 'win'. Any business can be considered as a game played against competitors, or even against customers. Economists have long used it as a tool for examining the actions of economic agents such as firms in a market.

The ideas behind game theory have appeared through-out history [1], apparent in the bible, the Talmud, the works of Descartes and Sun Tzu, and the writings of Charles Darwin [2]. However, some argue that the first actual study of game theory started with the work of Daniel Bernoulli, A mathematician born in 1700 [3]. Although his work, the "Bernoulli's Principles" formed the basis of jet engine production and operations, he is credited with introducing the concepts of expected utility and diminishing returns. Others argue that the first mathematical tool was presented in England in the 18th century, by Thomas Bayes, known as "Bayes' Theorem"; his work involved using probabilities as a basis for logical conclusion [3]. Nevertheless, the basis of modern game theory can be considered as an outgrowth of a three seminal works; a "Researches into the Mathematical Principles of the Theory of Wealth" in 1838 by Augustin Cournot, gives an intuitive explanation of what would eventually be formalized as *Nash equilibrium* and gives a dynamic idea of players best-response to the actions of others in the game. In 1881, Francis Y. Edgeworth expressed the idea of competitive equilibrium in a two-person economy. Finally, Emile Borel, suggested the existence of *mixed strategies*, or probability distributions over one's actions that may lead to stable play. It is also widely accepted that modern analysis of game theory and its modern methodological framework began with John Von Neumann and Oskar Morgenstern book [4].

We can say now that "Game Theory" is relatively not a new concept, having been invented by *John von Neumann* and *Oskar Morgenstern* in 1944 [4]. At that time, the mathematical framework behind the concept has not yet been fully established, limiting the concept's application to special circumstances only [5]. Over the past 60 years, however, the framework has gradually been strengthened and solidified, with refinements ongoing until today [6]. Game Theory is now an important tool in any strategist's toolbox, especially when dealing with a situation that involves several entities whose decisions are influenced by what decisions they expect from other entities.

In [4], John von Neumann and Oskar Morgenstern conceived a groundbreaking mathematical theory of economic and social organization, based on a theory of games of strategy. Not only would this reform economics, but the entirely new field of scientific inquiry it yielded has since been widely used to analyze a host of real-world phenomena from arms races to optimal policy choices of presidential candidates, from vaccination policy to major league baseball salary negotiations [6]. In addition, it is today established throughout both the social sciences and a wide range of other sciences.

Game Theory can be also defined as the study of how the final outcome of a competitive situation is dictated by interactions among the people involved in the game (also referred to as 'players' or 'agents'), based on the goals and preferences of these players, and on the strategy that each player employs. A *strategy* is simply a predetermined 'way of play' that guides an agent as to what actions to take in response to past and expected actions from other agents (i.e., players in the game).

In any game, several important elements exists, some of which are; *the agent*, which represents a person or an entity having their own goals and preferences. The second element, the *utility* (also called *agent payoff)* is a concept that refers to the amount of satisfaction that an agent derives from an object or an event. *The Game*, which is a formal description of a strategic situation, *Nash equilibrium*, also called *strategic equilibrium*, which is a list of strategies, one for each agent, which has the property that no agent can change his strategy and get a better *payoff*.

Normally, any game G has three components: a set of players, a set of possible actions for each player, and a set of utility functions mapping action profiles into the real numbers. In this chapter, the set of players are denoted as I, where I is finite with, $i = \{1,2,3,\ldots\ldots, I\}$. For each player $i \in I$ the set of possible actions that player i can take is denoted by A_i, and A, which is denoted as the space of all action profiles is equal to:

$$A = A_1 \times A_2 \times A_3 \times \ldots \times A_I \tag{1}$$

Finally, for each $i \in I$, we have $U_t : A \rightarrow R$, which denotes i's player utility function. Another notation to be defined before carrying on; suppose that $a \in A$ is a strategy profile and $i \in I$ is a player; and then $a_i \in A_i$ denote player i's action in a_i and a_{-i} denote the actions of the other $I - 1$ players.

In this chapter, some famous examples of games, some important definitions used in games and classifications of games are presented. Throughout this chapter, a mathematical proof is presented to show when mixed strategy games can be valid and invalid in different scenarios.

2. Examples of games

2.1 Prisoners' dilemma

In 1950, Professor Albert W. Tucker of Princeton University invented the Prisoner's Dilemma [7] and [8], an imaginary scenario that is without doubt one of the most famous representations of Game Theory. In this game, two prisoners were arrested and accused of a crime; the police do not have enough evidence to convict any of them, unless at least one suspect confesses. The police keep the criminals in separate cells, thus they are not able to communicate during the process. Eventually, each suspect is given three possible outcomes:

1. If one confesses and the other does not, the confessor will be released and the other will stay behind bars for ten years (i.e. -10);

2. If neither admits, both will be jailed for a short period of time (i.e. -2,-2); and
3. If both confess, both will be jailed for an intermediate period of time (i.e. six years in prison, -6).

The possible actions and corresponding sentences of the criminals are given in Table 1.

		2nd Criminal	
		Cooperate	*Defect*
1st Criminal	*Cooperate*	***-2, -2***	***-10, 0***
	Defect	***0, -10***	***-6, -6***

Table 1. Prisoners'' Dilemma game.

To solve this game, we must find the dominating strategy of each player, which is the best response of each player regardless of what the other player will play. From player one's point of view, if player two cooperates (i.e. not admitting), then he is better off with the defect (i.e. blaming his partner). If player two defects, then he will choose defect as well. The same will work with player two. In the end, both prisoners conclude that the best decision is to defect, and are both sent to intermediate imprisonment.

2.2 Battle of the sexes

Another well know game is the battle of the sexes, in which two couple argues where to spend the night out. In this example, she would rather attend an audition of Swan Lake in the opera and he would rather a football match. However, none of them would prefer to spend the night alone. The possible actions and corresponding sentences of the couple are given in Table 2.

		Female	
		Ballet	*Football*
Male	*Ballet*	***2, 4***	***0, 0***
	Football	***0, 0***	***4, 2***

Table 2. Battle of the Sexes game.

It is easy to see that both of them will either decide to go to the ballet or to the football match, as they are much better off spending the evening alone.

3. Nash Equilibrium

Definition: Nash Equilibrium exists in any game if there is a set of strategies with the property that no player can increase her payoff by changing her strategy while the other players keep their strategies unchanged. These sets of strategies and the corresponding payoffs represent the Nash Equilibrium. More formally, a Nash equilibrium is a strategy profile a such that for all $a_i \in A_i$,

$$U(a_i, a_{-i}) \geq U(\tilde{a}_i, a_{-i}) \tag{2}$$

Where ã, denotes another action for the player *i's* [1-3]. We can simply see that the action profile (defect, defect) is the Nash Equilibrium in the prisoners dilemma game and the actions profile (ballet, ballet) and (football, football) are the ones for the battle of the sexes game.

4. Pareto efficiency

Definition: Pareto efficiency is another important concept of game theory. This term is named after Vilfredo Pareto, an Italian economist, who used this concept in his studies and defined it as; "A situation is said to be Pareto efficient if there is no way to rearrange things to make at least one person better off without making anyone worse off" [9].
More Formally, an action profile $a \in A$ is said to be Pareto if there is no action profile $\tilde{a} \in A$ such that for all i,

$$U(a_i) \geq U(\tilde{a}_i) \tag{3}$$

In another word, an action profile is said to be Pareto efficient if and only if it is impossible to improve the utility of any player without harming another player.
In order to see the importance of Pareto efficiency, assume that someone was walking along the shore on an isolated beach finds a £20 bill on the sand. If bill is picked up and kept, then that person is better off and no one else is harmed. Leaving the bill on the sand to be washed out would be an unwise decision. However, someone might argue the fact that the original owner of the bill is worse off. This is not true, because once the owner loses the bill he is defiantly worse off. On the other hand, once the bill is gone he will be the same whether someone found it or it was washed out to the sea. This will lead us to another argument; assume there are two people walking on the beach and they saw the bill on the sand. Whether one of them will pick up the bill and the other will not get anything or they decide to split the bill between themselves. Who gains from finding the bill is quite different in those scenarios but they *all* avoid the inefficiency of leaving it sitting on the beach.

5. Pure and mixed strategy Nash Equilibrium

In any game someone will find pure and mixed strategies, a pure strategy has a probability of one, and will be always played. On the other hand, a mixed strategy has multiple purse strategies with probabilities connected to them. A player would only use a mixed strategy when she is indifferent between several pure strategies, and when keeping the challenger guessing is desirable, that is when the opponent can benefit from knowing the next move. Another reason why a player might decide to play a mixed strategy is when a pure strategy is not dominated by other pure strategies, but dominated by a mixed strategy. Finally, in a game without a pure strategy Nash Equilibrium, a mixed strategy may result in a Nash Equilibrium.
From the battle of the sexes game, we can see the mixed strategy Nash equilibria are the action profile (ballet, ballet) and (football, football). In order to drive that, we will assume first that the women will go to the ballet and the man will play some mixed strategy σ. Then the utility of playing this action will be $U_F = f(\sigma)$.
Then, $U_B = \sigma_B(4) + (1 - \sigma_B)(0)$, therefore in another word, the women gets '4' some percentage of the time and '0' for the rest of the time. Assuming the women will be going with her

partner to the football match, then $U_F = \sigma_B(0) + (1 - \sigma_B)(2)$, she will get '0' some percentage of the time and '2' for the rest of the time. Setting the two equations equal to each other and solving for σ, this will $\sigma_B = 1/3$. This means that in this mixed strategy Nash equilibrium, the man is going to the ballet third of the time and to going to the football match two-third of the time. Taking another look to the Table 2-2 , we can see that the game is symmetrical against the strategies, which means that the women will decide to go the ballet two-third of the time and third of the time to go to the football match.
In order to calculate the utility of each player in this game, we need to multiply the probability distribution of each action with by the user strategy, as shown in Table 3. We can simply see that the utility of both players is '4/3', which means that if they won't communicate with each other to decide where to go, they are both better-off to use mix strategies.

		Female	
		Ballet (2/3)	*Football (1/3)*
Male	*Ballet (1/3)*	2/9 **2, 4**	1/9 **0, 0**
	Football (2/3)	4/9 **0, 0**	2/9 **4, 2**

Table 3. Pure and Mixed Strategies, Battle of the Sexes example.

6. Valid and invalid mixed strategy Nash Equilibrium

This section shows how mixed strategies can be invalid with games in general forms. Recalling the prisoner's dilemma game from the previous section, where we going to solve the general class of the game by removing the numbers from the table and use the following variables;

		2nd Criminal	
		Cooperate	*Defect*
1st Criminal	*Cooperate*	***B, b***	***D, a***
	Defect	***A, d***	***C, c***

Table 4. Valid and Invalid Mixed Strategy Nash Equilibrium, Prisoners' Dilemma example.

Where we have, $A > B > C > D$ and $a > b > c > d$. We will simply start to solve this game the same way we did before, we will start looking for the dominate strategies. From the player one point of view, if player two cooperate then player one will not as $A > D$. If player two defect, then player one will defect as well as $C > D$. Doing the same thing for player two; if player one confess, then player two will defect as $a > d$. If player one defect, then player two will defect as well as $c > d$. Then, the only sensible equilibrium will be (Don't confess, Don't confess).
To make sure that there are no mixed strategy Nash equilibrium in this scenario, we need to find the utility of player two confessing as a function of some mixed strategy of player one. That is, some percentage of the time player two will get b and for the rest of the time will get d. Mathematically this will be; $U_C = \sigma_C(b) + (1 – \sigma_C)(d)$. Then, we do the same to find what the

utility of player two will be as function of player one mixed strategy. This can be shown as; $U_D = \sigma_C(a) + (1 - \sigma_C)(c)$. To find the mixed strategy, U_C must be equal to U_D, and that will lead us to the following equation;

$$\sigma_C = \frac{c-d}{b-d-a+c} \tag{4}$$

In order to proof that this is a valid mixed strategy Nash equilibrium, the following condition must be satisfied; $Pr(i) \in [0,1]$ (i.e. no event can occur with negative probability and no event can occur with probability greater than one). That is the probability that this strategy will happen is grater than zero and not less than one. For the first case, when $\sigma_C \geq 0$, the nominator and the denominator must be both positive or negative, otherwise, this mixed strategy will be invalid. Recalling our assumption, $a > b > c > d$m then the nominator must be grater than zero, the denominator must be grater than zero as well. That is $b + c - a - d > 0$, which can be re-arranged as $b + c > a + d$, at this point we can be sure whether this will give us the right answer of whether this is a valid mixed strategy or not as there will be some times where $b + c$ is grater than $a + d$ and some times where it is not. So, for the mixed strategy Nash equilibrium for this game does exist, σ_C must be less than or equal to one. This will lead us to the following equation:

$$\frac{c-d}{b-d-a+c} \leq 1 \tag{5}$$

That is $c - d \leq b - d - a + c$, which can be solved to $a \leq b$, which is not right as this violate or rule that $a > b$, so this is an invalid mixed strategy. Thus, we proved that there is no mixed strategy Nash equilibrium in this game and the two players will defect.

		Female	
		Ballet	*Football*
Male	*Ballet*	**A, b**	**C, c**
	Football	**C, c**	**B, a**

Table 5. Valid and Invalid Mixed Strategy Nash Equilibrium, Battle of the Sexes example.

On the other hand, if we work for the example of the Battle of the Sexes game. Table 5 shows the game in general format, were we removed the numbers again and used the following variables; $A \geq B \geq C \geq 0$ and $a \geq b \geq c \geq 0$. Following the same procedure we used in the previous example, we can solve for the man mixed strategy when his partener goes to watch the match, which will lead us to the following equality: $U_F = \sigma_F(b) + (1 - \sigma_F)(c)$, as the women get b some percentage of the time and get c the rest of the time. If she decides to go to the ballet, the equality becomes; $U_B = \sigma_F(c) + (1 - \sigma_F)(a)$. Now, taking these two equations to solve for the man mixed strategy, we can finally get: $\sigma_F = (a - c)/(a + b - 2c)$.
In order to prove that this mixed strategy is valid, the same condition used before must be satisfied, $Pr(i) \in [0,1]$. That is, $\sigma_F \geq 0$, we already have $a > c$, then the numerator is positive and greater than zero. For the denominator to be positive, $(a + b - 2c)$ must be positive. That is $a + b - 2c \geq 0$, which can be arranged as $a - c \geq c - b$, which proves that the denominator is positive as this is always true.

We must prove that $\sigma_F \leq 1$ to prove the validity of such mixed strategy. That means we must prove the following; $a - c \leq a + b - 2c$, which can be arranged to the following $c \leq b$, which is true as we already mentioned that $b \geq c \geq 0$.
Thus, we have proved that there exist three equilibriums in this game, the two players can go the Ballet or to the match together or each one of them can go to their preferred show with a probability of $(a - c)/(a + b - 2c)$.

7. Classification of game theory

Games can be classified into different categories according to certain significant features. The terminology used in game theory is inconsistent, thus different terms can be used for the same concept in different sources. A game can be classified according to the number of players in the game, it can be designated as a one-player game, two-player game or n-players game (where n is greater than '2'). In addition, a player need not be an individual person; it may be a nation, a corporation, or a team comprising many people with shared interests.

7.1 Non-cooperative and cooperative (coalition) games

A game is called non-cooperative when each agent (player) in the game, who acts in her self interest, is the unit of the analysis. While the cooperative (Coalition) game treats groups or subgroups of players as the unit of analysis and assumes that they can achieve certain payoffs among themselves through necessary cooperative agreements [10].
In non-cooperative games, the actions of each individual player are considered and each player is assumed to be selfish, looking to improve its own payoff and not taken into account others involved in the game. So, non-cooperative game theory studies the strategic choices resulting from the interactions among competing players, where each player chooses its strategy independently for improving its own performance (utility) or reducing its losses (costs). On the other hand, Cooperative game theory was developed as a tool for assessing the allocation of costs or benefits in a situation where the individual or group contribution depends on other agents actions in the game [11]. The main branch of cooperative games describes the formation of cooperating groups of players, referred to as coalitions, which can strengthen the players' positions in a game.
In Telecommunications systems, most game theoretic research has been conducted using non-cooperative games, but there are also approaches using coalition games [12]. Studying the selfishness level of wireless node in heterogeneous ad-hoc networks is one of the applications of coalition games. It may be beneficial to exclude the very selfish nodes from the network if the remaining nodes get better QoS that way [13].

7.2 Strategic and extensive games

One way of presenting a game is called the strategic, sometimes called static or normal, form. In this form the players make their own decisions simultaneously at the beginning of the game, the players have no information about the actions of the other players in the game. The prisoner's dilemma and the battle of the sexes are both strategic games.
Alternatively, if players have some information about the choices of other players, the game is usually presented in extensive, sometimes called as a game tree, form. In this case, the players can make decisions during the game and they can react to other players' actions.

Such form of games can be finite (one-shot) games or infinite (repeated) games [14]. In repeated games, the game is played several times and the players can observe the actions and payoffs of the previous game before proceeding to the next stage.

7.3 Zero-sum games

Another way to categorize games is according to their payoff structure. Generally speaking, a game is called zero-sum game (sometimes called if one gains, another losses game, or strictly competitive games) if the player's gain or loss is exactly balanced those of other players in the game. For example, if two are playing chess, one person will lose (with payoff '-1') and the other will win (with payoff '+1'). The win added to the loss equals zero. Given that sometimes a loss can be a gain, real life examples of zero-sum game can be very difficult to find. Going back to the chess example, a loser in such game may gain as much from his losses as he would gain if he won. The player may become better player and gain experience as a result of loosing at the first place.
In telecommunications systems, it is quite hard to describe a scenario as a zero-sum game. However, in a bandwidth usage scenario of a single link, the game may be described as a zero-sum game.

7.4 Games with perfect and imperfect information

A game is said to be a perfect information game if each player, when it is her turn to choose an action, knows exactly all the previous decisions of other players in the game. Then again, if a player has no information about other players' actions when it is her turn to decide, this game is called imperfect information game. As it is hardly ever any user of a network knows the exact actions of the other users in the network, the imperfect information game is a very good framework in telecommunications systems. Nevertheless, assuming a perfect information game in such scenarios is more suitable to deal with.

7.5 Games with complete and incomplete information

In games with "complete information", all factors of the game are common knowledge to all players. That is, each individual player is fully aware of other players in the game, their strategies and decisions and the payoff of each player. As a result, a complete information game can be represented as an efficient perfectly competitive game. On the other hand, in the "incomplete information" games, the player's dose not has all the information about other players in the game, which made them not able to predict the effect of their actions on others.
One of the very well known types of such games is the sealed-bid auctions, in which a player knows his own valuation of the good but does not knows the other bidders' valuation. A combination of incomplete but perfect information game can exist in a chess game, if one player knows that the other player will be paid some amount of money if a particular event happened, but the first player does not know what the event is. They both know the actions of each other, perfect information game, but does not know the payoff function of the other player, incomplete information game.

7.6 Rationality in games

The most fundamental assumption in game theory is rationality [15]. It implies that every player is motivated by increasing his own payoff, i.e. every player is looking to maximize his own utility. John V. Neumann and Morgenstern justified the idea of maximizing the expected payoff in their work in 1944 [4]. However, pervious studies have shown that humans do not always act rationally [16]. In fact, humans use a propositional calculus in reasoning; the propositional calculus concerns truth functions of propositions, which are logical truths (statements that are true in virtue of their form) [17]. For this reason, the assumption of rational behaviour of players in telecommunications systems is more justified, as the players are usually devices programmed to operate in certain ways.

7.7 Evolutionary games

Evolutionary game theory started its development slightly after other games have been developed [18]. This type of game was originated by John Maynard Smith formalization of evolutionary stable strategies as an application of the mathematical theory of games in the context of biology in 1973 [19]. The objective of evolutionary games is to apply the concepts of non-cooperative games to explain such phenomena which are often thought to be the result of cooperation or human design, for example; market information, social rules of conduct and money and credit. Recently, this type of games has become of increased interest to scientist of different background, economists, sociologists, anthropologists and also philosophers. One of the main reasons behind the interest among social scientists in the evolutionary games rather than the traditional games is that the rationality assumptions underlying evolutionary game theory are, in many cases, more appropriate for the modelling of social systems than those assumptions underlying the traditional theory of games [20].

8. Applications of game theory in telecommunications

Communications systems are often built around standard, mostly open ones, such as the TCP/IP (Transmission Control Protocol/Internet Protocol [21]) standard in which the internet is based. Devices that we use to access these systems are being designed and built by a diversity of different manufactures. In many cases, these manufacturers may have an incentive to develop products, which behave “selfishly” by seeking a performance advantage over other network users at the cost of overall network performance [22]. On the other hand, end users may have the ability to force these devices in order to work in a selfish manner. Generally speaking, the maximizing of a player’s payoff is often referred to as selfishness in a game. This is true in the sense that all the players try to gain the highest possible utility of their actions. However, a player gaining a high utility does not necessarily mean that the player acts selfishly. As a result, systems that are prepared to cope with users who behave selfishly need to be designed. If the designs of such systems are possible, designers should make sure that selfish behaviour within the system is unprofitable for individuals. When designing such system is not possible, they should be at least aware of the impact of such behaviour on the operation of the specified system.
One important thrust in these efforts focuses on designing high-level protocols that prevent users from misbehaving and/or provide incentives for cooperation. To prevent misbehaviour, several protocols based on reputation propagation have been proposed in the literature, e.g., [23], [24]. The mainstream of existing research in telecommunications networks focused on using non-cooperative games in various applications such as

distributed resource allocation [25], congestion control [26], power control [27], and spectrum sharing in cognitive radio, among others. This need for non-cooperative games led to numerous tutorials and books outlining its concepts and usage in communication, such as [28], [29]. Another thrust of research analyzes the impact of user selfishness from a game theoretic perspective, e.g., [22], [30]. Since the problem is typically too involved, several simplifications to the network model are usually made to facilitate analysis and allow for extracting insights. For example, in [22], the wireless nodes are assumed to be interested in maximizing energy efficiency. At each time slot, a certain number of nodes are randomly chosen and assigned to serve as relay nodes on the source- destination route. The authors derive a Pareto optimal operating point and show that a certain variant of the well known TIT-FOR-TAT algorithm converges to this point. In [22], the authors assume that the transmission of each packet costs the same energy and each session uses the same number of relay nodes. Another example is [30], which studies the Nash equilibrium of packet forwarding in a static network by taking the network topology into consideration. More specifically, the authors assume that the transmitter/receiver pairs in the network are always fixed and derive the equilibrium conditions for both cooperative and non-cooperative strategies. Similar to [22], the cost of transmitting each packet is assumed fixed. It is worth noting that most, if not all of, the works in this thrust utilize the repeated game formulation, where cooperation among users is sustainable by credible punishment for deviating from the cooperation point.

Cooperative games have also been widely explored in different disciplines such as economics or political science. Recently, cooperation has emerged as a new networking concept that has a dramatic effect of improving the performance from the physical layer [23], [24] up to the networking layers [25]. However, implementing cooperation in large scale communication networks faces several challenges such as adequate modelling, efficiency, complexity, and fairness, among others. In fact, several recent works have shown that user cooperation plays a fundamental role in wireless networks. From an information theoretic perspective, the idea of cooperative communications can be traced back to the relay channel [31]. More recent works have generalized the proposed cooperation strategies and established the utility of cooperative communications in many relevant practical scenarios, such as [25], [26] and [32]. In another line of work, in [27], the authors have shown that the simplest form of physical layer cooperation, namely multi hop forwarding, is an indispensable element in achieving the optimal capacity scaling law in networks with asymptotically large numbers of nodes. Multi-hop forwarding has also been shown to offer significant gains in the efficiency of energy limited wireless networks [28], [29]. These physical layer studies assume that each user is willing to expend energy in forwarding packets for other users. This assumption is reasonable in a network with a central controller with the ability to enforce the optimal cooperation strategy on the different wireless users. The popularity of ad-hoc networks and the increased programmability of wireless devices, however, raise serious doubts on the validity of this assumption, and hence, motivate investigations on the impact of user selfishness on the performance of wireless networks. The following chapters will be full of more details about the applications of game theory in wireless telecommunications systems, including applications of game theory in interface selections mechanisms, Mobile IPv6 protocol extensions, resource allocations and routing in Ad-Hoc wireless network and spectrum sharing in Cognitive Radio networks.

9. Summary

This chapter gives a detailed insight in the game theory definition, classifications and applications of games in telecommunications. Prisoners Dilemma and the Battle of the Sexes games have been discussed in details, showing different strategies from the players and discussing the expected outcome of such games. Nash Equilibrium and Pareto Efficient terms are discussed in details with detailed examples. Moreover, we have discussed mixed strategies in games and mathematically proved that a mixed strategy in Prisoners' Dilemma example does not exist. We have also proved that a mixed strategy exists in the battle of the sexes game. Finally, after classifying games into different categories, an introduces to the applications of game theory in Telecommunications.

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2

Auction and Game-Based Spectrum Sharing in Cognitive Radio Networks

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1. Introduction

One of the main reasons behind the concurrent increase in the demand for and congestion of Radio Frequency (RF) spectrum is the rapid development of radio networks of all kinds in our world, which has defiantly changed the public feeling about radio. Nowadays, almost everybody has a mobile phone and radio stations are literary everywhere. Someone can argue that our world is becoming a radio world where waves are weaving everywhere around the Earth. What's more, this congestion has created a battle between the public, private and military sectors over frequency ownership and has put a premium on the cost of spectrum. According to a recent research introduced by the FCC (Federal Communications Commission) and Ofcom, it was found that most of the frequency spectrum was inefficiently utilized [1-2]. The existing spectrum allocation process, denoted as Fixed Spectrum Access (FSA), headed for static long-term exclusive rights of spectrum usage [3] and shown to be inflexible [4]. Studies have shown, however, that spectral utilization is relatively low when examined not just by frequency domain, but also across the spatial and temporal domains [5]. Thus, an intelligent device aware of its surroundings and able to adapt to the existing RF environment in consideration of all three domains, may be able to utilize spectrum more efficiently by dynamically sharing spectral resources [6 and 7]. Since the 19th century, when the laws of electromagnetic have been discovered and described by the set of Maxwell's equations and technical devices been invented to produce and use these electromagnetic waves predicted by theory, man has added his own man-made waves to the natural ones [7].

It is fair to say that, from the very beginning of wireless telephony, maritime radio systems has always used shared channels [7-8]. For example, 2,182 KHz is used as a calling frequency as well as emergency signalling frequency and other frequencies are used as working frequencies. If two ships want to communicate, one should identify a working frequency and make a call. By specifying a channel or channels, that ships keep watch on, both emergency and establishing connections between ships can be facilitative. In fact, channel sharing was necessary and effective because of the lack of sufficient channels offered to every single ship and due to the fact that, the typical ship will require far less than a full channel of capacity [7-8]. Around the mid of 1970's, the FCC permitted land mobile operation on some of the lower UHF channels in several large cities, in order to expand land mobile services. One group of channels was made available to Radio Common Carriers (RCCs) to provide mobile service on a common carrier basis. The FCC adopted rules permitting open entry for these channels and requiring carriers to monitor the channels and select unused channel to carry each conversation. In essence, exclusivity was provided on a first come, first-served basis one conversation at a time [7-9].

Another example of spectrum sharing is the second generation of cordless telephone (CT2), developed by the British industry and government in the mid of 1980's. CT2 was designed to be used in both in home and in public and uses a pool of 40 channels. To establish a call, any equipment will automatically identify a vacant channel or a channel with the minimum interference and begins operation on that channel [7-8]. No one can ignore one of the main advantages of the radio, it can be used anywhere, at any time, capable of building links at very short distances as well as on a cosmic scale. Radio is a unique tool to connect men and things without any material medium. It is a wonderful tool for social progress. Having said all these facts about spectrum sharing, spectrum management can now be seen as a major goal for telecommunications efficiency. It is necessary that this natural and public resource be utilized for the profit of as many users as possible, taking care of the largest variety of needs.

If we want to talk about Cognitive Radio (CR), then we must mention Software Defined Radio (SDR), which is a transmitter in which operating parameters including transmission frequency, modulation type and maximum radiated or conducted output power can be altered without making any hardware changes. The sophistication possible in an SDR has now reached the level where a radio can possibly perform beneficial tasks that help the user, the network and help to minimize spectral congestion [7]. In order to raise an SDR's capabilities to make it known as a CR, it must support three major applications [7]:

Spectrum management and optimization.

Interface with a wide range of wireless networks leading to management and optimization of network resources.

Interface with human providing electromagnetic resources to aid the human in his and/or her activates.

We must begin with a few of the major contributions that have led us to today's CR developments, to truly recognize how many technologies have come together to drive CR technologies. The development of Digital Signal Processing (DSP) technologies arose due to the efforts of the research leaders [10-14], who taught an entire industry how to convert analog signal processes to digital processes. In the meantime, the simulation industry used in the radio industry was not only practical, but also resulted in improved radio communication performance, reliability, flexibility and increased value to the user [15-18].

The concept of CR emerged as an extension of SDR technology. Although, definitions of the two technology's are different, most radio expert agree with the fact that a CR device must have the following characteristic in order to be distinguished from an SDR one:

1. The named device should be aware of its environment.
2. The device must be able to change its physical behaviour in order to adapt to the changes of its current environment.
3. The device must be able to learn from its previous experience.
4. Finally, the device should be able to deal with situations unknown at the time of the device design. In another word, the device should be able to deal with any unexpected situations.

That being said, up to the authors knowledge, the idea of CR was first discussed officially in 1999 by [19]. It was a novel approach in wireless communications that the author describes it as "The point in which wireless personal digital assistants (PDA's) and the related networks are sufficiently computationally intelligent about radio resources and related computer-to-computer communications to detect user communications needs as a function of use context, and to provide radio resources and wireless services most appropriate to those needs." [19]. What's more, the work introduced in [19] can be considered one of the novel ideas which discussed CR technology. The work was based on the situation in which wireless nodes and the related networks are sufficiently computationally intelligent about radio resources and related computer-to-computer communication to detect the user communication needs as a function of use context and to provide resources and wireless resources most required. In another word, a CR is a radio that has the ability to sense and adapt to its radio environments. This work defined two basic characteristics of any CR device, which are cognitive capability and re-configurability. In order for the device to detect the spectrum parameters, the device should be able to interact with its environment. The spectrum needs to be analysed for spectrum concentration, power level, extent and nature of temporal and spatial variations, modulation scheme and existence of any other network operating in the neighbourhood. The CR device should be capable to adopt itself to meet the spectrum needs in the most optional method. The recent developments in the concept of software radios DSP techniques and antenna technology helped in this flexibility in CR devices design.

Finally, the intelligent support of CR's to the user arises by sophisticated networking of many radios to achieve the end behaviour, which provides added capability and other benefits to the user.

2. Game theory and spectrum sharing

Players in cooperative games try to maximize the overall profit function of everyone in the game in a fair fashion. This type of games has the advantage of higher total profit and better fairness. On the other hand, in non-cooperative or competitive games players try to maximize their own individual payoff functions. If such a game has a designer with preferences on the outcomes, it may be possible for the designer to decide on strategy spaces and the corresponding outcomes (*i.e.* the mechanism) so that the players' strategic behavior will not lead to an outcome that is far from desirable [20 and 21]. Recent studies have shown that despite claims of spectral insufficiency, the actual licensed spectrum remains unoccupied for long periods of time [8]. Thus, *cognitive radio* systems have been proposed [22] in order to efficiently exploit these spectral holes.

Previous studies have tackled different aspects of spectrum sensing and spectrum access. In [23], the performance of spectrum sensing, in terms of throughput, is investigated when the secondary users (SUs) share their instantaneous knowledge of the channel. The work in [24] studies the performance of different detectors for spectrum sensing, while in [25] spatial diversity methods are proposed for improving the probability of detecting the Primary User (PU) by the SUs. Other aspects of spectrum sensing are discussed in [26-27]. Furthermore, spectrum access has also received increased attention, e.g. [28-34]. In [28], a dynamic programming approach is proposed to allow the SUs to maximize their channel access time while taking into account a penalty factor from any collision with the PU. The work in [30] and [35-44] establishes that, in practice, the sensing time of CR networks is large and affects the access performance of the SUs. In [29], the authors model the spectrum access problem as a non-cooperative game, and propose learning algorithms to find the correlated equilibria of the game. Non-cooperative solutions for dynamic spectrum access are also proposed in [30] while taking into account changes in the SUs' environment such as the arrival of new PUs, among others.

Auctions of divisible goods have also received much attention [32] and [45-50]. Where the authors address the problem of allocating a divisible resource to buyers who value the quantity they receive, but strategize to maximize their net payoff (*i.e.* value minus payment). An allocation mechanism is used to allocate the resource based on bids declared by the buyers. The bids are equal to the payments, and the buyers are assumed to be in *Nash equilibrium*. When multiple SUs compete for spectral opportunities, the issues of fairness and efficiency arise. On one hand, it is desirable for an SU to access a channel with high availability. On the other hand, the effective achievable rate of an SU decreases when contending with many SUs over the most available channel. Consequently, efficiency of spectrum utilization in the system reduces. Therefore, an SU should explore transmission opportunities in other channels if available and refrain from transmission in the same channel all the time. Intuitively, diversifying spectrum access in both frequency (exploring more channels) and time (refraining from continuous transmission attempts) would be beneficial to achieving fairness among multiple SUs, in that SUs experiencing poorer channel conditions are not starved in the long run.

The objective of the work in this chapter is to design a mechanism that enables fair and efficient sharing of spectral resources among SUs. Firstly, we model spectrum access in cognitive radio networks as a repeated cooperative game. The theory and realization of cooperative spectrum sharing is presented in detail, where we assume that there is one PU and several SUs. We also consider the case of dynamic games, where the number of SUs changes. The advantages of cooperative sharing are proved by simulation. Secondly, we discuss the case of large number of SUs competing to share the offered spectrum and how the cooperative game will reduce the sellers and bidders revenue. Finally, we introduce a competitive auction and game-based mechanism to improve the overall system efficiency in terms of a better fairness in accessing the spectrum.

Throughout this chapter, an adaptive competitive second-price pay-to-bid sealed auction game is adapted as solution to the fairness problem of spectrum sharing between one primary user and a large number of secondary users in cognitive radio environment. Three main spectrum sharing game models are compared, namely optimal, cooperative and competitive game models introduced as a solution to the named problem. In addition, this chapter prove that the cooperative game model is built based on achieving Nash equilibrium between players and provides better revenue to the sellers and bidders in the game. Furthermore, the cooperative game is the best model to choose when the number of secondary users changes dynamically, but only when the number of competitors is low. As in practical situations, the number of secondary users might increase dramatically and the cooperative game will lose its powerful advantage once that number increases. As a result, the proposed mechanism creates a competition between the bidders and offers better revenue to the players in terms of fairness. Combining both second-price pay-to-bid sealed auction and competitive game model will insure that the user with better channel quality, higher traffic priority and fair bid will get a better chance to share the offered spectrum. It is shown by numerical results that the proposed mechanism could reach the maximum total profit for SUs with better fairness. Another solution is introduced in this chapter, which is done by introducing a reputation-based game between SUs. The game aims to elect one of the SUs to be a secondary-PU and arrange the access to other SUs. It is shown by numerical results that the proposed game managed to give a better chance to SUs to use the spectrum more efficiently and improve the PU revenue.

3. Assumptions and system model

3.1 PU's and SU's and allocation function

In the following sections, we consider a spectrum overlay-based cognitive radio wireless system with one PU and N SU's (as shown in Figure 6-1). The PU is willing to share some portion (b_i) of the free spectrum (F) with SU i. The PU asks each SU a payment of c per unit bandwidth for the spectrum share, where c is a function of the total size of spectrum available for sharing by the SU's. The revenue of SU i is denoted by r_i per unit of achievable transmission rate. A simple example is shown in Figure 1.

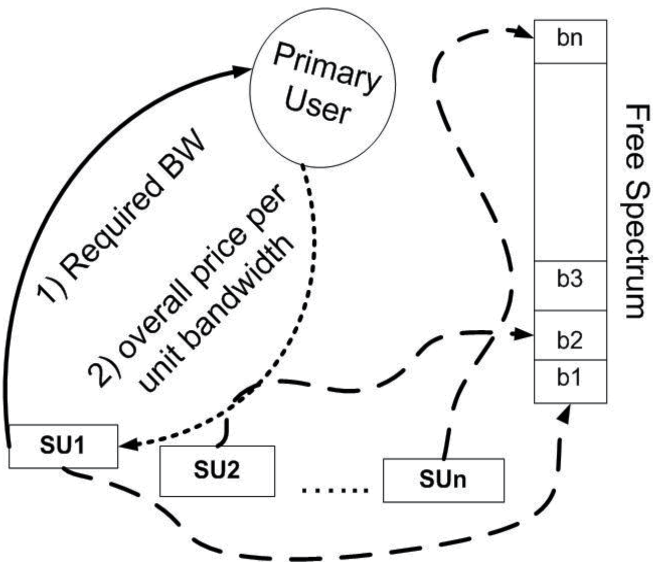

Fig. 1. System model for spectrum sharing.

Both centralized and distributed decision making scenarios are considers in this work. In the former case, each SU is assumed to be able to observe the strategies adopted by other users (*i.e.,* either the users have the ability to discuss their shares between them, or the PU sends update of each SU share). In the latter case, the adaptation for spectrum sharing is performed in a distributed fashion based on communication between each of the SUs and the PU only (*i.e.,* the secondary users are unable to observe the strategies and payoffs of each other).

3.2 Cost function, and wireless system model

A wireless transmission model based on adaptive modulation and coding (AMC) where the transmission rate can be dynamically adjusted based on channel quality is to be assumed in this chapter. With AMC, the signal-to-interference noise ratio (SINR) at the receiver is denoted as γ and equals to;

$$\gamma = \frac{p_i h_{ij}}{n_0 + \Sigma_{j \neq i} p_i h_{ij}} \tag{1}$$

Where h_{ij} is the channel gain from the user $j's$ transmitter to user $i's$ receiver, p_i is the transmitting power of the user i, and n_0 is the thermal noise level. The rate for user i (in bits/sec/Hz) is given by;

$$R_i = log_2(1+\gamma) \quad (2)$$

The spectral efficiency I_s of transmission by a secondary user can be obtained from [16];

$$I_s = log_2(1+K\gamma) \quad (3)$$

Where $k=1.5/\ (ln0.2/BER^{tar})$, BER^{tar} is the target bit-error-rate of the system. The pricing function [17] which the SU's pay is given by;

$$c(F) = y(b_1+b_2+\ldots+b_n)^z \quad (4)$$

y and z are assumed to be positive constants and greater than one so that the function in convex (*i.e.*, the function is continues and differentiable), knowing that B is the set of bids for all SU's (*i.e.*, B=$\{b_1, b_2, \ldots., b_n\}$). Now let us denote w as the worth of the spectrum to the PU. Then, the condition $c(F) > w \times \sum_{bj\in F} b_j$ must be satisfied in order to ensure that the PU is willing to share spectrum of size $b = \sum_{bj\in F} b_i d_j$ with the SU's (if it is equal, then PU will not gain any profit).

The overall revenue of any SU can be explained as the combination of the user revenue of achievable transmission rate, the spectral efficiency and the shared portion of the spectrum (*i.e.*, $r_i \times I_s \times b_i$). While the cost the user must pay is $b_i \times c(F)$. Then, the profit of every SU can be represented as;

$$\mu_i = r_i \times I_s \times b_i - b_i \times c(F) \quad (5)$$

The marginal profit of SU i can be obtained from;

$$\frac{d\mu_i(F)}{db_i} = r_i I_s - y(\Sigma_{b_j\in F} b_j)^z - yzb_i(\Sigma_{b_j\in F} b_j)^{z-1} \quad (6)$$

Knowing that, the optimal size of allocated spectrum to one SU depends on the strategies of other SU's are using. Nash equilibrium is considered as the solution of the game to ensure that all SU's are satisfied with it. By definition, Nash equilibrium of a game is a strategy profile with the property that no player can increase his payoff by choosing a different action, given the other players' actions. In this case, the Nash equilibrium is obtained by using the best response function, which is the best strategy of one player given others' strategies. Let ST_{-i} denote the set of strategies adopted by all except SU i (i.e., $ST_{-i} = \{st_j \mid j=1, 2, \ldots, N;\ j\neq i\}$ and $ST = ST_{-i} \cup \{st_i\}$). The best response function of SU i given the size of the shared spectrum by other SU's b_j, where $j\neq i$, is defined as follows;

$$BR_i = arg\ max_{bi}\ \mu_i\ (ST_{-i} \cup \{b_i\}) \quad (7)$$

Then the game is in *Nash Equilibrium* if and only if;

$$b_i = BR_i(ST_{-i}),\ \forall_i \quad (8)$$

4. Spectrum sharing strategies

Cognitive radio is an intelligent wireless communication system that is aware of its surrounding environment and can be used to improve the efficiency of frequency spectrum by exploiting the existence of spectrum holes [22]. Spectrum management in cognitive radio aims at meeting the requirements from both the primary user and the secondary users. There are three strategies in spectrum sharing optimal, competitive and cooperative models.

4.1 Optimal spectrum sharing model

The objective of optimal model is to maximize the profit sum, which may make some secondary users have no spectrum to share [28, 32 and 51]. Therefore, it is unfair for all secondary users. From equation 6-6, the total marginal profit function for all the SU's can be denoted as follows: $\frac{d\sum_{j=1}^{N}\mu_i(F(t))}{db_i(t)}$.

In order to get the solution of the biggest profit for all the secondary users, an optimal equation is built, as (6-9);

$$\textit{Maximize: } \sum_{j=1}^{N}\mu_i(F) \tag{9}$$

$$\textit{Subject to: } \mathrm{b_i} \geq 0,\ \forall\ b_i \in F$$

Our assumption works as follow, the initial sharing spectrum is $b_i(0)$ for the SU i, which is sent to the primary user. The PU adjusts the pricing function c, and then it is sent back to the SU. Since all secondary users are rational to maximize their profits, they can adjust the size of the requested spectrum b_i based on the marginal profit function. In this case, each secondary user can communicate with the primary user to obtain the differentiated pricing function for different strategies. The adjustment of the requested/allocated spectrum size can be modelled as a dynamic game [49] as follows:

$$b_i(t+1) = f(b_i(t)) = b_i(t) + \eta_i b_i(t)\frac{d\mu_i(F)}{db_i(t)} \tag{10}$$

Where $b_i(t)$ is the allocated spectrum size at time t to SU i and η_i is the adjustment speed parameter (*i.e.*, which can be expressed as the learning rate) of SU i. $f(.)$ denotes the self-mapping function. The SU can estimates the marginal profit function in the actual system by asking the price for share a spectrum from the PU of size $b_i(t) \pm\pi$, where π is a small number (*i.e.*, π is 0.0001). Simply after that the SU observes the response price from the PU $c^-(.)$ and $c^+(.)$ for $b_i(t)-\pi$ and $b_i(t)+\pi$, respectively. Then, the marginal profits for the two cases $\mu_i^{-}(t)$ and $\mu_i^{+}(t)$are compared and the marginal profit can be estimated from;

$$\frac{d\mu_i(.)}{db_i} = \frac{\mu_i^+(.) - \mu_i^-(.)}{2\pi} \tag{11}$$

The overall optimal profit can be estimated using equation (9).

4.2 Competitive spectrum sharing model

The main objective of competitive model is to maximize the profits of individual SU's by a game. The result is *Nash equilibrium*. In the distributed dynamic game, SU's may only be able to observe the pricing information from the PU; they cannot observe the strategies and profits of other SU's. The *Nash equilibrium* for each SU is built based on the interaction with the PU, similar to the case of the optimal sharing model. Since all SU's are rational to maximize their own profits, they can adjust the size of the requested spectrum b_i based on the marginal profit function (*i.e.,* equation (6)). In this case, each SU can communicate with the primary user to obtain different pricing function for different strategies. The adjustment of the requested/allocated spectrum size in competitive games show only a slight difference with optimal games, as each individual user is looking at improving his/her own profit. So equation (9) can be rewritten as;

$$\text{Maximize: } \mu_i(F) \tag{12}$$

$$\text{Subject to: } b_i \geq 0, \forall\, b_i \in F$$

In a similar way to the optimal game, an SU can estimate its marginal profit using the following equation:

$$\frac{d\mu_i(F(t))}{db_i(t)} = \frac{1}{2\pi}\{\mu_i(F_i(t)+\pi) - \mu_i(F_i(t)-\pi)\} \tag{13}$$

When $b_i\ (t + 1) = bi\ (t)$ is satisfied, the *Nash Equilibrium* points ($b_0, b_1, b_2, \ldots, b_N$) can be obtained.

4.3 Cooperative spectrum sharing model

As explained in previous section, in the model of competitive spectrum sharing, *Nash equilibrium* obtained at the maximum of the individual profit of SU. The result is not the best because they do not consider the interaction on other users. For cooperative spectrum sharing, the SU's can communicate with the consideration on the behaviour to other users.

In this chapter, we assume that players can reach in common by communicating with each other. Decreasing the size of sharing spectrum a little for all the SU's on *Nash equilibrium*, (*i.e.,* a factor σ_i $(0 < \sigma_i < 1)$ is multiplied on each SU strategy of *Nash equilibrium*). Although the size of shared spectrum has decreased, the cost which the PU charges to the SU decreases too, which results in the increase of the overall profit for all SU's and the total profits increase as well, but it might reduce the PU revenue.

SU's *Nash Equilibrium* strategy can be got from equation (10). All SU's will negotiate and multiply σ_i, the cooperative strategy is obtained (*i.e.,* $\sigma_1 b_1, \sigma_2 b_2, \ldots\ldots, \sigma_N b_N$). σ_i is chosen in such a way that both the overall and individual profit is maximized, which we called as the cooperative state;

$$\text{Maximize: } \textstyle\sum_{j=1}^{N} \mu_i(F) \text{ and } \mu_i(F) \tag{14}$$

$$\text{Subject to: } b_i \geq 0, \forall\, b_i \in F$$

However, we need to raise the problem of instability of this model. It is possible that one or more SUs may deviate from *Nash equilibrium*. For example, suppose *u1* to be the first SU to share the spectrum and want to deviate, its profit may increase by setting its marginal profit function of equation (6) to zero. If another SU *u2* does not change its strategy, the profit of *u2* will decrease. Therefore, any SU has the motive to deviate from cooperative state. In order to solve this problem, a mechanism needs to be applied to encourage the SUs not to deviate

from the Nash state by computing the long term profit of the SU. Suppose SU i is looking deviate from the Nash state, while SU j ($j \neq i$) is still in the named state. Before SU i deviate, it will compute the long term profit. The mechanism will multiply the future profit of SU i (if decided to deviate) with a weight ε_i ($0 < \varepsilon_i < 1$), which would make the profit in future stages are not higher than that of the previous stages, which means that the current profit is more valuable than future stages.
For any SU i, μ_i^{Ns}, μ_i^{N}, μ_i^{d} denotes the profits of Nash state, *Nash Equilibrium* and deviation, respectively. There are two cases: one is that they all in Nash at all stages, no SU to deviate from the optimal solution, the long term profit of any SU i is shown in equation (15). The other case is that SU i deviates from the optimal solution at the first stage, it will be in *Nash equilibrium* state in the following stages, and the long term profit of SU i is shown in equation (16).

$$\mu_i^{Ns} + \sigma_i \mu_i^{Ns} + \sigma_i^2 \mu_i^{Ns} + \ldots = \frac{1}{1-\sigma_i} \mu_i \tag{15}$$

$$\mu_i^{d} + \sigma_i \mu_i^{N} + \sigma_i^2 \mu_i^{N} + \ldots = \mu_i^{d} + \frac{\sigma_i}{1-\sigma_i} \mu_i \tag{16}$$

The Nash state will be maintained if the long-term profit due to adopting the state is higher than that caused by deviation.

$$\frac{1}{1-\sigma_i} \mu_i > \mu_i^{d} + \frac{\sigma_i}{1-\sigma_i} \mu_i$$

i.e.,

$$\sigma_i \geq \frac{\mu_i^{d} - \mu_i^{Ns}}{\mu_i^{d} - \mu_i^{N}} \tag{17}$$

From equation (15), we know that the Nash state will be kept because of low long term profit for the SU who wants to deviate. The weights σ_i are the vindictive factors to inhabit the motive of leaving the cooperative state.

5. Dynamic cooperative model

In reality, the number of SUs may change. Sometimes there are more secondary users to apply for the spectrum offered by the primary user, and sometimes the secondary users have finished the communication and drop out of the spectrum as it has taken up. For example, let us suppose that there are two SUs, which have been in Nash state. Now there is another (new) SU to apply for the offered spectrum. We assume that the PU has no more spectrums to share. This will lead us to one solution, which is that the two SUs should make some of their spectrums exist to the newcomer.
During the process of reallocating, an adaptive method is applied with the following requirements. The total profit for all the SUs should be the biggest and it should be fair for the reallocation. Being prior users it is rational for them to have priority in spectrum allocation than those who comes later. In order to keep the total profit to maximum, those

with better channel quality could take up more spectrum space. Therefore, the SUs with better channel quality could stop spectrum retreating earlier than those with worse channel quality. When the SUs reach optimal solution, the fairness will not be as good as the three SUs getting into Nash state directly. The reason is that these SUs coming at different time do not have the same priorities.

When SUs have finished the communication and exited the spectrum they had shared, an adaptive method is applied. A fixed part of the spectrum is allocated to the remaining SUs for each step. It is possible for SUs with better channel quality acquire more spectrum in order to make the total profit bigger.

6. Simulation results

6.1 Static game (two SU's only in the game)

In this section, we will consider a CR environment with one PU and two SUs sharing a frequency spectrum of 20MHz to 40MHz. The system has the following settings; for the pricing function, $c(F)$, we use $y=1$ and $z=1$. The worth of spectrum for the PU is assumed to be one (i.e. $w=1$). The revenue of a SU per unit transmission rate is $r_i = 10$, $\forall i$. The target average BER is $BER^{tar} = 10^{-4}$. The initial value is $b_i(0)= 2$. The adjustment speed parameter $\eta_i =0.09$. The SNR for SUs u_1 and u_2 are denoted by γ_1, γ_2 where γ_1 =11dB, γ_2=12dB.

6.1.1 Optimal and competitive models

As explained in the previous section, the total profit is represented by $\mu(B) = \mu_1(B) + \mu_2(B)$. In Figure 2, the total profits in optimal model arrived at its biggest value 228.7333 when (b_1, b_2) = (4.1, 15.6).

The trajectories of optimal model and competitive model are shown in Figure 3, (with γ_1 =11dB, γ_2=12dB), the initial value is (2, 2) for the two models. In competitive model, the

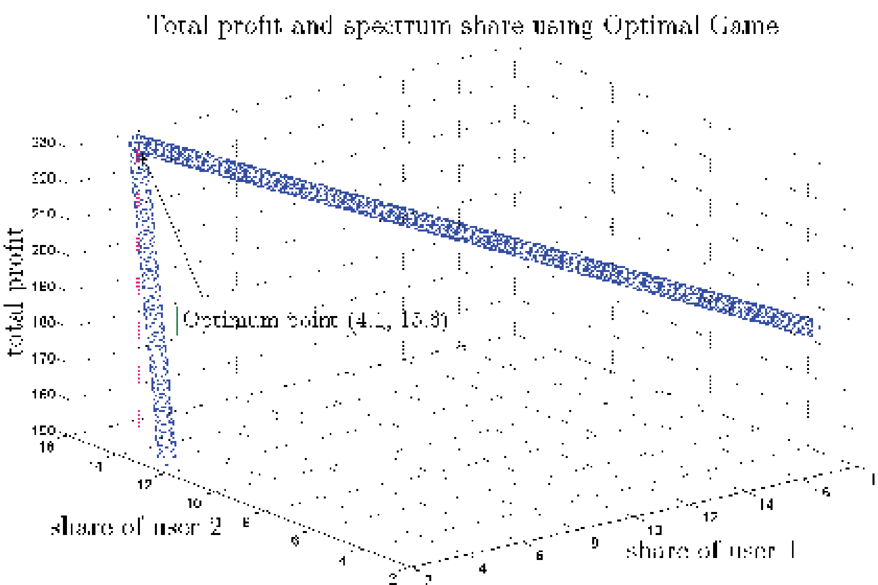

Fig. 2. Total profit and spectrum share using optimal game.

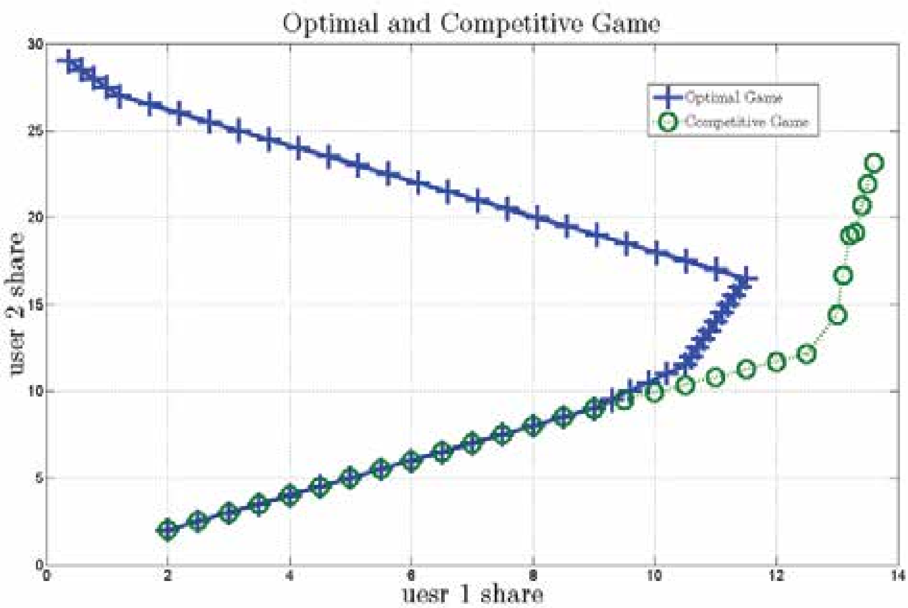

Fig. 3. Optimal and Competitive games

shared spectrum is determined by a game, where the two SUs have been in *Nash equilibrium*. In our simulation, the *Nash equilibrium* is at (14.2591, 24.1302). The sum of spectrum sharing is 11.3893 with the total profit of 228.2378.

It can be seen that the total profit for optimal model is higher than that of competitive model obviously. But one SU has no spectrum sharing for the optimal model, which means the lack of fairness. The advantage of competitive model is fair with a lower profit sum.

6.1.2 Cooperative spectrum sharing game

Based on the Nash equilibrium, we set the weight σ_i in the range of [0.5, 1]. In order to keep the fairness, we assume $| \sigma_1 - \sigma_2 | \leq 1$ to guarantee the size of sharing spectrum is similar for both two SUs. Two SUs got their *Nash equilibrium* at (18.2591, 19.1302). At σ_1 =0.70, σ_2 =0.80, the total profit of 234.4963. Compared with the competitive model, we found that the shared spectrum in cooperative model is less than that of competitive model; it has a bigger total profit than that of *Nash equilibrium*, as shown in Figure 3.

The reason is that we set ($\sigma_1 b_1$, $\sigma_2 b_2$) as the strategies to share the spectrum, the price is lower, and the total profit will increase. Now, let us suppose the SU u_1 deviates from the optimal solution. The strategy of SU u_2 does not change. SU u_1 adopts the strategy based on the marginal profit function. The profit for the two SUs will change when SU u_1 deviated. The comparison of the individual profit in cooperative model, competitive model and deviation is shown in Figure 4. The total profit for the SUs is shown in Figure 5. γ_1 is a variable, which changes in the range of 8~11dB, γ_2 =12dB.

It can be seen that μ_1, μ_2 are bigger in the cooperative model, compared with the competitive model. Therefore, the total profit is bigger too in the cooperative model. When SU *u1* deviates from the cooperative state, μ_1 is higher, and μ_2 is lower, and the total profit is lower (*i.e.* the amount of μ_1 increasing is smaller than that of μ_2 decreasing) as well.

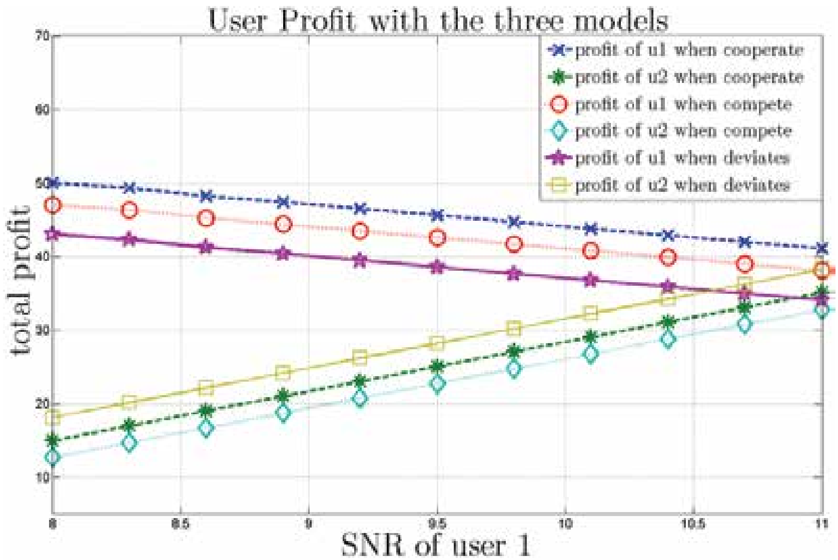

Fig. 4. Total profit with different modes.

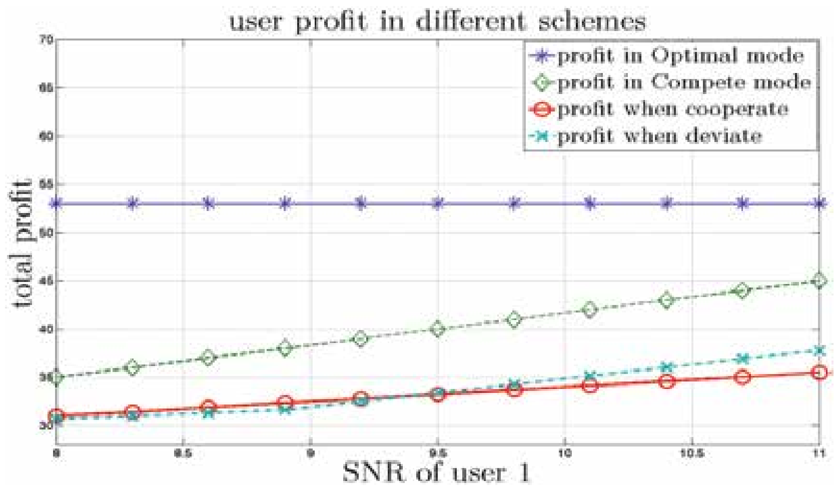

Fig. 5. User Profit with different modes.

6.1.3 Dynamic spectrum sharing game

The pervious results were based on the two SUs. The analyzing method is similar for more SUs. In practice, the number of SUs may change. For example, there is another secondary user denoted by u_3 looking to apply for the offered spectrum. We assume that the channel quality for u_3 is the same with secondary user u_2 (γ_1 is a variable, γ_2=γ_3 =12dB). There is no more free spectrum for the primary user to share with others. The previously mentioned adaptive method is applied in the allocation of spectrum. First u_1 and u_2 exit a fixed ratio of spectrum to u_3, and the total profit is computed. If the total profit could increase, the process will go on. If the total profit decreases, the SU with a better channel state will stop the process of exit. The trajectory of the process is shown in Figure 6. In addition, the corresponding total profit is shown in Figure 6-7. When a new SU applies for spectrum sharing, it would converge to the point of (3.418948, 5.4642, 0.4936). The total profit is 62.3421, which is a little bigger than the case with two SUs. When the third SU exits the spectrum, an adaptive method is applied to reallocate the spectrum. The left two SUs converge to (2.2148, 5.9393) with a total profit of 73.9867, as shown in Figure 6-8.

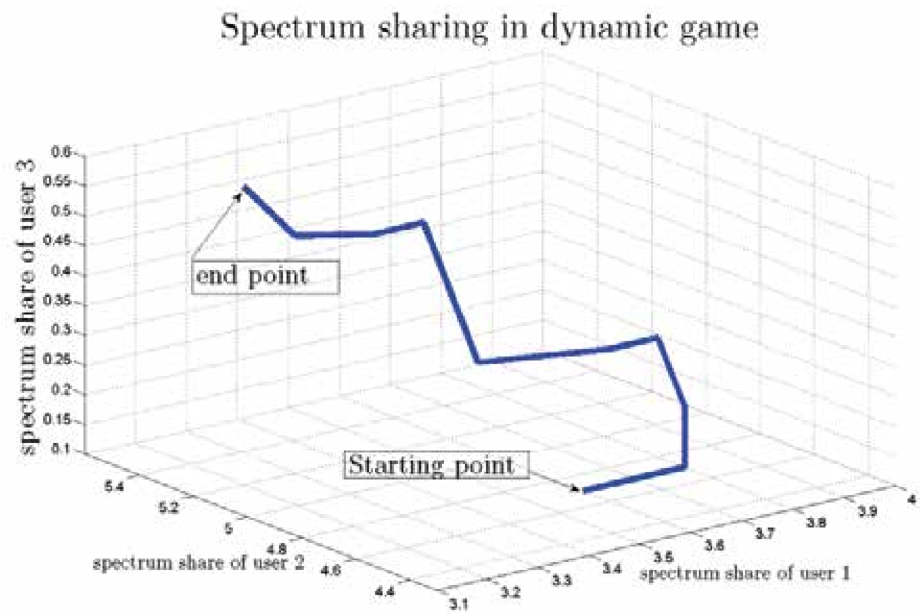

Fig. 6. Spectrum sharing in dynamic game.

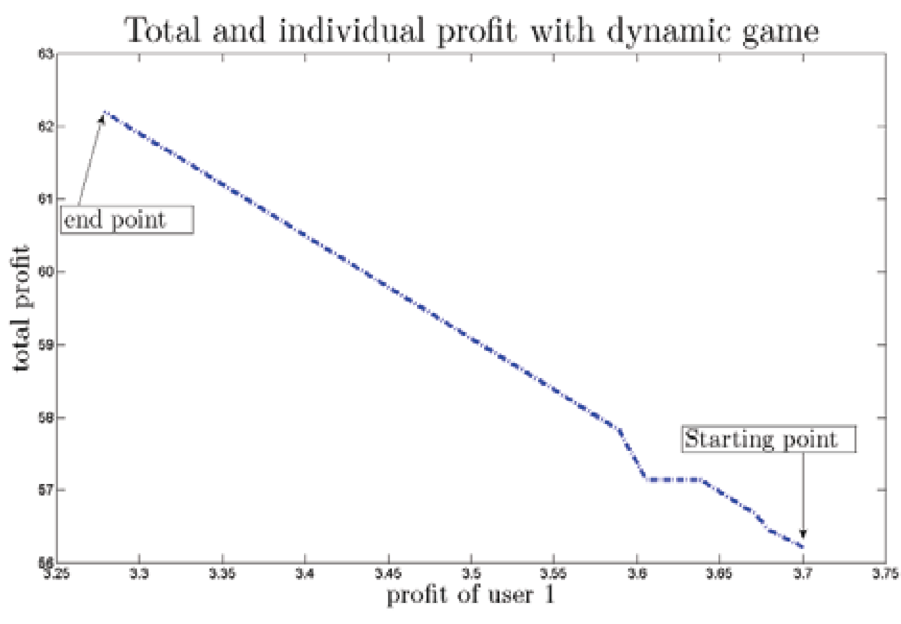

Fig. 7. Dynamic game and user profit.

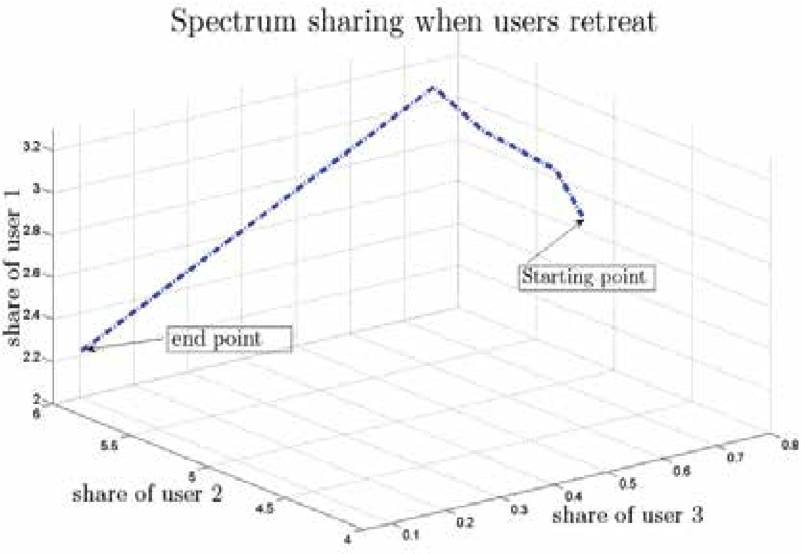

Fig. 8. Spectrum Share when user retreats.

7. Is the cooperative game visible?

So far we have discussed three game models to solve the problem of spectrum sharing in CR systems. We proved that the optimal game would improve the overall profit of the players in the game, which might lead to unfair distribution of the offered spectrum. The competitive game shows a lower overall profit, but gives a better share to the user with better channel quality, who ask for a share earlier and stays active for longer period (*i.e.,* a higher priority as compared to new comers). Finally, the cooperative game gives the best overall individual profit and it is the best way to insure a fair share between multiple users in any CR system. However, does the cooperative game model works in an actual CR system?

In practical CR environment, the communication between competitors (*i.e.,* players) is very hard to achieve. Individual users tend to contact the PU and ask for service [49], users can only observe the pricing function form the PU, but not the strategies and profits of other users. Nevertheless, achieving a cooperative scheme between the SUs (either, the PU forces the SU to get a fair share or using the model mentioned earlier) would improve both the seller and users revenue. Let us use the same assumption used in the previous section, where a PU have a 30MHz of free spectrum to offer to a group of users. The cooperative mode will work when the number of players is relatively small, so each player can discuss a fair share with the rest of the players. However, when the number of SUs increases, let say 20 or more SUs, the cooperative mode will not be useful anymore. If the PU or the users in such a scenario would decide to use the cooperative mode, the individual profit and share will be very low as compared to competitive game, taking into account the channel quality, user need and priority.

In order to solve such a problem, two solutions are proposed in the following sections. Firstly, a second-price pay-to-bid (or sometimes called as pay-as-bid) sealed auction mechanism is introduced to insure a fair competitive game between SUs. Secondly, reputation-based auction game is introduced as non-cooperative game to assign a SU to be a secondary-PU between other SUs. More details in the following sections:

7.1 Pay-to-Bid competitive auction

The allocation mechanism works as follows, let $W= [w_1, w_2, \ldots, w_n]$ be the non-negative bids (*i.e.*, user valuation) that the SU will pay in order to get a share of the offered spectrum and let $X= [x_1, x_2, \ldots, x_n]$ be the amount of the spectrum per unit bandwidth they are allocated as a result. We assume that the PU will announce the auction per unit bandwidth, for example the SUs will offer a bid for every 1MHz they will be allocated.
This allocation is made according to a cost-based allocation mechanism τ, so that with the given payment w, the allocation to SU i is given by $x_i = \tau_i (w)$, as shown in Figure 6-9. c will be assumed to be the reserved price of the PU, any SU bidding less than that will be withdrawn from the auction.
In order to reflect user i's valuation of the offered spectrum, a simple valuation function is proposed:

$$v_i = I_s \times up_i \tag{18}$$

Where v_i is user i's valuation to the offered spectrum per unit bandwidth, and up_i defines how much the user needs to get the desired share of the spectrum, which is a function of user traffic priority (tp_i) and the channel SNR (γ_i);

$$up^i= tp_i \times \gamma_i \tag{19}$$

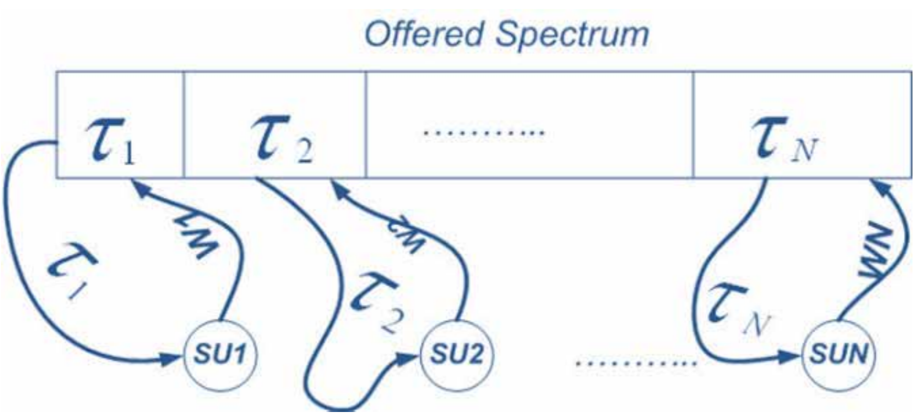

Fig. 9. Pay-to-bid allocation mechanism.

The user valuation can be interpreted that user i uses the importance of his traffic and the channel quality (already known to all users) as a ruler to set his bid in the auction. This valuation measures the SU (if he wins the auction) capabilities to bid more for the offered spectrum keeping in mind the capacity of his channel. We can see that when the channel condition is good (according to equation (3)), the user will be more willing to increase his bid. As a result, a higher bid would be expected from him/her and vice versa.
We must mention that the auction mechanism is designed in such a way that v_i does not represent the real price that an SU has to pay during the auction. Simply it is an interpretation of the strategic situation that a node is facing. In fact v_i reflects the relationship between the user valuation and the channel condition. Additionally, since the

channel coefficient k is a random variable with a known distribution to each user, the distribution of the valuation v_i is also known (according to their relationship shown in equation (16)). This means that v_i lies in the interval $[v_{min}, v_{max}]$. We defined *Bid* as the bid space in the auction, $\{bid_1, bid_2, \ldots, bid_N\}$, which represent the set of possible bids submitted to the PU. We can simply assign bid_0 to zero without loss of generality, as it represents the null bid. Accordingly, bid_1 is the lowest acceptable bid, and bid_N is the highest bid. The bid increment between two adjacent bids is taken to be the same in the typical case. In the event of ties (i.e. two bidders offer the same final price), the object would be allocated randomly to one of the tied bidders.

To find the winner of the first-price sealed-bid pay-to-bid auction, a theoretical model is defined based on the work of [52]. The probability of detecting a bid bid_i is denoted as ξ_1, the probability of not participating in the named auction will be denoted as ξ_0. Then the vector ξ, which equals to $(\xi_1, \xi_2, \ldots, \xi_N)$, denotes the probability distribution over *Bid*, where ($\sum_{i=0}^{N} \xi_i = 1$). Now we introduce the cumulative distribution function, which is used to find out whether a user i will bid with bid_i or less, $\sum_{j=0}^{i} \xi_j = \xi$, all of them are collected in the vector ξ. Then, any rational potential bidder with a known valuation of v_i faces a decision problem of maximizing his expected profit from winning the auction; *i.e.;*

$$\max_{<bid_i \in Bid>} (v_i - bid_i) Pr(winning \mid bid_i) \quad (20)$$

The equilibrium probability of winning for a particular bid b_i is denoted as θ_i, and these probabilities are collected in ϑ, $(\vartheta_0, \vartheta_1, \vartheta_2, \ldots, \vartheta_n)$. Using ξ, the elements of the vector ϑ can be calculated. We can easily find that ϑ_0 is known to be zero, as if any bidder submitted a null bid to the source, he is not going to win. We can calculate the remanning elements of ϑ as it can be directly verified that the following constitute a symmetric, Bayes-*Nash equilibrium* [53] of the auction game:

$$\varphi_i = \frac{\xi_i^n - {\xi_{i-1}}^n}{n(\xi_i^n - {\xi_{i-1}}^n)} \quad \forall i = 0,1,2,\ldots,n \tag{21}$$

We used the notation of Bayes-*Nash equilibrium* as defined in [53], there approach is to transform a game of incomplete information into one of imperfect information, and any buyer who has incomplete information about other buyers' values is treated as if he were uncertain about their types. From equation (21), we can see that the numerator is the probability that the highest bid is exactly equal to bid_i, while the denominator is the expected number of users how are going to submit the same bid (*i.e., bid_i*). For any user in the game, the best response will be to submit a bid which satisfies the following inequality;

$$(v_i - bid_i)\vartheta_i \geq (v_j - bid_j)\vartheta_j \qquad \forall j \neq i$$

The above inequality shows that user i's profit is weakly beat any other user j's profit. The above inequality is the discrete analogue to the equilibrium first-order condition for expected-profit maximization in the continuous-variation model [52], which takes the form of the following ordinary differential equation in the strategy function $\varnothing(v_i)$;

$$\varnothing'(v_i) + \varnothing(v_i)\frac{(n-1)f(v_i)}{F(v_i)} = v_i \frac{(n-1)f(v_i)}{F(v_i)} \tag{22}$$

Where $f(v_i)$ and $F(v_i)$ are the probability density and cumulative distribution function of each bidder valuation respectively. We assume that they are common knowledge to bidders along with n, the number of bidders in the system. The reserve price is denoted by c, (In many instance, sellers reserve the right not to sell the object if the price determined in the auction is lower than some threshold amount [53], say $c > 0$), and the above differential equation has the following solution;

$$\varnothing(v_i) = v_i - \frac{\int_r^{v_i} F(u)^{n-1} du}{F(v_i)^{n-1}} \tag{23}$$

In the case of the first-price sealed-bid auction, the bidder i will submit a bid of $bid_i = \varnothing(v_i)$ in equilibrium and he will pay a proportional price to his bid if he wins. On the other hand, for the second-price sealed-bid auction, a user I will submit his valuation truthfully. This is because the price a user has to pay if he wins the auction is not the winning bid but the second highest one. Therefore, there is nothing to drive a user to bid higher or lower than his true valuation to the data offered by the server. In this case, $bid_i = v_i$, shown in equation (18), and the payment process is the same as in the first-price auction. Once the winner has been announced, the PU will send an update message to all the SUs with the second highest price they need to pay in order to gain access. All SUs must pay the winning bid per unit bandwidth. To insure that the winner will get a higher priority than the rest of competitors, PU will send the winning bid to everyone and treat their replies according to the first bid was offered by the SUs in the first place.

This mechanism will offer a better competition in terms of fairness between players, the user with a better channel quality, a higher priority traffic and honest valuation will get a much better chance than other users to gain access to his/her desired share. Moreover, the named mechanism will improve the seller and winners revenue as compared to the optimal and cooperative game models.

Finally, next we will test the named mechanism with similar scenario assumptions as in the previous section. We are comparing three models; first, when the spectrum is offered to the users using a cooperative game. Second, using a similar setting but with a competitive game and finally a competitive second-price pay-to-bid sealed auction. We will study the effects in two simple scenarios; one, a SU (named u_1) who is competing with other bidders to get a share of the spectrum since the PU announce the auction. Two, a new comer is joining the game (the newcomer will join the game as the eleventh user onward) and how the introduced mechanism will improve his/her revenue, taking into account that the new comer has an excellent channel quality and a fair bid.

Figure 10, proofs what we discussed in section 6.1.3 in terms of individual user revenue. Although the cooperative games shows a better start (*i.e.,* when the number of bidders is low), the cooperative game tries to improve the player's revenue and keep a fair share between all bidders. This would cause a sharp decrease in the seller revenue when the number of bidders increases. On the other the competitive game takes into account the channel condition and the user ability to grab his/her share before the others, that's why it shows better revenue when compared to the cooperative model.

For the second scenario, Figure 11 shows the dramatic improvement in the newcomer revenue; keeping in mind that his/her priority is rather high. Clearly, the introduced mechanism helped in improving spectrum share in terms of fairness, massively improving the players' revenue when compared to the other models and gives the PU a better deal by using the second-price sealed-auction.

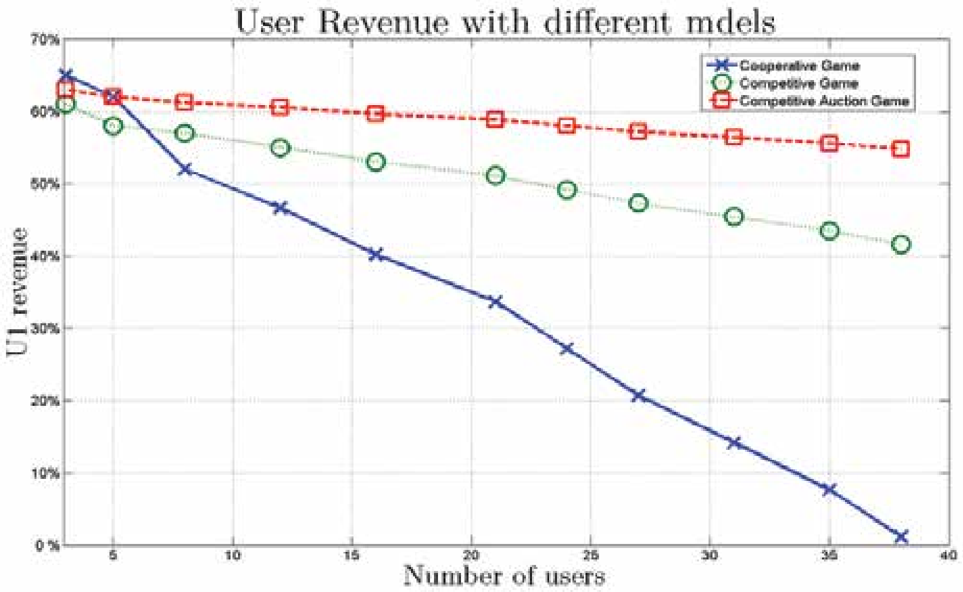

Fig. 10. SU revenue vs. number of users with different models.

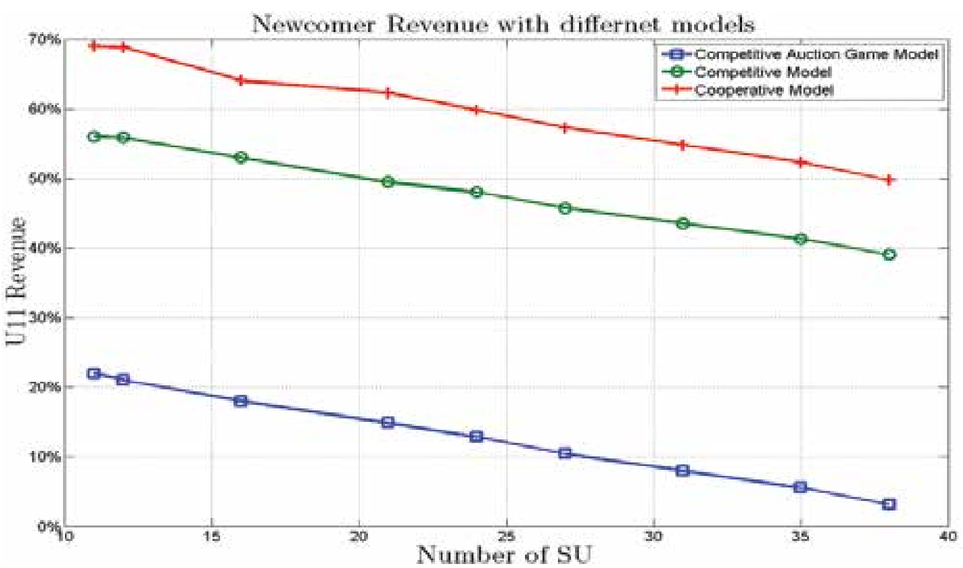

Fig. 11. Newcomer revenue vs. number of users.

7.2 Reputation-based non-cooperative auction games

With this game, PU will assign the spectrum to the winner of the second-price sealed auction process. The revenue of the PU will not change, as using the second-price auction insures that all bidders will bid around the real value of the offered spectrum. The winner of the auction will be a new PU between the rest of the SUs, and will have the right to decide whether to share the spectrum with the rest or not. However, a penalty factor is introduced to insure that not only paying more will guarantee a share of the spectrum but also reputation will be combined with each bid. This factor will be forwarded to the PU and will show whether the winner of the last auction was popular or not, which is done by helping other SUs to share the offered spectrum.

In this section we will represent the infinitely repeated version of game G by G^{∞} (*i.e. this is the case when G is going to be played over and over again in successive time periods*). We are assuming that the PU is offering a single frequency band to be shared by other SU's. However, if the PU is planning to offer more bands then the proposed mechanism must be repeated for the other bands between the secondary users. We will define the user reputation as R which will depends on user performance during any time period t as well as in prior time periods. Reputation of player i in some time period t is denoted by R_t^i. Formally, we define node reputation as follows:

$$R_t^i = R_{t-1}^i(1-\propto) + w \times \propto \quad 0 \leq \propto \leq 1, \quad t \geq 2 \tag{24}$$

Where $\propto$ is the history of the user, it depends on the user reputation in the previous periods according to user behaviour. "w" is equal to "1" when player i at time t is interested in sharing the offered spectrum and "0" otherwise. Therefore, 0 ≤ R_t^i ≤ 1, i.e. the reputation value of each player varies between "0" and "1" (including) ($R_t^i \in [0,1]$). Moreover, the reputation value of all players is equal to "0" when t = 0. A high value of $\propto$ means the more importance is assigned to a player's need in sharing the spectrum with the PU (higher priority) during the current period than its previous need record, and vice versa. Thus, when $\propto$ is high, a user with even low reputation value in the current time period t, can significantly improve his/her reputation when it realises that it needs a better share of the spectrum.

As was defined the Nash equilibrium case earlier, the evaluation of the Nash equilibrium of the repeated game G^{∞} will be engaged. By finding the Nash equilibrium of G^{∞} it leads to the deduction of the Nash equilibria of G. The proposed incentive mechanism is based on a player's links reputation R. The benefit of which is that a player draws from the system to its contribution, the benefit is a monotonically increasing function of a player's contribution. Thus, this is a non-cooperative game among the players, where each player with high priority traffic wants to maximize his/her utility. The classical concept of Nash equilibrium points a way out of the endless cycle of speculation and counter-speculation as to what strategies the players should use. The intent is to deduce a symmetric Nash equilibrium because all the players belong to the same population/network (i.e., assume the same role) and it is therefore easier (i.e., require no coordination among players) to achieve such an equilibrium. If the players in a game either do not differ significantly or are not aware of any differences among themselves (i.e., if they are drawn from a single homogeneous population) then it is difficult for them to coordinate and a symmetric equilibrium, in which every player uses the same strategy, is more compelling.

The argument of a single homogeneous population implies that all the peers in a CR network have equivalent responsibilities and capabilities as everybody else. We assume that if the player chooses the action {*want to share*}, this will assign him a probability of p, and if the player chooses the action {*does not want to share*}, this will assign one a probability of $1 - p$.

It must be mentioned that in the action profile, a time and money saving Nash equilibrium case is defined, if all players choose the action {*does not want to share*}. As this will mean that, players are not interested in sharing the spectrum for the entire communication time. That is to say, users have low priority traffic and accessing the spectrum will be by chance, players will not compete to send their data and will not offer more money to the PU to get the spectrum. If any other player *i* decided to switch to the action {*want to share*}, its payoff will be – *C* which is less than a payoff of "0" that the node gets when decided not to share the spectrum. An undesirable Nash equilibrium case is generated, if all the players choose the action {*want to share*}. This is easy to see because all nodes will have to compete against each other again, this will waste time and the winner will be the PU, as one of the SU's should pay more to share the offered spectrum.
The expected payoff of any player in period *t* when it selects the action {*want to share*} is:

$$p(-C + R_t^{share} \times U) \tag{25}$$

This payoff is denoted as $Payoff_{share}$, *U* is the nodes utility. Similarly, the payoff for any player selects the action {*does not want to share*} will be:

$$(1-p)(R_t^{dontshare} \times U) \tag{26}$$

This will be denoted as $payoff_{don'tshare}$. It is easy to show that the term $R_t^{share} \times U$ captures the notation that the probability of SU becoming a secondary PU by sharing the offered spectrum is directly proportional to node's reputation.
R_t^{share} is player *i* reputation when he/she wants to share the offered spectrum at time *t* (*i.e.* *w* = 1 in equation (24)), and $R_t^{don'tshare}$ is player *i* reputation when he/she decides to take the action {*does not want to share*} at the same time period *t* (i.e. *w* = 0 in equation (24)), from equation (24), we can get:

$$R_t^{share} = R_{t-1}(1-\propto) + \propto$$

and

$$R_t^{dontshare} = R_{t-1}(1-\propto) \tag{27}$$

Generally, each player's expected payoff in equilibrium is his/her expected payoff to any of its actions that he/she uses with positive probability. The above useful characterization of mixed-strategy Nash equilibrium yields to:

$$payoff_{share} = payoff_{don'tshare} \tag{28}$$

Using equations 6-25, 6-26, and 6-27;

$$p(-C + (R_{t-1}(1-\propto) + \propto) \times U) = (1-p)(R_{t-1}(1-\propto) \times U) \tag{29}$$

Solving equation 9 to get the final value of *p*;

$$p = \frac{R_{t-1} \times U \times (1-\propto)}{-C + 2R_{t-1} \times U \times (1-\propto) + U \times \propto} \tag{30}$$

It must be mentioned that the value p obtained above is not a constant, but varies in each time interval depending upon a node's reputation at the end of the previous time interval *t -1*.

Finally, the mixed strategy pair (p, $1 - p$) for actions { *want to share*, *does not want to share*} respectively, is a mixed strategy Nash equilibrium for the players (i.e. nodes in the network). Assuming no collusion among nodes, if all the other nodes follow the above strategy, then the best strategy for any node is to also to follow one of the above strategies. Actually, this is a symmetric mixed strategy Nash equilibrium for any G, as well as G^{∞}. In fact, it is a more stable equilibrium than the one in which no node is interested in sharing the offered spectrum. This is caused by two reasons. First, when none of the SUs is interested in sharing the spectrum, the network is not useful to any user. Second, in real-time scenarios, users that derive finite utility from altruism would always send some messages irrespective of how much they obtain in return. Therefore, it is unlikely to have a scenario in which no node is looking to contact the PU to share the spectrum.

7.3 Properties of the proposed Nash Equilibrium

In this section, we will present some of the interesting properties of the Nash equilibrium derived in the section above

7.3.1 Simplicity of calculating the Nash Equilibrium

In section '6.7.2', we have calculated the probability of achieving the equilibrium point between the SUs. This was based on which node will decide to share the spectrum with the PU and become a secondary PU. In each round of the game (or time period t) players decide whether they should ask to share the offered spectrum or not, based on their reputation at the end of the prior time period. This probability, as one can see, does not remain constant from one period to another. Moreover, it depends on a player's reputation at the end of the last time period. Players can calculate their reputation using equation (24), since they know precisely their actions at each round of the game. Thus, determining the Nash equilibrium strategy is fairly straightforward for any player. However, it must be noted that there is an inherent assumption that nodes are serviced based on their current reputation.

Figure 12, shows how players' reputations change in every time interval depending on their Nash strategy. At the beginning of the communication time, both, player 1 and 2 are competing with each other to guarantee access to the offered spectrum. However, player 1 uses the spectrum but at the same time managed to help player 2 (i.e. player 1 will be the secondary PU and will manage the access of players 2 and 3 to the offered spectrum). Player 3 shows his interest in the offered spectrum after the third time interval, and managed to use the spectrum once both player 1 and 2 finished using it or they are not interested anymore in sharing it. The figure shows the players (nodes) reputation values $0 \le R_t^i \le 1$ over ten time intervals.

On the other hand, Figure 13 below shows the same result but over a longer time period, around nine hundred time intervals. Similarly, three nodes are competing with each other, player one with the highest reputation and player three with the lowest. Player 1 will act as the secondary PU over the other two users (i.e. player 2 and 3). In this figure we used a random matrix generator to show different reputations when player 1 is interested to share the spectrum for 80% of the time, player 2 for 50% of the time and player 3 for 8% of the time only.

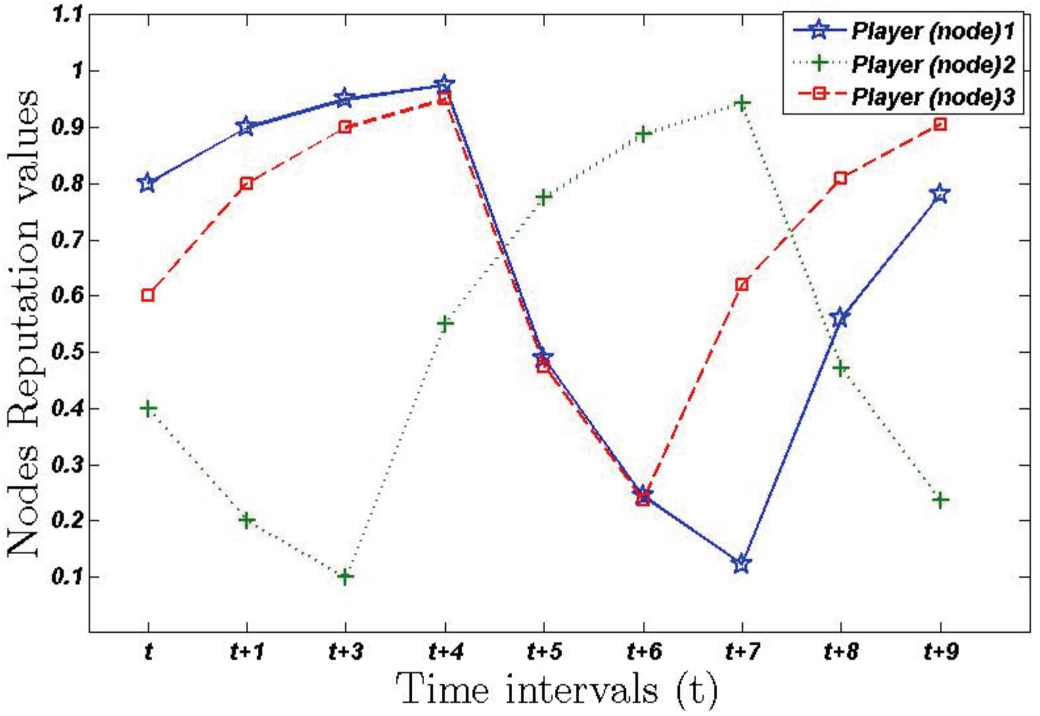

Fig. 12. Change in player's reputation controlled by their Nash equilibrium strategies.

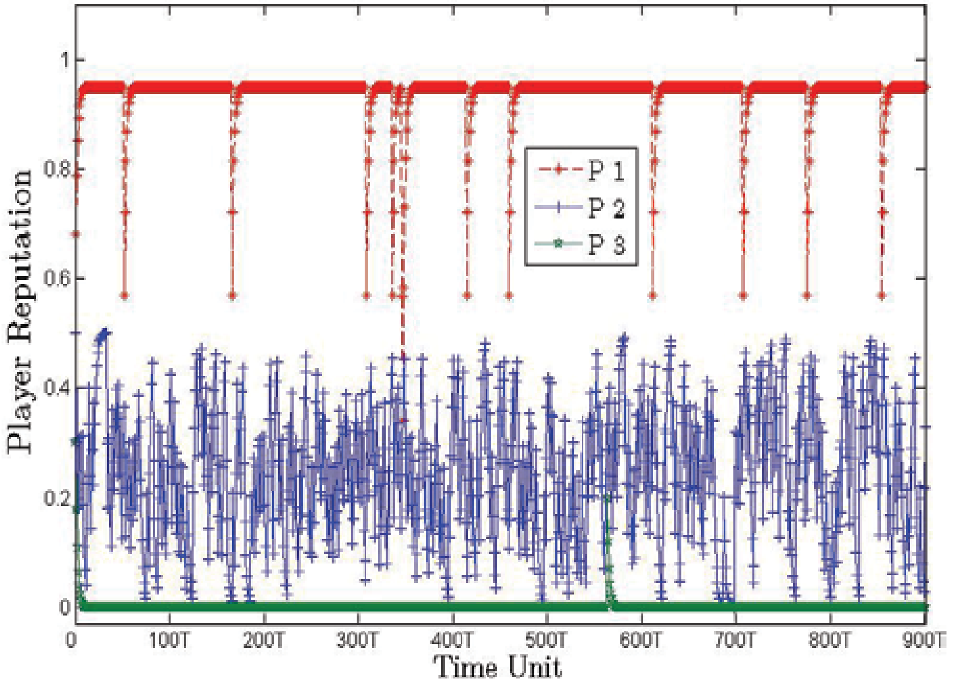

Fig. 13. Changing player reputation over a longer time period.

7.3.2 Addressing the spectrum to the right user

The simple game theoretic model presented in the previous sections, wherein node reputation is used as a basis for deciding who will share the offered spectrum, predicts that it is in every peer's best interest to serve others. This includes the nodes that are not interested to share the spectrum at the current time period. Our simulations support this behaviour as we found that the total service received by a node is balanced by the total service that it has to offer to others, as shown in Figure 12.

7.3.3 Addressing the problem of competitive sharing

An important property of the equilibrium emerges from equation (30) that predicts the probability with which one node will be a secondary PU and it should serve others. If we set the value of C in away such that, $C <<< U$ (i.e. C can be ignored from equation (6-30)), then equation (6-30) becomes:

$$p = \frac{R_{t-1}(1-\propto)}{2R_{t-1}(1-\propto)+\propto} \tag{31}$$

That would lead us to the conclusion that $p < 0.5$. Then, Nash equilibrium of the proposed game predicts that players should help each other less than 50 percent of the time when PU offers the spectrum. This, although it appears to be very restrictive, is a consequence of the fact that all nodes are selfish and are better off trying to share the spectrum than serving others. Intuitively, if a node knows that everyone else in the network behaves selfishly, i.e., provide as little service as possible, then the best strategy for the named node cannot be to serve others most of the time (i.e., with probability greater than 0.5).

7.3.4 Fairness and equal sharing of cost and spectrum

We concluded from the previous section that serving with a priority of less than 50 percent (i.e. when $C <<< U$) is an optimal point, the observer can notice that the overall system efficiency is severely reduced. This is because most of the nodes in the network act selfishly and at least half of the service requests from other nodes are not fulfilled. On the other hand, this equilibrium strategy provides fairness in the sense that the cost of system inefficiency is not burn by a single node (i.e. has one positive side), but it is shared among all nodes. This is because each node's request is likely to be turned down by the serving node (i.e. selfish secondary PU). In this work, we assume that if a node's request at one node is turned down, the node tries at some other candidate node capable of serving the request. On average, the probability that a node's request is successfully served in a time period is proportional to its current reputation.

7.3.5 Decreasing α for a better share of the spectrum

Figure 14 shows the effects of $\propto$ on the reputation probability of the nodes in the case where the node is not interested in sharing the spectrum. On the other hand, the node in figure 15 is looking to keep its share of the spectrum (derived from equation (27)).

As can be seen from Figures 14 and 15, a lower value of α shifts the reputation probability curve upwards. However, that all depends on whether the node is interested in using the offered spectrum or not. If the node is looking to give its share of the spectrum to other nodes, a low value of $\propto$ will gradually help the node to lose its share, however a high value of $\propto$ will guarantee a faster release of the spectrum. This is true for Figure 15 as well, which

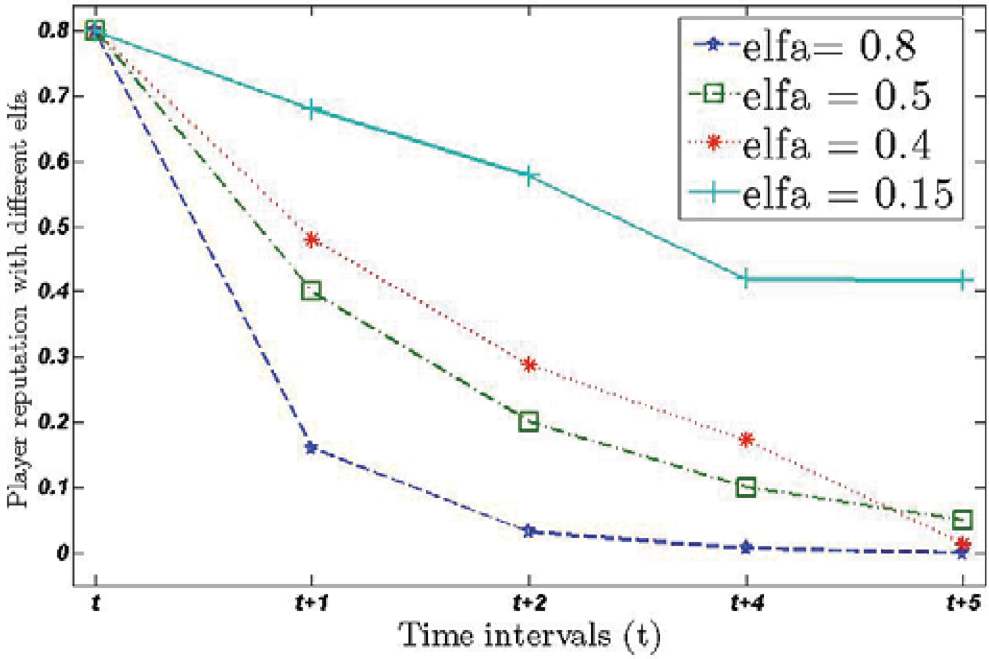

Fig. 14. Players reputation with respect to α and the node is not interested in sharing the offered spectrum.

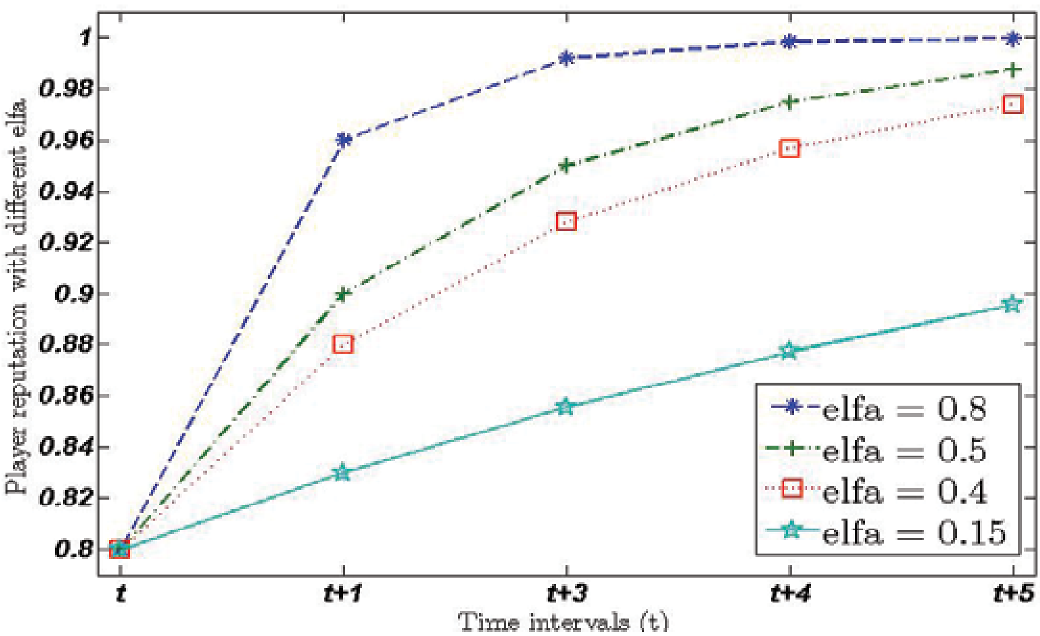

Fig. 15. Players reputation with respect to α and the node is definitely interested in sharing the offered spectrum from the PU.

is to be expected since $\propto$ determines how much importance is given to a node's current performance as compared to its past service record. A low value of $\propto$ (i.e., giving more importance to nodes past actions up to the current time period *t)* means that nodes need to continually provide service to be able to maintain high reputation and access spectrum offered from the PU. If however $\propto$ is high, nodes can easily increase their reputation in any period in which they provide service to other nodes. This is irrespective of how cooperative they have been in the past with regards to providing service to others. Therefore a simple way to improve the system efficiency is to set $\propto$ as low as possible.

8. Summery

Cognitive radio is regarded as the key technology for next generation of wireless network. Dynamic spectrum sharing is one of the most important problems related to Cognitive Radio networks. Based on the competitive spectrum sharing on game theory, an adaptive competitive game and auction-based spectrum sharing mechanism is presented in this chapter. The advantages over the optimal, cooperative and competitive modes have been proved by simulation. A general solution for the instability problem has been proposed and an adaptive method is used for the changing number of secondary users by using cooperative game model when the number of users is small. Another solution to such a problem is presented by using a non-cooperative game model combined with second-price auction to choose a secondary primary user. The decision is based on user reputation and user's valuation of the offered spectrum. We have the solution with maximum total profit and better fairness in spectrum sharing. We have discussed how the increase of competitors would affects the fairness of spectrum sharing and proved that the proposed mechanism offers better revenue to the seller and the bidders in terms of fairness.

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3

Game Theory in Wireless Ad- hoc Opportunistic Radios

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1. Introduction

In this chapter we explain how we use game theory application in wireless communication ad-hoc network. The application of mathematical analysis to the study of wireless communication ad hoc networks has met with limited success due to the complexity of mobility, traffic models and the dynamic topology. A scenario based UMTS TDD opportunistic cellular system with an ad hoc behaviour that operates over UMTS FDD licensed cellular network is considered. We describe how ad hoc opportunistic radio can be modeled as a game and how we apply game theory based Power Control in ad-hoc opportunistic radio

2. Game theory

Game theory is a field of applied mathematics that describes and analyzes interactive decision situations. It provides analytical tools to predict the outcome of complex interactions among rational entities, where rationality demands strict adherence to a strategy based on perceived or measured results. The main areas of application of game theory are economics, political science, biology and sociology. From the early 1990s, engineering and computer science have been added to this list. We limit our discussion to non-cooperative models that address the interaction among individual rational decision makers. Such models are called "games" and the rational decision makers are referred to as "players." In the most straightforward approach, players select a single action from a set of feasible actions. Interaction between the players is represented by the influence that each player has on the resulting outcome after all players have selected their actions. Each player evaluates the resulting outcome through a payoff or "utility" function representing her objectives.

There are two ways of representing different components (players, actions and payoffs) of a game: normal or strategic form, and extensive form. Here we will focus on the normal form representation.

Formally, a normal form of a game G is given by

$$G = \{ N, A, \{u_i\}\} \tag{1}$$

where $N=\{1,2,...,n\}$ is the set of players (decision makers), A_i is the action set for player i, $A = A_1 \times A_2 \times,...,\times A_n$ is the Cartesian product of the sets of actions available to each player, and

$\{u_i\}=\{u_1,\ldots,u_n\}$ is the set of utility functions that each player i ,wishes to maximize, where $u_i: A\rightarrow \mathbf{R}$. For every player i, the utility function is a function of the action chosen by player i, a_i and the actions chosen by all the players in the game other than player i, denoted as $\mathbf{a}_{-i}$. Together, a_i and $\mathbf{a}_{-i}$ make up the action tuple $\mathbf{a}$. An action tuple is a unique choice of actions by each player. From this model, steady-state conditions known as *Nash equilibria* can be identified. Before describing the Nash equilibrium we define the best response of a player as an action that maximizes her utility function for a given action tuple of the other players. Mathematically, $\bar{a}$ is a best response by player i to $\mathbf{a}_{-i}$ if

$$\bar{a} \in \{\arg\ \max u_i\ (a_i, \mathrm{a}_{-i})\} \tag{2}$$

Nash equilibrium (NE) is an action tuple that corresponds to the mutual best response: for each player i, the action selected is a best response to the actions of all others. Equivalently, a NE is an action tuple where no individual player can benefit from unilateral deviation. Formally, the action tuple

$\mathbf{a}^* = (a_1^*, a_2^*, a_3^*, \ldots, a_n^*)$ is a NE if $u_i(a_1^*, \mathbf{a}_{-i}^*) \geq (a_1^*, \mathbf{a}_{-i}^*)$ for all $\forall\ a_i \in A_\mathrm{i}$ and for all $\forall\ i \in N$. (3)

The action tuples corresponding to the Nash equilibria are a consistent prediction of the outcome of the game, in the sense that if all players predict that Nash equilibrium will occur then no player has any incentive to choose a different strategy. There are issues with using the Nash equilibrium as a prediction of likely outcomes (for instance, what happens when multiple such equilibria exist?). There are also refinements to the concept of Nash equilibrium tailored to certain classes of games. A detailed discussion of these is outside the scope of this deliverable. There is no guarantee that a Nash equilibrium, when one exists, will correspond to an efficient or desirable outcome for a game (indeed, sometimes the opposite is true). Pareto optimality is often used as a measure of the efficiency of an outcome. An outcome is Pareto optimal if there is no other outcome that makes every player at least as well off while making at least one player better off.
Mathematically, we can say that an action tuple

$\mathbf{a} = (a_1, a_2, a_3, \ldots, a_n)$ is Pareto optimal if and only if there exists no other action tuple

$\mathbf{b} = (b_1, b_2, b_3, \ldots, b_n)$ such that $u_i(\mathbf{b}) \geq (\mathbf{a})$ for $\forall\ i \in N$, and

for some $k \in N$ $u_k(\mathbf{b}) \geq u_k(\mathbf{a})$.

3. Game theory in wireless communication

There is a significant amount of work in wired and wireless networking that make use of game theory. The strategic situations in wireless networking the players have to agree on sharing or providing a common resource in a distributed way, our approach focuses on the theory of *non-cooperative games*.
Cooperative games require additional signalization or agreements between the decision makers and hence a solution based on them might be more difficult to realize. In a non-cooperative game, there exist a number of decision makers, called *players*, who have potentially conflicting interests. In the wireless networking context, the players are the *users* or *network operators* controlling their devices. In compliance with the practice of game theory, we assume that the players are *rational*, which means that they try to maximize their *payoffs* (or utilities). This assumption of rationality is often questionable, given, for example,

the altruistic behaviour of some animals. Herbert A. Simon was the first one was to question this assumption and introduced the notion of *bounded rationality* . But, we believe that in computer networks, most of the interactions can be captured using the concept of rationality, with the appropriate adjustment of the payoff function. In order to maximize their payoff, the players act according to their *strategies*. The strategy of a player can be a single *move* or a set of moves during the game.

We take an intuitive top-down approach in the protocol stack to select the examples in wireless networking as follows. Let us first assume that the time is split into *time steps* and each device can make one move in each time step.

In the first game called the *Forwarder's Dilemma*, we assume that there exist two devices as players, p_1 and p_2. Each of them wants to send a packet to his destination, dst_1 and dst_2 respectively, in each time step using the other player as a forwarder. We assume that the communication between a player and his receiver is possible only if the other player forwards the packet. We show the Forwarder's Dilemma scenario in Figure 1. If player p_1 forwards the packet of p_2, it costs player p_1 a fixed *cost* $0 < C << 1$, which represents the energy and computation spent for the forwarding action. By doing so, he enables the communication between p_2 and dst_2, which gives p_2 a *benefit* of 1. The payoff is the difference of the benefit and the cost. We assume that the game is symmetric and the same reasoning applies to the forwarding move of player p_2. The dilemma is the following: Each player is tempted to drop the packet he should forward, as this would save some of his resources; but if the other player reasons in the same way, then the packet that the first player wanted to send will also be dropped. They could, however, do better by mutually forwarding each other's packet. Hence the dilemma.

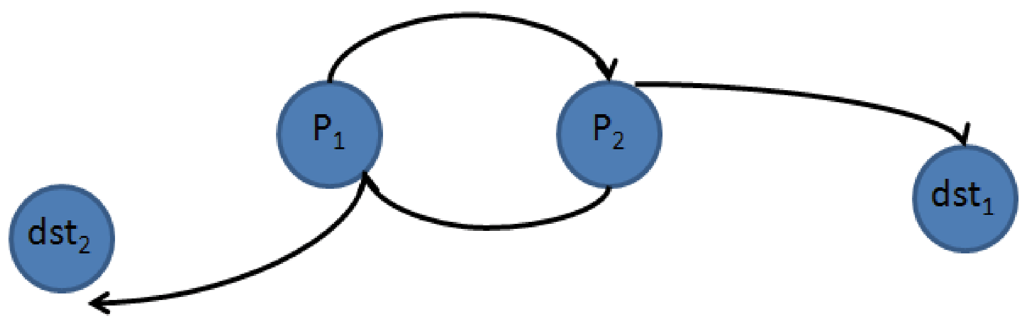

Fig. 1. The network scenario in the Forwarder's Dilemma game.

In the second example, called *Joint Packet Forwarding Game*, we present a scenario, in which a source *src* wants to send a packet to his destination *dst* in each time step. To this end, he needs *both devices* p_1 and p_2 to forward for him. Similarly to the previous example, there is a forwarding cost $0 < C << 1$ if a player forwards the packet of the sender. If both players forward, then they each receive a benefit of 1 (e.g., from the sender or the receiver). We show this packet forwarding scenario in Figure 2.

The third example, called *Multiple Access Game*, introduces the problem of medium access. Suppose that there are two players p_1 and p_2 who want to access a shared communication channel to send some packets to their receivers re_1 and re_2. We assume that each player has one packet to send in each time step and he can decide to access the channel to transmit it or to wait. Furthermore, let us assume that p_1, p_2, re_1 and re_2 are in the power range of each

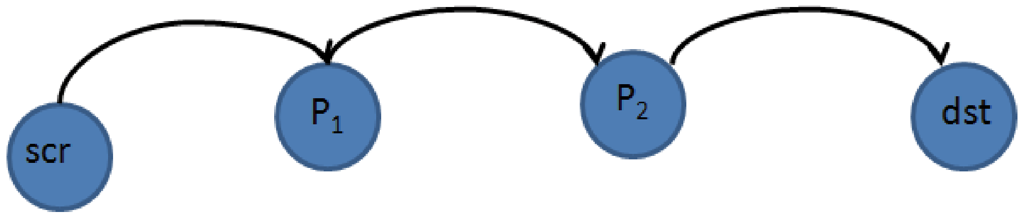

Fig. 2. The Joint Packet Forwarding Game.

other, hence their transmissions mutually interfere. If player p_1 transmits his packet, it incurs a sending cost of $0 < C << 1$. The packet is successfully transmitted if p_2 waits in that given time step (i.e., he does not transmit), otherwise there is a collision. If there is no collision, player p_1 gets a benefit of 1 from the successful packet transmission. The framework presented by Cagalj *et al.* in is a generalized version of the Multiple Access Game.

In the last example, called the *Jamming Game*, we assume that player p_1 wants to send a packet in each time step to a receiver re_1. In this example, we assume that the wireless medium is split into two channels x and y according to the Frequency Division Multiple Access (FDMA) principle. The objective of the *malicious* player p_2 is to prevent player p_1 from a successful transmission by transmitting on the same channel in the given time step. In wireless communication, this is called *jamming*. Clearly, the objective of p_1 is to succeed in spite of the presence of p_2. Accordingly, he receives a payoff of 1 if the attacker cannot jam his transmission and he receives a payoff of -1 if the attacker jams his packet. The payoffs for the attacker p_2 are the opposite of those of

player p_1. We assume that p_1 and re_1 are synchronized, which means that re_1 can always receive the packet, unless it is destroyed by the malicious player p_2. Note that we neglect the transmission cost C, since it applies to each payoff (i.e., the payoffs would be $1\text{-}C$ and $-1\text{-}C$) and does not change the conclusions drawn from this game.

The Jamming Game models the simplified version of a game-theoretic problem presented by Zander .We deliberately chose these examples to represent a wide range of problems over different protocol layers (as shown in Figure 3). There are indeed fundamental differences between these games as follows. The Forwarder's Dilemma is a symmetric *nonzero-sum game*, because the players can mutually increase their payoffs by cooperating (i.e., from zero to $1\text{-}C$). The conflict of interest is that they have to provide the packet forwarding service for each other. Similarly, the players have to establish the packet forwarding service in the Joint Packet Forwarding Game, but they are not in a symmetric situation anymore. The Multiple Access Game is also a nonzero-sum game, but the players have to share a common resource, the wireless medium, instead of providing it. Finally, the Jamming Game is a *zero-sum game* because the gain of one player represents the loss of the other player. These properties lead to different games and hence to different strategic analyses.

3.1 Cognitive radio

In information times, the increase of wireless equipments makes the spectrum to be the most essential and important resources. Now the wireless networks are regulated by a fixed spectrum assignment policy. However, according to Federal Communications Commission (FCC), a large portion of the assigned spectrum is used sporadically and geographically, so the serious problem is the inefficiency usage. This restriction of the tradition spectrum policy necessitates a new technology to exploit the spectrum available opportunities which is called—cognitive radio.

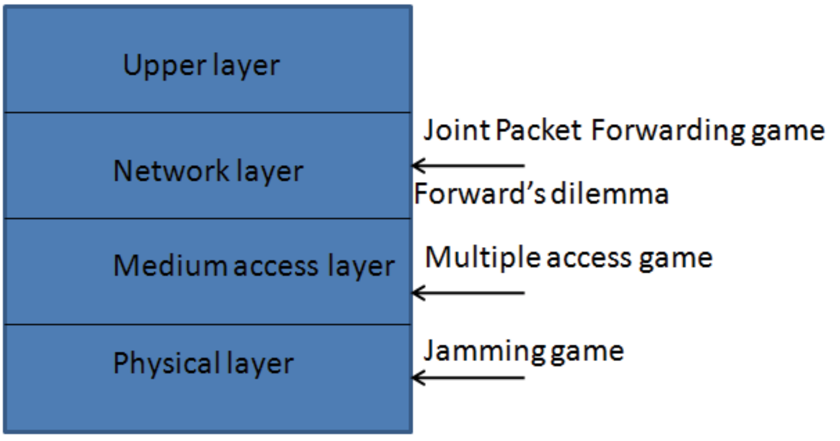

Fig. 3. The classification of the examples according to protocol layers.

A "cognitive radio" is a radio that can change its transmitter parameters base on interaction with the environment in which it operates . It is characterized by cognitive capability and reconfigurability. The cognitive capability refers to the capture and sense of the information from the radio environment by monitoring the power and
capturing the temporal and spatial variations. The reconfigurability enables the radio to be dynamically programmed by the radio knowledge representation language (RKRL) to select the best spectrum and appropriate operating parameters. Therefore, the cognitive radio can enhance the flexibility through the cognitive cycle, which has three main steps: radio-scene analysis, channel state estimation and predictive modeling, transmit power control and spectrum management . The cognitive cycle is pictured in Figure 4.

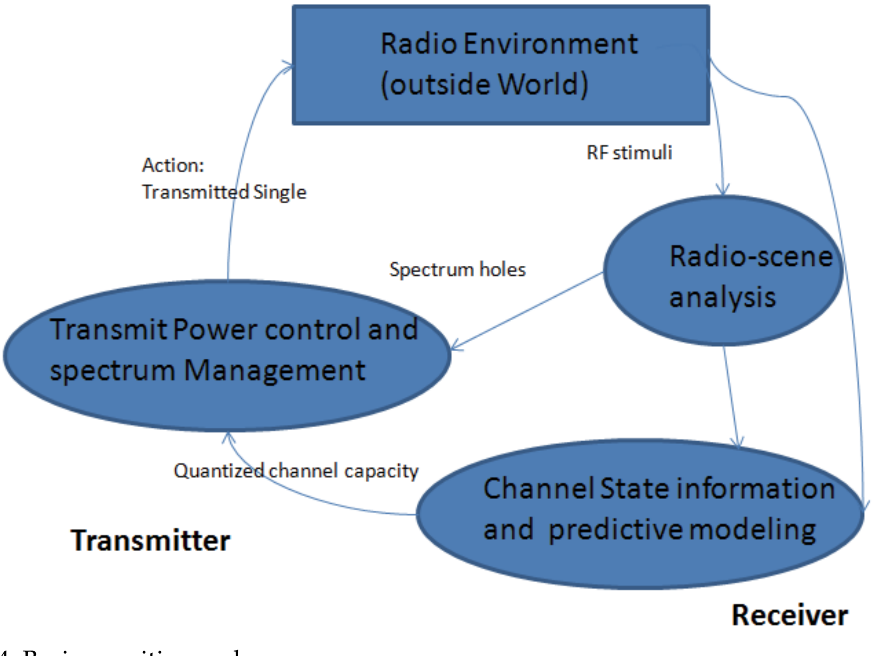

Fig. 4. Basic cognitive cycle

Transmit-power control is necessary for the cognitive radio system to broaden the scope of its applications and enhance the performance. It would have to operate under two limitations on network resources: the interference temperature limit imposed by regulatory agencies, and the availability of a limited number of spectrum holes depending on usage. In a multiuser cognitive radio environment, all the users operate in a decentralized manner; they are characterized by cooperation and competition. In such a case, information theory and game theory could be applied to exercise control over the transmit power.

4. Game theory in wireless ad- hoc opportunistic radios

Wireless communications play a very important role in military networks and networks for crisis management, which are characterised by their ad hoc heterogeneous structure. An example of a future network can be seen in Figure 5. This illustrates a range of future wireless ad hoc applications. In the heterogeneous ad hoc network, it is difficult to develop plans that will cope with every eventuality, particularly hostile threats, due to the temporary nature. Thus, dynamic management of such networks represents the ideal situation where the new emerging fields of cognitive networking and cognitive radio can play a part. Here we assume a cognitive radio 'is a radio that can change its transmitter parameters based on interaction with the environment where it operates', and additionally relevant here is the radio's ability to look for, and intelligently assign spectrum 'holes' on a dynamic basis from within primarily assigned spectral allocations. The detecting of holes and the subsequent use of the unoccupied spectrum is referred to as opportunistic use of the spectrum. An Opportunistic Radio (OR) is the term used to describe a radio that is capable of such operation .We use the opportunistic radio system which was proposed that shares the spectrum with an UMTS cellular network. This is motivated by the fact that UMTS radio frequency spectrum has become, in a significant number of countries, a very expensive commodity, and therefore the opportunistic use of these bands could be one way for the owners of the licenses to make extra revenue.

The OR system exploits the UMTS UL bands, therefore, the victim device is the UMTS base station, likely far from the opportunistic radio, whose creates local opportunities. These potential opportunities in UMTS FDD UL bands are in line with the interference temperature metric proposed by the FCC s Spectrum Policy Task Force. The interference temperature model manages interference at the receiver through the interference temperature limit, which is represented by the amount of new interference that the receiver could tolerate. As long as OR users do not exceed this limit by their transmissions, they can use this spectrum band. However, handling interference is the main challenge in CDMA networks, therefore, the interference temperature concept should be applied in UMTS licensed bands in a very careful way.

The UMTS is a DS-CDMA system, thus all users transmit the information spreaded over 5 MHz bandwidth at the same time and therefore users interfere with one another. Figure 6 shows a typical UMTS FDD paired frequencies. The asymmetric load creates spectrum opportunities in UL bands since the interference temperature (amount of new interference that the UMTS BS can tolerate) is not reached.

In order to fully exploit the unused radio resources in UMTS, the OR network should be able to detect the vacant channelization codes using a classification technique. Thus the OR network could communicate using the remaining spreading codes which are orthogonal to the used by the UMTS network. However, classify and identify CDMA's codes is a very computational intensive task for real time applications.

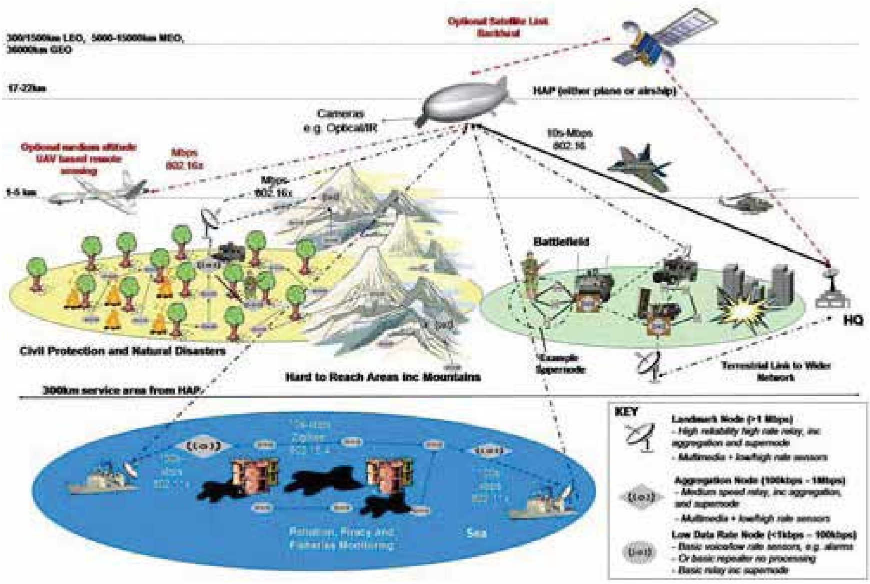

Fig. 5. Ad-hoc future network

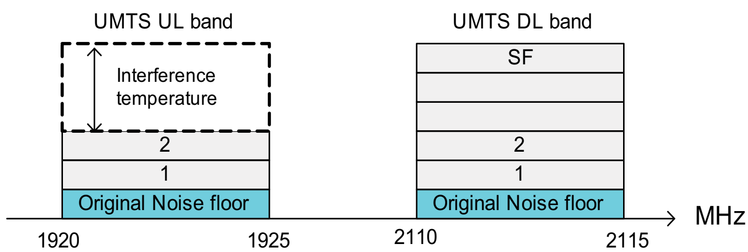

Fig. 6. UMTS FDD spectrum bands with asymmetric load

Moreover, synchronization between UMTS UL signals and the OR signals to keep the ortogonality between codes will be a difficult problem. Our approach is to fill part of the available interference temperature raising the noise level above the original noise floor. This rise is caused by the OR network activity, which aggregated signal is considered AWGN (e.g CDMA, MC-CDMA, OFDM).We consider a scenario where the regulator allows a secondary cellular system over primary cellular networks. Therefore we consider opportunistic radios entities as secondary users. The secondary opportunistic radio system can use the licensed spectrum provided they do not cause harmful interference to the owners of the licensed bands i.e., the cellular operators.Specifically we consider as a primary cellular network an UMTS system and as secondary networks an ad hoc network with extra sensing features and able to switch its carrier frequency to UMTS FDD frequencies. Figure 7 illustrates the scenario where an opportunistic radio network operates within an UMTS cellular system.

We consider an ah hoc OR network of M nodes operating overlapped to the UMTS FDD cell. The OR network acts as a secondary system that exploit opportunities in UMTS UL bands. The OR network has an opportunity management entity which computes the maximum allowable transmit power for each OR node in order to not disturb the UMTS BS.

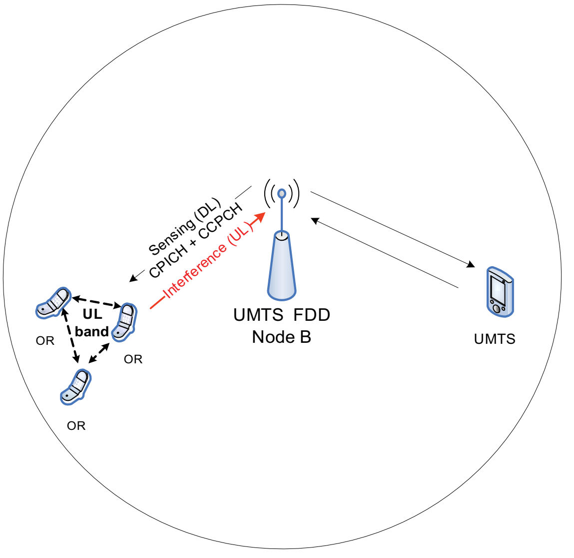

Fig. 7. Ad hoc ORs networks operating in a licensed UMTS UL band

4.1 The opportunities network with ad-hoc topology

The opportunistic network, showed in Figure 8, will interface with the link level simulator through LUTs.

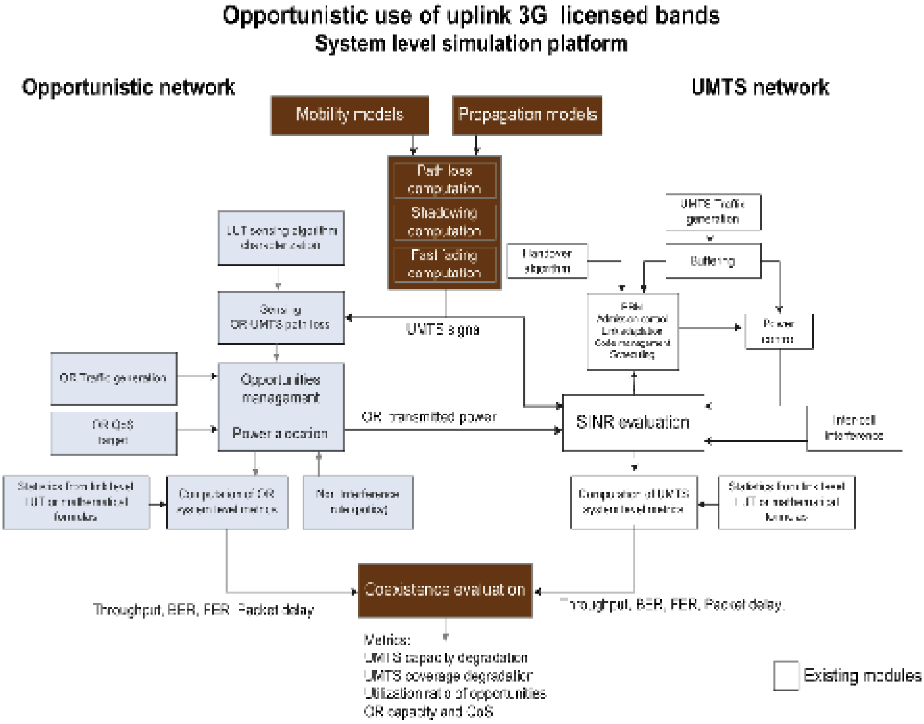

Fig. 8. Block diagram of the system level platform

The propagation models developed for the UMTS FDD network will be reused, and the entire channel losses (slow and fast fading) computed. The outputs will be the parameters that usually characterize packet transmissions: Throughput, BLER and Packet Delay. The LUT sensing algorithm characterization block contains the cyclostationary detector's performance, i.e. the output detection statistic, *d*, as a function of the SNR measured at the sensing antenna for different observation times [6].The sensing OR-UMTS path loss block estimates the path loss between UMTS BS and the OR location through the difference between the transmitted power and the estimated power given by cyclostationary detector (LUT sensing algorithm characterization block output). The OR traffic generation block contains real and non-real time service traffic models. OR QoS block defines the minimum data rate, the maximum bit error rate and the maximum transmission delay for each service class. The non-interference rule block compute the maximum allowable transmit power without disturbing the UMTS BS applying a simple non-interference rule (according to policy requirements).In the following, we briefly explain the opportunistic network blocks that was designed and implemented, using a C++ design methodology approach.
First of all, we assume that the OR knows a priori the UMTS carrier frequencies and bandwidths, which has been isolated and brought to the baseband. In order to get the maximum allowable power for OR communications the OR nodes need to estimate the path loss from its location to the UMTS BS, i.e., the victim device. The opportunistic user is interested in predefined services which should be available every time. This motivates the proposal of defining a set of usable radio front end parameters in order to support the demanded services classes under different channel conditions. Basically, at the beginning of each time step the opportunistic radio requires certain QoS guarantees including certain rate, delay and minimum interference to the primary user (non interference rule policy).
The opportunistic network has an opportunity management entity which computes the maximum allowable transmit power for each opportunistic node in order the aggregated interference do not disturb the UMTS BS. The aggregated transmit power allowed to the opportunistic network can be computed using a simple non-interference rule

$$10\log\left(\sum_{k=1}^{K}10^{\frac{P_{OR}(k)+G_{OR}+G_{BS}-\hat{L}p(k)}{10}}\right)\leq 10\log\left(10^{\frac{Nth+\mu}{10}}-10^{\frac{Nth}{10}}\right)-\Gamma \tag{4}$$

Where G_{OR} is the OR antenna gain, G_{BS} is the UMTS BS antenna gain, Lp is the estimated path loss between the OR node and the UMTS BS, K is the Number of ORs, performed by a sensing algorithm, and Nth is the thermal noise floor. μ is a margin of tolerable extra interference that, by a policy decision, the UMTS BS can bear. Finally, Γ is a safety factor to compensate shadow fading and sensing s impairments. Notice if the margin of tolerable interference μ=0 the OR must be silent. Γ is a safety factor margin (e.g. 6-10 dB) to compensate the mismatch between the downlink and uplink shadow fading and others sensing's impairments. The margin of tolerable interference is defined according to policy requirements.
Employing scheduling algorithms, we can provide a good tradeoff between maximizing capacity, satisfying delay constraint, achieving fairness and mitigating interference to the primary user. In order to satisfy the individual QoS constraints of the opportunistic radios, scheduling algorithms that allow the best user to access the channel based on the individual priorities of the opportunistic radios, including interference mitigation, have to be considered. The objective of the scheduling rules is to achieve the following goals:

- Maximize the capacity;
- Satisfy the time delay guarantees;
- Achieve fairness;
- Minimize the interference caused by the opportunistic radios to the primary user.

A power control solution is required to maximize the energy efficiency of the opportunistic radio network, which operates simultaneously in the same frequency band with an UMTS UL system. Power control is only applied to address the non-intrusion to the services of the primary users, but not the QoS of the opportunistic users.

A distributed power control implementation which only uses local information to make a control decision is of our particular interest. Note that each opportunistic user only needs to know its own received SINR at its designated receiver to update its transmission power. The fundamental concept of the interference temperature model is to avoid raising the average interference power for some frequency range over some limit. However, if either the current interference environment or the transmitted underlay signal is particularly non uniform, the maximum interference power could be particularly high.

Following we are going to explain why we consider Ad-hoc topology for the opportunistic radio system in cellular scenario.Mobile ad-hoc network is an autonomous system of mobile nodes connected by wireless links; each node operates as an end system and a router for all other nodes in the network. Mobile ad-hoc network fits for opportunistic radio because the following features:

Infrastructure

MANET can operate in the absence of any fixed infrastructure. They offer quick and easy network deployment in situations where it is not possible. Nodes in mobile ad-hoc network are free to move and organize themselves in an arbitrary fashion. This scenario is fit in the Opportunities in UMTS bands which are local and may change with OR nodes movement and UMTS terminals activity.

Dynamic Topologies

Ad hoc networks have a limited wireless transmission range. The network topology which is typically multi-hop may change randomly and rapidly at unpredictable times, and may consist of both bidirectional and unidirectional links which fits the typical short range opportunities which operate on different links in UMTS UL bands.

Energy-constrained operation

Some or all of the nodes in a MANET may rely on batteries or other exhaustible means for their energy. For these nodes, the most important system design criteria for optimization of energy conservation. This power control mechanisms for energy conversion (power battery) also helps to avoid harmful interference with the UMTS BS.

Reconfiguration

Mobile ad-hoc networks can turn the dream of getting connected "anywhere and at any time" into reality. Typical application examples include a disaster recovery or a military operation. As an example, we can imagine a group of peoples with laptops, in a business meeting at a place where no network services is present. They can easily network their machines by forming an ad-hoc network. In our scenario OR network reconfigure itself, as the interference coming from licensed users (PUs) causes some links being dropped. Ad hoc multi hop transmission allows decreases the amount of the OR's transmitted power and simultaneously decreases the interference with the UMTS BS.

Bandwidth-constrained, variable capacity links

Wireless links will continue to have significantly lower capacity. In addition, the realized throughput of wireless communications after accounting for the effects of multiple access, fading, noise, and interference conditions, etc. is often much less than a radio's maximum transmission rate This constrained also fit in our scenario where maximum transmission rate of ORs is less than the UMTS base station after the effects of multiple access, fading, noise and interference conditions.

Security

Mobile wireless networks are generally more prone to physical security threats than are fixed cable nets. The increased possibility of eavesdropping, spoofing, and denial-of-service attacks should be carefully considered. Existing link security techniques are often applied within wireless networks to reduce security threats. As a benefit, the decentralized nature of network control in MANETs provides additional robustness against the single points of failure of more centralized approaches. By using this property of MANETs, we avoid single point failure in ORs.

4.2 Co-existence analysis of single opportunities Radio link

We consider the simplest case where a single OR link operates within a UMTS FDD cell. Simulations were carried out to compute the coexistence analysis between the OR link and the UMTS network. The main parameters used for the simulations are summarized in Table 1. We consider an omnidirectional cell with a radius of 2000 meters. Each available frequency, in a maximum of 12, contains 64 primary user terminals. Each of these primary users receives the same power from the UMTS base station (perfect power control). We assume the primary users data rate equal to 12.2 kbps (voice call); the E_b/N_o target for 12.2 kbps is 9 dB. Thus, and since the UMTS receiver bandwidth is 3840 kHz, the signal to interference ratio required for the primary users is sensibly -16 dB. There is (minimum one) opportunistic radio in the cell coverage area, which has a transmitted power range from -44 to 10 dBm. The opportunistic radio duration call is equal to 90 seconds. We furthermore consider load characteristics.

Simulation results for a single UMTS frequency

In order to calculate Cumulative Distribution Function (CDF) for the interference at UMTS BS we consider 64 UMTS licensed UMTS terminals in each cell (with radius equal to R= 2000 m), as shown in the following Figure 9. The OR receiver gets interference from the PUs located in the central UMTS cell and in 6 adjacent cells. The ORs are within an ad-hoc network service area (with radius equal to R= 100 m); the OR receiver is 10 m away from the OR transmitter. The OR transmitter is constrained by the non-interference rule.
Based on the capacity's Shannon formula, the OR's link capacity that can be achieved between two OR nodes is given by:

$$C_{Mbps} = B\log_2\left(1+\frac{L_2 P_{OR\_Tx}}{Nth + I_{UMTS}}\right) \qquad \begin{array}{l} B = 5 \text{ MHz} \\ Nth = -107 \text{ dBm} \end{array} \tag{4}$$

Where B=5 MHz, L_2 is the path loss between the *OR_Tx* and the *OR_Rx*, *Nth* is the average thermal noise power and I_{UMTS} is the amount of interference that the UMTS terminals cause on the *OR_Rx*. On the other hand, the total interference at the UMTS BS caused by the OR activity can not be higher than the UMTS BS interference limit, -116 dBm.

Parameter Name	Value
UMTS system	
Time transmission interval (T_{ti})	2 ms
Cell type	Omni
Cell radius	2000 m
Radio Resource Management	
Nominal bandwidth (W)	5 MHz
Maximum number of available frequencies ($N_{[max]}$)	12
Data rate (R_b)	12.2 kbps
E_b/N_o target	9 dB
SIR target (γ)	-16 dB
Spreading factor	16
Spectral noise density (N_o)	-174 dBm/Hz
Step size PC	Perf. power ctrl
Channel Model	Urban
Carrier frequency	2 GHz
Shadowing standard deviation (σ)	8 dB
Decorrelation length (D)	50 m
Channel model	ITU vehicular A
Mobile terminals velocity	30 km/h
Primary User (PU)	
Number of primary user(s) terminals per cell/frequency (K)	64
Sensibility/Power received	-117 dBm
UMTS BS antenna gain	16 dBi
Noise figure	9 dB
Orthogonally factor	0
Opportunistic Radio (OR)	
Number of opportunistic radio(s) in the cell coverage area	2
Maximum/Minimum power transmitted ($P_{o\,[max/min]}$)	10/-44 dBm
Antenna gain	0 dBi
Duration call	90 s

Table 1. Main parameters used for the simulations

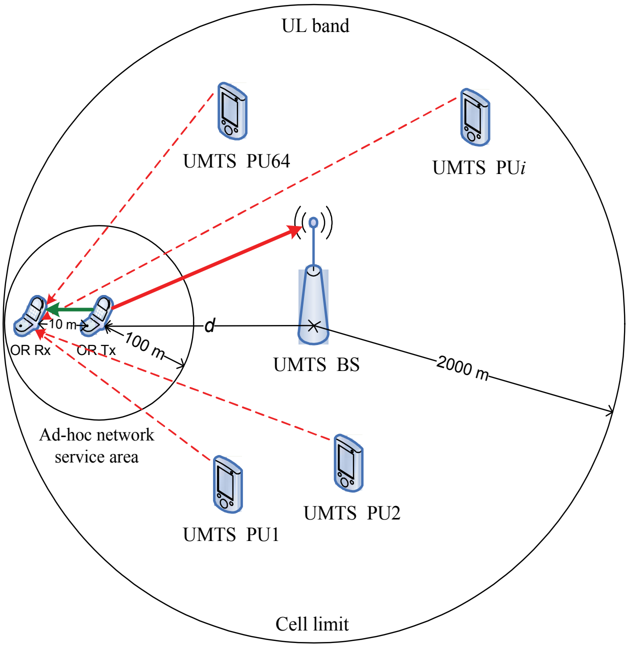

Fig. 9. Ad-hoc Single Link scenario

The following Figure 10 shows the CDF of the interference computed at the UMTS BS due the OR network activity. The results show that an 8 Mbps OR's link capacity is guaranteed for approximately 98% of the time without exceeding the UMTS BS interference limit (-116 dBm). However, this percentage decreases to 60% when an OR link with 32 Mbps is established identical in every UMTS cellular system and the frequencies are close enough so that the same statistical models apply.

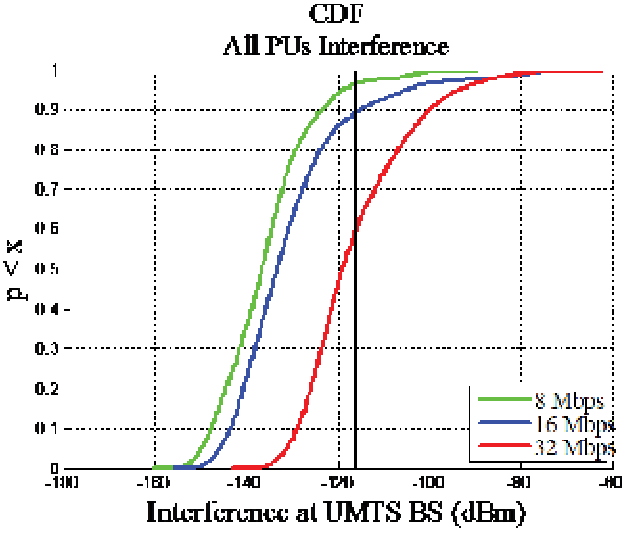

Fig. 10. Interference at UMTS BS

5. Game theory in opportunitics radio

A wireless ad hoc network is characterized by a distributed, dynamic, self-organizing architecture. Each node in the network is capable of independently adapting its operation based on the current environment according to predetermined algorithms and protocols. So, we are choosing analytical models to evaluate the performance of ad hoc networks with opportunists radio access have been scarce due to the distributed and dynamic nature of such networks. Game theory offers a suite of tools that may be used effectively in modeling the interaction among independent OR nodes in an ad hoc network. We are choosing analytical models to evaluate the performance of ad hoc networks with opportunists radio access have been scarce due to the distributed and dynamic nature of such networks. Game theory offers a suite of tools that may be used effectively in modeling the interaction among independent OR nodes in an ad hoc network.

For over a decade, game theory has been used as a tool to study different aspects of computer and telecommunication networks, primarily as applied to problems in traditional wired networks. In the past three to four years there has been renewed interest in developing networking games, this time to analyze the performance of wireless ad hoc networks (ORs). Since the game theoretic models developed for ad hoc networks focus on distributed systems, results and conclusions generalize well as the number of players (ORs) is increased. It is also of interest to investigate how selfish behavior by individual nodes (ORs) may affect the performance of the UMTS system as a whole.In a game, players (ORs) are independent decision makers whose payoffs depend on other players' (OR) actions. Nodes (OR) in an ad hoc network are characterized by the same feature. This similarity leads to a strong mapping between traditional game theory components and elements of an ad hoc network. Table 2 shows typical components of an ad hoc networking game. Game theory can be applied to the modeling of an ad hoc network at the physical layer (distributed power control), link layer (medium access control) and network layer (packet forwarding). Applications at the transport layer and above exist also, although less pervasive in the literature. A question of interest in all those cases is that of how to provide the appropriate incentives to discourage selfish behavior. Selfishness is generally detrimental to overall network performance; examples include a node's increasing its power without regard for interference it may cause on its neighbors (layer 1), a node's immediately retransmitting a frame in case of collisions without going through a backoff phase (layer 2), or a node's refusing to forward packets for its neighbours (layer 3).

Components of a game	**Elements of an ad hoc network**
Players	Nodes in the network
Strategy	Action related to the functionality Being studies(e.g. the decision to forward packets or not, the setting of power level, the selection of waveform/modulation scheme)
Utility function	Performance metrics(e.g. throughput, delay, target signal-to noise ratio)

Table 2. Typical mapping of ad hoc network components to a game

5.1 Using game theory as power control

Transmit-power control is necessary for the opportunistic radio system to broaden the scope of its applications and enhance the performance. It would have to operate under two limitations on network resources: the interference temperature limit imposed by regulatory agencies, and the availability of a limited number of spectrum holes depending on usage. In a multiuser opportunistic radio (ORs) environment, all the users operate in a decentralized manner; they are characterized by cooperation and competition. In such a case, game theory could be applied to exercise control over the transmit power. Distributed power control may be adopted by a node (OR). From a physical layer perspective, performance is generally a function of the effective signal-to-interference-plus-noise ratio (SINR) at the node(s) of interest. When the nodes in a network respond to changes in perceived SINR by adapting their signal, a physical layer interactive decision making process occurs. This signal adaptation can occur in the transmit power level and the signaling waveform (modulation, frequency, and bandwidth). The exact structure of this adaptation is also impacted by a variety of factors not directly controllable at the physical layer, including environmental path losses and the processing capabilities of the node(s) of interest. A game theoretic model for physical layer adaptations can be formed using the parameters listed in Table 3.
From Table 2 , the stage game for interactive physical layer adaptations can be modeled as

$$G = \{ N, \{ \mathbf{P_j} \times \mathbf{\Omega_j} \}, \{u_j (\mathbf{P}, \boldsymbol{\omega}, H)\} \tag{5}$$

Symbol	Meaning	Symbol	Meaning
N	The set of decision making nodes in the network;{1,2,…n}	**P**	The power space (R^n) formed from the Cartesian product of all $\mathbf{P}_j$ $\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2 \times \ldots \times \mathbf{P}_n$
hij	The link gain from node i to j. Note this may be the function of waveform selected	**p**	A power profile vector) from **P** formed as **p** =($p_1, p_2, \ldots p_n$)
H	The network link gain matrix. $H = \begin{bmatrix} 1 & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & 1 & & & \vdots \\ h_{31} & & & & \vdots \\ \vdots & & & \ddots & \vdots \\ h_{n1} & h_{n2} & & & 1 \end{bmatrix}$	Ω_j	The set of waveform know by node j.
		ω_j	A waveform chosen by j from Ω_j
		Ω	The waveform space formed by the Cartesian product of all Ω_j . $\Omega = \times_{j\in N} \Omega_j$
$\mathbf{P}_j$	The set of power level available to node j. This is presumde to be a subset of real number line.	ω	A waveform profile (vector) from Ω formed as $\omega = \omega_1, \omega_2, \ldots \omega_n$
p_j	A power level chosen by j from $\mathbf{P}_j$.	$u_j(p, \omega, H)$	The utility derived by j.

Table 3. Game theoretic model for OR ad hoc networks

For a general game, each OR node, j, selects a power level, p_j, and a waveform, ω_j, based on its current observations and decision making process. Distributed power control systems permit each OR radio to select p_j, but restrict $\mathbf{\Omega}_j$ to a singleton set; distributed waveform adaptation systems (interference avoidance) restrict the choice of p_j, but allow ω_j to be chosen by the physical layer.
Power control, though closely associated with cellular networks and is implemented in OR ad hoc network that operated in the same bands that the primary user UMTS system We now model the power control algorithm suggested as a normal form game. Note that a similar approach can be followed to model the other distributed algorithms as games, with each game involving a different utility function. We adopt the notation in Table 3.For most game models, the game theoretic equivalent of a distributed algorithm's steady state is a *Nash equilibrium* (NE). An action vector (or alternative vector) a is said to be a NE if equation (1) is satisfied.

$$u_i(\mathbf{a}) \geq u_i(b_i, \mathrm{a}_{-i}) \ \forall\ i \in N, b_i \in N \tag{6}$$

Consider a DS-CDMA system with a centralized receiver where all OR nodes other than the centralized receiver are adjusting their transmitted power levels in an attempt to maximize their signal-to interference- plus-noise ratio (SINR) as measured at the receiver. Here our set of players are the OR nodes (other than the centralized receiver); the action sets are the available power levels (presumably a finite number of power levels) all OR player's utility functions are given by equation (7)

$$u_i(\mathrm{p}) = h_i p_i \ / \ ((1/\mathrm{K}) \sum_{j \in N \setminus i} h_k p_k + \sigma) \tag{7}$$

where p_i is the transmitted power of node i, K is the statistical estimate of the spreading factor, h_i is the gain from a node to the receiver, and σ is the noise at the receiver.

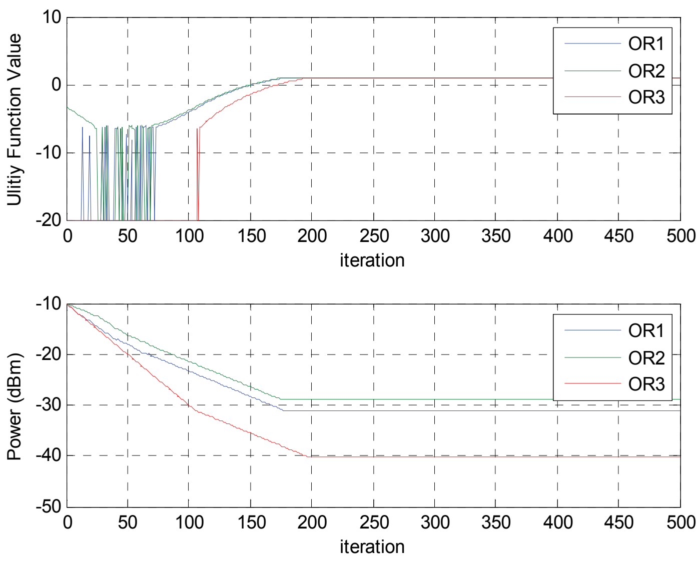

Fig. 11. 3 OR node closer to the UMTS system

As would be indicated by intuition, the unique Nash equilibrium for this game is the power vector where all OR nodes transmit at maximum power. This is an undesirable outcome as (6) capacity is greatly diminished due to near-far problems (unless the nodes are all at the same radius from the receiver as shown in the Figure 11 and Figure 12 where OR node are closer and far away from the UMTS system), equation (2) the resulting SINRs are unfairly distributed (the closest node will have a far superior SINR(as shown in the Figure 11) to the furthest node(as shown in the Figure 11 and (12) battery life would be greatly shortened. However, this outcome is Pareto optimal as any more equitable power allocation will reduce the utility of the closest node, and any less equitable allocation will reduce the utility of the disadvantaged nodes. In this scenario Pareto optimality actually misleads the analyst with respect to the desirability of the outcome.

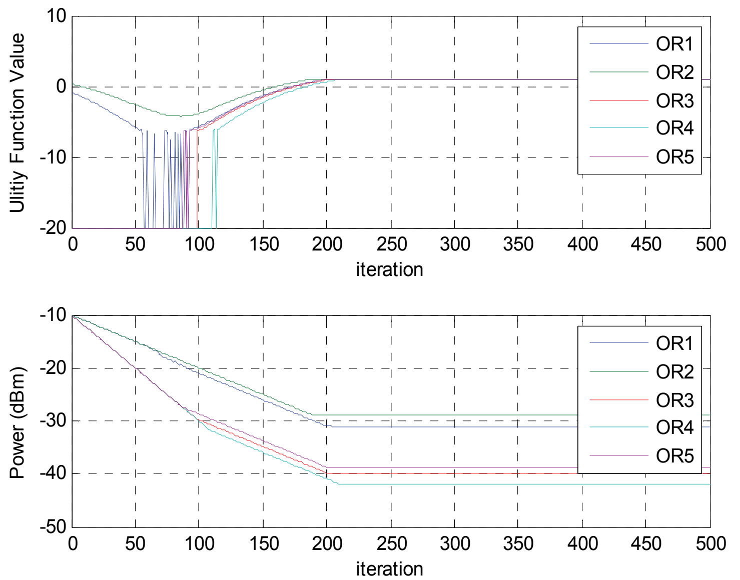

Fig. 12. 5 OR node far away to the UMTS system

6. Conclusion

Emerging research in game theory based power control applied to ad hoc opportunist networks shows much promise to help understand the complex interactions between OR nodes in this highly dynamic and distributed environment. Also, the employment of game theory in modeling dynamic situations for opportunist ad hoc networks where OR nodes have incomplete information has led to the application of largely unexplored games such as games of imperfect monitoring. Ad hoc security using game theory is the future area of research in ORs we have considered an ah hoc behavior in the opportunists radio (ORs) and suggested that by implementing ah hoc features in the ORs will improve the overall performance of system.

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4

Reliable Aggregation Routing for Wireless Sensor Networks based on Game Theory

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1. Introduction

Wireless integrated sensor networks, which include collecting, managing data and communication, are used more and more widely for their low cost and convenient deployment. Nowadays the research concerning each aspect of sensor networks is fairly active. Data Aggregation mechanism is one of the key problems in sensor networks. By considering the data transmission delay and overall network energy efficiency, this chapter develops a game-theoretic model of real-time reliable aggregation (RA-G) mechanism for wireless sensor networks.

Based on the study of related literatures, first of all in this chapter, the research status of WSN, the system architecture, the characteristics, and the critical technologies are summarized, current typical routing algorithms of WSN are classified and introduced one by one. Taking the implicit collaborative imperative for sensors to achieve overall network objectives (accomplish real-time collection tasks effectively) subject to individual resource consumption into account, this paper proposes a game-theoretic model of reliable data aggregation architecture in wireless sensor networks, defines a multi-tier data aggregation architecture in which semantic based aggregation and average computation aggregation is performed in sensor-level and node-level aggregation respectively. All nodes that detect the same target join the same logic group. Each selected group leader uses game-theoretic model which tradeoffs between energy dissipation and data transmission delay to determine the degree of aggregation. To meet the real-time constraints and balance the energy consumption between nodes, a decision-making model based on game theory which takes delay compensation into account is proposed in the data-relaying stage.

The simulation results show that the use of reliable data aggregation architecture can reduce the total transmission overhead of WSN, make the network more energy-efficient and prolong the lifetime of sensor network. On the other hand, the game-theoretic model used in group-level aggregation and data-relaying stage balance the tradeoffs between the energy dissipation and the timeliness of data transmission; therefore, also RA-G data aggregation mechanism is reliable.

2. Wireless sensor networks

Wireless sensor network is a data-centric wireless self-organizing network [1] consisting of a large number of integrated sensors, data processing unit, as well as short-distance wireless communication module. From the 21st century, sensor networks attracted academic, military and industry with great concern. The United States and Europe have launched a lot of research programs about wireless sensor networks and obtain the corresponding progress. The development of specific communication protocols and routing algorithm is the first issue of current field of wireless sensor networks need to be resolved.

2.1 Wireless sensor network architecture

The architecture of Wireless sensor network is shown in Figure 1.1 [2], wireless sensor network systems often include sensor nodes, Sink gateway nodes and the management nodes. A large number of sensor nodes deploy randomly inside of or near the monitoring area (sensor field), having ability of compositing networks through self-organization. Sensor nodes monitor the collected data to transmit along other sensor nodes by-hop. During the process of transmission, monitored data may be handled by multiple nodes, get to Sink gateway node after a multi-hop routing, and finally reach the management node through the Internet or satellite. The user configures and manages the wireless sensor network with the management node, publish monitoring missions and collect monitoring data.

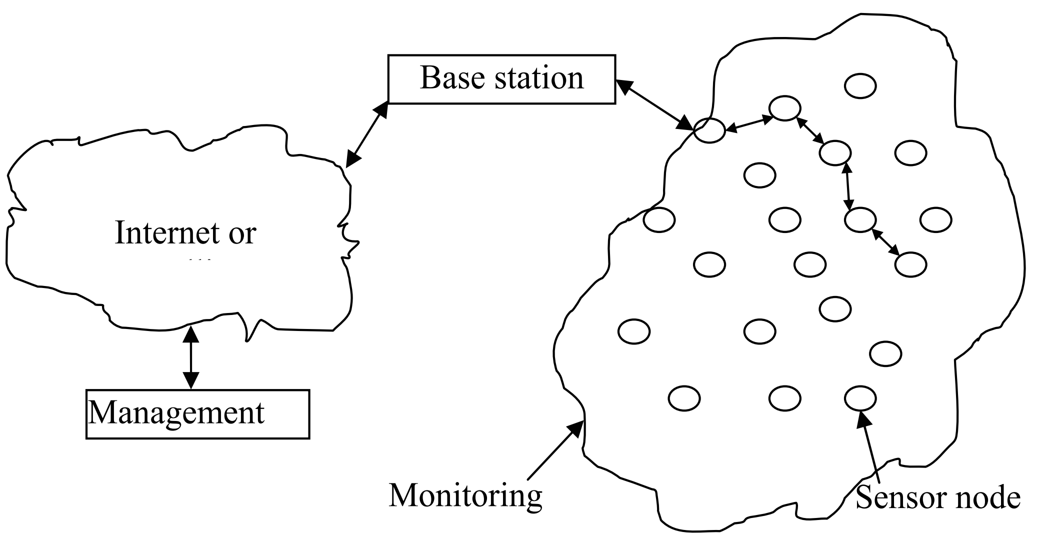

Fig. 1.1 Wireless Sensor Network Architecture

Sensor node is usually a tiny embedded system. It's processing power, storage capacity and communications capability is relatively weak, and the energy limited by carrying batteries. Sensor node consists of four parts [3] which are the sensor modules, processor modules, wireless communication module and power supply modules. Sensor module is responsible for the collection of information and the conversion of data in the area of monitoring; processor module responsible for controlling the operation of the sensor nodes, storage and processing their own collected data and the data sent by other nodes; wireless communication module is responsible for communicating wireless with other sensor nodes, exchanging controlled information, and sending and receiving collected data ; energy supply module provide the energy required to run for the sensor nodes, usually with a miniature battery.

Sensor nodes will be constricted by the limited supply of energy, communications capacity, computing and storage capacity, when achieving a variety of network protocols and applications. The features of sensor network are as follow:

1. Large-scale network [1, 2];
2. Self-organizing network [4];
3. Dynamic nature of networks;
4. Reliable network;
5. The application-specific networks;
6. The data-centric network [1, 2].

As a new research hot spot of information today, wireless sensor networks involve interdisciplinary field of study, and there are a lot of key technologies and researches to be found. The following list only some of the key technologies [1, 3, 5].

1. Network topology control. A good network topology generated automatically by topology control, is able to improve the routing protocol and the efficiency of MAC protocol and lay the foundation for many aspects such as data fusion, time synchronization and targeting, which will help to save the nodes and energy to extend the survival period of network. Therefore, the topology control is one of the core technology researches in wireless sensor networks.
2. Network protocol. Sensor network protocol is responsible for making all the independent nodes form a multi-hop data transmission network. The current study focused on network-layer protocols and data link layer protocol. Network layer routing protocols determine the transmission path of monitoring information; media access control of data link layer used to build the underlying infrastructure and control the communication process and work style for sensor nodes .
3. Network security. Ensuring the confidentiality of implementing the mandate, the reliability of data generation, the efficiency of data fusion and the security of data transmission is content which security issues in wireless sensor networks need to take full account of.
4. The time synchronization. Time synchronization is a key mechanism of sensor network systems needed to work together.
5. Location technology. Location information of sensor node is an integral part of the collected data. Determining the location of the incident or the node position of data collected is the most basic functions of sensor networks. Positioning mechanism must satisfy the self-organization, robustness, capacity-efficient, distributed computing requirements.
6. Data fusion. Sensor networks are constrained by energy. Reducing the amount of data can save energy effectively. Therefore in the process of collecting data from various sensor nodes, we can use computing and storage capacity of the local nodes to deal with the integration of data and to remove redundant information, thereby to achieve the purpose of saving energy.
7. Data management. From the view of data storage, sensor networks can be regarded as a distributed database. As a database method for data management in sensor networks, the logical view of data stored in the network can be separated from the realization of the network, making users of sensor networks need to only care about the logical structure of data query, no need to care about implementation details.

2.2 Comparative analysis of routing protocols of Wireless sensor network

After many years' efforts of national researchers, sensor network routing protocol algorithm has quite a number of results. According to the routing protocol algorithm, the network structure [10] can be divided into three categories as a flat routing, hierarchical routing and location-based routing; according to protocol operations rules, it can be divided into routing consultations, multi-path routing, QoS routing, query routing, etc. (Table 1.1 below). The following are introduced one by one by category.

Classification according to the Structural of network	Flat Routing	Directed Diffusion, SPIN, Rumor routing
	Hierarchical routing	LEACH, PEGASIS, EEN&APTEEN
	Location-based Routing	GAF, GEAR
Classification according to the protocol operation	Consultation route	SPIN, Directed Diffusion
	Multi-path routing	Directed Diffusion, SPIN, SPEED
	QoS Routing	SPEED
	Query Routing	Directed Diffusion, Rumor routing

Table 1.1 Classification of routing protocols of wireless sensor network

2.2.1 Protocol based on network structure

1. Flat routing protocols

In the flat multi-hop wireless sensor networks, flat routing protocols generally require each node to play the same role. Multi-sensor nodes implement acquisition of data synergistically. The studies for data-centric routing strategy have shown that energy can be saved through collaboration of multi-node operation and the elimination of redundant data, such as: SPIN [7-8] and Directed Diffusion [9-10]. Both protocols promote the other protocol design following a similar idea (i.e. data-centric routing method).

SPIN (Sensor Protocols for Information via Negotiation) [7-8]: W. Heinzelman and others made a class of adaptive SPIN routing protocol. The protocol assumes that all nodes in the network are potential Sink nodes, and each node can disseminate information to the other nodes in the network. It just needs to send the data which other nodes does not have. In addition, SPIN protocol classes also use the data negotiation strategies and resources adaptive algorithm. The node running SPIN protocol is assigned with each high-level data meta-data descriptor used to describe their data collected completely. Implementing the meta-data consultation before any data to be sent, to ensure that no redundant information transmit in the network. In addition, SPIN protocols have right to access the current energy level of each node, and adjust the running mold of protocol according to the residual energy level of node. Meta-data negotiation strategies of SPIN protocol solve the existing typical problems of the diffusion, thus improving energy efficiency and saving energy. However, the data broadcasting mechanism of SPIN protocol class can not guarantee that the data can transmit to the destination node.

Directed Diffusion [9-10]: C. Intanagonwiwat and others propose a new communication model of data acquisition for sensor networks, called directed diffusion. As a data-centric (DC data-centric) and application-aware communication model, directed diffusion protocol requires all of the data generated by sensor nodes named with attribute value pairs. The main idea of the model DC is a purposes to eliminate redundancy and minimize the amount of data transfer through data fusion of different sources nodes and re-routing, thus saving energy and extending the life of the network system. DC routing policy can find the path from multiple sources nodes to a single destination node and take the operation of redundant data fusion in the net. Comparing SPIN protocol, the capability of directed diffusion protocol to adapt to the environment in mobile applications is weak. In addition, the DC communication model may not apply to the application which requires a sustained data transmission to Sink node, and the query and data-matching work may require additional overhead.

2. Hierarchical routing protocols

Hierarchical or clustering routing strategy, first proposed in the wired network, is a better scalability and communication efficient routing. Hierarchical routing reduce the amount of data transmitting to Sink node through the implementation of data fusion, reduce energy consumption of each node within the cluster, and it is an effective solution to improve energy efficiency. Hierarchical routing mainly constituted by two levels: one level is used to create clusters and select the cluster head node, another level is used to integrate and process the collected data and routing data.

LEACH (Low Energy Adaptive Clustering Hierarchy) [11] [12-13]: W. Heinzelman and others propose a hierarchical clustering routing algorithm for sensor networks. It is a clustering routing protocol using distributed cluster formation technique. LEACH select a number of sensor nodes randomly acting as cluster head nodes (CHs, Cluster-Heads), so that all nodes take turns to act as cluster head nodes to bear the cost of energy evenly. In the LEACH protocol, the cluster head node integrate the data collected by all non-cluster head node (non-CHS, non-Cluster-Heads) which belong to it, and then sent the integrated data packets to the Sink node to reduce transmission volume of data. Table 1.2 compares SPIN, LEACH and Directed Diffusion routing technology according to the different parameters. It can be seen from the table that directed diffusion protocol is an energy-efficient routing of compromise due to the use of network processing and optimization path method.

	SPIN protocol	LEACH protocol	Directed Diffusion protocol
Optimal path	No	No	Yes
Internet Life	Well	Well	Well
Resource-aware	Yes	Yes	Yes
The use of meta-data	Yes	No	Yes

Table 1.2 SPIN, LEACH and Directed Diffusion protocol comparisons

TEEN (Threshold-sensitive Energy Efficient sensor Network Protocol) [14] and APTEEN (Adaptive Periodic Threshold-sensitive Energy Efficient sensor Network protocol) [15]: these two kinds of hierarchical routing protocols are proposed for time-critical data acquisition application. In the TEEN protocol, sensor nodes collect information constantly, but the process of data transfer is less. A cluster head node send a hard threshold (collection attributes), and a soft-threshold (can lead a change of sensed attribute value range for the node open the transmitter to transmit data) to its members. Only when the sensed attribute value in the context is in the range of interest, it will be allowed to transfer data.

The simulation results of TEEN and the APTEEN show that these two types of protocol are better than LEACH protocol in operational performance. It is proved by Experiment, according to energy consumption and network lifetime, the performance of APTEEN is between LEACH and TEEN. TEEN provide the best performance because it reduces the number of transmissions. The major shortcomings of these two protocols are the increase of the cost and complexity which is related to the formation of a multi-level class, the realization of the methods based on threshold functions and how to deal with the increase's cost of attribute based on named query methods.

3. GIS-based routing protocol

In this type of routing protocol, sensor nodes depend on the location information to address. The distance between neighbor nodes can be estimated by the arrived signal strength. The relative coordinates of neighbor nodes are get through the exchange of information between the nodes [16-17, 18]. In other words, if the node equipped with small low-power GPS receiver [19], nodes can get location information through communications with satellite directly using GPS. To conserve energy, without uncertain situation, some strategy based location information requires the nodes go to sleep. Make as many nodes as possible in sleep, so that the network can save more energy. The problem of designing table of the sleep cycle scheduling with a fixed way for each node are discussed in [19-20].

2.2.2 Protocol-based protocol operation

1. Negotiation-based routing protocol

These protocols using advanced data descriptors reduce the amount of data transmission through consultation to eliminate redundant data. Communication decision-making is made also based on the resources available to them. SPIN protocol suite [11-12] are examples of routing protocols based on negotiated. Motives of consultation are: to avoid the defects of diffusion, which will produce the problems of information explosion and overlap, so the node will receive multiple copies of the same data. This operation will consume more energy, bandwidth, and to spend more processing time due to send the same data to different nodes. The important ideal of negotiation-based routing protocol is to eliminate duplicate information, avoid redundant information sending to the next node or Sink node and do a series of operation in consultations before sending the actual data.

2. Multi-path routing protocols

In order to improve network performance, such protocols will use multi-path data routing rather than a single path. The fault-tolerant of protocol according to exist possibility of other alternative path when the basic path between source node and destination node fail. Increase of the fault tolerance get from maintaining the multi-path between the source node and destination nodes, with the ever-increasing cost of energy consumption and traffic generated. The paths of choice maintain its vitality through sending the message periodically, so increasing network reliability and fault tolerance is obtained through maintaining a number of alternative paths available with increasing cost.

3. QoS-based routing protocol

Once considering the performance QoS when address data, network has to strike a balance between power and data quality. Especially when the node to send data to Sink node, the network has to meet some QoS criteria, such as: delay, data accuracy, bandwidth utilization rate and so on.

4. Routing protocol based on query

Such routing protocols are characterized by: the destination node transmit a query through the Internet for collecting data needed to complete tasks, then after a node that owns the data match the query, we send the data back to the node starting the query, which is the destination node. Usually these queries are described by natural language or high-level query language. All nodes have a table consisted of query mandates they received. After receiving a query, they send the data matching with the queries. Directed diffusion protocol [7] is an example of this kind of routing. In the communication model of directional diffusion, Sink node sends interested information to all nodes. Once the interest spread through the network, the gradient is established which is from the source node to Sink nodes. When the source node has the data of the interest, the source node send data along the interest gradient path. To reduce energy consumption, it implements the routing after data fusion.

We provide an overview of a variety of routing algorithms above according to different classification, compare similar routing algorithm and point out their advantage and disadvantage.

3. An overview of game theory

Strictly speaking, the game theory is not a branch of economics. It is a methodology, whose scope of application is not limited to economics. Political science, military, diplomatic, international relations, public choice, criminology are related to game theory. Many scholars have already introduced game theory into the field of communication, including flow control, routing algorithms, power control. Game theory, also translated as game theory [21], is to study the decision when the behavior of decision-making body makes a direct interaction, as well as the balance of this decision-making.

Presentation of a complete game problem requires at least three basic elements: player, strategy set, and payoff function.

1. Player

Player is the immediate parties involved in game. He is the main maker of decision-making and strategy of game. In a different game, the player means different which can be individual, group or collective, but these organizations or groups must be for a common goal and interests to participate in game. Player should know clearly their own goals and interests and always take the best strategy to achieve their maximum effectiveness and interests in the game.

2. Strategy set

In a game, a practical, feasible and complete action which is available for participants to chooses to be called a strategy. Strategy set is all the possible set of strategies taken by player. It is the tools and instruments for player to play, and each set should be set at least two different strategies. Strategies from each strategy set in game forming a game situation.

3. Payoff function

When strategy set adopted by all players is determined, they have their own "payoff function" or "profit function". Payoff function express the level of the income or utility can be get from the game by player, which is the function of strategy for all players. Different strategies may lead to different benefits, which is the thing each player really cares about.

In game theory, one of the important bases for each player to make a rational decision-making is the amount of his possible profits, which is an insider need to calculate carefully the profit function. The structure and values of profit function will undoubtedly affect the player's behavior, thus also affect the final outcome of the game. As a result, the determination of profit function is a very important matter in game theory study.

Considering different point of view for game, a player can have all kinds of profit function which is not unique.

3.1 Nash equilibrium

Game theory is a mathematical tool used to study the decision when the behavior of decision-making body makes a direct interaction, as well as the balance of this decision-making. In other words, it is decision-making problems and balance issues when a choice involved in a subject is impacted by the choices of other subjects and return to influence the choice of other subjects. The most basic components of game theory is the game concept, using the formula is expressed as $G = \langle N, A, \{u_i\} \rangle$, where G is a specific game, $N = \{1, 2, \ldots, n\}$ is a limited set of participants (decision makers), A_i is a collection of optional behavior of the participant i, $A = A_1 \times A_2 \times \ldots \times A_n$ is behavior space, $\{u_i\} = \{u_1, u_2, \ldots u_n\}$ is the maximum effectiveness (objective) of function set which participants hope to. Each objective function of participant u_i is a function of the special action a_i selected by a participant i, but also the functions of the action a_{-i} chosen by all the other players in this game. That is to say the individual objective function depends not only on its own choice, but also on other participants' choices. Game may include some additional components, such as the information and communication mechanisms [21] which each participant can make use of.

For the game, the basic concept of steady state is the Nash equilibrium. In the Nash equilibrium, there is no node which can improve its objective function value through unilaterally deviating from the value of the state. For example: a^* is the steady state, only if:

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \forall a_i \in A_i, \forall i \in N \,. \tag{1.1}$$

These steady states can predict the output of distribution algorithms. Strategy a_i^* is a "best" strategy chosen by participant i in the face of opponents; this is true for all participants. Game result is "stable", which means that no participant has a incentive to deviate from this choice unilaterally; in a sense, Nash equilibrium is a "no regrets" solution of game.

Another expression for Nash equilibrium is sometimes very useful. For any $a_{-i} \in A_{-i}$, we define the best set of participants:

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})\} \text{, for all } a_i' \in A_i \tag{1.2}$$

In general, B_i is called the "best response function" of the participants, so we can define Nash equilibrium to a strategy vector $(a_1^*, \ldots a_n^*)$, where $a_i^* \in B_i(a_{-i}^*), \forall i \in N$.

A very important point is: in many cases, the concept of the solution of a game exists logically. In fact, the concept of Nash equilibrium is used widely because it exists in many games.

3.2 Incentive theory

Motivation theory [22-23] is one of the most important applications for the game theory in economics, which have a wide range of applications in all fields. It reveals the asymmetric information as an important role played in economics. The main analytical framework for incentive theory is made in the principal-agent relationship model. In this relationship, there is a principal and one or more agents, as agents have the expertise or unique information which a principal does not have, or simply because the client not has the time and energy to deal with certain things, the principal delegate an agent to deal with certain matters which originally belongs to his power or responsibility.

4. The model of data fusion based on game theory

In this section, the idea of game theory will be introduced to the wireless sensor networks to model RA-G (Reliable Aggregation based on Game theory) for delay and the energy efficiency of nodes integration of the data fusion mechanism. By the introduction of wireless sensor networks, we can see that the network node has features of severe restrictions on bandwidth resources, energy, storage capacity and computing. In the integration phase, each intermediate node want integrate sufficient data packets before sending data to minimize their consumption of energy required to send data. The more integration nodes collect data packets, the more accurate for the description of monitored goals, that is the accuracy of the information; but on the other hand, collecting more data packets need to wait for the longer integration time, which will lead that the final information delay received by network users would greatly increase. This situation is intolerable for real-time target tracking system. This shows that the above-mentioned factors in the network are contradictory. For the node, it want to save as much as possible the energy of their own bandwidth resources, and for the network, the delay is a key issue, that is to say the nodes and the interests of network exist contradictions; when the fusion node transmit fused data packets to the sink node, there is another issue to be considered. As each node in each period play different role and with different status, in data transmission phase, nodes have to weigh their own needs to send data and to forward data services for other nodes. On the one hand, when the node need to send data, other nodes can provide forwarding services; the other hand, each node try to forward the data as less as possible for the other nodes in order to reduce power consumption. But if all nodes are not willing to forward data for other nodes, then the connectivity of network will decline sharply and reliable real-time transmission of data packets can not be guaranteed, and ultimately affect the overall performance of the network seriously – which is also a contradiction between nodes and the interest of network.

Game theory is a good mathematical tool in dealing with such a conflict of interest. The following section will build a determination model of intermediate nodes integration based on game theory for the real-time target / event monitoring system, and make some preliminary attempts on node incentive mechanism.

4.1 Real-time target / event monitoring system

Real-time target / event monitoring [24] system consists of hundreds of tiny sensor nodes, which can monitor and track goals efficiently and real-timely within the monitoring region, and distinguish the targets. The result will be reported to end-users via satellite or cable network by sink node. This section used the integration of hierarchical models [25] to achieve efficient use of energy. If the particle size of integration is too small, a lot of useful information of the collected raw data may be premature loss; however, if the particle size of integration is too large, it will make wireless sensor networks consume excessive energy for transmitting data and maybe cause serious network congestion and loss of information. Therefore, in this section, real-time target / event monitoring system use a mechanism of hierarchical integration to solve the above problems, as shown in Figure 1.2 for the schematic of hierarchical integration.

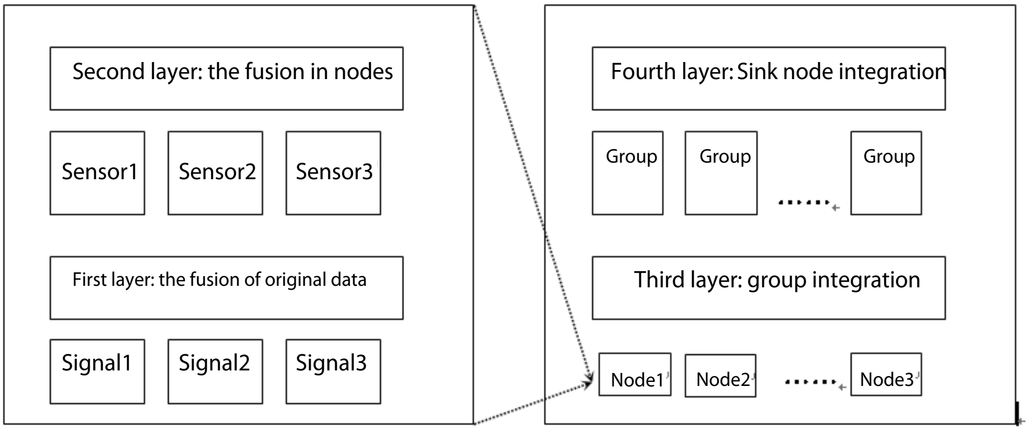

Fig. 1.2 the schematic of hierarchical integration

The first layer is about the fusion of original data. Data collected by sensor are the original input of the entire network. The integration in this layer provides the basis processing for the information of the tracked targets / monitoring of events in the network. Data fusion of this level must meet the following requirements: 1) meet the real-time constraints; 2) be able to handle a large number of input data. In order to enhance energy-saving effect of the integration operation, this layer operation of data fusion is semantics-based integration. By extracting the semantic of raw data collected from sensors to achieve higher efficiency integration.

The second layer is fusion in node level. Each sensor node integrates several different types of sensors. After collecting self-confidence vector of different sensors, nodes do the further integration. It will calculate the average of all nodes' confidence vector, and then forming a single node-level confidence vectors. Semantics of sensor data should be extracted and fuse at the node level, classification module of perception algorithm and the node level need to cache and deal with selected data. Here the processing time require in a reasonable range.

The third layer is group integration. When the node level fusion gets monitored results, we began to estimate related information of the current target, and should uniquely determine the monitored objectives in logic. During the preliminary estimate, we should let the collected information about the target location of each node use their confidence vector as the weight to the average all the monitoring value. This involve an issue is when and where the estimated calculation of such a collection should be done. Representation about the target is a classic problem. There are already a number of centralized or distributed algorithms of temporal and spatial correlation to achieve. In this system, there are two related mechanisms used in this layer.

1. The fusion method based on logic group

In the target / event monitoring system, there are two main tasks which are to collect relevant information of objectives and to represent goals. A simple solution is sending the monitoring results, the location and other information of all the nodes to a central base station, to estimate the current location and other information based on the location information [26-27] of all nodes sending the information and other related information collected, and in the process, to the use of space-time related algorithm to give and maintain the coherence for the sole objective. But the efficiency of this centralized mechanism is low both for energy consumption and delay. Sending the large amount of data report to the base station will cause excessive energy consumption, and if the target is far away from the base station will greatly increase the delay. In order to avoid the shortcomings of the above mechanism, using a distributed mechanism is a solution. Processing the data near the monitored target / event, and then sent fused information to the base station for further operations.

2. Balance of energy and delay based on Game Theory

In the group fusion layer, managing the node need to wait for some time to gather the data report of members in group, and integrate these reports, then forward to the Sink nodes through other nodes. In this process, there is a variable parameter need to be considered, degree of aggregation DOA, which is a direct expression to show whether the management node has received a sufficient number of reports of group members. That is to say the management node doesn't operate the fusion before receiving to a member of sufficient DOA data reports in group. In the management nodes, the problem of balance description need to be considered are as follows: For the management node, the larger DOA values means the more members' data report can be collected to fuse, and then sent data packets of once fusion. It compared with the situation of smaller DOA, obviously management node can save more energy consumption on sending data and is conducive to reducing the load nodes of transmission; while for the network users, the goals of real-time monitoring are the ultimate goals of the network. If the DOA value is so large that the producing delays beyond the limits of real-time systems, it will inevitably harm the interests of Internet users, resulting in unavailable purpose of real-time monitoring for target. In above process, the interests between the nodes and network create a conflict, which is needed to use some mechanism to guide the behavior of nodes in order to balance the interests of both.

From the above description we can see this game model's participants are nodes and networks, which should be a two-game model with incomplete information. Supposing energy saving through the data fusion by management node is E_p, while the wait time of fusion which is the increased delay for participants in network is T_{aggr}. Now we come to quantitative analysis the impact of DOA for E_s and T_{aggr}.

1. Energy savings in fusion

In real system, due to the impact of various factors, such as sensing range, target movement model and the node density, doing the analysis is difficult. Here we make some simplifying assumptions to do approximate analysis. Suppose sensing range of sensor nodes is a circular area with a radius R. The target moves forward with uniform speed along the straight line, and nodes in an unlimited sensor network are uniformly distributed.

Figure 1.3 shows the schematic diagram of target and monitored region. The red star represents the position of target. The sensor node in the circular can sense this target then forming a logical group. The sensor nodes with the dark mark are the managed nodes of logical group. Supposing the number of members nodes in group are n_g. If the value of DOA is 1, that is, don't do the operation of fusion in the management node. So for the management node, the energy consumption of sending group members required for data reporting is showed as follows:

$$E_{T-woaggr} = n_g \cdot (lE_{elec} + l\varepsilon d^r) \quad (1.3)$$

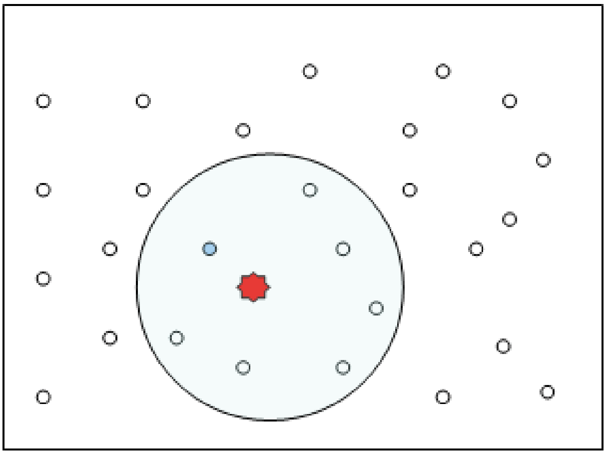

Fig. 1.3 Monitored region

Where, l is length of a data packet. E_{elec} is the energy consumption per bit data for sending or receiving circuit. The constant ε is related with the transmission channel model used. ε_{fs} is the free-space transmission, the corresponding r is 2. ε_{amp} is the multi-path fading transmission, the corresponding r is 4. When the distance d between the transmitter and receiver is less than the threshold value d_0, we use free-space transmission model. On the contrary if d is more than or d_0, we use multi-path fading model. When the management nodes do fusion of data, the value of DOA is a positive integer more than 1 and not more than n_g. At this point, the DOA data reporting of the members' nodes in group will be received and integrated by management nodes. Thus the energy consumption of sending the members' data reporting in group by nodes is:

$$E_{T-aggr} = (n_g - DOA + 1) \cdot (lE_{elec} + l\varepsilon d^r) \tag{1.4}$$

We can draw the conclusion from the above two equations, when $2 \le DOA \le n_g$, the percentage of energy savings by managed node is:

$$E_P = 1 - \frac{E_{T-aggr}}{E_{T-woaggr}} = 1 - \frac{n_g - DOA + 1}{n_g} \tag{1.5}$$

The above equation reveals the relationship between the saved energy obtained by data fusion in management nodes and DOA. In the game model discussed in this chapter, we define E_P as the benefits obtained by management nodes through the integration.

Definition 1.1 The proceeds of management nodes in Game model of group-level fusion are as follows:

$$X_I = E_P = 1 - \frac{n_g - DOA + 1}{n_g} = \frac{DOA - 1}{n_g} \tag{1.6}$$

2. The impact of convergence on the network delay

After management node generates its own data or receives the data reports of group member, it doesn't transmit them immediately but wait for a while to obtain sufficient data reporting, then do the fusion of data and transmit the fused data packet. The management nodes in this article can integrate a number of its data-reporting received through data fusing and processing into a new isometric data reporting, and the computing time of integration is much smaller than the data transmission time. Therefore we ignore this data-processing delay.

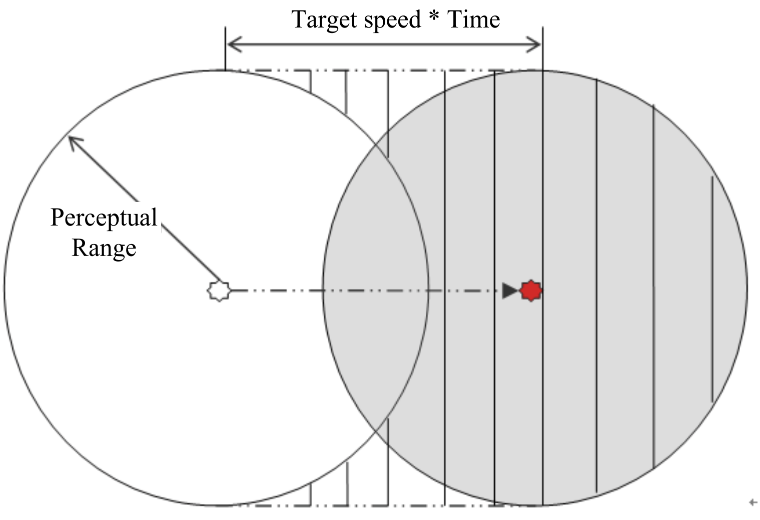

Fig. 1.4 Schematic diagram of the moving target trajectory

In Figure 1.4, goals move with speed *TS* for some time *T*, the target's perception range is *SR*. White and gray circular area represents the perception of the region of the target mobile before and after moving respectively. Nodes in the vertical shaded area are the existent new sensor nodes perceived after targets begin to move. The management node in the shadow need collect DOA data packet of members' nodes to start data fusion. The delay for that is as follows:

$$T_{aggr} = \frac{DOA}{2 \cdot SR \cdot TS \cdot D} \tag{1.7}$$

If the density *D* and sensing range *SR* of nodes in the network are determined, we can see that here the delay is related with DOA and moving speed *TS* of target. In the game model of this chapter, the longer time the integration of the management nodes are waiting for, more negative for real-time targets of the network. So we define the delay brought by integration as the penalty factor of the network for the management node, while getting the energy gains, management node must pay the price. Internet users can guide the behavior of management nodes through the definition of punish to make it operate in reasonable range.

Definition 1.2 The cost of the delay of the management node in group-level integration is showed as follows,

$$C_I = DOA^{f(TS)} \tag{1.8}$$

Where *f*(*TS*) is a function of target move speed, and the output is a positive number between 0 and 1. In the real-time monitoring system, the faster movement of the target the shorter the time is needed for information monitored to send to the Sink node. Here $f(TS)$ is the function of the urgency for sending the data reporting to the node, and it increase with the accretion of the target moving speed. So the expression of $f(TS)$ is showed as follows,

$$f(TS) = \text{Monitored target speed} \ / \ \text{The greatest possible speed of target} \qquad \text{(Eq.1.9)}$$

From equation 1.9 we can see that the output increase with the target moving speed increases, that is to say the targeted information monitored has the higher degree of urgency.

3. The definition of game model

Definition 1.3 From the above analysis, we can define the utility function of management nodes in a balanced game for the energy and delay as follows,

$$U_I = X_I - C_I = \frac{DOA-1}{n_g} - DOA^{f(TS)} \tag{1.10}$$

At this point, GA-G(Group Aggregation based on Game theory) can be described as follows.

1. Participants

In the game the two sides of the conflict of interest are manage nodes and network users.

2. Strategy

Management nodes evaluate the urgency of this monitoring information through the related information of goal monitored by the nodes of the members in group and themselves, which is the output value of the function $f(TS)$. It increases with the moving speed of target increases, which show the higher the degree of urgency for the information; in the game of this article, the management node as to networks can take the value of DOA which is more conducive to its own energy savings to carry out the operations of integration; while for networks, through avoiding the excessive delay, using penalty for delay to constraint the behavior of management node, punishment is harder as the intensity of target information increased, so that it is better for a high degree emergency information can be transmitted to the Sink node with the smaller delay.

3. The expression of utility function as follows:

$$\max U_I = \max\left(\frac{DOA-1}{n_g} - DOA^{f(TS)}\right) \tag{1.11}$$

Where the constraint condition is the value of DOA can not exceed the number of members' nodes in group n_g and no less than 2. Because in the real network, if the set for DOA over n_g, the management node will never do the operations of integration; and the values of nodes should be the value when the utility of nodes to take the largest value of DOA.

Therefore, the optimal value of DOA as follows:

$$DOA_{opt} = \underset{2 \le DOA \le n_g}{\arg\max}\left\{\frac{DOA-1}{n_g} - DOA^{f(TS)}\right\} \tag{1.12}$$

4. Qualitative Analysis of Game Model

In above model, the constraint condition is $2 \le DOA \le n_g$. Considering the case when DOA take 1, there is equivalent to introduce no group-level fusion mechanism, therefore, no data integration operation of the management node is involved in. That is, all nodes perceiving objectives transmit its data to Sink node through multi-hops after collecting the required data. There are not considerations for the balance of energy consumption and delay, therefore there is no such thing as a balanced solution; when $DOA \geq 2$, group-level integration mechanisms began to play its role and need to balance the energy consumption and delay in the management nodes. In this game model, the benefit of management node is $(DOA - 1)/n_g$. During a target / event monitoring process, sensor network nodes which perceiving the same target / event form a logical group. In the initial stage of group, the nodes can know the information of other neighbor nodes in group through interaction, and in a short period of time, the node monitoring of the goals / event is determined, that is to say n_g is certain. In this context, we can see the benefits of management nodes increase with the value of DOA increases. Meaning mapping to the network is that the more data reporting of members' node is collected, the management node can save more energy in transmitting data. Here it also implies a network parameter, the quality of information. If management node collects more data reporting of member' node in group, more accurate description of the targets / event then for monitoring is shown. When the members of the group increase, that is to say n_g increases, the management node consequentially increase the corresponding value of DOA, in order to obtain substantial benefits. It is good for both the energy savings and the accuracy of the information, and useful for the management node; in order to avoid excessive selfish of management node and setting too large values of DOA to get own interest which will lead to the large transmission delay of information, the network need to set the penalty factor to constrain the behavior of the management node. It is expressed as the second one $DOA^{f(TS)}$ in this model. While getting the benefit through the operations of integration, management node has to pay the appropriate price. The greater value of DOA, the delay will be greater, which means that while getting more revenue, management node also suffer the more punishment from the network. And in the real-time monitoring system, the moving speed of target/event is also the factors that must be considered. If the moving speed of goal is fast, then the propagation delay of information will be small. In the model, the index $f(TS)$ of DOA is an adjustment factor for the corresponding speed. $f(TS)$ will increase with the moving speed of the monitored target increases. When monitoring a fast moving target, the costs paid by the management node are higher than monitoring a low moving target. At this time, if the management node takes the greater the value of DOA, the punishment received grow faster, which is negative for the management node on the contrary. At this time, for the management node and network, the balance effectiveness is $\max((DOA\text{-}1)/n_g - DOA^{f(TS)})$. From the above discussion, we can see that the game model of management node can adjust the value of DOA according to the actual situation in the network to reach the balance between the interests of two sides, thereby improving overall effectiveness.

4.2 Game model of data packet forwarding

After fusing the collected data reporting of the members in group, management nodes need to forward packets through other nodes to the Sink node. In traditional routing in wireless sensor, we assume that all nodes are selfless, that is, when each sensor node receives a request of forwarding, it will accept the request and forward the received data packets. In order to extend the life cycle of sensor networks, this chapter describes a approach which use the self-serving nature of the nodes to balance the energy consumption of the network, making the energy consumption of network nodes in a balance state and the result is that the whole network will not split quickly.

We use game theory to solve the following conflicts of interest. The nodes in wireless sensor network are rational, which means there is certain selfishness and their actions are driven by self-interest. On the one hand, each node hopes that other nodes can't provide services of forwarding when it send data; the other hand, each node wants as little as possible on forwarding data for other nodes to reduce energy consumption. However, if all nodes are not willing to forward data for other nodes, then the connectivity of network will be a sharp decline, and even become non-connection; Moreover, the application background of this section is a real-time monitoring, so how to balance the energy while does not to cause too large delay is also a problem needed to be solved.

The game model of final stage for forwarding data described as follows:

1. Game participants

Game in the stage of data forwarding is defined as an extended two-person incomplete information game. The game participants are nodes and networks. For each node in the network, supposing the total number of transmitted data packets sent by this node to other nodes is $R_i(t)$, the number of successfully transmitted data packets sent by the network nodes for this node is $T_i(t)$; of these, $T_i(t)$ present that the number of successfully transmitted data packets of node i forwarded by other nodes in the network until the time t; $R_i(t)$ present that the number of transmitted data packets of node i forwarded by other nodes in the network until the time t. $f(TS)\cdot\lambda$ is the available delay compensation for agreeing to forward data packets.

2. The strategy set

This phase of the game is the extend game. For the extended game, the game participants can not predetermine a complete program of action. Participants' operations of every step are chosen based on the behavior of other participants before. In the game of this chapter, for this participant in network, the action of the node which can be taken includes accepting the forwarding request of the network to forward the data packets. At this time, the node can get the delay compensation from network. In a certain extent, such a mechanism encourage the nodes accept a forwarding request to reduce the forwarding delay of data packets; or deny the forwarding request of the network, which need to pay a certain price at the same time. Because the node refuse to forward the request means that a certain amount of delay is brought to the network. The action of the relative node can be taken include accepting the forwarding request of the node to forward the data packets, or refusing the forwarding request of node. Whether the node or network, decision of whether to accept the other's forwarding request is based on whether the other side forward a sufficient number of data packets for themselves and the corresponding delay compensation.

3. Utility function

From the perspective of each node, when the network forward packets successfully for this node, it means that the node obtain interest from the network. When the node accepts the forwarding request of the network to forward data packets for the network, it means that the node pay costs for the network. As the average number of hops α crossed by the exchange of data between the nodes and Sink nodes are no less than l, the benefits received after every successfully sending a own data packet is α times than the loss for forwarding a data packet for the network. This encourage the nodes in network involving in data forwarding; in addition, though the node's utility function is less than zero, if the node agree to forward the data packet, then it will get awards from the network, which is delay compensation, to encourage the node forwarding data; however, if the node refuse to forward data packets, then it will don't get the value of delay compensation, as a punishment to nodes from network.

As a result, the mathematical expression of utility function in the model of DR-G (Data Relaying base on Game Theory) is as follows:

$$U'(T_i(t),R_i(t)) = \alpha \times T_i(t) - R_i(t) + f(TS)\cdot\lambda$$

From above equation, we can introduce a decision function of node forwarding as follow, which is used to determine whether forward data for the other nodes.

$$\Delta'\left(T_i\left(t\right),R_i\left(t\right)\right) = \begin{cases} 1, & \alpha\times T_i(t)-R_i(t)+f(TS)\cdot\lambda\geq 0 \\ 0, & \alpha\times T_i(t)-R_i(t)+f(TS)\cdot\lambda< 0 \end{cases}$$

Where, $\alpha$is the average number of hops crossed by transmitting a data packet to the sink node, $f(TS)\cdot\lambda$ is the available delay compensation for agreeing to forward data packets. When the value of $\Delta(T_i(t),R_i(t))$ is 1, the intermediate node *i* agrees to forward; when the value of $\Delta(T_i(t),R_i(t))$ is 0, the node *i* refuses to forward.

4.3 Nash equilibrium of Game Theory model

The game model of forwarding a wireless sensor network's data packet was defined in the previous section, and in this section we will discuss that model. The main analysis of the content is that during the network operation the game model which was proposed above plays the role of the energy consumption of a balanced between the nodes with the passage of time. In which the delay compensation is different with the different target. Each goal is randomly independent of each other. The previous goals will not influence of the characteristics of a next target. Therefore, in the discussion does not involve the delay compensation of the model.

Wireless sensor networks which using the sensor nodes for forwarding decision function, for the network nodes *i*, there are

$$\lim_{t\to\infty}\sup\delta_i(t)\leq\frac{1}{\alpha+1} \tag{1.13}$$

In which, $\delta_i(t)$ means that until the time *t*, the proportion of the number of packets which send data packets of its' own successfully the proportion among total which the node *i* had sent, that

$$\delta_i\left(t\right)=\frac{T_i(t)}{T_i(t)+R_i(t)} \tag{1.14}$$

When the node's utility function value is zero, that: $\alpha \times T_i(t) - R_i(t) = 0$ The corresponding network participants' utility function value is also zero because it is a zero-sum game. At this point, if the network node received the packet request, it will refuse to forward. When $t \to \infty$, only after the node *i* had been forwarded at least αdata packets for the network, the network will re-forward the data for the node *i*. Before this there is $\alpha \times T_i(t) \leq R_i(t)$, added $T_i(t)$ both sides of this inequality, that

$$\alpha \times T_i(t) + T_i(t) \leq R_i(t) + T_i(t) \tag{1.15}$$

Into

$$\frac{T_i(t)}{T_i(t)+R_i(t)} \le \frac{1}{\alpha+1}, \quad \text{that is } \delta_i(t) \le \frac{1}{\alpha+1}$$

When $\alpha \times T_i(t) \ge R_i(t)$, the node i will forward data for other nodes, there are $(T_i(t)+1) \times \alpha \ge R_i(t)$. From this inequality can be derived $\alpha \cdot T_i(t) + \alpha + T_i(t) \ge R_i(t) + T_i(t)$, both sides are divided $T_i(t) + R_i(t)$, that

$$\frac{\alpha \cdot T_i(t)}{T_i(t)+R_i(t)} + \frac{\alpha}{T_i(t)+R_i(t)} + \frac{T_i(t)}{T_i(t)+R_i(t)} \ge 1 \tag{1.16}$$

Merger the first and third items of the left on the inequality, that

$$(\alpha+1)\delta_i(t) + \frac{\alpha}{T_i(t)+R_i(t)} \ge 1 \tag{1.17}$$

Then $\delta_i(t) \ge \frac{1}{\alpha+1} - \frac{\alpha}{(\alpha+1)(T_i(t)+R_i(t))}$, when $t \to \infty$, $\lim_{t\to\infty} \frac{1}{T_i(t)+R_i(t)} = 0$ and α is a finite integer, so there are

$$\lim_{t\to\infty} \delta_i(t) = \frac{1}{\alpha+1}$$

As can be seen from the above analysis, with the operation of the network over time, the network and the nodes converged at the Nash equilibrium point gradually, the two sides return to equilibrium. For the time $t \to \infty$, even the network gradually closed to the most advantage point of the overall performance, it will not affect the balance of return for various participants.

4.4 The application of model in forwarding process

This section will introduce how to use the game model for forwarding data packets by node to make the decision-making. Under considering the delay, we can do a better balance for the energy consumption of wireless sensor networks.

The previous routing algorithms of wireless sensor networks assume that when the node receives the data packets of other nodes in the network and requests its forwarding, the node will unconditionally accept the request and forward the data packet. In DR-G model, however, the node will priority to consider its own interest, and determine whether to forward packets through the decision-making function of the node forwarding.

To ensure that the data packet of the node is transmitted toward Sink node, in the network initialization phase, each sensor node adjust the distance between itself and the Sink nodes according to the received initialization message sent by Sink node, and set their level, while the Sink node is in the most "shallow" layer of the network (i.e., hop-count = 0). Adoption of this mechanism has the following advantages,

1. To guarantee a source node sends sensor data to Sink node directionally;
2. To adapt to characteristics of rapid changes in wireless sensor network topology. When the node failure, its child nodes can rapidly select the other nodes in the same floor as the parent node, without additional routing overhead;
3. Selected routing paths avoid routing loop issue.
4. Network topology is more stable. As shown in Figure 1.5.

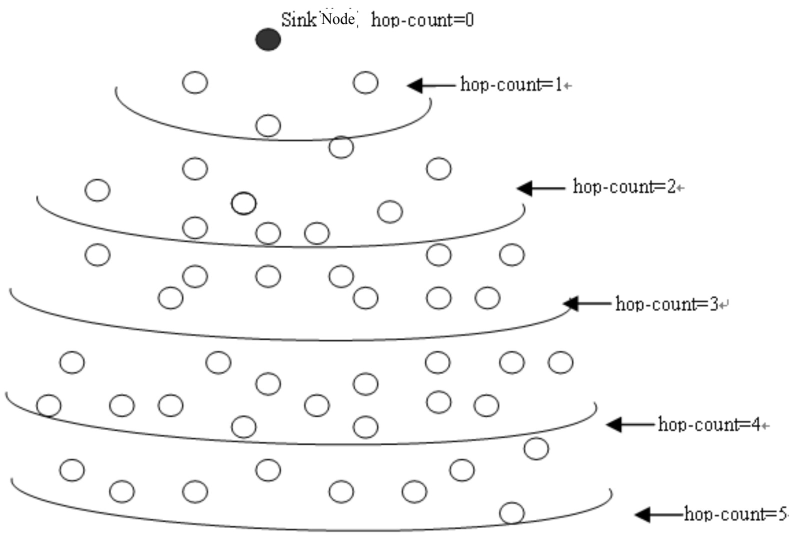

Fig. 1.5 Schematic diagram of layered wireless sensor network

For any one node in the network, the object requesting it to send the data packet includes two aspects: the data packet from the upper layer of the routing protocol, required to send to the other nodes in the network; the data packet which other nodes in network request for this node.

1. Send own data

When the node has demands for sending the data, first of all to send the request message to its previous direction neighbor nodes, the so-called previous direction is the nodes in wireless sensor network which is more shallow than their level, while the deeper nodes is not conducive to transmit data packets toward the Sink node due to the farther distance from the Sink nodes. After the previous direction neighbor nodes receive the message for requesting data, the node does the forwarding decisions according to the game model DR-G. The neighbor nodes which agreed to forward will returns a value of the feedback information with a utility function to the node requesting to send data. The node will choose the neighbor nodes of largest utility for data transmission. After data transmission, the node will have an additional one to $T_i(t)$, while the neighbor nodes of forwarding the data plus one to $R_i(t)$.

2. The other nodes request for forwarding data

When the node i receives the forwarding request of data packet, the first to determine by using the node forwarding decision-making function adding delayed compensation, if the output is 1, the node i will sent back a information of agreeing to forward data packets to the requesting node, and incidentally add the value of $U'(T_i(t), R_i(t))$ in this information. After receiving data packets needed to be transmitted and forwarding successfully, it will plus 1 to the value of $R_i(t)$, while the node which requests to forward data packets will plus 1 to its value of $T_i(t)$. If the output $\Delta'(T_i(t), R_i(t))$ is 0, then the node will refuse to forward packets for the network.

We can see from the above procedure, the node using the DR-G model to do the decision-making of forwarding is with full autonomy. When a node on the path aware that it has forwarded too much data packets for the network, the cost of the node for the utility function is too large, then the node will refuse to forward data packets, which can prevent leaving networks prematurely because of their large own energy consumption, which will also affect the normal data packet forwarding. At the same time, the introduction of delay compensation makes the node to forward data for the network during decision-making process, thus ensuring the data packet transmitted in real time.

5. Simulation and performance comparison and analysis

Through the front of the narrative, we know that wireless sensor networks consist of a large number of tiny sensor nodes deployed in the monitoring region, and forming a network system of multi-hop, self-organization by the methods of wireless communication. As the system is relatively complex, the study of wireless sensor networks is not easy to use the method of experimental analysis. TinyOS provides a powerful development language NesC, a comprehensive component library and network protocol stack. It is a architecture of component based, can quickly achieve a variety of applications, and use mainly in wireless sensor networks. In this chapter, we use the simulation tools TOSSIM embedded in the TinyOS to simulate, and do the performance of comparative analysis mainly from these two aspects of energy consumption and delay.

We use the application simulation platform TOSSIM whose open-source is based on TinyOS, and compare this reliable data fusion model RA-G to the classical data fusion routing DD and TEEN in wireless sensor networks in performance simulation. The operating system of experimental background is the virtual environment Cygwin of UNIX running on the Windows platform. In this section, we compare the data fusion model RA-G to the classical data fusion routing DD and TEEN in wireless sensor networks in performance simulation to measure the performance of RA-G.

Figure 1.6 compares the average energy consumption of the three methods in the network having 100 nodes in 2000s. As can be seen, in the beginning, the energy consumption of the integration model RA-G based on game theory is almost similar with DD and TEEN. However, with the operations of network, DD and TEEN gradually higher than the energy consumed by RA-G, such advantage will increase as the size of the network which becomes more apparent. This is mainly due to with the increases in network size, the interested proliferation of DD algorithm, the enhancement of multi-path and a cluster reconstruction work which require all nodes in the whole network to participate in TEEN algorithm will consume a large amount of energy. While in the RA-G, the energy consumption is mainly used by the node of participating target perception and needed to collect and integrate data, thus the average energy consumption rise marginally. Thus, RA-G can also well adapt to the changes in network size.

Figure 1.7 shows the comparison of the number of survival nodes in three methods with the simulation time of 1000s. When the simulation reaches 450 seconds or so later, the nodes of TEEN algorithm die quickly. As can be seen, the energy balance method of TEEN algorithm has played a certain role in energy balance, but the price is a little higher. In the DD, due to after increasing transmission delay in the shortest path, the data collected will forward along this path to the Sink node, which leads to the energy consumption between the nodes in network is extremely unbalanced, so the death rate of the node is faster. In the RA-G, the problem of the energy balance is fully taken into account. The results can be seen from the comparison, RA-G fusion model can effectively extend the network's normal working hours, to achieve the purposes of energy balance.

Figure 1.8 compares the real-time performance of RA-G model to DD and TEEN. Each curve is the average delay of data for the three method transfer under different network size when the running time is 1000s. Can be seen from the figure, with the increases of network size, the delay in DD and RA-G shows a rising trend, which is the same principle of the average energy consumption. Because the larger the network size, the path returning to Sink node for the data packet-by-hop is longer, and the delay in the transfer process will have a corresponding increase naturally.

However, since TEEN uses a hierarchical structure of the network for data fusion method, the time waiting for the cluster head's fusion is mainly delay, which is determine by the number of the node from the cluster. Although DD algorithm is use the enhanced shortest delay path for the data forwarding, the network is a better real-time performance in the early, so the data packet which forwarded through the enhanced path will get to the sink

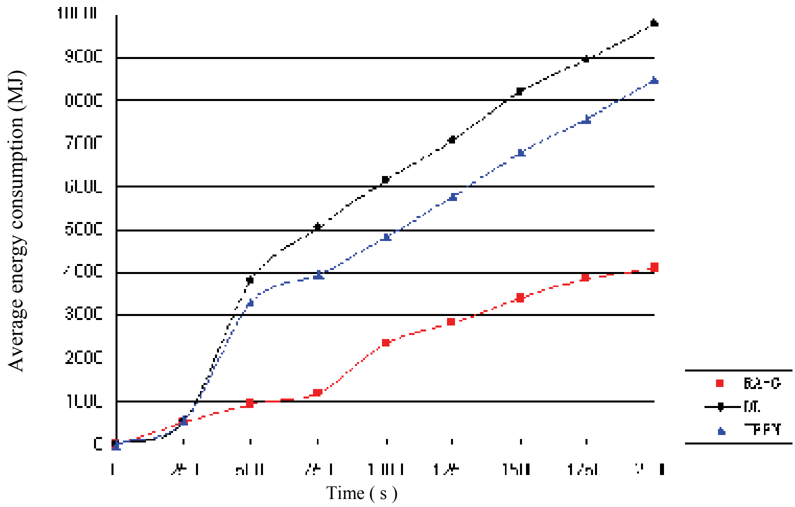

Fig. 1.6 Average energy consumption comparisons

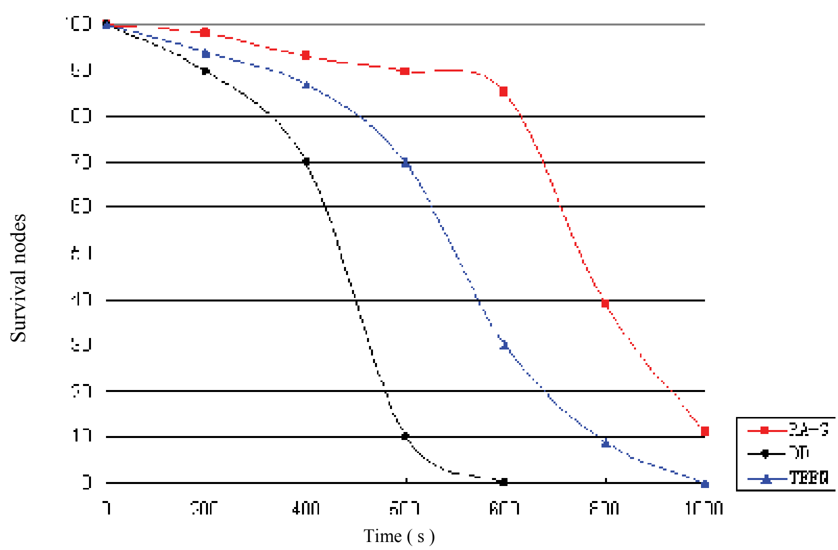

Fig. 1.7 Comparison of the number of survival nodes

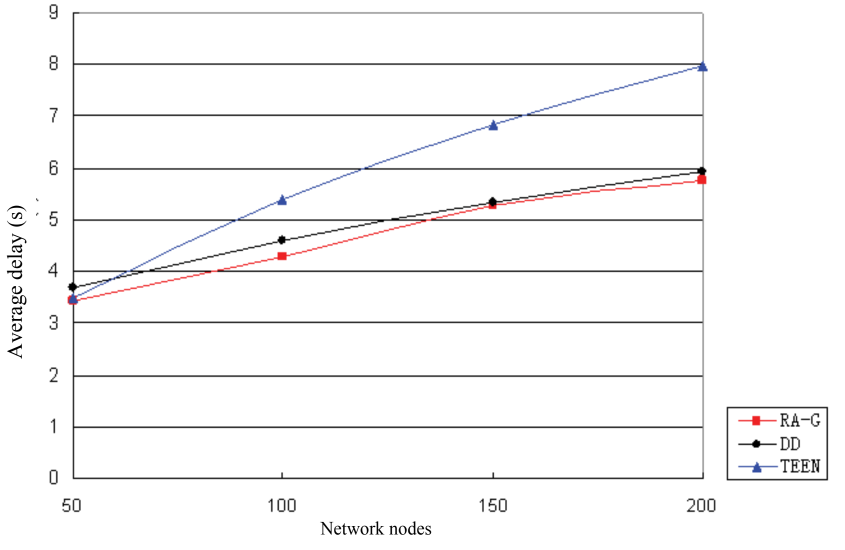

Fig. 1.8 Comparison of three methods delay

node with the shortest delay. However, with the network operations, the nodes on the enhance path consumed the energy too fast so that the lowest delay path can no longer continue to assume the task of forwarding data packets, the network had to choose another sub-optimal path to transfer data. The number of nodes which may be involved in data packet transmission is reduced, that will result in the delay become longer for data packet forwarding after the network operated for a period of time. So, taking into account the long-term stable operation of the network, DD algorithm does not highlight the real-time performance. Among the three methods, DD algorithm has large power consumption, and there is no mechanism for balanced energy consumption, the network's life cycle is shorter than TEEN and RA-G. The TEEN curve increases as the network grew rapidly. It can be concluded by observing and analyzing, that there is a delay less from the cluster head forwards the data packet to the sink node in TEEN algorithm. In the RA-G fusion mechanism, the data packets are forwarded to the sink node through multi-hop. According to game model to determine the process of forwarding, then the node use utility function to conduct merit-based routing. During this period it will bring some data packet transmission delay, the TEEN algorithm does not involve multi-hop data packet forwarding. So, TEEN data packet transfer delay is less than RA-G. But this is at the expense of a cluster head node's energy consumption. In the TEEN, the time waiting for the cluster head's fusion is always longer, because after the cluster head node allocated time slot to the cluster members, whether the members of the node want to send data or not, the other nodes are waiting for their time slot to sending data. This would give the system the too much of unnecessary delay. This trend will be more evident as the number of network nodes is increasing. As the network operation, due to TEEN need to do the cluster reorganization and the head cluster rotation in the whole network periodically, and each reorganization of cluster need to broadcast the new threshold, which will bring a lot of energy consumption to networks, in particular the head cluster node has a heavier burden. From the figure 1.12, we can see the death rate of the nodes of TEEN is faster than the RA-G in the latter part of the mechanism in the network. It has a negative impact for the reliability of the network. The accelerated death of the nodes lead to the network does not work, and the real-time reliable performance of a whole network degrades. While the RA-G can use GA-G fusion model in the group management node to dynamically determine the waiting time of regulation, and data packet forwarding game model DR-G can well balance energy consumption of each node in network while considering the delay. And multi-layer fusion mechanism can greatly reduce the traffic load of the network, effectively extend the life cycle of the network, and thus the data packet transmission delay can be stability in a long period.

From the above analysis we can see that in the network, the energy and latency are two interdependent and mutually constraining factors, only one aspect to be considered is not enough. RA-G fusion model consider both tow aspects at the same time and using the idea of game theory to build a balance model, effectively improve the network's overall performance.

6. Summary

This chapter primarily focuses on a reliable structure of data fusion RG-A of wireless sensor networks. Wireless sensor networks as a major form of mobile computing and treatment, so its position can not be replaced by other networks. Study on the Reliability about the Routing protocol for wireless sensor networks, which is the key to ensure that access to network robustness and reliability. It has a very high value and research value.

RG-A integration model is built based on game theory model for data fusion layer by layer. Nodes and network can be seen as rational actors and the two aspects of a conflict in game. Their utility function according to rational reasoning, through the game to balance the network parameters of the various constraints, so as to achieve a state of balanced, eventually achieved the purpose that to balance a real-time network and energy expenditure of the node. Not only improved the energy efficiency of the network but also to meet the target / event monitoring system for real-time reliability requirements.

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Inductive Game Theory: A Basic Scenario

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1. Introduction

1.1 General motivations

In game theory and economics it is customary to assume, often implicitly and sometimes explicitly, that each player has well formed beliefs/knowledge of the game he plays. Various frameworks have been prepared for explicit analyses of this subject. However, the more basic question of where a personal understanding of the game comes from is left unexplored. In some situations such as parlour games, it might not be important to ask the source of a player's understanding. The rules of parlour games are often described clearly in a rule book. However, in social and economic situations, which are main target areas for game theory, the rules of the game are not clearly specified anywhere. In those cases, players need some other sources for their beliefs/knowledge. One ultimate source for a player's understanding is his individual experiences of playing the game. The purpose of this paper is to develop and to present a theory about the origin and emergence of individual beliefs/knowledge from the individual experiences of players with bounded cognitive abilities.

People often behave naturally and effectively without much conscious effort to understand the world in which they live. For example, we may work, socialize, exercise, eat, sleep, without consciously thinking about the structure of our social situation. Nevertheless, experiences of these activities may influence our understanding and thoughts about society. We regard these experiences as important sources for the formation of an individual understanding of society.

Treating particular experiences as the ultimate source of general beliefs/knowledge is an inductive process. Induction is differentiated from deduction in the way that induction is a process of deriving a general statement from a finite number of observations, while deduction is a process of deriving conclusions with the same or less logical content with well-formed inference rules from given premises. Formation of beliefs/knowledge about social games from individual experiences is typically an inductive process. Thus, we will call our theory *inductive game theory*, as was done in Kaneko-Matsui [18]. In fact, economic theory has had a long tradition of using arguments about learning by experiences to explain how players come to know the structure of their economy. Even in introductory microeconomics textbooks, the scientific method of analysis is discussed: collecting data, formulating hypotheses, predicting, behaving, checking, and updating. Strictly speaking, these steps are applied to economics as a science, but also sometimes, less scientifically, to ordinary peoples' activities.

Our theory formalizes some part of an inductive process of an individual decision maker. In particular, we describe how a player might use his experiences to form a hypothesis about the rules and structure of the game. In the starting point of our theory, a player has little *a priori* beliefs/knowledge about the structure of the particular game. Almost all beliefs/knowledge about the structure of the particular game are derived from his experiences and memories.

A player is assumed to follow some regular behavior, but he occasionally experiments by taking some trials in order to learn about the game he plays. One may wonder how a player can act regularly or conduct experiments initially without any beliefs or knowledge. As mentioned above, many of our activities do not involve high brow analytical thoughts; we simply act. In our theory, some well defined *default action* is known to a player, and whenever he faces a situation he has not thought about, he chooses this action. Initially, the default action describes his regular behavior, which may be interpreted as a norm in society. The experimental trials are not well developed experiments, but rather trials taken to see what happens. By taking these trials and observing resulting outcomes from them, a player will start to learn more about the other possibilities and the game overall.

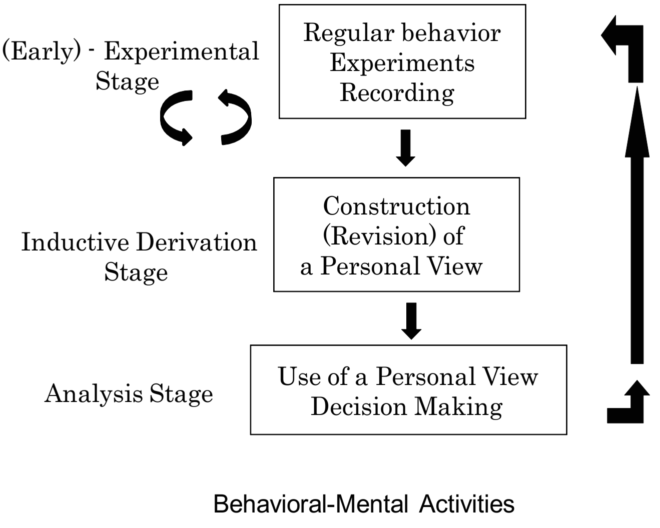

Fig. 1.1. Three stages of inductive game theory

The theory we propose has three main stages illustrated in Fig.1.1: the (early) *experimentation stage*; the *inductive derivation stage*; and the *analysis stage*. This division is made for conceptual clarity and should not be confused with the rules of the dynamics. In the experimentation stage, a player accumulates experiences by choosing his regular behavior and occasionally some alternatives. This stage may take quite some time and involve many repetitions before a player moves on to the inductive stage. In the inductive derivation stage he constructs a view of the game based on the accumulated experiences. In the analysis stage, he uses his derived view to analyze and optimize his behavior. If a player successfully passes through these three stages, then he brings back his optimizing behavior to the objective situation in the form of a strategy and behaves accordingly.

In this paper, we should stop at various points to discuss some details of each of the above stages. Since, however, our intention is to give an entire scenario, we will move on to each stage sacrificing a detailed study of such a point. After passing through all three stages, the player may start to experiment again with other behaviors and the experimentation stage starts again. Experimentation is no longer early since the player now has some beliefs about the game being played. Having his beliefs, a player may now potentially learn more from his experiments. Thus, the end of our entire scenario is connected to its start.

While we will take one player through all the stages in our theory, we emphasize that other players will experiment and move through the stages also at different times or even at the same time. The precise timing of this movement is not given rigorously. In Section 7.2 we give an example of how this process of moving through these stages might occur. We emphasize that experiments are still infrequent occurrences, and the regular behavior is crucial for a player to gain some information from his experiments. Indeed, if all players experiment too frequently, little would be learned.

We should distinguish our theory from some approaches in the extant game theory literature. First, we take up the type-space approach of Harsanyi [10], which has been further developed by Mertens-Zamir [24] and Brandenburger-Dekel [4]. In this approach, one starts with a set of parameter values describing the possible games and a description of each player's "probabilistic" beliefs about those parameters. In contrast, we do not express beliefs/knowledge either by parameters or by probabilities on them. In our approach, players' beliefs/knowledge are taken as structural expressions. Our main question is how a player derives such structural expressions from his accumulated experiences. In this sense, our approach is very different.

Our theory is also distinguished from the fields with the titles of evolution/learning/ experiment (cf., Weibull [31], Fudenberg-Levine [7], Kalai-Lehrer [12], and more generally, Camerer [5]) and the case-based decision theory of Gilboa-Schmeidler [8]. Those theories are typically interested in adjustment/convergence of actions to some equilibrium; they do not address questions on how a player learns the rules/structure of the game. Some of them extend payoff functions to fit predictions by the theory to observed experimental results. Case-based decision theory looks more similar to ours. This theory focuses on how a player uses his past experiences to predict the consequences of an action in similar games. Unlike our theory, it does not discuss the emergence of beliefs/knowledge on social structures.

Rather than the above mentioned literature, our theory is reminiscent of some philosophical tradition on induction. Both Francis Bacon [2] and Hume [11] regard individual experience as the ultimate source of our understanding nature, rather than society. Our theory is closer to Bacon than Hume in that the target of understanding is a structure of nature in Bacon, while Hume focussed on similarity. In this sense, the case-based decision theory of Gilboa-Schmeidler [8] is closer to Hume. Another point relevant to the philosophy literature is that in our theory, some falsities are inevitably involved in a view constructed by a player from experiences and each of them may be difficult to remove. Thus, our discourse does not give a simple progressive view for induction. This is close to Thomas Kuhn's [22] discourse of scientific revolution (cf. also Harper-Schulte [9] for a concise survey of related works).

1.2 Treatments of memories and inductive processes

Here, we discuss our treatment of memory and induction in more detail. A player may, from time to time, construct a personal view to better understand the structure of some objective game. His view depends on his past interactions. The entire dynamics of a player's interactions in various objective games is conceptually illustrated in the upper diagram of Fig.1.2. Here, each particular game is assumed to be described by a pair $(\Gamma, \mathfrak{m})$ of an n-person objective extensive game Γ and objective *memory functions* $\mathfrak{m} = (\mathfrak{m}_1,...,\mathfrak{m}_n)$. Different superscripts here denote different objective games that a player might face, and the arrows represent the passing of time. This diagram expresses the fact that a player interacts in different games with different players and sometimes repeats the same games.

We assume that a player focuses on a particular game situation such as $(\Gamma^1, \mathfrak{m}^1)$, but he does not try to understand the entire dynamics depicted in the upper diagram of Fig.1.2. The situation $(\Gamma^1, \mathfrak{m}^1)$ occurs occasionally, and we assume that the player' behavior depends only upon the situation and he notices its occurrence when it occurs. By these assumptions, the dynamics are effectively reduced into those of the lower diagram of Fig.1.2. His target is the particular situation $(\Gamma^1, \mathfrak{m}^1)$. In the remainder of the paper, we denote a particular situation $(\Gamma^1, \mathfrak{m}^1)$ under our scrutiny by $(\Gamma^o, \mathfrak{m}^o)$, where the superscript "o" means "*objective*". We use the superscript i to denote the inductively derived personal view $(\Gamma^i, \mathfrak{m}^i)$ of player i about the objective situation $(\Gamma^o, \mathfrak{m}^o)$.

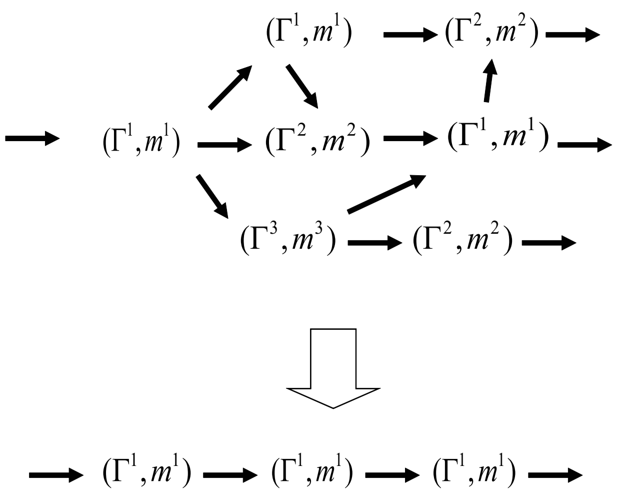

Fig. 1.2. Various social situations

The *objective* memory function $\mathfrak{m}_i^o$ of player i describes how the raw experiences of playing Γ^o are perceived in his mind. We refer to these memories as *short-term* memories and presume that they are based on his observations of *information pieces* and *actions* while he repeatedly plays Γ^o. The "information pieces" here correspond to what in game theory are typically called "information sets", and they convey information to the player about the set of available actions at the current move and perhaps some other details about the current environment. Our use of the term "piece" rather than "set" is crucial for inductive game theory and it is elaborated on in Section 2.

An objective short-term memory $\mathfrak{m}_i^o(x)$ for player i at his node (move) x consists of sequences of pairs of information pieces and actions as depicted in Fig.1.3. In this figure, a single short-term memory consists of three sequences and describes what, player i thinks, might have happened prior to the node x in the current play of Γ^o. In his mind, any of these

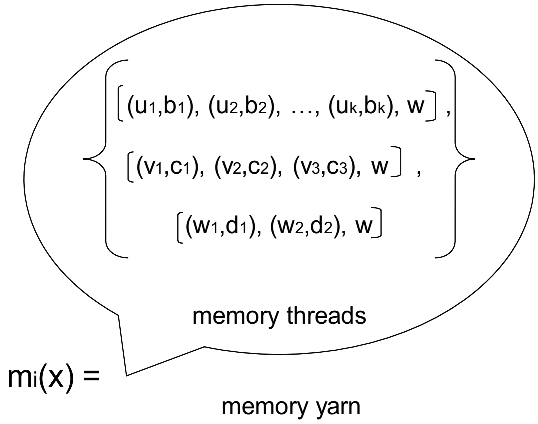

Fig. 1.3. Local memory - short-term memory

sequences could have happened and the multiplicity may be due to forgetfulness. We will use the term *memory thread* for a single sequence, and *memory yarn* for the value ("set of memory threads") of the memory function at a point of time.

One role of each short-term memory value $\mathfrak{m}_i^o(x)$ is for player i to specify an action depending upon the value while playing Γ^o. The other role is the source for a *long-term memory*, which is used by player i to inductively derive a personal view (Γ^i, $\mathfrak{m}^i$).

The objective record of short-term memories for player i in the past is a long sequence of memory yarns. A player cannot keep such an entire record; instead, he keeps short-term memories only for some time. If some occur frequently enough, they change into long-term memories; otherwise, they disappear from his mind. These long-term memories remain in his mind as *accumulated memories,* and become the source for an inductive derivation of a view on the game. This process will be discussed in Section 3.

The induction process of player i starts with a *memory kit*, which consists of the set of *accumulated threads* and the set of *accumulated yarns*. The accumulated threads are used to inductively derive a subjective game Γ^i, and the yarns may be used to construct his subjective memory function $\mathfrak{m}^i$. This inductive process of deriving a personal view is illustrated in Fig.1.4.

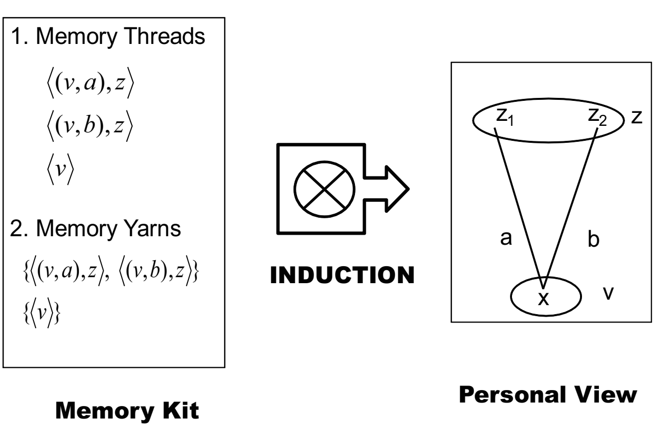

Fig. 1.4. Inductive derivation

In this paper, we consider one specific procedure for the inductive process, which we call the *initial-segment procedure*. This procedure will be discussed formally in Section 4.

1.3 The structure of the present paper

This paper is divided into three parts:

Part I: Background, and basic concepts of inductive game theory. Sections 1 - 3. Section 1 is now describing the motivation, background, and a rough sketch of our new theory. We will attempt, in this paper, to give a basic scenario of our entire theory. The mathematical structure of our theory is based on extensive games. Section 2 gives the definition of an extensive game in two senses: strong and weak. This distinction will be used to separate the objective description of a game from a player's subjective view, which is derived inductively from his experiences. Section 3 gives an informal theory of accumulating long-term memories, and a formal description of the long-term memories as a memory kit.

Part II: Inductive derivation of a personal view. Sections 4 - 6. In Section 4, we define an inductively derived personal view. We do not describe the induction process entirely. Rather, we give conditions that determine whether or not a personal view might be inductively derived from a memory kit. Because we have so many potential views, we define a direct view in Section 5, which turns out to be a representative of all the views a player might inductively derive (Section 6).

Part III: Decision making using an inductively derived view. Sections 7 - 9. In this part, we consider each player's use of his derived view for his decision making. We consider a specific memory kit which allows each player to formulate his decision problem as a 1-person game. Nevertheless, this situation serves as an experiential foundation of Nash equilibrium. This Nash equilibrium result, and more general issues of decision making, are discussed in Sections 7 and 8.

Before proceeding to the formal theory in Section 2, we mention a brief history of this paper and the present state of inductive game theory. The original version was submitted to this journal in January 2006. We are writing the final version now two and a half years later in July 2008. During this period, we have made several advancements in inductive game theory, which have resulted in other papers. The results of the present paper stand alone as crucial developments in inductive game theory. Nevertheless, the connection between the newer developments and this paper need some attention. Rather than to interrupt the flow of this paper, we have chosen to give summaries and comments on the newer developments in a postscript presented as Section 9.3.

2. Extensive games, memory, views, and behaviour

To describe a basic situation like $(\Gamma^1,\mathfrak{m}^1)$ in Fig.1.2, we will use an n-person extensive game Γ^1 and memory functions $\mathfrak{m}^1 = (\mathfrak{m}_1^1, ..., \mathfrak{m}_n^1)$. We follow Kuhn's [21] formulation of an extensive game to represent Γ^1, except for the replacement of information sets by information pieces.[1] This replacement is essential for inductive game theory. We use extensive games in the strong and weak senses to model the objective game situation and

[1] There are various formulations of extensive games such as in von Neumann-Morgenstern [32], Selten [30], Dubey-Kaneko [6], Osborne-Rubinstein [27] and Ritzberger [29]. Those are essentially the same formulations, while Dubey-Kaneko [6] give a simultaneous move form.

the inductively derived view of a player, which are given in Section 2.1. The memory functions $\mathfrak{m}_1^1$, ..., $\mathfrak{m}_n^1$ will be described in Section 2.2. Then, we formally define an objective description $(\Gamma^1,\mathfrak{m}^1)$ and a personal view $(\Gamma^i,\mathfrak{m}^i)$ of player i in Section 2.2. In Section 2.3 we give a formal definition of a behavior pattern (strategy configuration) for the players.

2.1 Extensive games

Our definition of an extensive game in the strong sense differs from that of Kuhn [21] mainly in that the information sets of Kuhn are replaced by information pieces. This difference is essential from the subjective point of view, though it is less essential from the objective point of view. An extensive game in the weak sense differs more substantially from an extensive game of Kuhn.

For notational simplicity, we sometimes make use of a function with the empty domain, which we call an *empty function*. When the empty domain and some (possibly nonempty) region are given, the empty function is uniquely determined.

Definition 2.1 (Extensive games). An *extensive game in the strong sense* $\Gamma = ((X,<),(\lambda,W),\{(\varphi_x,A_x)\}_{x\in X},(\pi,N),h)$ is defined as follows:

K1(*Game Tree*): $(X,<)$ is a finite forest (in fact, a tree by K14);

K11: X is a finite non-empty set of nodes, and $<$ is a partial ordering over X;

K12: the set $\{x \in X : x < y\}$ is totally ordered with $<$ for any $y \in X$;[2]

K13: X is partitioned into the set X^D of *decision nodes* and the set X^E of *endnodes* so that every node in X^D has at least one successor, and every node in X^E has no successors;[3]

K14: X has the smallest element x_0, called the *root*.[4]

K2(*Information Function*): W is a finite set of information pieces and $\lambda : X \to W$ is a surjection with $\lambda(x) \neq \lambda(z)$ for any $x \in X^D$ and $z \in X^E$;

K3(*Available Action Sets*): A_x is a finite set of *available actions* for each $x \in X$;

K31: $A_x = \emptyset$ for all $x \in X^E$;

K32: for all $x, y \in X^D$, $\lambda(x) = \lambda(y)$ implies $A_x = A_y$;

K33: for any $x \in X$, φ_x is a bijection from the set of immediate successors[5] of x to A_x;

K4(*Player Assignment*): N is a finite set of players and $\pi: W \to 2^N$ is a player assignment with two conditions;

K41: $|\pi(w)| = 1$ if $w \in \{\lambda(x) : x \in X^D\}$ and $\pi(w) = N$ if $w \in \{\lambda(x) : x \in X^E\}$;

K42: for all $j \in N$, $j \in \pi(w)$ for some $w \in \{\lambda(x) : x \in X^D\}$;

K5(*Payoff functions*): $h = \{h_i\}_{i\in N}$, where $h_i : \{\lambda(x) : x \in X^E\} \to R$ is a payoff function for player $i \in N$.

Bijection φ_x associates an action with an immediate successor of x. Game theoretically, it names each branch at each node in the tree. When x is an endnode, φ_x is the empty function. Since A_x is empty, too, by K31, φ_x is a bijection.

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2 The binary relation $<$ is called a *partial ordering* on X iff it satisfies (i)(irreflexivity): $x \not< x$; and (ii)(transitivity): $x < y$ and $y < z$ imply $x < z$. It is a *total* ordering iff it is a partial ordering and satisfies (iii)(totality): $x < y$, $x = y$ or $y < x$ for all $x, y \in X$.

3 We say that y is a *successor* of x iff $x < y$, and that y is an *immediate* successor of x, denoted by $x <^I y$, iff $x < y$ and there is no $z \in X$ such that $x < z$ and $z < y$.

4 A node x is called the *smallest* element in X iff $x < y$ or $x = y$ for all $y \in X$.

5 The reason for the bijection from immediate successor to actions, rather than from actions to immediate successors will be found in $K33_0$ below.

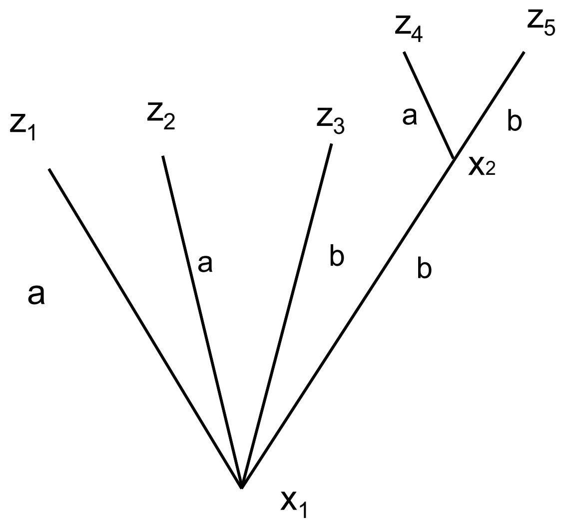

Fig. 2.1. Violation of condition K33.

When K14 (root) is dropped, and K33 (bijection) and K5 (payoffs) are replaced by the following weaker requirements, we say that Γ is an *extensive game in the weak sense*:

K33$_0$: for any $x \in X$, φ_x is a function from the set of immediate successors of x to A_x.

K5$_0$: $h : \{\lambda(x) : x \in X^E\} \to R$ is a payoff function for player i.

Since X may not have the smallest element, $(X,<)$ is not necessarily a tree. However, $(X,<)$ is divided into several connected parts. We can prove that each maximal connected subset of $(X,<)$ is a tree. Thus, $(X,<)$ is a class of trees, i.e., a forest. For any $x \in X$, there is a unique *path* to x, i.e., each maximal set $\{x_1, ..., x_{m+1}\}$ with $x_t < x_{t+1}$ for $t = 1, ...,m$ and $x_{m+1} = x$. When x is an endnode, we will call the path to x a *play*.

In an extensive game in the weak sense, an action a at a node x may not uniquely determine an immediate successor. See Fig.2.1, which will be discussed as a derived view in Section 4.1. The converse, however, that an immediate successor determines a unique action, does hold by K33$_0$. Thus, we can define: $x <_a^I y$ iff $x <^I y$ and $\varphi_x(y) = a$, which means that y is an immediate successor of x via action a. Then, we define $x <_a y$ iff there is some y' such that $x <_a^I y'$ and ($y' = y$ or $y' < y$).

We will use an extensive game in the strong sense as an objective description of a social situation we target, e.g., $\Gamma^o = \Gamma^1$ in Fig.1.2. An extensive game in the weak sense will be used for a personal view inductively derived from experiences. The latter differs from the former in several respects, besides the one mentioned above. First, we take the payoffs as personal and assume that a player's personal view does not include the payoffs of other players. Hence, condition K5 is weakened to K5$_0$ for a personal view. Dropping the root assumption and weakening K33 are more substantial changes. We will see in Section 4 why such changes are needed when we derive a personal view.

For an extensive game in the weak or strong sense, condition K32 implies that the set of available actions at a node x is determined by the information piece $w = \lambda(x)$. Thus, we may write A_w or $A_{\lambda(x)}$ rather than A_x.

An extensive game in the strong sense is the same as that given in Kuhn [21], except that we use information pieces W, rather than information sets. When the structure of Γ is known, information sets are defined by information pieces, i.e., $\{x : \lambda(x) = w\}$ for $w \in W$. In this sense, our definition of an extensive game is essentially the same as Kuhn's formulation from the objective point of view. However, the replacement of information sets by information pieces is substantive from the subjective point of view for our inductive game theory.

For the purpose of comparisons, we first mention the standard interpretation of the theory of extensive games due to Kuhn [21] (also, cf., Luce-Raiffa [23], Section 3.6). The interpretation is summarized as follows:

(**Full cognizance**): each player is fully cognizant of the game structure;

(***Ex Ante*** **decision**): each player makes a strategy choice before the actual play of the game.

Under (i), when a player receives an information piece w, he can infer the information set $\{x : \lambda(x) = w\}$ corresponding to piece w. Interpretation (i) is usually assumed so as to make (ii) meaningful. This will be discussed in the end of this subsection.

In the inductive context described in Section 1, the assumption (i) is dropped. Instead, players learn some part of the game structure by playing the game. Early on, a player may not infer at all the set of possible nodes having information piece w. To explain such differences, we use one small example of an extensive game, which we will repeatedly use to illustrate new concepts.

Example 2.1. Consider the extensive game depicted in Fig.2.2. It is an example of a 2-person extensive game. Player 1 moves at the root x_0, and then at the node x_3 if it is reached. Player 2 moves at x_1 or x_2 depending on the choice of player 1 at x_0. The information function assigns $\lambda(x_0) = w$, $\lambda(x_1) = \lambda(x_2) = v$, $\lambda(x_3) = u$. At the endnodes, z_1, z_2, z_3, z_4, z_5, the information function is the identity function, i.e., $\lambda(z_t) = z_t$ for $t = 1, ..., 5$. At endnode z_4 the payoffs to players 1 and 2 are $(h_1(z_4), h_2(z_4)) = (0, 1)$.

In Kuhn's interpretation, each player has the knowledge of the game tree. In Fig.2.2, for example, when player 2 receives information piece v, he can infer that either x_1 or x_2 is possible, which means that he knows the information set $\{x_1, x_2\}$.

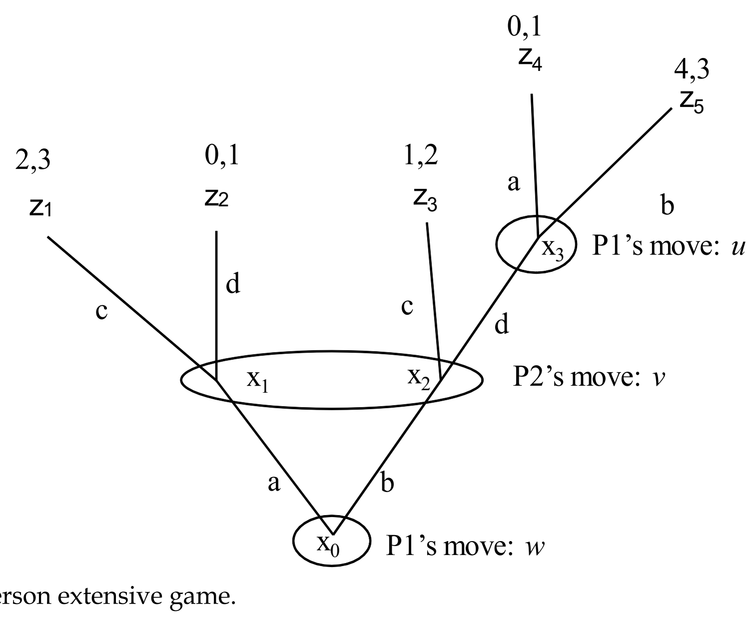

Fig. 2.2. 2-person extensive game.

Under the inductive interpretation, when player 2 receives information piece v, he may not come to either of the conclusions mentioned in the previous paragraph. He might not even be aware of the existence of player 1 - - player 1 may think that the structure could be like Fig.2.1. In such a case, piece v does not imply the information set $\{x_1, x_2\}$ and the choices by player 1 either. Thus, in the inductive situation, receiving information piece v may be totally different from knowing the corresponding information set.

The above consideration suggests that there are multiple interpretations of the knowledge a player gets from an information piece. Here, we specify the minimal content a player gets from each information piece w in Γ:

M1: the set A_w of available actions;

M2: the value $\pi(w)$ of the player assignment π if w is a decision piece;

M3: his own payoff $h_i(w)$ (as a numerical value) if w is an endpiece.

These are interpreted as being written on each piece w. These conditions will be discussed further when we consider some specific memory functions in Section 2.2 and the inductive derivation of a view in Section 4.

Let us return to (i) and (ii) of the standard interpretation of an extensive game given by Kuhn [21]. In our inductive game theory, since we drop the cognizance assumption (i), the *ex ante* decision making of (ii) does not make sense before an individual constructs a view of the game. We presume that until he constructs a view, he follows some regular behavior and makes occasional trials in an effort to learn the game he is playing. At some point of time, he will try to construct a view based on his accumulated memories of his experiences. Once a view is constructed, it may then be used by the player to construct an optimal strategy for future plays.

2.2 Memory functions and views

It is standard in the literature of extensive games to describe the memory ability of a player in terms of information sets (cf. Kuhn [21]). This does not separate the role of an information piece (set) as information transmission from the role of an individual memory capability. In our inductive game theory, the treatment of various types of memories is crucial, and thus, we need an explicit formulation of individual memories in addition to an extensive game. For this reason, we introduce the concept of a memory function, which describes short-term memories of a player within a play of an extensive game.

A memory function expresses a player's short-term memory about the history of the current play of a game. Let $\Gamma = ((X,\prec), (\lambda,W), \{(\varphi_x,A_x)\}_{x\in X}, (\pi,N), h)$ be an extensive game in the weak or strong sense. Recall that for each node $x \in X$, there is a unique path to x which is denoted by $\langle x_1, ..., x_{m+1}\rangle$ with $x_{m+1} = x$. Also, the actions taken at $x_1, ..., x_m$ on the path to x are uniquely determined, i.e., for each $t = 1, ...,m$, there is a unique $a_t \in A_{x_t}$ satisfying $\varphi_{x_t}(x_{t+1}) = a_t$. We define the *complete history of information pieces and actions up to* x by

$$\theta(x) = \langle(\lambda(x_1),a_1),...,(\lambda(x_m),a_m),\lambda(x_{m+1})\rangle. \tag{2.1}$$

The history $\theta(x)$ consists of observable elements for players, while the path $\langle x_1, ..., x_{m+1}\rangle$ to x consists of unobservables for players. Memories will be defined in terms of these observable elements.

A short-term memory consists of memory threads, which look somewhat like the historical sequence $\theta(x)$. However, we allow a player to be forgetful, which is expressed by incomplete threads or multiple threads. Formally, a *memory thread* is a finite sequence

$$\mu = \langle (v_1, a_1), ..., (v_m, a_m), v_{m+1} \rangle, \tag{2.2}$$

where

$$v_t \in W,\ a_t \in A_{v_t} \text{ for all } t = 1,...,m \text{ and } v_{m+1} \in W. \tag{2.3}$$

Each component (v_t, a_t) ($t = 1, ...,m$) or v_{m+1} in μ is called a *memory knot*. A finite nonempty set of memory threads is called a *memory yarn*. See Fig.1.3 for an illustration of these concepts. Now, we have the definition of a memory function.

Definition 2.2 (Memory Functions). We say that a function $\mathfrak{m}_i$ is a *memory function* of player i iff for each node $x \in X_i = \{x \in X : i \in \pi \cdot \lambda(x)\}$, $\mathfrak{m}_i(x)$ is a memory yarn satisfying:

$$w = \lambda(x) \text{ for all } \langle \xi, w \rangle \in \mathfrak{m}_i(x). \tag{2.4}$$

The memory function $\mathfrak{m}_i$ gives a memory yarn consisting of a finite number of memory threads at each node for player i. The multiplicity of threads in a yarn describes uncertainty at a point in time about the past.

In Fig.1.3, the memory yarn $\mathfrak{m}_i(x)$ consists of three memory threads. The first one is a long one, the second and third are memory threads of short lengths. Condition (2.4) states that the tails of any memory threads at a node x are identical to the correct piece $w = \lambda(x)$. This is interpreted as meaning that the player correctly perceives the current information piece.

Here, we mention four classes of memory functions and one specific one. In the first memory function, which is the self-scope perfect-recall memory function, player i recalls what information he received during the current game and what actions he took, but nothing about the other players. For this example, we define player i's own history: For a node $x \in X_i$, let $\theta(x) = \langle \lambda(x_1), a_1), ..., (\lambda(x_m), a_m), \lambda(x_{m+1}) \rangle$, and let $\langle x_{k1}, ..., x_{kl}, x_{kl+1} \rangle$ be the i-part of $\langle x_1, ..., x_m, x_{m+1} \rangle$, i.e., the maximal subsequence of nodes in the path $\langle x_1, ..., x_m, x_{m+1} \rangle$ to x satisfying $i \in \pi \cdot \lambda(x_{k_t})$ for $t = 1, ..., l+1$. Then we define *player i's* (objective) *history of information pieces and actions up to x* by

$$\theta(x)_i = \langle (\lambda(x_{k_1}), a_{k_1}), ..., (\lambda(x_{k_l}), a_{k_l}), \lambda(x_{k_{l+1}}) \rangle. \tag{2.5}$$

(1) Self-scope[6] perfect-recall memory function: It is formulated as follows:

$$\mathfrak{m}_i^{spr}(x) = \{\theta(x)_i\} \text{ for each } x \in X_i. \tag{2.6}$$

With the memory function $\mathfrak{m}_i^{spr}$, player i recalls his own information pieces and actions taken in the current play of the game. This memory function will have a special status in the discourse of this paper. In the following, we call $\mathfrak{m}_i^{spr}$ the *SPR function*.

In Fig.2.2, the SPR function $\mathfrak{m}_1^{spr}$ for player 1 is given as:

$$\begin{aligned} \mathfrak{m}_1^{spr}(x_0) &= \{\langle w \rangle\}, \text{ and } \mathfrak{m}_1^{spr}(x_3) = \{\langle (w,b), u \rangle\}; \\ \mathfrak{m}_1^{spr}(z_t) &= \{\langle (w,a), z_t \rangle\} \text{ for } t = 1,2, \text{ and } \mathfrak{m}_1^{spr}(z_3) = \{\langle (w,b), z_3 \rangle\}; \\ \mathfrak{m}_1^{spr}(z_4) &= \{\langle (w,b), (u,a), z_4 \rangle\} \text{ and } \mathfrak{m}_1^{spr}(z_5) = \{\langle (w,b), (u,b), z_5 \rangle\}. \end{aligned} \tag{2.7}$$

6 We have chosen the name self-scope to mean that he has only himself in his his scope. Of course we allow for perfect recall memory functions where the player has other player's in his scope.

At node x_3, player 1 receives piece u and recalls his choice b at w. By the minimal requirement M1, he knows the available actions $A_w = \{a, b\}$ and $A_u = \{a, b\}$. Without adding any other source than $\mathfrak{m}_1^{spr}$, player 2 does not appear in the scope of player 1. It will be discussed that Fig.2.1 is an inductively derived view in this example.

The next example is the Markov memory function. As its name suggests, a player recognizes only the present piece and forgets all after he moves.

(2) Markov memory function: It is formulated as

$$\mathfrak{m}_i^M(x) = \{\lambda(x)\} \text{ for each } x \in X_i. \tag{2.8}$$

It gives only the present information piece. Nonetheless, by the minimal requirement M1, the player can extract his available action set $A_{\lambda(x)}$ whenever he receives an information piece $\lambda(x)$.

For both $\mathfrak{m}_i^{spr}$ and $\mathfrak{m}_i^M$, we would have no difficulty in presuming that each player only receives his own information pieces and gets the minimal information described by M1, M2 and M3. As we will see now, some other memory functions provide a player with information about some other players' information pieces and actions. The first such memory function is the perfect-information memory function.

(3) Perfect-information memory function: This is formulated as

$$\mathfrak{m}_i^{PI}(x) = \{\theta(x)\} \text{ for each } x \in X_i. \tag{2.9}$$

Recall that $\theta(x)$ is given by (2.1). Thus, if player i has this memory function, he recalls the perfect history even including the other players' pieces. By M1 and M2, he also knows the available actions and the player who moves at each decision piece.

There are at least two possible interpretations of how he comes to know the perfect history. One interpretation is that player i observes other players' moves as the game is played. Another interpretation is that player i's information pieces contain the complete history, i.e., $\theta(x)$ is written on piece $\lambda(x)$. Under either interpretation, a player gets more than the minimal amount of information described in M1-M3.

The next memory function typically gives a player less information than the perfect information memory function.

(4) Classical memory function: This memory function is formulated as

$$\mathfrak{m}_i^C(x) = \{\theta(y) : y \in X_i \text{ and } \lambda(y) = \lambda(x)\} \text{ for each } x \in X_i. \tag{2.10}$$

Observe that this function gives player i the set of complete histories up to nodes with his current information piece. The multiplicity of memory threads can be interpreted as some ambiguity about the past. This memory function can also be interpreted in the ways suggested for $\mathfrak{m}_i^{PI}$. We should mention yet another interpretation which is the motivation for the name "classical" memory function. In this interpretation, player i knows the structure of the extensive game. Consequently, he can infer the set of possible complete histories compatible with the present information piece. The classical memory function together with this interpretation is less compatible with our inductive game theory than the other memory functions. Since it is still mathematically allowed and is closer to the classical game theory, we consider it.

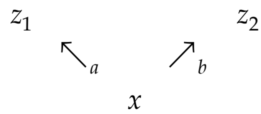

Fig. 2.3. False memory

The general definition of a memory function allows it to even involve false components. We give one example of false memories using the following simple extensive game. Consider the 1-person extensive game (Γ, $\mathfrak{m}_1$) depicted as Fig.2.3 with the identity information function.

A false memory function $\mathfrak{m}_1$ is given as:

$$\mathfrak{m}_1(x) = \{\langle x \rangle\}, \mathfrak{m}_1(z_1) = \{\langle (x,a), z_1 \rangle\} \text{ and } \mathfrak{m}_1(z_2) = \{\langle (x,a), z_2 \rangle\}. \tag{2.11}$$

This $\mathfrak{m}_1$ takes a false value at z_2, at which player 1 incorrectly recalls having chosen a at x though he actually chose b at x.

Having described an extensive game and memory functions, we now have the basic ingredients for objective descriptions and subjective personal views.

(Objective description): A pair (Γ^o,$\mathfrak{m}^o$) is called an *objective description* iff Γ^o is an extensive game in the strong sense and $\mathfrak{m}^o = (\mathfrak{m}_1^o, \ldots, \mathfrak{m}_n^o)$ is an n-tuple of memory functions in Γ^o.

We use the superscript o to denote the objective description. We will put a superscript i to denote a personal view of player i.

(Personal view): A pair (Γ^i,$\mathfrak{m}^i$) is a *personal view* for player i iff Γ^i is an extensive game in the weak sense specifying only the payoff function of player i, and $\mathfrak{m}^i$ is a memory function for player i in Γ^i.

A personal view (Γ^i,$\mathfrak{m}^i$) of player i describes the game player i believes he is playing. Since his belief is based on his experiences, we do not include the memory functions or payoffs of other players. We regard payoff values and memory values as personal.[7]

2.3 Behavior patterns

Let $\Gamma = ((X,<),(\lambda,W),\{(\varphi_x, A_x)\}_{x \in X},(\pi,N),h)$ be an an extensive game in the weak or strong sense and let $\mathfrak{m}_i$ be a memory function for player $i \in N$. The extensive game and memory function may be either the objective description or a personal view. We give a definition of a behavior pattern to be applied to both cases.

We say that a function σ_i on the set of nodes $X_i^D := \{x \in X^D : i \in \pi \cdot \lambda(x)\}$ is a *behavior pattern* (*strategy*) of player i iff it satisfies conditions (2.12) and (2.13):

$$\text{for all } x \in X_i^D,\ \sigma_i(x) \in \{a \in A_x : \varphi_x(y) = a \text{ for some } y \in X\}; \tag{2.12}$$

$$\text{for all } x, y \in X_i^D,\ \mathfrak{m}_i(x) = \mathfrak{m}_i(y) \text{ implies } \sigma_i(x) = \sigma_i(y). \tag{2.13}$$

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[7] As stated several times, we regard this as an alternative assumption adopted in the present discourse. This can be extended to include other players as we have done in Kaneko-Kline [17].

Condition (2.12) means that a behavior pattern σ_i prescribes an action leading to some decision node. This slightly complicated statement is required since Γ may be of the weak sense[8]. Condition (2.13) means that a strategy depends upon local memories.
These are standard conditions for the definition of a strategy. We denote, by Σ_i, the *set of all behavior patterns* for player i in Γ. We say that an n-tuple of strategies $\sigma = (\sigma_1, ..., \sigma_n)$ is a *profile of behavior patterns*.
We use the term behavior pattern (strategy) to acknowledge that the behavior of a player may initially represent some default behavior with no strategic considerations. Once, a player has gathered enough information about the game, his behavior may become strategic. This will be discussed in a remark in the end of Section 3.2.
In order to evaluate a behavior pattern, we introduce the concepts of compatible endnodes and compatible endpieces. All evaluations of strategies in this paper will be done in terms of compatible endpieces. Each behavior profile $\sigma = (\sigma_1, ..., \sigma_n)$ determines the set of *compatible endnodes*:

$$\begin{aligned} z(\sigma) \quad &= \{z \in X^E : \theta(z) = \langle(\lambda(x_1), \sigma(x_1)), ..., (\lambda(x_k), \sigma(x_k)), \lambda(x_{k+1})\rangle \\ &\quad \text{for the path } \langle x_1, ..., x_k, x_{k+1}\rangle \text{ to } z\}. \end{aligned} \tag{2.14}$$

Thus, the actions in the history $\theta(z)$ were prescribed by the behavior profile $\sigma = (\sigma_1, ..., \sigma_n)$.
Each behavior profile σ also determines the set of *compatible endpieces*:

$$\lambda(\sigma) = \{w : \lambda(x) = w \text{ for some } x \in z(\sigma)\}. \tag{2.15}$$

When Γ is an extensive game in the strong sense, $z(\sigma)$ and $\lambda(\sigma)$ are singleton sets. However, for extensive games in the weak sense, these sets may have multiple elements.

3. Bounded memory abilities and accumulation of local memories

In this section, we first define a domain of accumulation of short-term memories. This definition is based on the presumption that a player has a quite restricted memory capability. Theoretically, however, there are still many other possibilities. In Section 3.2, we will give one informal theory about the accumulation of short-term memories as long-term ones. This informal theory suggests a particular domain which we call the *active domain*, which turns out to be linked to Nash equilibrium behavior, as will be shown in Section 7.2. Informal and premathematical discussions of this type are intended to provoke further discussions and debates over the appropriate domain(s) for consideration.

3.1 The objective recurrent situation and domains of accumulation of memories

Let an extensive game $\Gamma^o = ((X^o, <^o), (\lambda^o, W^o), \{(\varphi^o_x, A^o_x)\}_{x \in X}, (\pi^o, N^o), h^o)$ in the strong sense and a profile $\mathfrak{m}^o = (\mathfrak{m}^o_1, ..., \mathfrak{m}^o_n)$ of memory functions be the description of the objective situation. The present purpose is to consider the accumulation of memories from playing in $(\Gamma^o, \mathfrak{m}^o)$ repeatedly.
From the objective point of view, an individual player i has been experiencing short-term memories:

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[8] If φ_x is a surjection, then $\{a \in A_x : \varphi_x(y) = y \text{ for some } y \in X\} = A_x$. However, since a personal view may satisfy only $K33_0$, we require this condition.

$$\cdots \left| \; \mathfrak{m}_i(x_1^t),\ldots,\mathfrak{m}_i(x_{\ell_t}^t) \; \right| \; \mathfrak{m}_i(x_1^{t+1}),\ldots,\mathfrak{m}_i(x_{\ell_{t+1}}^{t+1}) \; \left| \cdots \right. \qquad (3.1)$$
$$(\Gamma^o,\mathfrak{m}^o) \text{ at } t \qquad\qquad (\Gamma^o,\mathfrak{m}^o) \text{ at } t+1$$

where $\langle x_1^t,\ldots,x_{\ell_t}^t\rangle$ is the realized sequence of player i's nodes in the occurrence of $(\Gamma^o, \mathfrak{m}^o)$ at time t. Due to bounded memory, player i will only accumulate some part of these as long-term memories.

In the extensive game $(\Gamma^o, \mathfrak{m}^o)$, the *domain of accumulation* for player i is a nonempty subset D_i of the set $X_i^o = \{x \in X^o : \pi^o(x) \ni i\}$ of nodes for player i. Player i is *relevant in his own domain* D_i iff D_i contains at least one decision node for player i. This definition will be important later in this paper.

A *memory kit* $(T_{D_i}, \mathcal{Y}_{D_i})$ for domain D_i is given by

$$T_{D_i} = \bigcup_{x\in D_i} \mathfrak{m}_i^o(x); \text{and } \mathcal{Y}_{D_i} = \{\mathfrak{m}_i^o(x) : x \in D_i\}. \qquad (3.2)$$

A memory kit is determined by both the domain of accumulation D_i and the objective memory function $\mathfrak{m}_i^o$ of player i. It will be the source for an inductive construction of a personal view. The set T_{D_i} of memory threads is used to construct a skeleton of the tree for a personal view. The set $\mathcal{Y}_{D_i}$ of yarns is used to construct a perceived memory function. Mathematically speaking, the latter set gives the former, but we keep those two sets to emphasize that they have different usages.

For a memory kit, we assume that player i has accumulated some incidences of short-term memories as both threads and yarns. However, a kit includes neither a full record of short-term memories nor frequencies. In Section 3.2, we will discuss one rationale for this treatment.

Here, we give three domains of accumulation. The first two are trivial ones, and the third example is the one we are going to explore in this paper.

(1): Full domain: This is simply given as the entire set $D_i^F = X_i^o$ of player i's nodes. When the game is small, is repeated often enough and also when the accumulation ability of player i is strong enough, this domain may be appropriate.

(2): Cane domain: A cane domain is a complete set of nodes for player i on one play. Formally, let $\langle x_0,\ldots,x_m\rangle$ be the path to an endnode x_m. Then the *cane domain of player i to x_m* is given as $\{x_0,\ldots,x_m\} \cap X_i^o$. A cane domain may arise if every player behaves always following some regular behavior pattern with no deviations.

Now, let $\sigma^o = (\sigma_1^o,\ldots,\sigma_n^o)$ be a profile of behavior patterns in the extensive game $(\Gamma^o, \mathfrak{m}^o)$. Then, this σ^o determines a unique path to an endnode. Hence, the cane domain for player i is uniquely determined, which is denoted by $D_i^c(\sigma^o)$. Using this concept, we can define the active domain relative to a profile of behavior patterns.

(3): Active domain: The *active domain* relative to a profile $\sigma^o = (\sigma_1^o,\ldots,\sigma_n^o)$ of behavior patterns for player i is given as

$$D_i^A(\sigma^o) = \bigcup_{\sigma_i\in\Sigma_i^o} D_i^c(\sigma_i,\sigma_{-i}^o). \qquad (3.3)$$

Here, Σ_i^o is the *set of all behavior patterns* for player i in (Γ^o, $\mathfrak{m}^o$) and $(\sigma_i, \sigma_{-i}^o)$ is the profile obtained from σ^o by substituting σ_i for σ_i^o in σ^o. That is, the active domain $D_i^A(\sigma^o)$ is the set of nodes for player i that are reached by unilateral deviations of player i.

For a unified treatment of the above domains, we introduce one definition. We say that a domain D_i for player i is *closed* iff D_i is expressed as some union of cane domains of player i. The above three examples of domains are closed. A domain which is not closed is the set X^{oE} of endnodes.

Example 3.1. Let us continue with the example of Fig.2.2. Let the regular behavior be given by $\sigma_1^o(x_0) = \sigma_1^o(x_3) = a$ and $\sigma_2^o(x_1) = \sigma_2^o(x_2) = c$. The cane domain and active domain of player 1 determined by σ^o are given as

$$D_1^c(\sigma^o) = \{x_0, z_1\} \text{ and } D_1^A(\sigma^o) = \{x_0, z_1, z_3\}. \tag{3.4}$$

The full domain is simply given as $D_1^F = X_1^o = \{x_0, x_3, z_1, z_2, z_3, z_4, z_5\}$.

The memory kit of player 1 depends also on his objective memory function $\mathfrak{m}_1^o$. For the three domains mentioned above, the Markov and SPR memory functions, we have a total of six memory kits. We mention two and leave the reader to consider the other four.

For the SPR function $\mathfrak{m}_1^o = \mathfrak{m}_1^{spr}$ and the cane domain, we have $T_{D_1^c(\sigma^o)} = \{\langle w\rangle, \langle (w,a), z_1\rangle\}$, and $\mathcal{Y}_{D_1^c(\sigma^o)} = \{\{\langle w\rangle\}, \{\langle (w,a), z_1\rangle\}\}$.

For the Markov memory function $\mathfrak{m}_1^o = \mathfrak{m}_1^M$ and the active domain, we have $T_{D_1^A(\sigma^o)} = \{\langle w\rangle, \langle z_1\rangle, \langle z_3\rangle\}$ and $\mathcal{Y}_{D_1{}^A(\sigma^o)} = \{\{\langle w\rangle\}, \{\langle z_1\rangle\}, \{\langle z_3\rangle\}\}$.

3.2 An informal theory of behavior and accumulation of memories

Our mathematical theory starts with a memory kit. Behind a memory kit, there is some underlying process of behavior and accumulation of short-term memories. We now describe one such underlying process informally, which justifies the active domain of accumulation. This description is given in terms of some informal postulates.

(1): Postulates for behavior and trials: The first postulate is the rule-governed behavior of each player in the recurrent situation ..., (Γ^o, $\mathfrak{m}^o$), ..., (Γ^o, $\mathfrak{m}^o$), ...

Postulate BH1 (Regular behavior): Each player typically behaves regularly followinghis behavior pattern σ_i^o.

Player i may have adopted his regular behavior for some time without thinking, perhaps since he found it worked well in the past or he was taught to follow it. Without assuming regular behavior and/or patterns, a player may not be able to extract any causal pattern from his experiences. In essence, learning requires some regularity.

To learn some other part than that regularity experienced, the players need to make some trial deviations. We postulate that such deviations take place in the following manner.

Postulate BH2 (Occasional deviations): Once in a while (infrequently), each player unilaterally and independently makes a trial deviation $\sigma_i \in \Sigma_i^o$ from his regular behavior σ_i^o and then returns to his regular behavior.

Early on, such deviations may be unconscious or not well thought out. Nevertheless, a player might find that a deviation leads to a better outcome, and he may start making deviations consciously in the future. Once he has become conscious of his behavior-deviation, he might make more and/or different trials.

The set of trial deviations for a player is not yet well specified. In the remainder of this paper, we explore one extreme case where he tries every possible behavior. The following postulate is made for simplicity in our discourse and since it connects our theory to standard game theory.

Postulate BH3 (All possible trials): Each player experiments over all his possible behaviors.

Postulate BH3 is an extreme case that each player tries all his alternative behaviors. We do not take this as basic. The choice of a smaller set of trial deviations is very relevant, since a player might not have prior knowledge of his available behaviors.

(2): Epistemic postulates: Each player may learn something through his regular behavior and deviations. What he learns in an instant is described by his short-term memory. For the transition from short-term memories to long-term memories, there are various possibilities. Here we list some postulates based on bounded memory abilities that suggest only the active domain of accumulation.

The first postulate states that if a short-term memory does not occur frequently enough, it will disappear from the mind of a player. We give this as a postulate for a cognitive bound on a player.

Postulate EP1 (Forgetfulness): If experiences are not frequent enough, then they would disappear from a player's mind.

This is a rationale for not assuming that a player has a full record of short-term memories, as well as for the term "short-term memory". This explains also the assumption that he cannot keep the relative frequency of a short-term memory: It may remain for some short periods, but if it is not reinforced by other occurrences or the player is very conscious, they may disappear from his mind, i.e., many disappear. This means that a memory remaining after some time loses relative positions with other memories and is isolated. Hence, it is difficult to calculate its frequency relative to others.

In the face of the cognitive bound, only some memories become lasting. The first type of memories that become lasting are the regular ones since they occur quite frequently. The process of making a memory last by repetition is known as habituation.

Postulate EP2 (Habituation): A short-term (local) memory becomes lasting as a long-term memory in the mind of a player by habituation, i.e., if he experiences something frequently enough, it remains in his memory as a long-term memory even without conscious effort.

By EP2, when all players follow their regular behavior patterns, the short-term memories given by them will become long-term memories by habituation.

The remaining possibilities for long-term memories are the memories of trials made by some players. We postulate that a player may consciously spend some effort to memorize the outcomes of his own trials.

Postulate EP3 (Conscious memorization effort): A player makes a conscious effort to memorize the result of his own trials. These efforts are successful if they occur frequently enough relative to his trials.

Postulate EP3 means that when a player makes a trial deviation, he also makes a conscious effort to record his experience in his long-term memory. These memories are more likely to be successful if they are repeated frequently enough relative to his trials. Since the players are presumed to behave independently, the trial deviations involving multiple players will occur infrequently, even relative to one player's trials. Thus, the memories associated with multiple players' trials do not remain as long-term memories. This has the implication that our experiential foundation is typically incompatible with the subgame perfect concept of Selten [30], which will be discussed again in Section 9.

In sum, postulates EP1 to EP3 and BH1 to BH3 suggest that we can concentrate on the active domain of a player.

Some other domains such as a cane domain and the full domain might emerge as candidates in slightly different situations. For example, if no trials are made, then EP2(Habituation) gives the cane domain corresponding to σ^o. Alternatively, if the game is small enough and if it is repeated enough, then each player has experienced every outcome. And if he has an ability to recall all the incidences, then we would get the full domain. The additional assumption of full recall seems plausible for very small games.

Remark (Default decision and all the possible behaviors): One may criticize our treatments in that:

(1) σ_i^o has the total domain X_i^o and

(2) σ_i varies over the entire Σ_i^o of (3.3),

since these might conflict with the assumption of no *a priori* knowledge of the structure of the game for player *i*.

We can answer (1) by interpreting one action at every decision node as a default action. When a player receives an unknown (unfamiliar) information piece, he just takes the default action. This assumption avoids a player's need to plan for his behavior over the entire domain.

We take (2) as a legitimate criticism, particularly, when the game is large. We have chosen (3.3) as a working assumption in this paper.

4. Inductively derived views

In this section, we give a definition of an inductively derived (personal) view, which we abbreviate as an i.d.view. Here, player *i* uses only his memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ as a summary of his experiences to construct an i.d.view. Before the definition, we talk about our basic principles to be adopted in this paper. After the definition, we will consider various examples to see the details of the definition.

4.1 Observables, observed, and additional components

The central notion in inductive game theory is the process of inductive inferences. An inductive inference is distinguished from a deductive inference in that the former allows some generalization of observations by adding some hypothetical components, while the latter changes expressions following well-formed inference rules and keeps the same or less contents. A player, *i,* having a memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ may add some hypothetical components to the kit in his inductive process to develop a personal view.

The need for this addition of hypothetical components may be found in the assumption that a player can only observe some elements of the objective extensive game Γ^o. As remarked in Section 2.2, only information pieces and actions are observable for each player, while nodes are hypothetical and unobservables. In addition, many or some pieces and actions do not end up in the memory kit. Pieces and actions only along some of the paths in a game tree are more likely observed for players. Moreover, the bounds on their memory capabilities will allow them to accumulate memories of only some of what they have observed. The memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ for player *i* is the collection of observed parts effectively remaining in the mind of player *i*.

Since player i describes his view (Γ^i, $\mathfrak{m}^i$) as an extensive game in the weak sense with a memory function, he needs to invent a tree structure by adding hypothetical nodes. In this sense he already goes beyond deductive inferences. To construct a coherent view, a player may add other components, e.g., more information pieces, actions, and possible histories to his memories. In this paper, however, we adhere to the basic principle that only elements in the memory kit (T_{Di}, $\mathcal{Y}_{Di}$) can be used as the observables in (Γ^i, $\mathfrak{m}^i$). In Section 4.2, we will adopt a specific inductive process called the *initial-segment procedure* and use this procedure to define an i.d.view. With this procedure, a player forms the underlying skeletal structure of his view by adding hypothetical nodes.

4.2 Definition and examples

Now, consider the recurrent situation of (Γ^o, $\mathfrak{m}^o$) illustrated in Fig.1.2. Here, $\Gamma^o = ((X^o, <^o), (\lambda^o, W^o), \{(\varphi_x^o, A_x^o)\}_{x\in X^o}, (\pi^o, N^o), \{h_i^o\}_{i\in N^o})$ is an extensive game in the strong sense and $\mathfrak{m}^o = (\mathfrak{m}_1^o, ..., \mathfrak{m}_n^o)$ is an n-tuple of memory functions. Recall that a personal view is given as a pair (Γ^i, $\mathfrak{m}^i$), where $\Gamma^i = ((X^i, <^i), (\lambda^i, W^i), \{(\varphi_x^i, A_x^i)\}_{x\in X^i}, (\pi^i, N^i), h^i)$ is an extensive game in the weak sense specifying only the payoff function h^i of player i and $\mathfrak{m}^i$ is a memory function for player i in that game. We assume that player i uses his memory kit (T_{Di}, $\mathcal{Y}_{Di}$) in the sense of (3.2) to construct his personal view (Γ^i, $\mathfrak{m}^i$).

Strictly speaking, we will not consider the precise process of inductive derivation of a view (Γ^i, $\mathfrak{m}^i$). Instead, we consider possible candidates of (Γ^i, $\mathfrak{m}^i$) for the result of inductive derivation. For the definition of such a candidate, we need a bridge between (T_{Di}, $\mathcal{Y}_{Di}$) and (Γ^i, $\mathfrak{m}^i$). We can think of various procedures to have such bridges, but we will use one procedure, called the *initial-segment procedure,* as stated in Section 4.1. It will become clear shortly why we have chosen this name.

First, for a given candidate (Γ^i, $\mathfrak{m}^i$), we define the set $\Theta(\Gamma^i)$ of possible histories in Γ^i:

$$\Theta(\Gamma^i) = \{\theta^i(y) : y \in X^i\}, \tag{4.1}$$

where $\theta^i(y) = \langle(w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ is the complete history up to y in Γ^i. With the initial-segment procedure, we will connect $\Theta(\Gamma^i)$ with T_{Di}.

For the sake of rigor, we make the following definitions. First, a subsequence of $[(w_1, a_1), ..., (w_m, a_m)]$ is simply defined in the standard manner by regarding each (w_t, a_t) as a component of the sequence. Second, $\langle(w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ is said to be a *subsequence* of $\langle(v_1, b_1), ..., (v_k, b_k), v_{k+1}\rangle$ iff $[(w_1, a_1), ..., (w_m, a_m), (w_{m+1}, a)]$ is a subsequence of $[(v_1, b_1), ..., (v_k, b_k), (v_{k+1}, a)]$ for some a. A supersequence is defined in the dual manner. We say that $\langle(w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ is a *maximal sequence* in a given set of sequences iff there is no proper supersequence in that set. An *initial segment* of $\langle(w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ is a subsequence of the form $\langle(w_1, a_1), ..., (w_k, a_k), w_{k+1}\rangle$ and $k \leq m$.

Now, we can define the set of initial segments of memory threads in T_{Di} as:

$$T_{D_i}^* := \{\langle\xi, w\rangle : \langle\xi, w\rangle \text{ is an initial segment of some maximal sequence in } T_{D_i}\}. \tag{4.2}$$

We require $\Theta(\Gamma^i)$ to be the same as $T_{D_i}^*$ for Γ^i to be inductively derived from T_{Di}. This is why the following is called the *initial-segment procedure*. A player uses all his initial segments in T_{Di} to construct the histories in Γ^i.

We now give the full set of requirements for an inductively derived personal view based on the initial-segment procedure. As mentioned above, we will give a more general definition of an i.d.view in another paper, which will allow for other inductive procedures (see Section 9.3). In the following definition, we assume that player i is relevant in his own domain D_i, i.e., D_i contains at least one decision node of player i.

Definition 4.1 (Inductively derived view). A personal view (Γ^i, $\mathfrak{m}^i$) for player i is *inductively derived* from the memory kit (T_{D_i}, $\mathcal{Y}_{D_i}$) iff

P1(Construction of an extensive game): Γ^i is an extensive game in the weak sense satisfying:

(a)(Preservation of the informational structure): $\Theta(\Gamma^i) = T^*_{D_i}$;

(b)(Action sets): $A^i_x = A^o_{\lambda^i(x)}$ for each $x \in X^i$;

(c)(Player assignment at decision nodes): $\pi^i \cdot \lambda^i(x) = \pi^o \cdot \lambda^i(x)$ for all $x \in X^{iD}$;

(d)(Own Payoffs): $h^i \cdot \lambda^i(x) = h^o_i \cdot \lambda^i(x)$ for each $x \in X^{iE}$;

P2(Construction of a memory function): $\mathfrak{m}^i$ is a memory function on $X^i_i = \{x \in X^i : i \in \pi^i \cdot \lambda^i(x)\}$ satisfying:

(a)(Preservation of memory yarns): $\{\mathfrak{m}^i(x) : x \in X^i_i\} \subseteq \mathcal{Y}_{D_i}$;

(b)(Internal consistency): $\theta^{\,i}(x) \in \mathfrak{m}^i(x)$ for any $x \in X^i_i$;

(c)(Dependence up to observables): if $\theta^{\,i}(x) = \theta^{\,i}(y)$, then $\mathfrak{m}^i(x) = \mathfrak{m}^i(y)$.

We abbreviate an inductively derived view as an *i.d.view*.

For an i.d.view, the extensive game Γ^i is constructed based on the set $T^*_{D_i}$ of initial segments of maximal memory threads in T_{D_i}. P1a states that the game tree is based on $T^*_{D_i}$. Conditions P1b, P1c, P1d are the minimum requirements M1, M2, M3 stated in Section 2.1. By P1c and K42, the player set for Γ^i is determined as

$$N^i = \{j \in N^o : j \in \pi^i \cdot \lambda^i(x) \text{ for some } x \in X^{iD}\}.$$

Since λ^i is a surjection from X^i to W^i by K2, and since $\Theta(\Gamma^i) = T^*_{D_i}$ by P1a, we have $W^i \subseteq W^o$. Hence, P1b and P1c are well-defined. For the well-definedness of P1d, it should hold that for any $x \in X^{iE}$, the associated piece $\lambda^i(x)$ is an endpiece in the objective game Γ^o.

The personal memory function $\mathfrak{m}^i$ is constructed based on the set $\mathcal{Y}_{D_i}$ of memory yarns. This principle explains condition P2a, while player i is not required to use all of them. Condition P2b states that each yarn $\mathfrak{m}^i(x)$ should contain the complete history $\theta^{\,i}(x)$. The reason for this is that (Γ^i,$\mathfrak{m}^i$) is now in the mind of player i and can be seen by player i as the objective observer. Still, P2b is one alternative among several possible internal consistency requirements. Condition P2c is more basic, stating that his subjective memory yarns should include no elements additional to what, he believes, have been observed in the play in his view Γ^i.

An analogy with a jigsaw puzzle may help understand the above definition of an i.d.view. Treating the memory threads as the picture on each piece and memory yarnsas pieces in a jigsaw puzzle, a player tries to reconstruct an extensive game, though his memory kit may be very incomplete and does not allow him to reach a meaningful view.

To see how an i.d.view is obtained, we look at several examples.

Example 4.1 (SPR function $\mathfrak{m}^{spr}_1$). For this memory function, any i.d.view will be a 1-person game played by player 1, even if the objective game (Γ^o, $\mathfrak{m}^o$) involves multiple players.

Consider this memory function on the cane domain described in Example 3.1. The memory kit is given as

$$T_{D_1^c(\sigma^o)} = \{\langle w\rangle, \langle (w,a), z_1\rangle\}, \text{ and } \mathcal{Y}_{D_1^c(\sigma^o)} = \{\{\langle w\rangle\}, \{\langle (w,a), z_1\rangle\}\}.$$

Then $T_{D_1^c(\sigma^o)} = T^*_{D_1^c(\sigma^o)}$, and an i.d.view is given as Fig.4.1. It consists of the set of nodes $X^1 = \{y_0, y_1\}$, $\lambda^1(y_0) = w$, $\lambda^1(y_1) = z_1$, $\pi^1(w) = \pi^1(z_1) = \{1\}$, $h^1(z_1) = 2$ and his memory function is given as $\mathfrak{m}^1(y_0) = \{\langle w\rangle\}$ and $\mathfrak{m}^1(y_1) = \{\langle (w,a), z_1\rangle\}$. Since $A^1_{y_0} = A^o_w = \{a,b\}$ by P1b, condition K33 (bijection requirement) is violated, but $K33_0$ is satisfied.

$$\begin{array}{c} y_1 \\ \uparrow_a \\ y_0 \end{array} \qquad\qquad\qquad \begin{array}{cc} y_1 & y_1' \\ \uparrow_a & \uparrow_a \\ y_0 & y_0' \end{array}$$

Fig. 4.1. Cane. Fig. 4.2. Duplicated.

Now, let us observe that some multiplicity of i.d.views is involved in Definition 4.1, which is caused by the use of hypothetical elements of nodes. In the original game (Γ^o, $\mathfrak{m}^o$) as well as in the derived game (Γ^1, $\mathfrak{m}^1$), the nodes are unobservable and auxiliary. We can use different symbols for y_0 and y_1 without changing the informational structure of the game; the cane with nodes y_0' and y_1' differs from the cane of Fig.4.1. This causes also another type of multiplicity; the game having the duplication of (Γ^1, $\mathfrak{m}^1$) described in Fig.4.2 satisfies all the requirements of Definition 4.1. We will introduce the concept of a game theoretic p-morphism in Section 6 as a means for dealing with those types of multiplicity.

The definition of an inductive derivation based on the initial-segment procedure may not work to deliver an i.d.view. Here, we give two negative examples and one positive one.

Example 4.2 (Markov memory function $\mathfrak{m}_i^M$: General failure). Let player i have the Markov memory function $\mathfrak{m}_i^M = \mathfrak{m}_i^o$.. Suppose that player i is relevant in his domain D_i in Γ^o, i.e., D_i has at least one decision node y. Let $\lambda^o(y) = w$. Since $\mathfrak{m}_i^M$ is the Markov memory function, we have $T^*_{D_i} = T_{D_i} = \{\langle v\rangle : \lambda^o(x) = v \text{ and } x \in D_i\}$. This prevents player i from having an i.d.view, since all elements in $T^*_{D_i}$ have no successors but $\lambda^o(y) = w$ cannot have a payoff, i.e., P1d cannot be satisfied.

Example 4.3 (Perfect information memory function $\mathfrak{m}_1^{PI}$: Full recoverability). Let player 1 have the perfect-information memory function $\mathfrak{m}_1^{PI}$ and let the domain be the full domain $D_1^F = X_1^o$ in the game of Fig.2.2. In this case, player 1 can reconstruct the objective game Γ^o from his memory kit, except for player 2's payoffs and memory function. This full-recoverability result can be generalized into any game.

When player i has the classical memory function $\mathfrak{m}_i^C$ and the full domain D_i^F, we have also the full-recoverability result. When the domain D_i is smaller than D_i^F, we may encounter some difficulty.

Example 4.4 (Classical memory $\mathfrak{m}_1^C$ with the cane domain: failure). Let player 1 have the classical memory function $\mathfrak{m}_1^C$ on the cane domain $D_1^c = D_1^c(\sigma^o) = \{x_0, z_1\}$ of (3.4) in Example 3.1. Then $T_{D_1^c} = \{\langle w\rangle, \langle (w,a), v\rangle, \langle (w,b), v\rangle\}$; one candidate for an i.d.view is described as Fig.4.3, which violates conditions K2 and K31. Thus, there is no i.d.view in this case.

$$
\begin{array}{ccc}
z_1 & & \\
\uparrow_c & & \\
y_1 : v & & y_2 : v \\
& \nwarrow_a \quad \nearrow_b & \\
& y_0 : w &
\end{array}
$$

Fig. 4.3. Failure with $\mathfrak{m}_1^C$

5. Direct views

In Section 4, we gave the definition of an inductively derived view for a given memory kit $(T_{Di}, \mathcal{Y}_{Di})$ and found that there may be many i.d.views for each $(T_{Di}, \mathcal{Y}_{Di})$. In this section, we single out one of those views which we call the direct view. We will argue that it has a special status among i.d.views or simply among views. Here, we give some results for a direct view to be an i.d.view. In Section 6, we will show that our analysis of direct views is sufficient to describe the game theoretic contents of any i.d.view.

A direct view for a given memory kit $(T_{Di}, \mathcal{Y}_{Di})$ is constructed by treating each thread in $T^*_{D_i}$ as a node in the derived game. As in Section 4, we assume that player i is relevant in his own domain D_i.

Definition 5.1 (Direct view). A *direct view* $(\Gamma^d, \mathfrak{m}^d) = ((X^d, <^d), (\lambda^d, W^d), \{(\varphi^d_x, A^d_x)\}_{x \in X^d},$ $(\pi^d, N^d), h^d\%), \mathfrak{m}^d)$ from a memory kit $(T_{Di}, \mathcal{Y}_{Di})$ is defined in the following manner:

d1: $X^d = T^*_{D_i}$;

d2: $\langle \xi, v \rangle <^d \langle \eta, w \rangle$ iff $\langle \xi, v \rangle$ is a proper initial segment of $\langle \eta, w \rangle$;

d3 (Information function): $\lambda^d \langle \xi, v \rangle = v$ for all $\langle \xi, v \rangle \in X^d$; and $W^d = \{v : \langle \xi, v \rangle \in X^d$ for some $\xi\}$;

d4 (Action sets): $A^d_{\langle \xi, v \rangle} = A^o_v$ for all $\langle \xi, v \rangle \in X^d$; and if $\langle \xi, v \rangle \in X^{dD}$, then $\varphi^d_{\langle \xi, v \rangle} \langle \xi, (v, a), u \rangle = a$ for each immediate successor $\langle \xi, (v, a), u \rangle$ of $\langle \xi, v \rangle$;

d5 (Player assignment): $\pi^d(v) = \pi^o(v)$ for all $\langle \xi, v \rangle \in X^{dD}$; and $\pi^d(v) = N^d$ for all $\langle \xi, v \rangle \in X^{dE}$, where $N^d = \{j : j \in \pi^o(v)$ for some $\langle \xi, v \rangle \in X^{dD}\}$;

d6 (Payoff function): for any $\langle \xi, v \rangle \in X^{dE}$, if $\lambda^o(x) = v$ for some $x \in X^{oE}$, then $h^d(v) = h^o_i(v)$; and otherwise, $h^d(v)$ is arbitrary;

d7 (Memory function): for any node $\langle \xi, v \rangle$ in X^d_i, if some $\mathfrak{y} \in \mathcal{Y}_{D_i}$ contains $\langle \xi, v \rangle$, then $\mathfrak{m}^d \langle \xi, v \rangle$ is such a $\mathfrak{y} \in \mathcal{Y}_{D_i}$; and otherwise, $\mathfrak{m}^d \langle \xi, v \rangle = \{\langle \xi, v \rangle\}$.

In the following, $\Gamma^d = ((X^d, <^d), (\lambda^d, W^d), \{(\varphi^d_x, A_x{}^d)\}_{x \in X^d}, (\pi^d, N^d), h^d)$ defined by d1to d6 is called a *direct structure*, and $\mathfrak{m}^d$ defined by d7 is a *direct memory function*.

Condition d6 has an arbitrariness if some $\langle \xi, v \rangle \in X^{dE}$ does not come from an endpiece in Γ^o. If this is avoided, i.e., a direct structure is an extensive game in the weak sense, it is uniquely determined. Condition d7 may still allow multiple memory functions.

A direct view $(\Gamma^d, \mathfrak{m}^d)$ for $(T_{Di}, \mathcal{Y}_{Di})$ may not be a personal view; specifically, conditions K2 and K31 may be violated. Example 4.4 violates K2 and K31, and also, when the objective

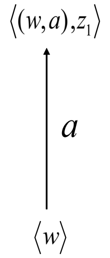

Fig. 5.1. Unique direct view.

memory function is the Markov, a direct view always violates K31. In Theorem 5.2, we will give a condition for a direct view to be a personal view as well as an i.d.view.

Another important comment is about the avoidance of additional hypothetical components such as nodes. It is directly constructed from the components in the memory kit, focusing the initial segments of memory threads in T_{Di}. Consequently, the complete history up to each node $x \in X^d$ is the same as x itself, which is stated as Lemma 5.1.

Lemma 5.1. For any direct structure Γ^d, $\theta^d(x) = x$ for all $x \in X^d$.

Proof. Let $x \in X^d$. By d1, $x = \langle \xi, v \rangle = \langle (w_1, a_1), ..., (w_k, a_k), v \rangle$ is an initial segment of a maximal thread in T_{D_i}. The path to $\langle \xi, v \rangle$ is $\langle w_1 \rangle$, $\langle (w_1, a_1), w_2 \rangle$, ..., $\langle (w_1, a_1), ..., (w_{k-1}, a_{k-1}), w_k \rangle$, $\langle \xi, v \rangle$. The complete history up to $\langle \xi, v \rangle$ is the sequence $\langle (w_1, a_1), ..., (w_{k-1}, a_{k-1}), (w_k, a_k), v \rangle$, which is x itself. ■

Let us now look at an example of a direct view.

Example 4.1 (continued): In Fig.4.1 and Fig.4.2, we gave two examples of i.d.views for player 1. This example has a unique direct view, which is given in Fig.5.1 and is an i.d.view with the memory function $\mathfrak{m}^i(x) = \{x\}$ for all $x \in X^d$.

Now, we give conditions for a direct view to be an i.d.view. Recall the assumption that player i is relevant for his own domain D_i.

Theorem 5.2 (Conditions for a direct view to be I.D.): Let $(T_{D_i}, \mathcal{Y}_{Di})$ be a memory kit.

(i): The direct structure Γ^d for $(T_{D_i}, \mathcal{Y}_{Di})$ is uniquely determined and is an extensive game in the weak sense satisfying P1a-P1d if and only if for any maximal $\langle \xi, v \rangle$ in $T^*_{D_i}$, $v = \lambda^o(x)$ for some $x \in X^{oE}$.

(ii): Let Γ^d be a direct structure for T_{Di}. There there is a direct memory function $\mathfrak{m}_d$ for Γ^d satisfying P2a-P2c if and only if for any $\langle \xi, w \rangle \in T^*_{D_i}$ with $i \in \pi^o(w)$,

$$\text{there is an } x \in D_i \text{ such that } \langle \xi, w \rangle \in \mathfrak{m}^o_i(x). \tag{5.1}$$

This theorem will be proved at the end of this section. The part (i) states that a condition for the unique determination of a direct structure is that every maximal thread in $T^*_{D_i}$ occurs at an endnode in the objective game. The part (ii) gives a necessary and sufficient condition for a direct memory function prescribed by d7 to satisfy P2a-P2c. When both of these conditions are satisfied, there is a direct view that is i.d., but there is still, however, some arbitrariness in the memory function, which allows for multiple direct views. This is shown by Example 5.1.

Example 5.1. Consider the objective 1-person sequential move game of Fig.5.2. Here, the information function is given by $\lambda^o(y_t) = v$ for $t = 1, 2$, and it is the identity function everywhere else. Suppose that the domain of accumulation is the full domain $D^F_1 = X^o_1 = X^o$.

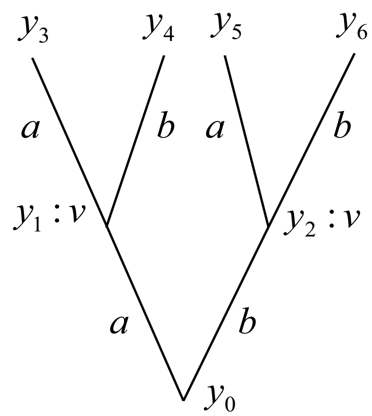

Fig. 5.2. 1-person game.

Let the objective memory function $\hat{\mathfrak{m}}_1^o$ be defined by:

$$\hat{\mathfrak{m}}_1^o(y_t) = \begin{cases} \{\theta^o(y_t)\} & \text{if } t \neq 1,2; \\ \{\langle(y_0,a),v\rangle,\langle(y_0,b),v\rangle,\langle v\rangle\} & \text{if } t = 1; \\ \{\langle(y_0,a),v\rangle,\langle(y_0,b),v\rangle\} & \text{if } t = 2. \end{cases} \tag{5.2}$$

In this example, the direct structure Γ^d is uniquely determined, which has the same structure as Fig.5.2 consisting of nodes $\theta^o(y_1)$, ..., $\theta^o(y_6)$. However, a memory function has some arbitrariness at the nodes $\theta^o(y_1)$ and $\theta^o(y_2)$. For example, assigning the memory $\mathfrak{m}^d(\theta^o(y_1)) = \mathfrak{m}_1^o(y_2)$ and $\mathfrak{m}^d(\theta^o(y_2)) = \mathfrak{m}_1^o(y_1)$, together with $\mathfrak{m}^d(\theta^o(y^t)) = \mathfrak{m}_i^o(y^t)$ for $t \neq 1$ and $t \neq 2$, gives one i.d.direct view. In this view, the player mixes up his memories at y_1 and y_2. In Section 8.2, we will see how this mixing up may create some difficulties. Another view is where he assigns his memory yarns correctly. Still two other views are obtained if he assigns one memory yarn to each of those nodes.

We now introduce two conditions on a memory function, that we will use in combination with Theorem 5.2 to provide a sufficient condition for the uniqueness of a direct view.

(Recall of past memories - RPM): for all $x, y \in X_i^o$, if $\langle \xi, w\rangle \in \mathfrak{m}_i^o(x)$ and $x <^o y$, then $\langle \xi, w\rangle$ is a proper initial segment of some $\langle \eta, v\rangle \in \mathfrak{m}_i^o(y)$.

(Single thread yarns - STY): $|\mathfrak{m}_i^o(x)| = 1$ for all $x \in X_i^o$.

The first condition states that every memory thread occurring at a node x of player i will occur as a subsequence of a thread at any later node y of player i. This is interpreted as meaning that player i recalls what past memories he had in the current play of the game. The second condition is simply that each yarn consists of a single thread.

The following corollary gives a sufficient condition for the unique determination of a direct view, which guarantees that it is an i.d.view.

Corollary 5.3. Let D_i be a closed domain, and let $(T_{D_i}, \mathcal{Y}_{D_i})$ be a memory kit determined by a memory function $\mathfrak{m}_i^o$ satisfying RPM and STY. Furthermore, suppose the latter part, (5.1), of Theorem 5.2.(ii). Then, the direct view $(\Gamma^d, \mathfrak{m}^d)$ is uniquely determined by d1-d7, and $\mathfrak{m}^d(x) = \{x\}$ for all $x \in X_i^d$. Moreover, $(\Gamma^d, \mathfrak{m}^d)$ is an i.d.view.

It is straightforward to check that the SPR function $\mathfrak{m}_i^{spr}$ and the perfect-information memory function $\mathfrak{m}_i^{PI}$ on a closed domain satisfy the conditions of Corollary 5.3. Thus, in those cases, we can speak of a unique direct view. We prove this corollary after proving Theorem 5.2.

Proof of Theorem 5.2.(i) (If): Suppose that for any maximal $\langle \xi, v\rangle$ in $T_{D_i}^*$, $v = \lambda^o(x)$ for some $x \in X^{oE}$. Under this supposition, we first show that the direct structure is a uniquely determined extensive game in the weak sense.

Let Γ^d be a direct structure satisfying d1 to d7. First, observe that the verification of each of K11 to K13 is straightforward by d1, d2, the non-emptiness of D_i and the finite number of threads for each yarn of the memory function $\mathfrak{m}_i^o$. Condition K2 follows from K2 for Γ^o, d1, d2, d3, condition (2.3) for $\mathfrak{m}_i^o$, and the supposition of the *if* part. Condition K31 also follows from the supposition of the *if* part together with K31 on Γ^o and d4. Conditions K32 and K33$_0$ follow from d1, d2, d3, and d4. K4 uses d5 and d6. Finally, condition K5$_0$ follows from d6. The supposition of the *if* part implies the payoff function h_i^d is uniquely determined by d6. Thus, we have shown that the direct structure Γ^d is determined uniquely as an extensive game in the weak sense.

Next we show that P1a holds. By Lemma 5.1, $\Theta(\Gamma^d) = X^d$, and by d1, $X^d = T_{D_i}^*$. Hence, $\Theta(\Gamma^d) = T_{D_i}^*$. The other parts of P1 follow immediately from the definition of a direct structure.

(Only-if): Suppose that there is a maximal $\langle \xi, v\rangle$ in $T_{D_i}^*$ and $v = \lambda^o(x)$ for some $x \in X^{oD}$. By K33 for Γ^o, $A_x^o \neq \emptyset$. By d4, we have $A_{\langle \xi, v\rangle}^d = A_x^o \neq \emptyset$. However, $\langle \xi, v\rangle \in X^{dE}$ since $\langle \xi, v\rangle$ is maximal in $T_{D_i}^*$. Hence, K31 is violated for Γ^d, and thus Γ^d is not an extensive game in the weak sense.

(ii)(If): Suppose that for any $\langle \xi, w\rangle \in T_{D_i}^*$ with $i \in \pi^o(w)$, there is an $x \in D_i$ such that $\langle \xi, w\rangle \in \mathfrak{m}_i^o(x)$. Then we can define $\mathfrak{m}^d\langle \xi, w\rangle = \mathfrak{m}_i^o(x)$. This is a direct memory function of player i for the direct structure Γ^d, since it associates a memory yarn from $\mathcal{Y}_{D_i}$ to each $\langle \xi, w\rangle \in T_{D_i}^* = X_i^d$. Then, P2a and P2b are satisfied since by Lemma 5.1, $\theta^d\langle \xi, w\rangle = \langle \xi, w\rangle$. Finally, $\mathfrak{m}^d$ satisfies P2c, since by Lemma 5.1, $\theta^d\langle \xi, w\rangle = \theta^d\langle \eta, v\rangle$ implies $\langle \xi, w\rangle = \langle \eta, v\rangle$.

(Only-if): If $\mathfrak{m}^d$ is a direct memory function for Γ^d, then the result follows by P2a and P2b for $\mathfrak{m}^d$. ■

Proof of Corollary 5.3. The right-hand side of Theorem 5.2.(i) is equivalent to that if $\langle \xi, w\rangle \in T_{D_i}^*$ and $\lambda^o(x) = w$ for some decision node $x \in D_i$, then $\langle \xi, w\rangle$ is not maximal in $T_{D_i}^*$. Let $\langle \xi, w\rangle \in T_{D_i}^*$ and suppose that $\lambda^o(x) = w$ for some decision node $x \in D_i$. Then either $\langle \xi, w\rangle$ is a proper initial segment of some $\langle \eta, v\rangle \in T_{D_i}^*$, or $\langle \xi, w\rangle \in T_{D_i}$. In the first case, $\langle \xi, w\rangle$ cannot be maximal in $T_{D_i}^*$. Suppose that $\langle \xi, w\rangle \in T_{D_i}$. Then, $\langle \xi, w\rangle \in \mathfrak{m}_i^o(x')$ for some $x' \in D_i$. By K2, (2.4), and the supposition that $\lambda^o(x) = w$ for some decision node $x \in D_i$, it follows that x' must also be a decision node in D_i. Then, by closedness we have a $z \in D_i$ with $x' <^o z$. By RPM, there is a $\langle \eta, v\rangle \in \mathfrak{m}_i^o(z)$ such that $\langle \xi, w\rangle$ is a proper subsequence of $\langle \eta, v\rangle$. Thus, $\langle \xi, w\rangle$ is not maximal in $T_{D_i}^*$.

By Theorem 5.2.(i), the direct structure Γ^d is uniquely determined and is an extensive game in the weak sense satisfying P1a-P1d. It remains to show that the memory function $\mathfrak{m}^d(x) = \{x\}$ is the only memory function for Γ^d that satisfies P2. By the supposition in the corollary that for any $\langle \xi, w\rangle \in T_{D_i}^*$ with $i \in \pi^o(w)$, there is an $x \in D_i$, it follows by Theorem 5.2.(ii) that there is a direct memory function for Γ^d that satisfies P2. By STY, $\mathfrak{m}^d(x) = \{x\}$ is the only possible memory function for Γ^d. ■

6. Game theoretical *p*-morphisms: comparisons of views

In this section, we will show that for any i.d.view $(\Gamma^i, \mathfrak{m}^i)$, there is a direct i.d.view $(\Gamma^d, \mathfrak{m}^d)$ having the same game theoretical structure. This result reduces the multiplicity of i.d.views, and allows us to concentrate on the direct views for our analysis of i.d.views. For example, the existence of an i.d.view is equivalent to the existence of a direct i.d.view. This consideration will be possible by introducing the concept of a game theoretical *p*-morphism, which is a modification of a *p*-morphism in the modal logic literature (cf. Ono [26] and Blackburn-de Rijke-Venema [3]). We call it simply a *g*-morphism.

6.1 Definition and results

In the following definition, we abbreviate the superscript i for each component of $(\Gamma^i, \mathfrak{m}^i)$ and $(\hat{\Gamma}^i, \hat{\mathfrak{m}}^i)$ to avoid unnecessary complications.

Definition 6.1 (Game theoretical *p*-morphism): Let $(\Gamma, \mathfrak{m})$ and $(\Gamma^i, \mathfrak{m}^i)$ be personal views of player i. A function ψ from X to $\hat{X}$ is called a *g-morphism* (game theoretical *p*-morphism) iff

g0: ψ is a surjection from X to $\hat{X}$;

g1: for all $x, y \in X$ and $a \in A_x$, $x <_a y$ implies $\psi(x) \mathrel{\hat{<}_a} \psi(y)$;

g2: for all $\hat{x}, \hat{y} \in \hat{X}$, $y \in X$ and $a \in \hat{A}_{\hat{x}}$,

$\hat{x} \mathrel{\hat{<}_a} \hat{y}$ and $\hat{y} = \psi(y)$ imply $x <_a y$ and $\hat{x} = \psi(x)$ for some $x \in X$;

g3 (Information pieces): $\hat{\lambda} \cdot \psi(x) = \lambda(x)$ for all $x \in X$;

g4 (Action sets): $\hat{A}_{\psi(x)} = A_x$ for all $x \in X$;

g5 (Player assignment): $\hat{\pi} \cdot \hat{\lambda} \cdot \psi(x) = h \cdot \lambda(x)$ for all $x \in X$;

g6 (Payoff function): $\hat{h} \cdot \hat{\lambda} \cdot \psi(x) = h \cdot \lambda(x)$ for all $x \in X^E$;

g7 (Memory function): $\hat{\mathfrak{m}} \cdot \psi(x) = \mathfrak{m}(x)$ for all $x \in X_i$.

We say that $(\Gamma, \mathfrak{m})$ is *g-morphic* to $(\hat{\Gamma}, \hat{\mathfrak{m}})$, denoted by $(\Gamma, \mathfrak{m}) \to (\hat{\Gamma}, \hat{\mathfrak{m}})$, iff there is a *g*-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

A *g*-morphism ψ compares one personal view to another one. When a *g*-morphism exists from $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$, the set of nodes in Γ is mapped onto the set of nodes in $\hat{\Gamma}$, while the game theoretic components of $(\Gamma, \mathfrak{m})$ are preserved. Since ψ is a surjection from X to $\hat{X}$, we cannot take the direct converse of g1, but we take a weak form, g2, which requires that the image $(\hat{\Gamma}, \hat{\mathfrak{m}})$ should not have any additional structure. In sum, the mapping ψ embeds $(\Gamma, \mathfrak{m})$ into $(\hat{\Gamma}, \hat{\mathfrak{m}})$ without losing the game structure. Nevertheless, a *g*-morphism allows a comparison of quite different games.

In the modal logic literature, the concept of a *p*-morphism is used to compare two Kripke models and their validities. As mathematical objects, Kripke models and extensive games have some similarity in that their basic structures are expressed as some graphs (or trees) (cf., Ono [26] and Blackburn at el [3]). In our case, the other game theoretical components including a memory function are placed on the basic tree structure. Therefore, we require our *g*-morphism to preserve those components, i.e., g3-g7. It will be seen that this concept is useful for comparisons of i.d.views for a given memory kit.

Let us consider a few examples to understand *g*-morphisms.

Example 6.1 (Infinite number of p.v.'s *g*-morphic to a given p.v.). Given a personal view $(\Gamma, \mathfrak{m})$, we can construct a larger personal view by simply replicating $(\Gamma, \mathfrak{m})$. The replicated game with twice as many nodes is *g*-morphic to $(\Gamma, \mathfrak{m})$; for example, Fig.4.2 is obtained from

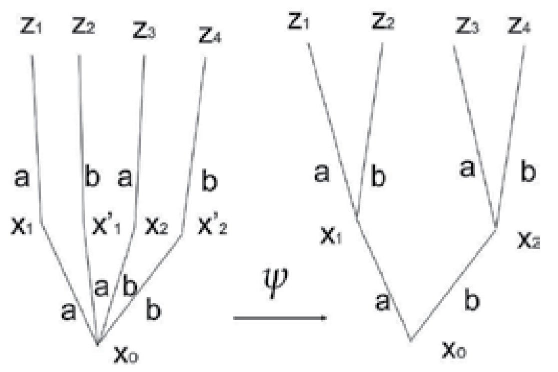

Fig. 6.1. Non-trivial g-morphism.

Fig.4.1 by replication. By this method, we can construct personal views of any size that are g-morphic to $(\Gamma, \mathfrak{m})$. Thus, there are an infinitely many personal views g-morphic to $(\Gamma, \mathfrak{m})$.

The following is a less trivial example than the above.

Example 6.2. Fig.6.1 gives a g-morphism between two 1-person games, where the memory function for each personal view is assumed to be the perfect-information memory function $\mathfrak{m}^{PI}$. Define ψ as the identity mapping everywhere except $\psi(x_1') = x_1$ and $\psi(x_2') = x_2$. This ψ is a g-morphism from the left game to the right game.

Here, we give an example where two i.d.views have no g-morphisms. The fact is caused by attached memory functions.

Example 6.3 (Negative example). Consider the objective description of Example 5.1. In this case, the player has four distinct direct views, each of which is an i.d.view. The direct structure is uniquely determined, but there are four possible direct memory functions. No g-morphisms are admitted between each pair of direct views.

Now, we show that a g-morphism fully preserves the i.d.property. All the results presented here will be proved in Section 6.2.

Theorem 6.1 (Preservation of the i.d. property). Suppose that $(\Gamma, \mathfrak{m})$ is g-morphic to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then, $(\Gamma, \mathfrak{m})$ is an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$ if and only if $(\hat{\Gamma}, \hat{\mathfrak{m}})$ is an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$.

It follows from this theorem and Example 6.1 that if a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$ admits at least one i.d.view, then there are, in fact, an infinite number of i.d.views for $(T_{D_i}, \mathcal{Y}_{D_i})$. Thus, we should consider which i.d.views are more appropriate than others. We will see that the direct views have a special status among the i.d.views. Before that, we give the following simple but basic observations, which can be proved just by looking at the definitions carefully.

Lemma 6.2.(1): The g-morphic relation $\rightarrow$ satisfies reflexivity and transitivity.

(2): Suppose that $(\Gamma, \mathfrak{m}) \leftarrow\rightarrow (\hat{\Gamma}, \hat{\mathfrak{m}})$, i.e., $(\Gamma, \mathfrak{m}) \rightarrow (\hat{\Gamma}, \hat{\mathfrak{m}})$ and $(\Gamma, \mathfrak{m}) \leftarrow (\hat{\Gamma}, \hat{\mathfrak{m}})$. Then the g-morphism ψ from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$ satisfies

g0*: ψ is a bijection from X to $\hat{X}$;

g1*: for all $x, y \in X$ and $a \in A_x$, $x <_a y$ if and only if $\psi(x) \mathrel{\hat{<}_a} \psi(y)$.

By (1), the relation $\leftarrow\rightarrow$ is an equivalence relation over personal views. We can use this relation to consider the equivalence classes of personal views. Any two views in one equivalence class are isomorphic in the sense of g0*, g1* and g3-g7, where g2 is included in g1*. These two views are identical in our game theoretical sense except for the names of nodes.

In the next theorem we show that every i.d.view is g-morphic to a direct view.

Theorem 6.3. (g-Morphism to a direct personal view). Let $(T_{Di}, \mathcal{Y}_{Di})$ be a memory kit. For each i.d.view $(\Gamma, \mathfrak{m})$, there is a direct view $(\Gamma^d, \mathfrak{m}^d)$ such that $(\Gamma^d, \mathfrak{m}^d)$ is a personal view and $(\Gamma, \mathfrak{m})$ is g-morphic to $(\Gamma^d, \mathfrak{m}^d)$.

The direct view $(\Gamma^d, \mathfrak{m}^d)$ given in Theorem 6.3 is also an i.d.view for $(T_{Di}, \mathcal{Y}_{Di})$ by Theorem 6.1. This has the implication that we can focus our attention on direct views without loss of generality. The following corollary states that the existence of an i.d.view is characterized by the existence of a direct i.d.view which in turn is characterized by Theorem 5.2.

Corollary 6.4. (Existence of an i.d.view). Let $(T_{Di}, \mathcal{Y}_{Di})$ be a memory kit. There is an i.d.view for $(T_{Di}, \mathcal{Y}_{Di})$ if and only if there is a direct view that is an i.d.view for $(T_{Di}, \mathcal{Y}_{Di})$.

6.2 Proofs of the results

First, we start with giving a simple observation.

Lemma 6.5. Let ψ be a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then $x \in X^D$ if and only if $\psi(x) \in \hat{X}^D$.

Proof. Let $x \in X^D$. Then x has an immediate successor. Thus, $A_x \neq \varnothing$ by K33$_0$, which implies $\hat{A}_{\psi(x)} \neq \varnothing$ by g4. By K31, $\psi(x) \in \hat{X}^D$. The converse follows by tracing back this argument starting with $\psi(x) \in \hat{X}^D$. ■

The next lemma translates g1 and g2 into the corresponding $\overline{\mathrm{g}}_1$ and $\overline{\mathrm{g}}_2$ in terms of the immediate successor relation $<^I_a$.

Lemma 6.6. Suppose that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then:

$\overline{\mathrm{g}}1$: for all $x, y \in X$ and $a \in A_x$, $x <^I_a y$ implies $\psi(x) \mathrel{\hat{<}^I_a} \psi(y)$;

$\overline{\mathrm{g}}2$: for all $\hat{x}, \hat{y} \in \hat{X}, y \in X$ and $a \in \hat{A}_{\hat{x}}$,

$\hat{x} \mathrel{\hat{<}^I_a} \hat{y}$ and $\hat{y} = \psi(y)$ imply $x <^I_a y$ and $\hat{x} = \psi(x)$ for some $x \in X$.

Proof. $\overline{\mathrm{g}}1$: Let $x <^I_a y$ for some $x, y \in X$. Now, on the contrary, suppose that $\psi(x) \mathrel{\hat{<}_a} \hat{z} \mathrel{\hat{<}_b} \psi(y)$ for some $\hat{z}$ and b. Then, by g2, there is some $z \in X$ such that $\psi(z) = \hat{z}$ and $z <_b y$. . By K12 for Γ, we have $x <_a z <_b y$ or $z <_b x <_a y$. The first case, $x <_a z <_b y$, is impossible since it contradicts $x <^I_a y$. In the second case, we have $\hat{z} \mathrel{\hat{<}_b} \psi(y)$ by g1, and then, by $\psi(x) \mathrel{\hat{<}_a} \hat{z}$, we have $\hat{z} \mathrel{\hat{<}} \hat{z}$ by the transitivity of K11 for $\hat{\Gamma}$, which contradicts the irreflexivity of K11 for $\hat{\Gamma}$. Thus, we must have $\psi(x) \mathrel{\hat{<}^I_a} \psi(y)$.

$\overline{\mathrm{g}}2$: Let $\hat{x} \mathrel{\hat{<}^I_a} \hat{y}$ and $\hat{y} = \psi(y)$ for some $\hat{x}, \hat{y} \in \hat{X}, y \in X$ and $a \in \hat{A}_{\hat{x}}$. By g2, there is some $x \in X$ such that $x <_a y$ and $\hat{x} = \psi(x)$. Now, on the contrary, suppose that $x <_a z <_b y$ for some z and b. Then, by g1, we have $\psi(x) \mathrel{\hat{<}_a} \psi(z) \mathrel{\hat{<}_b} \psi(y)$, which is a contradiction to $\hat{x} \mathrel{\hat{<}^I_a} \hat{y}$. Thus, we must have $x <^I_a y$. ■

The next lemma makes use of the previous one.

Lemma 6.7. Suppose that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then:

(1): If $\langle x_1, ..., x_m\rangle$ is a path in $(\Gamma, \mathfrak{m})$, then $\langle \psi(x_1), ..., \psi(x_m)\rangle$ is a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$ and $\theta(x_t) = \hat{\theta} \cdot \psi(x_t)$ for $t = 1, ...,m$.

(2): If $\langle \hat{x}_1, ..., \hat{x}_m\rangle$ is a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$, then there is a path $\langle x_1, ..., x_m\rangle$ in $(\Gamma, \mathfrak{m})$ such that $\psi(x_t) = \hat{x}_t$ and $\theta(x_t) = \hat{\theta}(\hat{x}_t)$ for $t = 1, ...,m$.

Proof.(1) Let $\langle x_1, ..., x_m\rangle$ be a path in $(\Gamma, \mathfrak{m})$. Then there are $a_1, ..., a_{m-1}$ such that $x_t <^I_{a_t} x_{t+1}$ for $t = 1, ..., m-1$. Thus, $\psi(x_t) \hat{<}^I_{a_t} \psi(x_{t+1})$ for $t = 1, ...,m-1$ by $\overline{g}1$ of Lemma 6.6. This means that $\langle\psi(x_1), ..., \psi(x_m)\rangle$ is a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$ and, by g3, $\theta(x_t) = \hat{\theta} \cdot \psi(x_t)$ for $t = 1, ...,m$.
(2) Let $\langle \hat{x}_1, ..., \hat{x}_m\rangle$ be a path in $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then there are $a_1, ..., a_{m-1}$ such that $\hat{x}_t \hat{<}^I_{a_t} \hat{x}_{t+1}$ for $t = 1, ...,m - 1$. Then, by g0, we can choose an $x_m \in X$ with $\psi(x_m) = \hat{x}_m$. Then, applying $\overline{g}2$ of Lemma 6.6 to the last pair $(\hat{x}_{m-1}, \hat{x}_m)$ and $\psi(x_m) = \hat{x}_m$, there is an $x_{m-1} \in X$ such that $\psi(x_{m-1}) = \hat{x}_{m-1}$ and $x_{m-1} <^I_{a_{m-1}} x_m$. Repeating this argument (exactly speaking, by mathematical induction), we construct $\langle x_1, ..., x_m\rangle$ with $x_t <^I_{a_t} x_{t+1}$ for $t = 1, ...,m - 1$ and $\psi(x_t) = \hat{x}_t$ for $t = 1, ..., t$. This is a path in $(\Gamma, \mathfrak{m})$ having the required properties. ■
We have the immediate result from Lemma 6.7 that the mapping ψ preserves the complete histories of information pieces and actions, and the values of the memory yarns.
Lemma 6.8. Suppose that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. Then:
(a): $\Theta(\Gamma) = \Theta(\hat{\Gamma})$;
(b): $\{\mathfrak{m}^i(x) : x \in X_i^i\} \subseteq \{\hat{\mathfrak{m}}^i(x) : x \in \hat{X}_i^i\}$.
Proof. (a) Lemma 6.7.(1) states that $\theta(x) = \hat{\theta} \cdot \psi(x)$ for all $x \in X$. Thus, $\Theta(\Gamma) \subseteq \Theta(\hat{\Gamma})$. Conversely, take any $\hat{x} \in \hat{X}$. Lemma 6.7.(2) states that there is an x such that $\theta(x) = \hat{\theta}(\hat{x})$. Thus, $\Theta(\hat{\Gamma}) \subseteq \Theta(\Gamma)$.
(b) By g7, we have $\{\mathfrak{m}^i(x) : x \in X_i^i\} = \{\hat{\mathfrak{m}}^i(x) : x \in \hat{X}_i^i\}$. The converse inclusion follows from the surjectivity of ψ by g0. ■
Now, we prove Theorem 6.1. Actually, we prove a more precise claim than the theorem: when there is a g-morphism ψ from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$, each of P1a-P1d and P2a-P2c for $(\Gamma, \mathfrak{m})$ is equivalent to the corresponding one for $(\hat{\Gamma}, \hat{\mathfrak{m}})$.
Proof of Theorem 6.1. Suppose that there is a g-morphism ψ from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$. As stated above, we prove that each requirement of P1a-P1d and P2a-P2c for $(\Gamma, \mathfrak{m})$ is equivalent to the corresponding one for $(\hat{\Gamma}, \hat{\mathfrak{m}})$.
P1a: By Lemma 6.8.(a), we have $\Theta(\Gamma) = \Theta(\hat{\Gamma})$. P1a holds for Γ, i.e., $T^*_{D_i} = \Theta(\Gamma)$, if and only if $T^*_{D_i} = \Theta(\hat{\Gamma})$, i.e., P1a for $\hat{\Gamma}$.
P1b: Let P1b hold for Γ, i.e., $A_x = A^o_{\lambda(x)}$. Consider any $\hat{x} \in \hat{X}$. Then we have some $x \in X$ with $\psi(x) = \hat{x}$. By g4, $\hat{A}_{\hat{x}} = A_x$. Thus, $\hat{A}_{\hat{x}} = A^o_{\lambda(x)}$. Since $\lambda(x) = \hat{\lambda}(\hat{x})$ by g3, we have $\hat{A}_{\hat{x}} = A^o_{\hat{\lambda}(\hat{x})}$. The converse can be proved similarly.
P1c: Suppose P1c holds for $\hat{\Gamma}$, i.e., $\hat{\pi} \cdot \hat{\lambda}(\hat{x}) = \pi^o \cdot \hat{\lambda}(\hat{x})$ for any $\hat{x} \in \hat{X}$. Let $x \in X$. By g3, g5 and P1c for $\hat{\Gamma}$, we have $\pi \cdot \lambda(x) = \hat{\pi} \cdot \hat{\lambda} \cdot \psi(x) = \pi^o \cdot \hat{\lambda} \cdot \psi(x) = \pi^o \cdot \lambda(x)$. Thus, we have P1c for Γ. The converse is similar.
P1d: Suppose P1d for Γ. Consider any $\hat{x} \in \hat{X}$. We should show $\hat{h} \cdot \hat{\lambda}(\hat{x}) = h_i^o \cdot \hat{\lambda}(\hat{x})$. By g3, g6 and P1d for Γ, we have $\hat{h} \cdot \hat{\lambda}(\hat{x}) = \hat{h} \cdot \hat{\lambda} \cdot \psi(x) = h \cdot \lambda(x) = h_i^o \cdot \lambda(x) = h_i^o \cdot \hat{\lambda} \cdot \psi(x) = h_i^o \cdot \hat{\lambda}(\hat{x})$. Thus, P1d for $\hat{\Gamma}$. The converse is similar.
P2a: By Lemma 6.8.(b), $\{\hat{\mathfrak{m}}(\hat{x}) : \hat{x} \in \hat{X}_i\} = \{\mathfrak{m}(x) : x \in X_i\}$. Hence, $\mathfrak{m}$ satisfies P2a if and only if $\hat{\mathfrak{m}}$ does.
P2b: By g7 and Lemma 6.7, $\mathfrak{m}$ satisfies P2b if and only if $\hat{\mathfrak{m}}$ does.
P2c: Suppose P2c for $\hat{\mathfrak{m}}$. Let $\hat{\theta}(\hat{x}) = \hat{\theta}(\hat{y})$. Since ψ is a surjection, we have some $x, y \in X$ such that $\psi(x) = \hat{x}$ and $\psi(y) = \hat{y}$. By Lemma 6.7, $\theta(x) = \hat{\theta}(\hat{x})$ and $\theta(y) = \hat{\theta}(\hat{y})$. Hence $\mathfrak{m}(x) = \mathfrak{m}(y)$ by

P2c for $\mathfrak{m}$. Then, by g7, $\hat{\mathfrak{m}}(\hat{x}) = \mathfrak{m}(x)$ and $\hat{\mathfrak{m}}(\hat{y}) = \mathfrak{m}(y)$. Thus, P2c holds for $\hat{\mathfrak{m}}$. The converse is similar. ■

The next target is to prove Theorem 6.3. We take two steps to have the assertion of the theorem: Under the supposition that $(\Gamma, \mathfrak{m})$ is an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$, (1) we can find a direct view so that it is a personal view; and (2) it is g-morphic to $(\Gamma, \mathfrak{m})$. The first part is given as a lemma, and the second is given as the proof of the theorem.

Lemma 6.9. Suppose that $(\Gamma, \mathfrak{m})$ is an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$. Then $(\Gamma^d, \mathfrak{m}^d)$ is a personal view where Γ^d is the unique direct structure for $(T_{D_i}, \mathcal{Y}_{D_i})$ and $\mathfrak{m}^d$ is defined by:

$$\text{for all } x \in X_i^d,\ \mathfrak{m}^d(x) = \mathfrak{m}(y_x) \text{ for some } y_x \in X_i \text{ satisfying } \theta(y_x) = x. \tag{6.1}$$

Proof. Let $(\Gamma, \mathfrak{m})$ be an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$. We first show the right hand side of Theorem 5.2.(i). This implies that Γ^d is the unique direct structure for $(T_{D_i}, \mathcal{Y}_{D_i})$ and Γ^d is an extensive game in the weak sense. We next show that (6.1) defines a memory function for Γ^d, from which it follows that $(\Gamma^d, \mathfrak{m}^d)$ is personal view.

Suppose, on the contrary, that there is some maximal thread $\langle \xi, v\rangle \in T^*_{D_i}$ such that $v = \lambda^o(x)$ for some $x \in X^{oD}$. Then, $A^o_v \neq \emptyset$ by K33 for Γ^o. Since $(\Gamma, \mathfrak{m})$ is an i.d.view for memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$, we have $\Theta(\Gamma) = T^*_{D_i}$ by P1a. Also, since $\langle \xi, v\rangle$ is maximal in $T^*_{D_i}$, there exists $y \in X^E$ such that $\theta(y) = \langle \xi, v\rangle$. Then, by P1b, $A_y = A^o_v \neq \emptyset$. This contradicts that y is an endnode in Γ. Hence, the right hand side of Theorem 5.2.(i) holds.

Now let us see that $\mathfrak{m}^d$ is defined by (6.1) is a memory function for Γ^d. By P1a, $W = W^d$. Then by c4 and P1b, $\mathfrak{m}^d$ is a memory function for Γ^d since $\mathfrak{m}$ is a memory function for Γ. ■

Proof of Theorem 6.3. Let $(\Gamma, \mathfrak{m})$ be an i.d.view for $(T_{D_i}, \mathcal{Y}_{D_i})$. By Lemma 6.9, $(\Gamma, \mathfrak{m})$ is a personal view, where Γ^d is the unique direct structure for $(T_{D_i}, \mathcal{Y}_{D_i})$ and $\mathfrak{m}^d$ is defined by (6.1). First we show that $(\Gamma^d, \mathfrak{m}^d)$ is a direct view. Since Γ^d is the unique direct structure, we need only to show that $\mathfrak{m}^d$ satisfies d7. Let $x \in X_i^d$. By (6.1) and P2b for $\mathfrak{m}$, $x = \theta(y_x) \in \mathfrak{m}(y_x) = \mathfrak{m}^d(x)$ for some $y_x \in X_i$.

We define the function ψ from $(\Gamma, \mathfrak{m})$ to $(\Gamma^d, \mathfrak{m}^d)$ by:

$$\psi(x) = \theta(x) \text{ for all } x \in X. (6.2)$$

The proof will be completed if we show that ψ is a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\Gamma^d, \mathfrak{m}^d)$.

g0: We have $X^d = T^*_{D_i}$ by d1, and also $\Theta(\Gamma) = T^*_{D_i}$ by P1a for $(\Gamma, \mathfrak{m})$. Thus, $X^d = \Theta(\Gamma)$ and so ψ is a surjection from X to X^d.

g1: Let $x < y$. Then, $\theta(x)$ is an initial segment of $\theta(y)$, i.e., $\psi(x) = \theta(x) <^d \theta(y) = \psi(y)$ by d2.

g2: Let $\hat{x} <^d \hat{y}$ and $\hat{y} = \psi(y)$. Then, $\hat{x}$ and $\hat{y}$ can be written as $\langle \xi, v\rangle$ and $\langle \eta, w\rangle$ respectively, and by d2, $\langle \xi, v\rangle$ is an initial segment of $\langle \eta, w\rangle$. Since $\hat{y} = \psi(y) = \theta(y) = \langle \eta, w\rangle$, and $\langle \xi, v\rangle$ is an initial segment of $\langle \eta, w\rangle$, we can find a unique x on the path to y with $\theta(x) = \langle \xi, v\rangle$. Thus, $x < y$ and $\psi(x) = \theta(x) = \hat{x}$.

For g3-g7 we will use the generic history $\theta(x) = \langle \xi, v\rangle$ for the node x in question.

g3: Let $x \in X$. Then $\psi(x) = \theta(x) = \langle \xi, v\rangle$. Hence, $\lambda^d \cdot \psi(x) = \lambda^d\langle \xi, v\rangle = v$ where the last equality follows from d3. Hence, we have shown that $\lambda^d \cdot \psi(x) = \lambda(x)$.

g4: Let $x \in X$. Then, by d4, $A^c_{\langle \xi, v\rangle} = A^o_v$. By P1b, we have $A_x = A^o_{\lambda(x)} = A^o_v$. Hence, $A^c_{\psi(x)} = A_x$.

g5: Let $x \in X$. By d3, $\pi^d \cdot \lambda^d \cdot \psi(x) = \pi^c(v)$. If $x \in X^D$, then by P1c, $\pi \cdot \lambda(x) = \pi^o \cdot \lambda(x) = \pi^o(v)$. Also, since $x \in X^D$, it follows by Lemma 6.2 that $\langle \xi, v \rangle \in X^{dD}$. Hence, by d5, $\pi^d(v) = \pi^o(v)$. Thus, for $x \in X^D$ we have the desired result that $\pi^d \cdot \lambda^d \cdot \psi(x) = \pi \cdot \lambda(x)$. Next consider $x \in X^E$. Then by K42, $\pi \cdot \lambda(x) = \{j : j \in \pi \cdot \lambda(y) \text{ for some } y \in X^D\}$. By Lemma 6.5, g0 and d5, it follows that this set is equivalent to $\pi^d(v)$.
g6: Let $x \in X^E$. By P1a and P1d, $v = \lambda(x) = \lambda^o(y)$ for some $y \in X^{oE}$, and $h(v) = h_i^o(v)$. By Lemma 6.2 and g3, $\psi(x) \in X^{dE}$ and $\cdot \psi(x) = v = \lambda^o(y)$ for some $y \in X^{oE}$. So, by d6, $h^d(v) = h_i^o(v)$. Hence, we have shown that $h^d \cdot \lambda^d \cdot \psi(x) = h \cdot \lambda(x)$.
g7: Let $x \in X^i$. Then by the definition of ψ, (6.1) and P2c for $\mathfrak{m}$, it follows that $\mathfrak{m}^d \cdot \psi(x) = \mathfrak{m}^d \cdot \theta(x) = \mathfrak{m}(y) = \mathfrak{m}(x)$. ■

7. Decision making and prescribed behavior in IGT

The inductive derivation of an individual view from past experiences is not the end of the entire scenario of our theory. The next step is to use an i.d.view for decision making and to bring the prescribed (or modified) behavior back to the objective situation. This is the third stage of Fig.1.1. Because this paper aims to present a basic and entire scenario of our theory, we will here concentrate on a clear-cut case. Specifically, we assume in this and next sections that the objective memory function $\mathfrak{m}_i^o$ for each player i is given as the SPR function $\mathfrak{m}_i^{spr}$, and that player i has the active domain $D_i^A(\sigma^o)$. Then, we will discuss how he can use the inductively derived view for his decision making as well as how the prescribed behavior helps his objective behavior. This gives an experiential foundation for Nash equilibrium.

7.1 Decision making using a personal view

Fig.7.1 describes the steps from experimentation (trial and error) to decision making using an i.d.view. One basic question is whether the i.d.view helps the player for his decision making, as well as whether the decision can be used in the objective situation when he brings it back there. In this and next sections, we will discuss these questions.
We assume that each player i:
(7a): is relevant in his own domain;
(7b): has the SPR function $\mathfrak{m}_i^o = \mathfrak{m}_i^{spr}$;
(7c): follows a behavior pattern σ_i^o;
(7d): accumulates memories over his active domain $D_i^A(\sigma^o)$.
(7e): adopts the direct view $(\Gamma^d, \mathfrak{m}^d)$.
Under these assumptions, it is already proved in Corollary 5.3 that there is a unique direct i.d.view for each player i. Now, we consider the case where player i adopts this direct i.d.view $(\Gamma^d, \mathfrak{m}^d)$.
Nevertheless, the direct structure Γ^d may not be an extensive game in the strong sense, which may create some complications in the following discourse. Thus, we make the following assumption to avoid it: for each player i,
(7f): for all $x, y \in D_i^A \cap X^{oE}$, $\theta^o(x)_i$ is not a proper subsequence of $\theta^o(y)_i$.
Under this assumption, the direct view $(\Gamma^d, \mathfrak{m}^d)$ is an extensive game in the strong sense, which will be stated in Lemma 7.1.

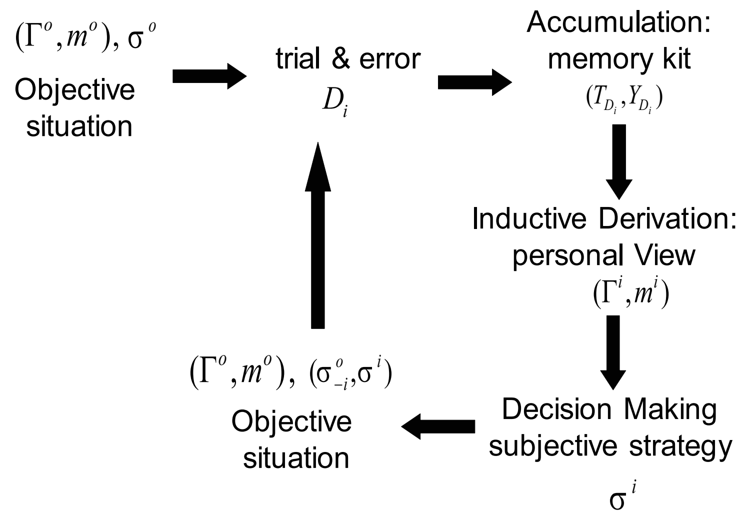

Fig. 7.1. Various Phases

Condition 7f is implied by Kuhn's [21] condition that each information piece for player i occurs at most once in each play in Γ^o, which was stated in terms of information sets in [21]. Fig.7.2.A, called the *absent-minded driver game* in Piccione-Rubinstein [28], with the SPR function $\mathfrak{m}_i^{spr}$ violates Condition 7f. In this case, $\langle$(E,e), 1$\rangle$ belongs to T_{D1} , but not to $T^*_{D_1}$ since $\langle$(E,c), (E,e), 1$\rangle$ is a proper supersequence of $\langle$(E,e), 1$\rangle$. Fig.7.2.B is the direct view but is not an extensive game in the strong sense.

The proofs of the results will be given in the end of this subsection.

Lemma 7.1. The direct view $(\Gamma^d, \mathfrak{m}^d)$ for $(T_{D_i}, \mathcal{Y}_{D_i}) = (T_{D_i^A(\sigma^o)}, \mathcal{Y}_{DA(\sigma^o)})$ is uniquely determined and is an i.d.view satisfying:

(a): Γ^d is a 1-person extensive game in the strong sense with $N^d = \{i\}$;

(b): $\mathfrak{m}^d$ satisfies P2a with equality, i.e., $\{\mathfrak{m}^d(x) : x \in X^d\} = \mathcal{Y}_{D_i^A(\sigma^o)}$.

For the consideration of utility maximization of a behavior pattern σ_i, player i needs to consider the sets of compatible endnodes for various behavior patterns. Recall from (2.15) that $\lambda(\sigma)$ denotes the set of compatible endpieces for a profile of behavior patterns $\sigma = (\sigma_1, ..., \sigma_n)$. Since Γ^d is a 1-person extensive game in the strong sense, the set of compatible endpieces will be a singleton set for each behavior pattern σ_i of player i. Consequently, we will use $\lambda^d(\sigma_i)$ here to denote the compatible endpiece in Γ^d for σ_i.

Then, player i has a subjective strategy σ_i^d in Γ^d to maximize h_d in the following sense:

$$h^d \cdot \lambda^d(\sigma_i^d) \geq h^d \cdot \lambda^d(\sigma_i) \text{ for all } \sigma_i \in \Sigma_i^d. \tag{7.1}$$

Once again, we emphasize that this decision is made in the personal view $(\Gamma^d, \mathfrak{m}^d)$ of player i, i.e., in the mind of player i. This conceptually differs from the payoff maximization in the objective situation, which is now the subject to be considered.

After the choice of the subjective strategy in (7.1), player i brings back σ_i^d to the objective situation $(\Gamma^o, \mathfrak{m}^o)$, adjusting his behavior pattern σ_i^o with σ_i^d . The adjustment from his objective behavior σ_i^d into σ_i^1 is as follows: for all $x \in X_i^o$,

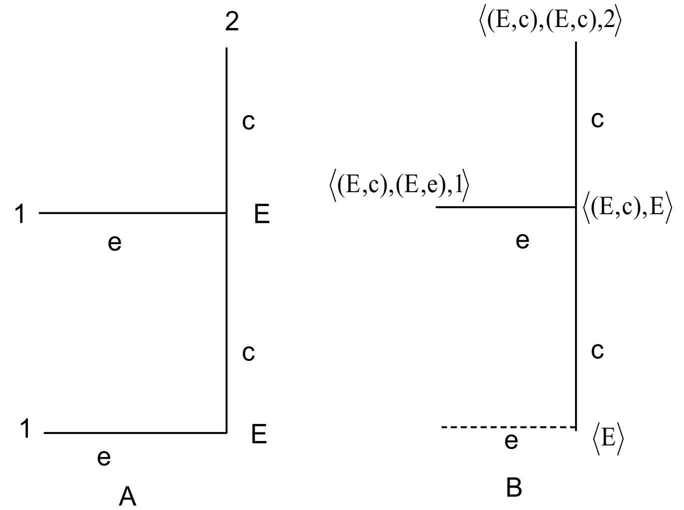

Fig. 7.2. Violation of condition 7f and the direct view

$$\sigma_i^1(x) = \begin{cases} \sigma_i^d\langle \xi, v\rangle & \text{if } \mathfrak{m}_i^o(x)\ (=\{\langle \xi, v\rangle\}) \in \mathcal{Y}_{D_i^A(\sigma^o)}; \\ \sigma_i^o(x) & \text{if } \mathfrak{m}_i^o(x) \notin \mathcal{Y}_{D_i^A(\sigma^o)}. \end{cases} \tag{7.2}$$

That is, player i follows σ_i^d whenever a memory yarn in $\mathcal{Y}_{D_i^A(\sigma^o)}$ occurs; and otherwise, he keeps the old behavior pattern. This adjustment produces a behavior pattern for player i in Γ^o, i.e., $\sigma_i^1 \in \Sigma_i^o$. The next theorem states that the modified strategy σ_i^1 of player i defined by (7.2) is objectively utility maximizing for player i in Γ^o when the other players follow their regular behavior σ_{-i}^o in Γ^o.

Before the next theorem, we give a small remark. Since the objective game Γ^o is also an extensive game in the strong sense, the set of compatible endpieces $\lambda^o(\sigma_i, \sigma_{-i}^o)$ will also be a singleton for player i's behavior pattern σ_i and the other players' behavior patterns σ_{-i}^o. We follow the convention of using $\lambda^o(\sigma_i, \sigma_{-i}^o)$ to denote the compatible endpiece, not the set of compatible endpieces.

Theorem 7.2 (One-person utility maximization in the *n*-person game): The strategy σ_i^1 defined by (7.2) satisfies the objective payoff maximization for player i, i.e.,

$$h_i^o \cdot \lambda^o(\sigma_i^1, \sigma_{-i}^o) \geq h_i^o \cdot \lambda^o(\sigma_i, \sigma_{-i}^o) \text{ for all } \sigma_i \in \Sigma_i^o. \tag{7.3}$$

We emphasize that this is not the utility maximization obtained directly in the objective situation. Instead, the utility maximization is made in his i.d.view (Γ^d, $\mathfrak{m}^d$), and then the modified strategy σ_i^1 is brought to the objective situation (Γ^o, $\mathfrak{m}^o$). It happens that it maximizes his objective utility function. This process of obtaining the objective utility maximization occurs only after many repetitions of collecting data to construct his view.

Thus, we have succeeded in having individual utility maximization in the well-defined form in both subjective and objective senses. Nevertheless, once we leave the case of 7a-7f, player i would have many difficulties at various steps in Fig.7.1. These problems will be discussed in Section 8.2 and in separate papers.

Proof of Lemma 7.1.(a): The condition N^d = {i} follows immediately since $\mathfrak{m}_i^o = \mathfrak{m}_i^{spr}$. By Corollary 5.3, it suffices to show that Γ^d satisfies K14 and K33.

K14: Since Γ^o is an extensive game in the strong sense, each strategy combination determines a unique play. Let $\langle x_1, ..., x_m\rangle$ be the unique play determined by σ^o, and let x_t be the first node of player i in this play, i.e., $i \in \pi^o \cdot \lambda^o(x_t)$ and $i \notin \pi^o \cdot \lambda^o(x_s)$ for all $s < t$. Then $\theta^o(x_t) = \langle(\lambda^o(x_1), \sigma_{j_1}^o(x_1)), ..., (\lambda^o(x_{t-1}), \sigma_{j_{t-1}}^o(x_{t-1})), \lambda^o(x_t)\rangle$ where $j_1, ..., j_{t-1}$ denote the players moving at $x_1, ..., x_{t-1}$ respectively. Let $(\sigma_i, \sigma_{-i}^o)$ be any other strategy combination where all the players other than player i choose according to σo. Then, the first t nodes in the play determined by this strategy combination must also be $x_1, ..., x_t$. Hence, for any play determined on the active domain, x_t is the first node of player i. Thus, x_t determines the smallest node $\theta^o(x_t)$ in X^d.

K33: We show that for each $\langle \xi, v\rangle \in X^{dD}$, the function $\varphi^d_{\langle \xi, v\rangle}$ defined in d4 is a bijection. Let $\langle \xi, v\rangle \in X^{dD}$ and let a be an arbitrary action in $A_{\langle \xi, v\rangle}$. Since $\langle \xi, v\rangle \in X^{dD}$ and the memory function is $\mathfrak{m}_i^{spr}$, we have $\langle \xi, v\rangle = \theta^o(x)_i$ for some $x \in X^{oi}$, and x is on the path determined by some $(\sigma_i, \sigma_{-i}^o)$. Consider the strategy σ'_{-i} defined by:

$$\sigma'_i(y) = \begin{cases} \sigma_i(y) & \text{if } \mathfrak{m}_i^o(y) \neq \mathfrak{m}_i^o(x); \\ a & \text{if } \mathfrak{m}_i^o(y) = \mathfrak{m}_i^o(x). \end{cases}$$

Since $\mathfrak{m}_i^o = \mathfrak{m}_i^{spr}$, it follows that $\mathfrak{m}_i^o(y) \neq \mathfrak{m}_i^o(x)$ for any $y \in X_i^{oD}$ with $y <^o x$. Hence x is on the play determined by $(\sigma'_i, \sigma_{-i}^o)$. Since the other players follow their strategies in σ^o, the action a determines a unique immediate successor x' of x with $\mathfrak{m}_i^{spr}(x') = \{\langle \xi, (v,a), u\rangle\}$. Then we find also an endnode z coming from x'. Then, $\langle \xi, (v,a), u\rangle$ is an initial segment of $\theta_i^o(z)$. By condition 7f, $\theta_i^o(z)$ is a maximal sequence in T_{D_i}. These mean that $\langle \xi, (v,a), u\rangle \in T_{D_i}^* = X^d$.
We can show similarly that a different action $a' \in A_{\langle \xi, v\rangle}$ determines a different immediate successor $\langle \xi, (v,a'), u'\rangle \in X^d$, so the mapping $\varphi^d_{\langle \xi, v\rangle}$ from $\langle \xi, (v,a), u\rangle$ to a is a bijection.

(b): Let $x \in D_i$. We show that $\mathfrak{m}_i^o(x) \in \{\mathfrak{m}^d(y) : y \in X^d\}$. Since $\mathfrak{m}_i^o = \mathfrak{m}_i^{spr}$, we have $T_{D_i} = T_{D_i}^*$. Since $\mathfrak{m}_i^o(x) = \{\theta^o(x)_i\}$, it follows that $\theta^o(x)_i \in T_{D_i}^* = X^d$. Corollary 5.3 states that the direct view $(\Gamma^d, \mathfrak{m}^d)$ exists uniquely and $\mathfrak{m}^d(y) = \{y\}$ for all $y \in X_i^d$. Hence, $\mathfrak{m}^d(\theta^o(x)_i) = \{\theta^o(x)_i\} = \mathfrak{m}_i^o(x)$. ■

Proof of Theorem 7.2. Consider any $\sigma_i \in \Sigma_i^o$. Recall that the endnode determined by $(\sigma_i, \sigma_{-i}^o)$ in Γ^o is denoted by $z(\sigma_i, \sigma_{-i}^o)$. Let $x = z(\sigma_i, \sigma_{-i}^o)$. Consider the history of player i at x, i.e., $\theta^o(x)_i = \langle(w_1, a_1), ..., (w_m, a_m), w_{m+1}\rangle$ with $w_{m+1} = \lambda^o(x)$, and also, let the corresponding history of nodes be given as $\langle x_1, ..., x_m, x_{m+1}\rangle$ with $x_{m+1} = x$. Then, $\lambda^o(x_t) = w_t$ and $\sigma_i(x_t) = a_t$ for all $t = 1, ..., m$. Hence, we choose a strategy τ_i^d having the property that $\tau_i^d\langle(w_1, a_1), ..., (w_{t-1}, a_{t-1}), w_t\rangle = \sigma_i(x_t)$ for $t = 1, ..., m$. Then, the compatible endpiece $\lambda^o(\sigma_i, \sigma_{-i}^o) = \{v\}$ is the same as $\lambda^d(\tau_i^d)$. Hence, $\lambda^o(\sigma_i^1, \sigma_{-i}^o) = \lambda^d(\tau_i^d)$. If we apply this procedure to σ_i^1, then we have σ_i^d satisfying (7.1). Hence, we have $\lambda^o(\sigma_i^1, \sigma_{-i}^o) = \lambda^d(\sigma_i^1)$.

By d7 and using the above result, we have $h_i^o \cdot \lambda^o(\sigma_i^1, \sigma_{-i}^o) = h^d \cdot \lambda^d(\sigma_i^d) \geq h^d \cdot \lambda^d(\tau_i^d) = h_i^o \cdot \lambda^o(\sigma_i, \sigma_{-i}^o)$. ■

7.2 An experiential foundation for Nash equilibrium

It is straightforward to extend Theorem 7.2 to all players relevant in their own domains and to obtain a Nash equilibrium. Here, we still state this theorem, since it gives one explanation of Nash equilibrium from the experiential viewpoint. For it, however, we need some more notation and one more definition.

First, since our discussion involves more than one i.d.view, we put subscript "i" to the direct i.d.view of player i, i.e., $(\Gamma_i^d, \mathfrak{m}_i^d)$. Second, for each player i who is relevant in his own domain, we define the *induced strategy* σ_i^d of σ^o to the direct i.d.view $(\Gamma_i^d, \mathfrak{m}_i^d)$ for $(T_{D_i^A(\sigma^o)}, \mathcal{Y}_{D^A(\sigma^o)})$ by: for all $\langle \xi, w\rangle \in X_i^d$,

$$\sigma_i^d\langle \xi, w\rangle = \sigma_i^o(x) \text{ for any } x \in X_i^o \text{ with } \theta_i^o(x) = \langle \xi, w\rangle. \tag{7.4}$$

The well-definedness of (7.4) is verified as follows. First, by the properties of the SPR function, for each $\langle \xi, w\rangle \in X_i^d$, there is an $x \in X_i^o$ such that $\theta_i^o(x) = \langle \xi, w\rangle$. Secondly, since $\theta_i^o(x) = \theta_i^o(y)$ implies $\mathfrak{m}_i^{spr}(x) = \mathfrak{m}_i^{spr}(y)$, the strategy defined by (7.4) does not depend upon the choice of x. Finally, we verify (2.12) and (2.13) for σ_i^d. The condition (2.12) follows from d4. Condition (2.13) is also satisfied since by Corollary 5.3, the direct memory function of player i is uniquely determined as $\mathfrak{m}_i^d\langle \xi, w\rangle = \{\langle \xi, w\rangle\}$.

Then we have the following theorem, which is a straightforward implication of Theorem 7.2

Theorem 7.3 (Experimental foundation for Nash equilibrium): A profile σ^o of behavior patterns is a Nash equilibrium in $(\Gamma^o, \mathfrak{m}^o)$ if and only if for each player $i \in N^o$ who is relevant in his domain $D_i^A(\sigma^o)$, the induced strategy σ_i^d of σ^o to the direct view $(\Gamma_i^d, \mathfrak{m}_i^d)$ for the memory kit $(T_{D_i^A(\sigma^o)}, \mathcal{Y}_{D_i^A(\sigma^o)})$ satisfies condition (7.1).

Recall that we have adopted the assumptions 7a-7f. Under these assumptions, each player makes his decision in his 1-person derived view. The theorem states that the behavior pattern σ^o is a Nash equilibrium in the the objective game $(\Gamma^o, \mathfrak{m}^o)$ if and only the induced strategy for each player i maximizes his utility in the direct view $(\Gamma_i^d, \mathfrak{m}_i^d)$. Thus, this theorem decomposes the Nash equilibrium in $(\Gamma^o, \mathfrak{m}^o)$ into utility maximizations in n one-person games.

As discussed in Section 3, the accumulation of $(T_{D_i^A(\sigma^o)}, \mathcal{Y}_{D_i^A(\sigma^o)})$ and the inductive derivation of $(\Gamma_i^d, \mathfrak{m}_i^d)$ need many repetitions of the game $(\Gamma^o, \mathfrak{m}^o)$. Also, in the present scenario, each player revises his behavior over $D_i^A(\sigma^o)$, and other players may be influenced by his revision, and may change their personal views. This revision process may continue. The above theorem describes a stationary state in the revision process.

The revision process may take a long time to reach a Nash equilibrium or even may not reach a Nash equilibrium. Furthermore, we did not explicitly consider the case where the players' trials and errors are restricted. If we take these limitations over experimentations, the above "Nash equilibrium" is understood as a Nash equilibrium relative to the restricted domains of actions.

In the above senses, Theorem 7.3 is one characterization of Nash equilibrium from the experiential viewpoint. In separate papers, we will discuss other characterizations of Nash equilibrium and/or difficulties arising for them. Finally, we give one example to suggest the nonconvergence of the process of revising behavior via constructed personal views. If the objective game $(\Gamma^o, \mathfrak{m}^o)$ has no Nash equilibria, then the above process does not converge. The following example has a Nash equilibrium.

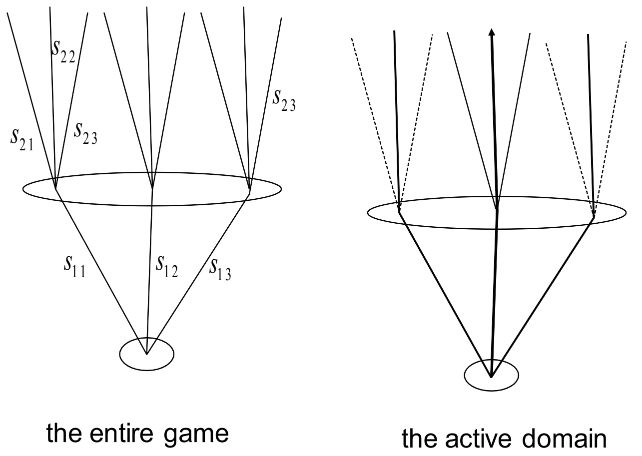

Fig. 7.3. Nonconvergence example

Example 7.1. (Nonconvergence): Consider the 2-person simultaneous game which is described as Fig.7.3 and its payoffs are given in Fig.7.4. The bold arrow is the regular path ($\mathbf{s}_{12}$, $\mathbf{s}_{22}$) and each player is presumed to have the SPR function.

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	$(3,3)^{NE}$	(2,2)	(2,2)
$\mathbf{s}_{12}$	(2,2)	(4,2)	(2,4)
$\mathbf{s}_{13}$	(2,2)	(2,4)	(4,2)

Fig. 7.4.

Player 1's direct i.d.view is the 1-person game summarized by the matrix form of Fig.7.5, and player 2's i.d.view is the 1-person game summarized in Fig.7.6.

$\mathbf{s}_{11}$	2
$\mathbf{s}_{12}$	4
$\mathbf{s}_{13}$	2

Fig. 7.5.

$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
2	2	4

Fig. 7.6.

In this case, player 1 maximizes his utility in his i.d.view by choosing $\mathbf{s}_{12}$. Thus, he has no incentive to change his objective behavior from the regular pattern. However, player 2 maximizes his utility in his i.d.view by changing from $\mathbf{s}_{22}$ to $\mathbf{s}_{23}$.

By this revision, the regular behavior becomes ($\mathbf{s}_{12}$, $\mathbf{s}_{23}$). After experiencing this pair as well as some trials, the personal views of the player's will be revised to the 1-person games summarized by the matrices of Fig.7.7 and Fig.7.8

$\mathbf{s}_{11}$	2
$\mathbf{s}_{12}$	2
$\mathbf{s}_{13}$	4

Fig. 7.7

$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
2	2	4

Fig. 7.8

With this new view, player 1 now finds that he should change his behavior, while player 2 does not. The revised behavior becomes ($\mathbf{s}_{13}$, $\mathbf{s}_{23}$). In this manner, the players move cyclically through the four regular behaviors depicted in the bottom right corner of Fig.7.9, and never converge to the Nash equilibrium ($\mathbf{s}_{11}$, $\mathbf{s}_{21}$).

	$\mathbf{s}_{21}$	$\mathbf{s}_{22}$	$\mathbf{s}_{23}$
$\mathbf{s}_{11}$	(3,3)	(2,2)	(2,2)
$\mathbf{s}_{12}$	(2,2)	(4,2)	$\rightarrow$(2,4)$\downarrow$
$\mathbf{s}_{13}$	(2,2)	$\uparrow$(2,4)	$\leftarrow$(4,2)

Fig. 7.9

8. *g*-Morphism analysis of decision making

In Section 6, we showed, using the concept of a *g*-morphism, that the direct view can be regarded as a representative one. On the other hand, in Section 7, we assumed that a player makes a decision using the direct view (Γ^d, $\mathfrak{m}^d$). Here, we apply the *g*-morphism analysis to the decision making of a player. The concept of a *g*-morphism helps us analyze decision making within some class of i.d.views. Here we do not restrict ourselves to the memory kits based on the SPR function $\mathfrak{m}_i^{spr}$ and on the active domain $D_i^A(\sigma^*)$. Although the *g* morphism analysis works well, we still find some difficulties in decision making with personal views and in transitions from subjective optimality to objective behavior.

8.1 Subjective optimality and *g*-morphism analysis

Let (Γ, $\mathfrak{m}$) be a personal view of player *i*. We assume that Γ satisfies $N = \{i\}$, i.e., it is a 1-person game. We call such a view a *purely personal view*.

We compare subjective optimality across *g*-morphic views of player *i*. For this purpose, let (Γ, $\mathfrak{m}$) and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be two purely personal views of player *i*, and let $\sigma_i \in \Sigma_i$ and $\hat{\sigma}_i \in \hat{\Sigma}_i$. Here, we follow the convention that each notion in $(\hat{\Gamma}, \hat{\mathfrak{m}})$ is distinguished from the corresponding one in (Γ, $\mathfrak{m}$) by the "cap", e.g., Σ_i and $\hat{\Sigma}_i$ are the sets of strategies of (Γ, $\mathfrak{m}$) and $(\hat{\Gamma}, \hat{\mathfrak{m}})$, respectively. We say that σ_i and $\hat{\sigma}_i$ are *endpiece-equivalent* iff

$$\lambda(\sigma_i) = \hat{\lambda}(\hat{\sigma}_i). \tag{8.1}$$

Recall that $\lambda(\sigma_i)$ is the set of compatible endpieces for σ_i, defined in (2.15). Endpiece-equivalent strategies σ_i and $\hat{\sigma}_i$ lead to the same endpieces in (Γ, $\mathfrak{m}$) and $(\hat{\Gamma}, \hat{\mathfrak{m}})$. When we have a *g*-morphism ψ from (Γ, $\mathfrak{m}$) to $(\hat{\Gamma}, \hat{\mathfrak{m}})$, we can carry over any strategy in (Γ, $\mathfrak{m}$) to $(\hat{\Gamma}, \hat{\mathfrak{m}})$ keeping endpiece-equivalence; and the converse needs one additional condition on (Γ, $\mathfrak{m}$).

The additional condition on (Γ, $\mathfrak{m}$) is as follows:

K33^S: for any $x \in X$, φ_x is a surjection from the set of immediate successors of x to A_x.

Condition K33^S is a weakening of K33, which requires φ_x to be a bijection. Under this condition on (Γ, $\mathfrak{m}$), we will have the converse that an endpiece-equivalent strategy is carried over from $(\hat{\Gamma}, \hat{\mathfrak{m}})$ to (Γ, $\mathfrak{m}$). The proofs will be given in the end of this subsection.

Theorem 8.1 (*g*-morphism and behavior). Let (Γ, $\mathfrak{m}$) and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be two purely personal views of player *i*, and let ψ be a *g*-morphism from (Γ, $\mathfrak{m}$) to $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

(a): Let $(\Gamma, \mathfrak{m})$ satisfy condition K33S. For each $\hat{\sigma}_i \in \hat{\Sigma}_i$, the function σ_i defined by (8.2) is a strategy in Σ_i and is endpiece-equivalent to $\hat{\sigma}_i$: for all $x \in X_i^D$,

$$\sigma_i(x) = \hat{\sigma}_i \cdot \psi(x). \tag{8.2}$$

(b): For each $\sigma_i \in \Sigma_i$, the function $\hat{\sigma}_i$ defined by (8.3) is a strategy in $\hat{\Sigma}_i$ and is endpiece-equivalent to σ_i: for each $\hat{x} \in \hat{X}_i^D$,

$$\hat{\sigma}_i(\hat{x}) = \sigma_i(x) \text{ for some } x \in X_i^D \text{ with } \psi(x) = \hat{x}. \tag{8.3}$$

In general, a g-morphism ψ embeds a larger game to a smaller game preserving certain game theoretical properties described in Definition 6.1. Assertion (a) converts a strategy from the smaller game to the larger game. A larger game may be too sparse to allow this conversion. Condition K33^S requires the larger game to be appropriately dense to allow it. On the other hand, (b) has no difficulty since the conversion of a strategy is along the g-morphism ψ in the direction from a larger game to a smaller game.

Condition K33^S itself may appear to be simply a mathematical condition for $(\Gamma, \mathfrak{m})$, though we already mentioned its game theoretical interpretation that each action leads to some consequence. In fact, this condition corresponds to one non-basic axiom called N3 (History-Independent Extension) in the theory of information protocols in Kaneko- Kline [16]. There, an information protocol with three non-basic axioms and two basic axioms is shown to be "equivalent" to an extensive game in the strong sense of the present paper. The other condition, K33^I, obtained from K33^S by replacing "surjection" by "injection" corresponds to another non-basic axiom in [16] called N2 (Determination). This axiom was shown, in Kaneko-Kline [17], to also have some important behavioral implications. Thus, these conditions, K33^S and K33^I are not only mathematically clear-cut, but also essential in the theory of extensive games in the strong and weak senses.

We should consider the implications of Theorem 8.1 in two respects. One is in terms of subjective optimality, and the other is about when player i brings back his modified behavior in the objective situation. From the viewpoint of g-morphisms, everything works well even in these respects. However, there are still some remaining difficulties in those two respects that are not captured by g-morphisms. These will be discussed in Section 8.2.

(1): *g*-morphism and subjective optimality: Since we do not assume that $\mathfrak{m}_i^o = \mathfrak{m}_i^{spr}$ and $D_i^o = D_i^A(\sigma^*)$, some i.d.views may be extensive games only in the weak sense. In such cases, the utility maximization (7.3) in Section 7 needs some modification. Here, we give one possible modification.

Let $(\Gamma, \mathfrak{m})$ be a purely personal view of player i. A strategy σ_i is *subjectively optimal* in $(\Gamma, \mathfrak{m})$ iff

$$\min_{w \in \lambda(\sigma_i)} h(w) \geq \min_{w' \in \lambda(\sigma_i')} h(w') \text{ for all } \sigma_i' \in \Sigma_i. \tag{8.4}$$

This is the maximin criterion for his decision making: The worst outcome compatible with this strategy is better than or equal to the worst outcome of any other strategy.

Corollary 8.2 (*g*-morphism and subjective optimality). Let $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$ be two purely personal views of player i, and let ψ be a g-morphism from $(\Gamma, \mathfrak{m})$ to $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

(a): Let $(\Gamma, \mathfrak{m})$ satisfy condition K33^S. If $\hat{\sigma}_i$ satisfies (8.4) in $(\hat{\Gamma}, \hat{\mathfrak{m}})$, then the endpiece-equivalent strategy σ_i defined by (8.2) satisfies (8.4) in $(\Gamma, \mathfrak{m})$.

(b): If σ_i satisfies (8.4) in (Γ, $\mathfrak{m}$), then the endpiece-equivalent strategy $\hat{\sigma}_i$ defined by (8.3) satisfies (8.4) in ($\hat{\Gamma}$, $\hat{\mathfrak{m}}$).

Again, we talk about the corollary in the context of i.d.views. By the results of Section 6, we can regard ($\hat{\Gamma}$, $\hat{\mathfrak{m}}$) as a direct one. By this result, we lose nothing in terms of subjective optimality by focusing on a direct view.

(2): *g*-morphism and objective behavior: After his decision making in an i.d.view, a player modifies his behavior pattern with his subjective strategy, and brings it back to the objective situation. This modification might depend upon the particular i.d.view of the player. In fact, we will show that the prescriptions for objective strategies are not different across *g*-morphic i.d.views. This implies that we can focus on the direct view even in the step of taking the prescription back to the objective world.

For the above consideration, we first modify (7.2) in the following way. Let (Γ, $\mathfrak{m}$) be a purely personal view of player *i* and let σ_i satisfy (8.4). We define the *prescribed* behavior of player *i* in (Γ^o, $\mathfrak{m}^o$) by: for all $x \in X_i^o$,

$$\sigma_i^1(x) = \begin{cases} \sigma_i(x') & \text{if } \mathfrak{m}_i^o(x) = \mathfrak{m}(x') \text{ for some } x' \in X; \\ \sigma_i^o(x) & \text{if } \mathfrak{m}_i^o(x) \neq \mathfrak{m}(x') \text{ for any } x' \in X. \end{cases} \tag{8.5}$$

This strategy prescribes the same behavior as (7.2) in the case of Section 7. The next corollary states that *g*-morphic views give the same prescriptions for behavior in the objective situation.

Corollary 8.3 (*g*-morphism and modified behavior). Let (Γ, $\mathfrak{m}$) and ($\hat{\Gamma}$, $\hat{\mathfrak{m}}$) be two purely personal views of player *i*, and let ψ be a *g*-morphism from (Γ, $\mathfrak{m}$) to ($\hat{\Gamma}$, $\hat{\mathfrak{m}}$).

(a): Let (Γ, $\mathfrak{m}$) satisfy condition K33^S. Let $\hat{\sigma}_i$ be a strategy in ($\hat{\Gamma}$, $\hat{\mathfrak{m}}$), and let σ_i be the endpiece-equivalent strategy defined by (8.2). Then σ_i and $\hat{\sigma}_i$ prescribe the same behavior to player *i* in (Γ^o, $\mathfrak{m}^o$).

(b): Let σ_i be a strategy in (Γ, $\mathfrak{m}$), and let $\hat{\sigma}_i$ be the endpiece-equivalent strategy defined by (8.3). Then σ_i and $\hat{\sigma}_i$ prescribe the same behavior to player *i* in (Γ^o, $\mathfrak{m}^o$), that is, the modified behaviors defined by (8.5) with σ_i and $\hat{\sigma}_i$ are the same.

In this corollary, we did not refer to the optimization condition (8.4). Of course, we can assume that σ_i in (a) or $\hat{\sigma}_i$ in (b) satisfies (8.4). Although Corollary 8.2 states that subjective optimality is invariant with personal views, subjective optimality may not guarantee, in general, the objective optimality of the prescribed behavior in contrast to Theorem 7.2.

Now we prove Theorem 8.1 and the corollaries. To prove (a) of Theorem 8.1, we first present the following lemma.

Lemma 8.4. Suppose that (Γ, $\mathfrak{m}$) satisfies K33^S. Let ψ be a g-morphism from (Γ, $\mathfrak{m}$) to ($\hat{\Gamma}$, $\hat{\mathfrak{m}}$). Then ψ satisfies: for all $\hat{x}, \hat{y} \in \hat{X}$, $x \in X$, and $a \in \hat{A}_{\hat{x}}$, if $\hat{x} \mathrel{\hat{<}_a^I} \hat{y}$ and $\hat{x} = \psi(x)$, then $x <_a^I y$ for some $y \in X$.

Proof. Let $\hat{x}, \hat{y} \in \hat{X}$, $a \in \hat{A}_{\hat{x}}$, and $x \in X$ with $\hat{x} = \psi(x)$ and $\hat{x} \mathrel{\hat{<}_a^I} \hat{y}$. By $\hat{x} = \psi(x)$ and g4, we have $\hat{A}_{\hat{x}} = A_x$. Thus, $a \in \hat{A}_{\hat{x}} = A_x$. So, by $K33^S$ on Γ, there is some $y \in X$ such that $x <_a^I y$. ■

Proof of Theorem 8.1.(a): Let $\hat{\sigma}_i \in \hat{\Sigma}_i$. Consider σ_i defined by (8.2). First, we show that σ_i is a function over X_i^D and satisfies (2.12) and (2.13) on (Γ, $\mathfrak{m}$).

Consider $x \in X_i^D$. By Lemma 6.5, we have $\psi(x) \in \hat{X}_i^D$. Thus, (8.2) assigns one action $\hat{\sigma}_i \cdot \psi(x)$ as $\sigma_i(x)$. Hence, σ_i is a function over X_i^D.

Next, we show (2.12) for σ_i. Let $\psi(x) = \hat{x}$ and $\hat{\sigma}_i(\hat{x}) = a$. Then, $\sigma_i(x) = \hat{\sigma}_i(\hat{x}) = a$ by (8.2). It suffices to show that $\varphi_x(y) = a$ some $y \in X$. By (2.12) for $\hat{\sigma}_i$, we have $\hat{\sigma}_i(\hat{x}) = \hat{\varphi}_{\hat{x}}(\hat{y}) = a$ for some $\hat{y} \in \hat{X}_i$, i.e., $\hat{x} \hat{<}_a^I \hat{y}$. By Lemma 8.4, we have $x <_a^I y$ for some $y \in X$, which implies $\varphi_x(y) = a$.

To prove (2.13) for σ_i defined by (8.2), consider $x, y \in X_i^D$ with $\mathfrak{m}(x) = \mathfrak{m}(y)$. Then, by g7, $\hat{\mathfrak{m}} \cdot \psi(x) = \mathfrak{m}(x) = \mathfrak{m}(y) = \hat{\mathfrak{m}} \cdot \psi(y)$. Since $\hat{\sigma}_i$ satisfies (2.13), we have $\sigma_i(x) = \hat{\sigma}_i \cdot \psi(x) = \hat{\sigma}_i \cdot \psi(y) = \sigma_i(y)$.

Next we show that the two strategies are endpiece-equivalent. This has two parts, $\lambda(\sigma_i) \subseteq \hat{\lambda}(\hat{\sigma}_i)$ and $\hat{\lambda}(\hat{\sigma}_i) \subseteq \lambda(\sigma_i)$. We show the former. The latter is proved in the same way.

First, let $w \in \lambda(\sigma_i)$. Then, there is a play $\langle x_1, \ldots, x_k, x_{k+1} \rangle$ in Γ with $\lambda(x_{k+1}) = w$ and $\theta(x_{k+1}) = \langle (\lambda(x_1), \sigma_i(x_1)), \ldots, (\lambda(x_k), \sigma_i(x_k)), \lambda(x_{k+1}) \rangle$. We denote $\psi(x_t)$ by $\hat{x}_t$ for $t = 1, \ldots, k+1$. By Lemma 6.7, $\langle \hat{x}_1, \ldots, \hat{x}_k, \hat{x}_{k+1} \rangle$ is a play in $\hat{\Gamma}$ and $\hat{\theta}(\hat{x}_{t+1}) = \theta(x_{k+1})$. By g3, $\hat{\lambda}(\hat{x}_t) = \lambda(x_t)$ for $t = 1, \ldots, k+1$, and by (8.2), $\hat{\sigma}_i(\hat{x}_t) = \sigma_i(x_t)$ for $t = 1, \ldots, k$. Hence, $\hat{\theta}(\hat{x}_{t+1}) = \langle (\hat{\lambda}(\hat{x}_1), \hat{\sigma}_i(\hat{x}_1)), \ldots, (\hat{\lambda}(\hat{x}_1), \hat{\sigma}_i(\hat{x}_k)), \hat{\lambda}(\hat{x}_{k+1}) \rangle$, which means $w \in \hat{\lambda}(\hat{\sigma}_i)$.

(b): Let $\sigma_i \in \Sigma_i$. We start by showing that $\hat{\sigma}_i$ defined by (8.3) is well-defined and satisfies (2.12) and (2.13) on $(\hat{\Gamma}, \hat{\mathfrak{m}})$.

Consider $\hat{x} \in \hat{X}_i^D$. Since ψ is a surjection by g0, $\psi(x) = \hat{x}$ for some $x \in X$. By Lemma 6.5, we have $x \in X_i^D$. Observe that there may be distinct $x, y \in X_i^D$. satisfying $\psi(x) = \psi(y) = \hat{x}$. Nevertheless, we can show that $\psi(x) = \psi(y)$ implies $\sigma_i(x) = \sigma_i(y)$, so that $\hat{\sigma}_i$ defined by (8.3) is well defined. To see this fact, observe that if $\psi(x) = \psi(y)$, then by g7, $\mathfrak{m}(x) = \mathfrak{m}(y)$, which together with (2.13) for σ_i implies $\sigma_i(x) = \sigma_i(y)$.

By (2.12) for σ_i, we have a $y \in X$ so that $\varphi_x(y) = \sigma_i(x)$. Let $\sigma_i(x) = a$. Then, $x <_a^I y$, so by Lemma 6.6, $\psi(x) \hat{<}_a^I \psi(y)$. Thus, $\varphi_{\hat{x}}(\psi(y)) = a$, which implies (2.12) for $\hat{\sigma}_i$.

Consider (2.13) for $\hat{\sigma}_i$. Let $\hat{x}, \hat{y} \in \hat{X}_i^D$ and $\hat{\mathfrak{m}}(\hat{x}) = \hat{\mathfrak{m}}(\hat{y})$. By g0 (surjection), we can find x and y so that $\psi(x) = \hat{x}$ and $\psi(y) = \hat{y}$. By g7, $\mathfrak{m}(x) = \mathfrak{m}(y)$. Hence, by (2.13) for σ_i and (8.3), we have $\hat{\sigma}_i(\hat{x}) = \hat{\sigma}_i(\hat{y})$.

It remains to check that $\hat{\sigma}_i$ and σ_i are endpiece-equivalent, which is shown in almost the same way as in the proof of (a) using (8.3) in place of (8.2). ∎

Proof of Corollary 8.2. We prove only (b). Let σ_i satisfy (8.4) in $(\Gamma, \mathfrak{m})$, and let $\hat{\sigma}_i$ be the endpiece-equivalent strategy defined by (8.3). By g3, g6, and endpiece-equivalence of σ_i and $\hat{\sigma}_i$, we have $\min_{\hat{w} \in \hat{\lambda}(\hat{\sigma}_i)} \hat{h}(\hat{w}) = \min_{w \in \lambda(\sigma_i)} h(w)$. For each $\hat{\sigma}_i' \in \hat{\Sigma}_i$, Theorem 8.1 guarantees that there is an endpiece-equivalent strategy $\sigma_i' \in \Sigma_i$ defined by (8.2) and $\min_{\hat{w}' \in \hat{\lambda}(\hat{\sigma}_i')} \hat{h}(\hat{w}') = \min_{w' \in \lambda(\sigma_i')} h(w')$.

Hence, since σ_i satisfies (8.4) in $(\Gamma, \mathfrak{m})$, we have, $\min_{\hat{w} \in \hat{\lambda}(\hat{\sigma}_i)} \hat{h}(\hat{w}) \geq \min_{\hat{w}' \in \lambda(\hat{\sigma}_i')} h(\hat{w}')$ for all $\hat{\sigma}_i' \in \hat{\Sigma}_i$. ∎

Proof of Corollary 8.3.(b): Let σ_i satisfy (8.4) in $(\Gamma, \mathfrak{m})$, and let $\hat{\sigma}_i$ be the strategy defined by (8.3). By Corollary 8.2, $\hat{\sigma}_i$ satisfies (8.4) in $(\hat{\Gamma}, \hat{\mathfrak{m}})$. We let $\sigma_i^1(x)$ and $\hat{\sigma}_i^1(x)$ denote the behavior prescribed by (8.5) in $(\Gamma, \mathfrak{m})$ and $(\hat{\Gamma}, \hat{\mathfrak{m}})$. respectively. Let $x \in X_i^o$. If $\mathfrak{m}_i^o(x) = \mathfrak{m}(x')$ for some $x' \in X$, then by g0 there is an $\hat{x}' \in \hat{X}$ where $\hat{x}' = \psi(x')$. By (8.3), $\hat{\sigma}_i(\hat{x}') = \sigma_i(x')$, so $\sigma_i^1(x) = \hat{\sigma}_i^1(x)$. If, alternatively, $\mathfrak{m}_i^o(x) \neq \mathfrak{m}(x')$ for any $x' \in X$, then $\sigma_i^1(x) = \sigma_i^o(x) = \hat{\sigma}_i^1(x)$. Part (a) is proved in almost the same way as (b). ∎

8.2. Difficulties involved in subjective thinking and in playing in the objective situation

In Section 7, we assumed that player i has the memory function $\mathfrak{m}_i^o = \mathfrak{m}_i^{spr}$ and the active domain $D_i^A(\sigma^o)$. Then, he succeeds in having the unique direct view, in finding an optimal strategy in $(\Gamma^d, \mathfrak{m}^d)$ as well as in bringing it back to the objective situation. However, if we drop these assumptions, then a subjectively optimal strategy may not help him behave properly in the objective situation. We can find many difficulties in decision making here, but we restrict ourselves to only some of them.

(1): Difficulty in subjective thinking: We start with a difficulty involved in subjective thinking. In Corollary 5.3, we gave a necessary and sufficient condition for a direct view to be unique and inductively derived. When the direct view is uniquely determined, the results of Section 6 state that it is essentially the smallest i.d.view. Also, the results of Section 8.1 imply that decision making is invariant to the choice of a personal view.

Problems may arise because of multiplicity of direct views for a given memory kit $(T_{D_i}, \mathcal{Y}_{D_i})$. In this case, player i faces a difficulty first in choosing an i.d.view.

In Example 5.1 there are four direct i.d.views, which all differ in terms of the memory function. Fig.8.1 gives two of those direct i.d.views with only the relevant memory yarns listed, and the payoffs are now attached. In Fig.8.1.A, the memory yarns are mixed up at the nodes $\langle (y_0, a), v\rangle$ and $\langle (y_0, b), v\rangle$ as $\mathfrak{m}_1^o(y_2)$ and $\mathfrak{m}_1^o(y_1)$, while the objective game has the same structure with the opposite assignment of $\mathfrak{m}_1^o(y_2)$ and $\mathfrak{m}_1^o(y_1)$. In Fig.8.1.B, he expects the same memory yarn $\mathfrak{m}_1^o(y_1)$ at each of his second decision nodes. In the view A, he does not use the memory yarn $\mathfrak{m}_1^o(y_2)$ in $\mathcal{Y}_{D1}$. This multiplicity of views causes some difficulty for the player in deciding which view to use for his decision making. His choice of a view may influence his decision making since, e.g., in the view A he can make different choices at $\langle (y_0, a), v\rangle$ and $\langle (y_0, b), v\rangle$, while in view B, he is required to make the same choice.

(2): Difficulty in objective optimality: Suppose that player 1 has chosen an direct i.d.view and a behavior pattern for it that is subjectively optimal in the sense of (8.4). Consider the direct view of Fig.8.1.A. One subjectively optimal strategy is defined by σ_1 choosing action a at the root node and the left node with $\mathfrak{m}_1^o(y_2)$, while choosing b at the right node with $\mathfrak{m}_1^o(y_1)$. When he modifies his regular behavior in the objective game by this strategy σ_1 and brings it back to the objective situation, he receives the payoff 0. Thus he fails to behave optimally in the objective situation.

Next, consider the view B. In this view, he has a subjectively optimal strategy prescribing the choice of b at all the decision nodes. If he takes this strategy to the objective world, he will receive the memory yarn $\mathfrak{m}_1^o(y_2)$, which he does not expect and, indeed, is not contained in his constructed personal view. Thus, the player finds a further difficulty with his view and a reason to revise his behavior or his view.

This difficulty is caused by the weak inclusion condition of P2a, allowing the possibility of $\{\mathfrak{m}^i(x) : x \in X_i^i\} \subsetneq \mathcal{Y}_{D_i}$. By strengthening P2a to equality, this difficulty could be avoided as in the view B. Nevertheless, the multiplicity of views remains, and so does the difficulty that a subjectively optimal strategy may not be objectively optimal.

Thus, when there are multiple direct i.d.views, player i may meet some difficulties both subjectively and objectively. Either of these difficulties gives a player a reason to revise his behavior or his view. In this paper, however, we do not consider those revisions.

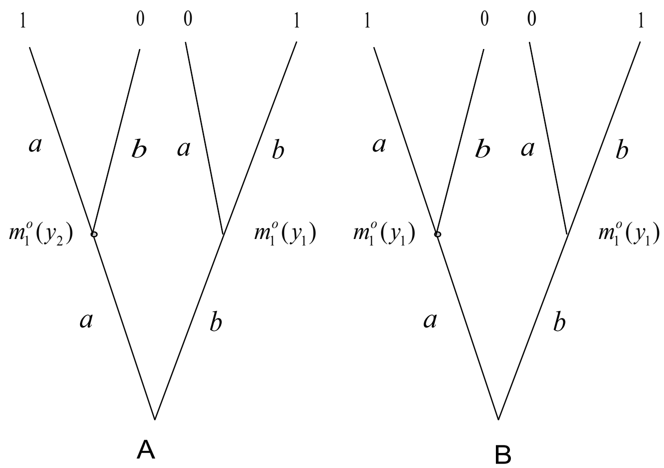

Fig. 8.1. Difficulty in objective optimality

9. Concluding comments

We have given a discourse of inductive game theory by confining ourselves to clear-cut cases. It would be, perhaps, appropriate to start this section with comments on our discourse. Then we will discuss some implications for extant game theory.

9.1 Comments on our discourse

We have made particular choices of assumptions and definitions for our discourse. One important methodological choice is to adopt extensive games in the strong and weak senses for objective and subjective descriptions. First, we will give some comments on this choice, and then, we will discuss the definition of an inductively derived view given in Section 4 based on the initial segment procedure.

As pointed out in Section 4, an extensive game contains observable and unobservable elements. The nodes with the successor relation are unobservable for the players and even for the outside observer, in which sense those are highly hypothetical. The components in a memory kit are all observables and actually observed. Thus, our definition of the inductive derivation of a personal view from a memory kit extends the observed observables by adding hypothetical elements. This may be interpreted as an "inductive" process of adding unobservable elements to observed data. However, this freedom of adding hypothetical elements leads us a proliferation of possible views. To prevent this proliferation, we need some criterion to choose a view from many possible ones. In this paper, we have used the concept of a g-morphism (game theoretical p-morphism) to choose a smallest one.

Conceptually speaking, the choice of a personal view is supposed to be done by a player, rather than us. While the definition of an inductive derivation allows many views, a player cannot construct a large one because of his bounded cognitive ability. Thus, the criteria of smallness and constructiveness are important from this point of view. The direct view defined in Section 5 has a constructive nature as well as being a smallest one for a given memory kit. In this sense, the direct view has a special status among those possible views.

Nevertheless, Definition 4.1 may admit no inductively derived views for a given memory kit, as characterized by Theorem 5.2. In fact, the initial segment procedure adopted in

Definition 4.1 still gives a strong restriction on the addition of hypothetical elements. If we allow more freedom in using hypothetical elements in an inductive derivation, we could avoid the nonexistence result. For example, if we allow a player to add "nature nodes" to his personal view, we could even avoid the use of an extensive game in the weak sense. On the other hand, this creates vast arbitrariness in inductive derivations; and we expect serious difficulties in finding natural criteria to narrow down the use of "nature nodes". Until we find natural criteria, we should refrain from the cheap use of "nature nodes".

The above conclusion may sound negative to any extension of our definition of an inductive derivation, but we have different opinions. We could actually have a more general procedure to construct a personal view than the initial segment procedure. Since this paper is intended to provide an entire scenario, we have chosen the initial segment procedure as a clear-cut case. In separate papers, we will discuss less restrictive definitions. See Section 9.3.

Another comment should be given on the choice of extensive games. In fact, we can avoid the adoption of extensive games; instead, the present authors ([16]) have developed a theory of *information protocols*, which avoids the use of nodes and describes game situations directly in terms of information pieces and actions together with a history-event relation. If we adopt this theory, then we could avoid a proliferation of personal views generated by the use of hypothetical nodes. In the theory of information protocols it may be easier to discuss extensions of inductive derivations. One reason for our adoption of extensive games here is their familiarity within our profession. The choice of extensive games makes the distinction between observables and unobservables explicit, which is another reason for our choice.

We expect gradual developments of inductive game theory to come about by deeper analysis and alternative approaches to the various stages mentioned in the diagram of Fig.1.1. By such gradual developments, we may find natural criteria for steps such as the use of nature nodes, and some experimental tests of inductive game theory.

9.2 Implications to extant game theory

It is a main implication of our discourse that a good individual view on society is difficult to construct from the experiential point of view: There are many places for a player to get stuck in his inductive process and analysis process. Nevertheless, we gave a characterization theorem of Nash equilibrium in Section 7. Here, we discuss some other implications to extant game theory and economics chiefly with respect to Nash equilibrium.

There are various interpretations of Nash equilibrium (cf. Kaneko [14], Act 4). Nash [25] himself described his concept from the viewpoint of purely *ex ante* decision making, but in economic applications, it is typically more natural to interpret Nash equilibrium as a strategically stable stationary state in a recurrent situation. The characterization given in Section 7 is along this line of interpretations, including also *ex ante* decision making in a player's constructed personal view.

To reach Nash equilibrium, which may not be the case, it takes a long time. Also, the process of trial and error may not allow all possible available actions. The Nash equilibrium reached should be regarded as a Nash equilibrium in the game with respect to the actually experienced domains. Thus, the characterization of Nash equilibrium in Section 7 should not merely be interpreted as a positive result. It means that the characterization would be obtained if all those processes go through well and if reservations about restrictions on trials are taken into account.

From the same point of view, the subgame perfect equilibrium of Selten [30] involves even deeper difficulties from our experiential point of view, which was already pointed out in

Kaneko-Matsui [18]. The reason is that subgame perfection requires higher order experimentations. When one player deviates from his regular behavior, other players in turn need, again, to make experimentations from regular behavior. This second or higher order experimentation is already problematic and violates some principles discussed in the informal theory in Section 3.2. In fact, a similar criticism is applied to Nash equilibrium, as already stated. Nash equilibrium itself is regarded as one limit notion, and subgame perfection is a higher limit one.

Taking the above criticism seriously, one important problem arises. The complexities, in a certain sense, of an inductively derived view as well as of experimentations are measured and restricted. In the epistemic logic context, Kaneko-Suzuki [20] introduced the concept of contentwise complexity, which measures complexity of a single instance of a game. This notion can be converted to our inductive game theory. Then, we will be able to give restrictions on individual views as well as experiments. In this manner, our inductive game theory will be developed in the direction of “bounded rationalities”.

We have restricted our attention to the purely experiential sources. In our society, usually, we have different sources of beliefs/knowledge such as from other people or through education. These suggest that a player may get more beliefs/knowledge on the social structure, but do not suggest that he can guess other people’s thinking, which has usually been assumed in the standard game theory (cf., Harsanyi [10] for incomplete information game and Kaneko [13] for the epistemic logic approach). At least, the assumption of common knowledge is far beyond experiences. If we restrict interpersonal thinking to very shallow levels, deductive game theory may have some connections to inductive game theory (cf. Kaneko-Suzuki [19] for such a direction of deductive game theory).

9.3 Postscript

By now, several new developments along the line of the scenario given in this paper have been made in Kaneko-Kline [15], [16], [17], and Akiyama-Ishikawa-Kaneko-Kline [1]. We use this postscript section to present some small summaries of those papers to help the reader catch up to the present state of inductive game theory.

The main concern of Kaneko-Kline [15] is the size of an inductively derived view for a player with bounded cognitive abilities. If the objective situation is too large, a player may have difficulty: 1) analyzing it strategically; and 2) accumulating enough experiences to have a rich view. The premise of that paper is that the number of experiences and the size of a view must be small for it to be managed by a player. The concept of “marking” some parts and actions as important was introduced in that paper and shown to be successful in allowing a player to obtain a manageable, though potentially biased, view.

As already mentioned in Section 9.1, Kaneko-Kline [16] introduced a new construct called an “information protocol”, based on “actions” and “information pieces” as tangible elements for each player rather than hypothetical non-tangible concepts such as nodes. This approach gives a more direct and simpler description of a game situation from the perspective of a player. It has another merit to classify extensive games in a more clear-cut manner. With an appropriate choice of axioms, it fully characterizes an extensive game in the weak and strong senses. It also enables us to avoid g-morphisms, since we have no multiplicity in i.d.views caused by hypothetical nodes and branches. The theory of information protocols has been adopted in our more recent research including Kaneko-Kline [17].

Kaneko-Kline [17] took up that task of constructing i.d.views with more partiality in a players memory. Accordingly, the definition of an i.d.view had to be weakened to admit a view. By these generalizations, the induction becomes less deterministic and we meet some multiplicity of consistent views with a given set of memories. The interactions between a player's i.d.view, his future behavior, and future views become the topics of this paper and also serve as potential sources for resolving the multiplicity problem.

Finally, Akiyama et al. [1] took a computer simulation approach in order to look into the process of experiencing and memorizing experiences in a one-person problem called "Mike's bike commuting". That paper tries to clarify the informal theory of behavior and accumulation of memories discussed in Section 3.2 of this paper. The simulation approach is based on finite experiences and accumulations of memories. The use of "marking" introduced in Kaneko-Kline [15] was found to be crucial for obtaining a rich enough view.

These developments are, more or less, consistent with the scenario spelled out in this paper and give more details into each step in the basic scenario. We are presently continuing our research along those lines making progress into experiential foundations of beliefs/knowledge on other players' thinking.

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6

Cooperative Logistics Games

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1. Introduction

Roughly speaking, Game Theory deals with analysing conflict and cooperation situations in which two or more rational and intelligent agents are involved. There are many real and theoretical situations which can be examined from the point of view of Game Theory. Therefore it is not difficult to find in the literature a rich variety of applications of Game Theory to many and very diverse fields of knowledge. In particular, Game Theory plays a significant role in Economics, but we can also find applications to Computer Science and Engineering.

Game Theory can be roughly divided into two main areas: cooperative and non cooperative games. The basic key for distinguishing between these two areas is whether it is possible or not to reach binding agreements. When binding agreements are possible, we are then faced with a cooperative situation. Thus, in a cooperative environment the concept of coalition plays an important role and very often the main goal is to achieve the cooperation of all agents. In this chapter we will assume that binding agreements among the agents are possible and therefore we will use the cooperative approach for analysing some logistics problems.

On the other hand, there are a number of theoretical and conceptual connections between Game Theory and Operations Research (OR). For example, we should mention the connection between the duality in mathematical programming and the minimax theorems for zero-sum games (see Raghavan, 1994); the linear complementary theory and the bi-matrix games (see Lemke, 1965), or the optimal control theory and the differential games (see Friedman, 1994) among others. Furthermore we can find applications of OR to Game Theory, for example the characterization of balanced games using the duality concept (Bondareva, 1963 and Shapley, 1967). Likewise, Game Theory contributes to completing the analysis of OR problems when there is more than one agent involved in the corresponding situation. Thus, after optimising a particular system by means of OR techniques, in which there are two or more agents involved, who have to collaborate in order to be able to achieve that optimal result, saying something about how to distribute the extra benefits or the costs saved by cooperation among those agents seems reasonable and necessary. Hence cooperative games can play a role in the complete analysis of the situation.

In the literature, not only we can find many OR problems studied from the point of view of cooperative games in the sense mentioned previously, but also OR problems analysed from a strategic or non-cooperative approach. However, in this chapter we are more interested in the cooperative approach. Some of the first OR situations studied using cooperative games are assignment problems (Shapley & Shubik, 1971), linear production problems (Owen, 1975), network flow problems (Kalai & Zemel, 1982) and minimum cost spanning tree problems (Claus & Kleitman, 1973 and Bird, 1976), obtaining the so-called assignment games, linear production games and so on. The games obtained from OR problems are usually called OR-games (see Borm et al., 2001 for a survey on this topic).

In general, the methodology to analyse an OR problem from a cooperative approach consists of associating a coalitional game to each problem or characteristic function form game summarising the gains or savings from cooperation for each possible coalition of the agents involved and, thereby, analysing different topics of Game Theory such as solution concepts, stability, etc. Thus we can try to answer the question posed before, namely, 'How to distribute the extra benefits or the costs saved by achieving cooperation among the different agents involved'.

Logistics include the analysis and management of many different situations which can be formulated or modelled as OR problems. Thus problems related to transportation, inventory, supply chain, distribution, location, routing or storage among others, arise frequently in logistics. One can also consider that all of these problems may have more than one agent involved, so a game theoretical approach could be used to tackle them either from a cooperative point of view or from a non cooperative point of view. In the literature we can find both approaches for the different logistics problems but we will concentrate our attention on the cooperative approach.

In this chapter we will only analyse two logistics problems from a cooperative point of view: transportation situations –and some related problems– and supply chain situations. The two problems selected are representative of a particular problem in logistics, such as the transportation of goods from stores or production sources to points of sale or distribution and a general problem, such as the supply chain which embraces many (or all) logistics tasks. Therefore we have selected one particular problem and a more general problem. In this sense it is possible to consider logistics as being a part of supply chain management but we have considered the supply chain inside logistics in order to be able to analyse separately different interesting optimisation problems under the same umbrella. On the other hand, we are aware that these two problems do not cover all possible logistics situations but we believe that the analysis of these problems together with the references provided throughout the chapter can provide a good starting point for the reader interested in this topic.

Finally, since we will use the cooperative approach to analyse the different problems and hence are interested in cooperation between the agents, then we will study the concept of coalitional stability represented by the core of the game. To this end, we will analyse the non-emptiness of the core of the corresponding game and therefore the existence of coalitional stable distributions. Likewise, we will explore other possible solution concepts and their relationship to the core of the game.

The rest of the chapter is organised as follows. In Section 2 we provide the basic definitions, concepts and solutions of cooperative games. We also describe the methodology for defining a cooperative OR game and introduce logistics games. Section 3 analyses the cooperative approach for transportation situations and some related problems which can arise in logistics situations. In Section 4 we review the literature for the cooperative approach for supply chain situations and explore the possibility to analyse from a cooperative standpoint supply chain situations without storage through two particular examples. Finally, in Section 5 we briefly revise the literature for other logistics games.

2. Preliminaries

In this section we formally introduce some basic definitions, concepts and solutions for cooperative games in order to provide the reader with all the necessary background to follow this chapter. Likewise, we present what we mean for Operations Research Games and the definition of logistics games.

2.1 Basic notions on cooperative games

First, a *cooperative game in characteristic function form* is a pair (N, v) where N is a finite set of agents called players and v is a function that associates to each set $S \subset N$ a real value $v(S)$ satisfying $v(\emptyset)=0$. This value $v(S)$ represents the joint gain that the agents in S can guarantee by themselves if they cooperate independently of what the agents in $N\backslash S$ could do. Therefore, in some sense, $v(S)$ measures the worth of coalition S. On the other hand, when the characteristic function represents costs instead of gains or benefits then we will denote it by c and we refer to *cost games*. Of course, it is possible to transform a cost game (N, c) in a benefit game through the so-called *savings game*. The definition of a savings game (N, v^c) associated with a cost game (N, c) is the following:

$$v^c(S) = \sum_{i \in S} c(i) - c(S) . \quad (1)$$

Therefore the savings game is simply the saved costs from cooperation with respect to all the individual costs. Thus, the savings game represents the gains of cooperation as opposed to acting separately.

We will denote by G^N the set of all (benefit or profit) games with set of players N and by CG^N the set of all cost games with set of players N. Furthermore, we will denote by G the set of all (benefit or profit) games and by CG the set of all cost games.

There are some properties of the characteristics function which, at first glance, if a game satisfies them, then it seems that cooperation is profitable for the agents and hence the possibility of cooperation exists. However, a more careful analysis is necessary as we will see later.

For profit or benefit games the properties are the following:

- *Monotonicity*: if $v(S) \le v(T)$ for all $S \subset T \subset N$.
- *Superadditivity*: if $v(S \cup T) \ge v(S) + v(T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$.
- *Convexity*: if $v(S \cup T) + v(S \cap T) \ge v(S) + v(T)$ for all $S, T \subset N$.

For cost games their counterparts can be written as:

- *Monotonicity*: if $c(T) \le c(S)$ for all $S \subset T \subset N$.
- *Subadditivity*: if $c(S \cup T) \le c(S) + c(T)$ for all $S, T \subset N$ such that $S \cap T = \emptyset$.
- *Concavity*: if $c(S \cup T) + c(S \cap T) \le c(S) + c(T)$ for all $S, T \subset N$.

Given a game (N, v) (resp. cost game (N, c)) a *distribution* or *allocation* for it is a vector $z \in \Re^N$ such that $\sum_{i \in N} z_i \le v(N)$ (resp. $\sum_{i \in N} z_i \ge c(N)$). We will denote by $z(S) = \sum_{i \in S} z_i$. A distribution z is called *efficient* if $z(N)=v(N)$ (resp. $z(N)=c(N)$).

A *solution* for G (resp. CG) is a map $\sigma{:}G \to \Re^N$ (resp. $\sigma{:}CG \to \Re^N$) such that $\sigma(N,v) \subset \Re^N$ for all $(N, v)\in G$ (resp. $\sigma(N,c) \subset \Re^N$ for all $(N, c)\in CG$) and $z(N)=v(N)$ (resp. $z(N)=c(N)$) for all $z\in \sigma(N,v)$. If σ is always a single point then it is called *value*, otherwise it is called a set-valued solution or simply a solution. A solution for a game is a set of efficient distributions of the total gain or cost. One of the most outstanding solutions is the *core*. The core of a game is the set of all coalitional stable distributions and, therefore, any coalition obtains at least what the members of it can achieve by themselves. In formulas for benefit/profit games and cost games respectively:

$$Core(N,v) = \{z \in \Re^N : z(S) \geq v(S) \text{ for all } S \subset N \text{ and } z(N) = v(N)\}\ . \tag{2}$$

$$Core(N,c) = \{z \in \Re^N : z(S) \leq c(S) \text{ for all } S \subset N \text{ and } z(N) = c(N)\}\ . \tag{3}$$

The distributions in the core of a game are interesting because there is no incentive for any coalition to reject them. However, the core of a game can be empty. The games with non-empty core are called *balanced*. (Shapley, 1971) proved that all convex games (resp. concave for the case of cost games) have a non-empty core and hence they are balanced.
On the other hand, another interesting set of distributions is the *imputation set*. It is defined as the set of all efficient and individually stable (or rational) distributions. In formulas for benefit/profit games and cost games respectively:

$$I(N,v) = \{z \in \Re^N : z_i \geq v(i) \text{ for all } i \in N \text{ and } z(N) = v(N)\}\ . \tag{4}$$

$$I(N,c) = \{z \in \Re^N : z_i \leq c(i) \text{ for all } i \in N \text{ and } z(N) = c(N)\}\ . \tag{5}$$

Given a game (N, v) the *marginal contribution* of player i to coalition S ($i\notin S$) is given by $v(S\cup i)-v(S)$ (resp. $c(S\cup i)-c(S)$). Based on this concept another outstanding solution for cooperative games is defined: the *Shapley value* (Shapley, 1953). For each player the Shapley value is the average of all her possible marginal contributions. The mathematical expression of the Shapley value is the following:

$$\begin{gathered} Sh_i(N,v) = \sum_{S\subset N, i\notin S} \gamma_n(S)\big[v(S\cup i) - v(S)\big],\ \forall i \in N \\ \text{where } \gamma_n(S) = \frac{s!(n-s-1)!}{n!} \text{ and } s = card(S). \end{gathered} \tag{6}$$

The Shapley always exists but does not belong to the core in general. However, (Shapley, 1971) proved that if the game is convex (resp. concave for cost games) then the Shapley value is always in the core of the game.
(Schmeidler, 1969) introduced a value, called *nucleolus*, which always belongs to the core of the game when it is non-empty. The definition of the nucleolus is based on the concept of *excess (or complaint) of a coalition with regard to a distribution*. Given a game (N, v) (resp. (N, c)), a coalition $S\subset N$ and a distribution z, the excess of coalition S with regard to distribution z is given by $e(S;z) = v(S) - z(S)$ (resp. $e(S;z) = z(S) - c(S)$). Likewise, we define $\theta(z)$ as the vector of all excesses with regard to z written in decreasing order. The nucleolus of a game (N, v) (analogously for a cost game (N, c)) is defined as

$$nu(N,v) = \{z \in I(N,v) : \theta(z) \leq_L \theta(x) \text{ for all } x \in I(N,v)\}, \tag{7}$$

where $\leq_L$ is the lexicographic order. Therefore, the nucleolus is the distribution that minimises the maximal excess or complaint of all coalitions.

There are a number of different solutions for cooperative games in characteristic function form. For this reason it is necessary to know which solutions are more suitable for a particular situation. One way to understand the solutions better is through the properties they satisfy. The main objective is to know which "reasonable" properties characterise each solution. Thus, depending on which properties are meaningful or important in a particular situation, we would be able to find out which solutions fit better too. Therefore, we can find many papers in the literature characterising solutions for cooperative games using different sets of properties.

2.2 Cooperative Operations Research Games (ORGs)

Consider a system where there are one or more agents interested in optimising it. One way to deal with this situation is to have recourse to Operations Research and we are then faced with an operations research problem. The simpler situation is when there is only one agent or decision-maker involved in the problem and, therefore, there is no conflict of interests. In that case the analysis of the system is completed on the procurement of one optimal solution for it using the appropriate optimisation techniques. However, it is not difficult to find that, on many occasions, there would be more than one agent or decision-maker involved in the system and, consequently, some kind of conflict of interests could arise. In that case, each agent could own or control one or more parts of the system and if they wanted to optimise the system then they should cooperate but, perhaps, they should agree on how to distribute the profits/benefits or saved costs among themselves. Therefore, the analysis of the systems does not end with the procurement of one optimal solution but it is necessary to go a step further in order to convince the agents involved to cooperate, most likely, via a good distribution of the profits or saved costs. One way to tackle this last step in the analysis is using cooperative games.

Given an operations research problem A in which there is a finite set N of agents involved, we define an associated cooperative game in characteristic function form (N, v^A) in the following way:

$$\begin{gathered} v^A(\varnothing)=0, \\ v^A(N)=Optval(A) \text{ and} \\ v^A(S)=Optval(A_S) \text{ for all } S\subset N, \end{gathered} \tag{8}$$

where $Optval(A)$ is the optimal value for problem A and $Optval(A_S)$ is the optimal value for problem A_S, where A_S is the problem obtained using only the parts of problem A owned or controlled by the agents in coalition S. In the case that problem A is a cost problem we can analogously define the cost game (N, c^A). These games are called (cooperative) *operations research games*. Furthermore, if the operations research problems are related to logistics situations then we will call them *cooperative logistics games*.

Once we have defined a cooperative game associated with an operations research problem, then we could obtain different answers to the question of how to distribute the profits/benefits or saved costs among the agents involved using the solutions defined for cooperative games, such as the core, the Shapley value, the nucleolus, etc. Note that if we only use the characteristic function of the game then we may lose some of the essence of the problem. However, it would also be possible to think of the primal and dual optimal solutions of the operations research problem to obtain distributions of the profits/saved costs among the agents involved. Thus, in the latter approach, we would be considering, in some manner, the particular features of the operations research problem. Of course, the choice of one approach or another will depend on the particular situation.

Two examples of solutions based on the primal optimal solutions of the corresponding operations research problems are the Bird solution for minimum cost spanning tree games (Bird, 1976) which is based on the application of the Prim algorithm (Prim, 1957) and the pairwise solutions for transportation games (Sanchez-Soriano, 2003 and 2006). The first is a solution based on an algorithm while the second are solutions based directly on the optimal solutions of the problem. Therefore, we have two different examples of how to use the Operations Research techniques to obtain the distribution of the total profits/saved costs among the agents taking part in the problem. In both cases the relationship between the solution and the core of the game is studied.

Another possibility is to deal with the optimal solutions of the dual problem. Two examples of this approach are (Shapley & Shubik, 1971) for assignment problems and (Owen, 1975) for linear production problems. In the first paper, the authors proved that the core of the game and the set of dual optimal solutions coincide. In the second paper, the inclusion of the set of distributions based on the dual optimal solutions in the core of the game is demonstrated. The set of distributions based on the dual optimal solutions is called the *Owen set* (van Gellekom et al., 2000).

3. Transportation, distribution and warehouse sharing games

In this section we will study some transportation problems from the point of view of cooperative games. We will start with the simplest transportation situation with only two types of agents (suppliers and demanders) which we call two-sided transportation problem. A problem of this kind describes three possible logistics situations of transportation of goods: producers-retailers, producers-wholesalers or wholesalers-retailers. In each case, the mathematical treatment of these is essentially the same. Secondly, we will analyse transportation situations with three types of agents (suppliers, intermediates and demanders) which we call three-sided transportation problems. A situation of this kind corresponds to producers-wholesalers-retailers distribution problems. Finally, we will study warehouse sharing problems in which the agents involved in the situation must share the warehouses in order to optimise their transportation profits/costs.

3.1 Two-sided transportation games

Basically, a two-sided transportation problem consists of two sets of agents, called producers and retailers, which produce and demand goods. Each producer produces a quantity of goods and each retailer demands a certain amount of goods. The transport of the goods from the producers to the retailers is costly (profitable) and, therefore, the main objective is to transport the goods from the producers to the retailers at minimum cost (at maximum profit). The way to achieve this objective is by means of cooperation, otherwise if each agent would make decisions on their own, then the final result of the transportation would be unpredictable and, perhaps, far from the optimal situation. Therefore, if cooperation is profitable then this should be promoted through a good distribution of the extra profits or saved costs.

Let P and R be the sets of producers and retailers respectively. We denote by p_i the production of goods of producer $i \in P$ and by d_j the demand of goods of retailer $j \in R$. The unitary cost (resp. benefit) of transportation from producer i to retailer j is denoted by c_{ij} (resp. b_{ij}). The mathematical model of this problem can be described by:

$$\begin{aligned} \min \quad & \sum_{i \in P} \sum_{j \in R} c_{ij} x_{ij} \\ s.t.: \quad & \sum_{j \in R} x_{ij} \le p_i, i \in P \\ & \sum_{i \in P} x_{ij} \ge d_j, j \in R \\ & x_{ij} \ge 0, i \in P, j \in R \end{aligned} \tag{9}$$

where x_{ij} is the number of units transported from producer i to retailer j.

Problem (9) has feasible solutions if it satisfies that $\sum_{i \in P} p_i \ge \sum_{j \in R} d_j$. However, if we consider that each transported unit lead up to a benefit b (large enough to compensate any unitary cost) then we can consider a maximisation problem with coefficients $b_{ij}=b-c_{ij}$ and relax the second block of constraints by changing the direction of the inequalities. This new problem has always got feasible solutions and that drawback is avoided. Therefore, from now on, we will consider transportation problems with benefits instead of costs. Consequently, the corresponding mathematical program is given by

$$\begin{aligned} \max \quad & \sum_{i \in P} \sum_{j \in R} b_{ij} x_{ij} \\ s.t.: \quad & \sum_{j \in R} x_{ij} \le p_i, i \in P \\ & \sum_{i \in P} x_{ij} \le d_j, j \in R \\ & x_{ij} \ge 0, i \in P, j \in R. \end{aligned} \tag{10}$$

Now, we can define a cooperative game in characteristic function form associated with each (benefit) transportation problem T. The set of players $N = P \cup R$ and the characteristic function v^T is defined following the general formulas given in (8). The game (N, v^T) is called *transportation game*. Transportation games are superadditive but not convex in general. Furthermore, the core of these games is always non-empty. On the other hand, if $(u; w)$ is an optimal solution for the dual problem of (10), then $((p_i u_i)_{i \in P}; (d_j w_j)_{j \in R}) \in Core(N, v^T)$. Therefore, the $Owenset(N, v^T)=\{((p_i u_i)_{i \in P}; (d_j w_j)_{j \in R}) \in \Re^{P \cup R}$: $(u; w)$ *is a dual optimal solution*$\}$ is contained in the core of the game. However, the core and the Owen set of transportation games do not coincide in general (see Sanchez-Soriano et al., 2001). In (Thompson, 1980) the extreme points of the Owen set that the author called "core" are studied.

In (Sanchez-Soriano, 2003 and 2006) the *pairwise solutions* for transportation games are introduced. These solutions are based directly on the optimal solutions of the corresponding transportation problem. Since transportation problems can have more than one optimal solution, the pairwise solutions are set-valued (but discrete). However, on many occasions, transportation problems have only one optimal solution and, hence, we could consider that pairwise solutions are "essentially" values. The philosophy behind the pairwise solutions is simply that the benefit obtained by each pair producer-retailer in an optimal solution is distributed between them in some way. The proportion of benefit achieved for a player in a pair producer-retailer will depend on the bargaining abilities of both or on their relative weight (power) in the whole transportation system. When we assume that nothing is known about the relative weights of the agents and, therefore, we could consider that they all have the same weight, then we obtain the *pairwise egalitarian solution*. Given a weight vector π, such that $\pi_k>0$ for all $k\in N$, and an optimal solution x^* for the corresponding problem (10), the pairwise solution associated with π and x^* is defined as follows:

$$\begin{aligned} ps_i(\pi,x^*) &= \sum_{j\in R} \frac{\pi_i}{\pi_i+\pi_j} b_{ij}x^*_{ij},\ i\in P \\ ps_j(\pi,x^*) &= \sum_{i\in P} \frac{\pi_j}{\pi_i+\pi_j} b_{ij}x^*_{ij},\ j\in R. \end{aligned} \tag{11}$$

The *pairwise solution with weight vector* π for the game (N, v^T) is defined as

$$PS^{\pi}(N, v^T)=\{ps(\pi,x^*)\in\Re^{P\cup R}: x^*\in Opt(T)\}, \tag{12}$$

where $Opt(T)$ is the set of all optimal solutions for the corresponding transportation problem T.

On the other hand, we could use a more general concept as the weight systems (Kalai & Samet, 1987) instead of a simple weight vector. A weight system on a set N is a pair (Σ, π) where Σ is a partition of N, $(N_1, N_2, \ldots, N_q)$, and π is a weight vector, whose coordinates are ordered in the same order as the partition. Such that the weight of agents in N_h is zero with respect to the agents in N_k if $h<k$. Inside of each N_h each agent has a positive weight. In this situation we can define the *pairwise solution with weight system* (Σ, π) for the game (N, v^T), $PS^{(\Sigma,\pi)}(N, v^T)$, analogously to (11) and (12). The pairwise solutions do not belong to the core of the game in general, but in (Sanchez-Soriano, 2006) it is proved that

$$Core(N,v^T) \subset \bigcup_{(\Sigma,\pi)} PS^{(\Sigma,\pi)}(N,v^T). \tag{13}$$

Therefore, each core allocation can be seen as a pairwise solution for particular weight systems but there are, in general, pairwise solutions which do not belong to the core of the corresponding transportation game.

Let us consider a transportation situation T with two producers (called A and B) and three retailers (called 1, 2, and 3). The productions of A and B are 12 and 15 units respectively and the demand of each retailer is 10 units. The unitary costs of transportation are $c_{A1}=3$, $c_{A2}=5$, $c_{A3}=6$, $c_{B1}=5$, $c_{B2}=4$ and $c_{B3}=3$. And the unitary benefit obtained by each good is 9. Solving the corresponding transportation problem (10), we obtain that the only optimal solution for the (benefit) transportation problem is $x_{A1}=10$, $x_{A2}=2$, $x_{B2}=5$, $x_{B3}=10$ and $x_{ij}=0$ otherwise. The characteristic function of the game (N, v^T) is the following:

$v^T(N)=153$; $v^T(A123)=68$, $v^T(B123)=85$, $v^T(AB12)=110$, $v^T(AB13)=120$, $v^T(AB23)=100$; $v^T(A12)=68$, $v^T(A13)=66$, $v^T(A23)=46$, $v^T(B12)=70$, $v^T(B13)=80$, $v^T(B23)=85$, $v^T(AB1)=60$, $v^T(AB2)=50$, $v^T(AB3)=60$; $v^T(A1)=60$, $v^T(A2)=40$, $v^T(A3)=30$, $v^T(B1)=40$, $v^T(B2)=50$, $v^T(B3)=60$; otherwise $v^T(S)=0$.

In this case, $Owenset(N, v^T)$={(48,75;20,0,10)}. We know that this allocation is in the core of the game but it seems unfair with retailer 2 since this player contributes significantly to the benefit of the grand coalition, in particular $v^T(N)-v^T(AB13)=33$. As for the core of the game, the segment comprised between the allocations (68,85;0,0,0) and (7,17;53,33,43) is contained in the core of the game. Therefore, $Core(N, v^T)$ is larger than $Owenset(N, v^T)$. Likewise, if we consider the following two weight systems (Σ^1,π^1)=({1,2,3},{A,B};(1,1,1,1,1)) and (Σ^2,π^2)=({A,B,1,3},{2};(1,1,53/7,43/7,1)), then we obtain the following two pairwise solutions $PS^{(\Sigma^1,\pi^1)}(N,v^T)=\{68,85;0,0,0)\}$ and $PS^{(\Sigma^2,\pi^2)}(N,v^T)=\{7,17;53,33,43\}$. On the other hand, if we simply consider π=(1,1;1,1,1), then we obtain the pairwise egalitarian solution $PS^{(1,1;1,1,1)}(N,v^T)$={(34,42.5;30,16.5,30)} which, in this example, belongs to the core of the game. Finally, if we consider the vector of weights π=(1,2;3,4,5), then we obtain the pairwise solution $PS^{(1,2;3,4,5)}(N, v^T)$={(16.60,25.48;45.00,23.07,42.86)} which does not belong to the core of the game.

3.2 Three-sided transportation games

A three-sided transportation problem consists of three sets of agents, called producers, wholesalers and retailers, which produce, store and demand goods. Each producer produces an amount of goods, each wholesaler has a capacity of storage and each retailer demands a certain amount of goods. The transport of the goods from the producers to the retailers via a wholesaler is costly (profitable) and, therefore, the main objective is to transport the goods from the producers to the retailers via the wholesalers at minimum cost (at maximum profit). We will call this situation the *distribution problem*. The same reasoning about the interest of cooperation and the benefit approach holds for these problems.

Let P, W and R be the sets of producers, wholesalers and retailers respectively. We denote by p_i the production of goods of producer $i \in P$, c_j the capacity of storage of wholesaler j and by d_k the demand of goods of retailer $k \in R$. The unitary benefit of transportation from producer i to retailer k via wholesaler j is denoted by b_{ijk}. The mathematical program that models this problem is the following:

$$\begin{aligned}
\max \quad & \sum_{i\in P}\sum_{j\in W}\sum_{k\in R} b_{ijk}x_{ijk} \\
s.t.: \quad & \sum_{j\in W}\sum_{k\in R} x_{ijk} \le p_i,\ i \in P \\
& \sum_{i\in P}\sum_{k\in R} x_{ijk} \le c_j,\ \ j \in W \\
& \sum_{i\in P}\sum_{j\in W} x_{ijk} \le d_k,\ k \in R \\
& x_{ijk} \ge 0,\ i \in P,\ j \in W, k \in R
\end{aligned} \tag{14}$$

where x_{ijk} is the number of units transported from producer i to retailer k via wholesaler j.

Now, we can define a cooperative game in characteristic function form associated with each distribution problem D. The set of players $N = P \cup W \cup R$ and the characteristic function v^D is defined following the formulas in (8). The game (N, v^D) is called *distribution game*.

On the one hand, in (Quint, 1991) it is shown that the core of m-sided assignment games can be empty, therefore if we consider that the goods are indivisible then distribution games can have empty cores. In this sense, there will be many distribution situations in which a core allocation is not possible. Furthermore, the Owen set could consist of non efficient allocations because the duality gap. However, we can always find reasonable allocations based on the primal optimal solutions of problem (14), defined analogously as pairwise solutions, which we call *triplewise solutions*.

On the other hand, if we consider that the goods are perfectly divisible then distribution games have non-empty cores since the Owen set of these games is always non-empty and it is contained in the core of the game. Of course, in distribution situations with perfectly divisible goods, it is also possible to consider the triplewise solutions as reasonable solutions.

We would like to point out that, in the case of two-sided transportation situation, we have not distinguished between indivisible and perfectly divisible goods because the constraint matrix in problem (10) is totally unimodular and therefore we can relax the indivisibility condition when necessary.

Finally, (Perea et al., 2008) study from a cooperative standpoint a class of distribution problems and prove that the corresponding cooperative games have non-empty core. Likewise, the authors introduce two new solutions which satisfy certain interesting properties related to fairness.

3.3 Warehouse sharing games

Now, we consider another situation, also related to transportation problems, in which there are two or more distribution systems, each of them consisting of producers, warehouses and retailers. In principle several producers and retailers could belong to different distribution systems but the warehouses can only belong to one distribution system. In this situation the distribution systems involved in the problem could share their warehouses in order to increase the efficiency of all systems considered as a whole. Therefore, if cooperation is profitable then this should be promoted through a good distribution of the extra profits or saved costs. A similar reasoning about the benefit approach holds for these problems. We will call these optimisation situations *warehouse sharing problems*.

Each distribution system faces the same optimization problem which is modelled as (14). Likewise, if two or more distribution systems collaborate then the corresponding optimisation problem is also modelled as (14). Therefore, we can approach this situation as an operations research game.

Let D be the set of distribution systems and P_i, W_i and R_i the sets of producers, warehouses and retailers in distribution system $i \in D$. We denote by p_{if} the production of goods of producer $f \in P_i$, c_{ig} the capacity of storage of warehouse $g \in W_i$ and by d_{ih} the demand of goods of retailer $h \in R_i$. The unitary benefit of transportation from producer $f \in \bigcup_{i \in D} P_i$ to retailer $h \in \bigcup_{i \in D} R_i$ via warehouse $g \in \bigcup_{i \in D} W_i$ is denoted by b_{fgh}. If one producer (resp. retailer) belongs to more than one distribution system then, when these distribution systems collaborate, the production (resp. demand) to take into account for that producer (resp. retailer) is the sum of its productions (resp. demands). As it is not difficult to see, the mathematical formulation of this problem is as (14).

Next, we can define a cooperative game in characteristic function form associated with each warehouse sharing problem WS. In this case, the set of players $N = D$ and the characteristic function v^{WS} is defined following the formulas in (8). The game (N, v^{WS}) is called *warehouse sharing game*.

In this kind of situation we can observe two levels of cooperation. On the one hand, we find the cooperation among producers, warehouses and retailers inside of a distribution system. And, on the other hand, we have the cooperation among the different distribution systems. It is obvious that if we are only interested in the warehouse sharing game then similar comments as in Sections 3.1 and 3.2 regarding the allocation of the extra benefits among the agents involved can be done. However, if we are interested in the two levels simultaneously considering the problem as a whole system then, perhaps, we may be dealing with a game with a priori unions or restricted cooperation and, consequently, we should take into account this fact in order to analyse this situation.
Finally, this situation can resemble the cooperation among supply chains with deterministic productions/demands and without penalties and, therefore, it could be considered within of the literature of supply chain games. However, we have considered its analysis more appropriate as an operations research game because the mathematical model describing this problem is close related to a three-sided transportation situation as we have shown. On the other hand, several papers, in which different levels of cooperation (horizontal, vertical or lateral) are analysed for transportation or supply chain situations, are (Cruijssen et al., 2007), (Mason et al., 2007) and (Simatupang & Sridharan, 2002).

4. Supply chain games

For researchers in Operations Research and Economics, supply chains represent one of the key issues which can be relied on. This section brings together a series of works, which present different paradigms and results related to cooperative game theory as applied to supply chain management. This comprises review oriented papers that look at the kind of methodologies that have been applied, in addition to theoretical papers discussing new developments and results. As a direct consequence of this, we hope that this section will serve as a source for current and future researchers in this field.
Moreover, another aim of this part is to show the applicability of cooperative game theory as a tool with which to analyse supply chains since a main feature of any supply chain is cooperation. In particular, the central contribution of cooperative game theory is related to determine a suitable allocation rule among the agents of that supply chain. However, we would like to point out that the use of cooperative game theory to analyse problems in supply chain management is a very recent development.

4.1 Definition of a supply chain

There are numerous definitions for the term “supply chain”. For example, (Christopher, 1998) defined this notion as “… network of organizations that are involved, through upstream and downstream linkages, in the different processes and activities that produce value in the form of products and services in the hands of the ultimate consumer”. Whereas (Ganeshan et al., 1999) define a supply chain as “a system of suppliers, manufacturers, distributors, retailers and customers where materials flow downstream from suppliers to customers and information flows in both directions”. On the other hand, supply chain management is defined as a set of management processes (Leng & Parlar, 2005). However, all definitions in the literature share the idea that supply chains are based on cooperation in order to obtain a higher benefit. In fact, (Thun, 2005) claims that, in the future, competition will take place between supply chains instead of between individual firms. In order to yield the benefits related to cooperation, contracts for vertical cooperation must be established within supply chains.

Nevertheless, the main drawbacks for the right supply chain management are two. First, trust can be seen as the most critical factor of cooperation between firms (Poirer, 1999). In this way, modelling supply chains via cooperative games can be important to analyse the impact of rationality on the final allocation (Thun, 2005). Secondly, there is a phenomenon commonly referred to as "the bullwhip effect", which was first observed at P&G concerning disposable diapers (Lee et al., 1997). Sharing information across the supply chain is a way to mitigate its negative effects (Thun, 2005).

4.2 Examples of supply chain games

In this section we show two examples of situations related to supply chain management. The first example is based on (Müller et al., 2002), while the second one is based on (Granot & Sosic, 2003).

Example 1. We consider the usual newsvendor game where each agent (store) faces a stochastic demand (of newspapers, for example). These demands are actually correlated, although this fact has usually been ignored in the literature seeking simplicity. We will take into account this feature of the game. So, any coalition of agents that faces a demand x and orders a quantity y of newspapers incurs a cost as follows,

$$\phi(y,x)=\begin{cases} h(y-x), & \text{if } y \geq x \\ \pi(x-y), & \text{if } y < x \end{cases}, \tag{15}$$

where h is the holding cost per unit of stocking more newspapers than are actually demanded, and π is the opportunity cost related to not ordering enough newspapers. Following with the description of the game, each agent i experiences a random demand X_i. For coalition $S \subset N$, we define the total demand as $X_S = \sum_{i \in S} X_i$. For technical purposes, we focus on random demands such that $E\left[\phi(y,X)\right] < \infty$. In this way, the optimal quantity ordered by S is $y_S^* = \arg\min_y E\left[\phi(y,X_S)\right]$ and coincides with the $\pi/(h+\pi)$ quantile of the distribution of the random variable X_S. Consequently, the value (cost) of the characteristic function of coalition S in this kind of game is defined as $C(S) = E\left[\phi\left(y_S^*, X_S\right)\right]$. Finally, let N be the finite set of agents. In this way, we are able to define a cooperative game as (N, C).

Example 2. In this example we briefly show a three-stage game of a supply chain consisting of n retailers, each of whom experiences a random demand for an identical product. Next we explain the different steps of the game. Before the demand is realised, each player orders her initial inventory in an independent way (first stage). After the demand is actually realised, each player decides how much of their residual stock they wish to share with the other retailers (second stage). In the final stage, a total profit should be allocated among the players due to the fact that residual stocks are transhipped to meet the joint demand. In this way, in the third (cooperative) stage, residual inventories are transhipped to meet residual demands, and the additional profit has to be allocated among the retailers. Obviously, this example excludes the possibility of storing at one or several shared warehouses.

4.3 Review of the literature on supply chain games

Many articles on supply chain management point towards the relevance of cooperation among the supply chain members in order to increase the supply chain benefits and the overall performance. However, only a few researchers so far have deployed cooperative game theory to analyse the stability and rationality of collaboration within a supply chain. Authors such as (Cachon & Netessine, 2004) have reviewed the literature describing supply chain and game theory concluding that "papers employing cooperative game theory have been scarce, but are becoming more popular". Something similar has been pointed out in other reviews such as (Leng & Parlar, 2005) and (Nagarajan & Sosic, 2008). This section is partially based on these good reviews. Nevertheless, we have added very recent publications on this issue which were not mentioned in those three reviews. On the other hand, for a specific review of the literature on inventory centralization we refer to (Meca & Timmer, 2008).

In 1961 (Chacko, 1961) analysed the impact of coalition formation between a multi-plant multi-product manufacturing company, two suppliers and several customers. Unfortunately, this paper did not become the starting point for the use of cooperative game theory in supply chain. Twenty years later, one can find a paper mixing supply chain management and cooperative game theory. (Jeuland & Shugan, 1983) explored the problem of coordination of the members of a channel, which includes as a particular case the manufacturer-retailer-consumer channel. They also proposed the form of the quantity discount schedule that results in optimum channel profits. (Kim & Hwang, 1989) studied how the supplier can formulate the terms of a quantity-discount pricing schedule, under the assumption that the supplier behaves in an optimal way. In particular, they show the formula for price and order size that maximises the sum of the profits of both agents and the corresponding allocation between the parties.

(Gerchak & Gupta, 1991) analysed the effectiveness of four popular schemes of cost allocation in the context of a continuous review order quantity reorder point (Q, r) inventory system with complete back ordering. They also proposed a proportional method that has the notable feature that any customer's post-centralization share of overheads does not exceed its costs without consolidation. Inspired by this paper, (Robinson, 1993) showed that the best allocation rule proposed in (Gerchak & Gupta, 1991) does not necessarily belong to the core. Furthermore, he also showed the formulation of the Shapley value for this game and proved that this allocation rule does actually belong to the core.

(Wang & Parlar, 1994) proposed a single-stage game to model a particular inventory problem where three retailers try to determine their optimal order amount. They assume stochastic demands and substitutable products. In this context, they determine the conditions that assure that the core of this game is non-empty.

So far the papers reviewed focus on horizontal cooperation in a supply chain. Nevertheless, there are papers devoted to vertical cooperation. One example is the paper by (Li & Huang, 1995). They explored the simple (monopolistic) buyer-seller channel from a cooperative approach. The authors showed the common incentives and the individual disincentives for cooperation. A rule, based on quantity discount, is also proposed to implement a profit sharing mechanism for achieving equal division of additional cooperative system profits.

In (Hartman & Dror, 1996) the cost allocation problem for the centralized and continuous-review inventory system is studied. They proposed three necessary criteria (stability, justifiability and polynomial computability) for appropriating selection of an allocation rule.

They showed that common allocation schemes may not meet the three criteria and introduced a method that meets them all. Following this line, (Hartman et al., 2000) considered a set of n stores with centralized ordering and inventory with holding and penalty costs. They showed the (restrictive) condition under such a cooperative game has a non-empty core and conjectured that the core is non-empty at least for independent demands. (Hartman & Dror, 2003) proved the non-emptiness of the core for a single period inventory game with n retailers experiences normally distributed, correlated individual demands. On the other hand, (Müller et al., 2002) proved a stronger result than that conjectured by (Hartman et al., 2000). In particular, they showed that the core of this type of games is always non-empty regarding the joint distribution of the stochastic demands. (Slikker et al., 2005) studied a more complex situation, called the general newsvendor game, where the agents could use transhipments after demand is satisfied. Their main result states that the general newsvendor game has a non-empty core.

(Anupindi et al., 2001) analysed a supply chain problem with n independent retailers of an identical item for consumption. Each agent experiences a random demand and must order their inventory before the demand is realised. After realising such a demand, some retailers might meet their residual demand by means of the other retailers' residual supplies. This game is very similar to example number 2 above. Nevertheless, it is played as a decentralised two-stage distribution model, whereas example 2 consists of three stages. In addition, (Anupindi et al., 2001) assumed that all retailers will share all their residual supply/demand in the second stage. Regarding the allocation schemes, these authors suggested an allocation rule based on a dual solution for the transhipment problem. This solution is always in the core of the game and, hence, it encourages the retailers not to form coalitions. Later, (Granot & Sosic, 2003) extended the two-stage model of (Anupindi et al., 2001) allowing each retailer to decide how much of their residual supply/demand they would like to share with others in a third and final stage. They found that allocations based on dual solutions will not induce the retailers to share their total residuals with others. Furthermore, they proved that the Shapley value is a value-preserving allocation scheme, i.e., it induces all the retailers to share their residual supply/demand in quantities that do not result in a decrease in the total additional profit.

We now turn to vertical cooperation in supply chain problems and consider the paper of (Raghunathan, 2003). This author studied a situation where a manufacturer and n retailers share demand information. The author used the Shapley value to analyse the expected manufacturer and retailer shares of the surplus generated from the cooperative game. Mainly, (Raghunathan, 2003) showed that higher demand correlation increases the manufacturer's allocation and has the opposite result on the retailers.

Under horizontal cooperation, (Meca et al., 2004) studied a simple inventory model with n retailers who experience deterministic demand. The firms can cooperate to reduce their ordering costs. This approach is called the basic inventory model because it forms the basis for a wide variety of inventory models. Also, the authors developed a proportional rule to allocate joint ordering cost among the retailers. They showed that this rule leads to an allocation in the core. For a more general study of holding games see (Meca, 2007).

(Hartman & Dror, 2005) studied the problem faced by the management of independent stores, with a similar product, of cost management for a centralised operation of their inventory. They modelled the centralised cost as a metric space obtained from the Cholesky factorisation of the corresponding covariance matrix. They considered two cooperative games, one based on optimal expected costs and another based on demand realisations. For the first game, they showed that when holding and penalty shortage costs are identical and normally distributed demands, the corresponding game has a non-empty core. Unfortunately for the second game, they showed that even in the case of identical holding and penalty costs the game might have an empty core.

(Klijn & Slikker, 2005) analysed a location-inventory model with m customers and n distribution centres. Under this context, they proved the emptiness of the corresponding cooperative game when demand processes are identically and independently distributed.

(Reinhardt & Dada, 2005) considered a problem with n firms who collaborate by pooling their critical resources in order to make their cost structure more efficient. They proposed to use the Shapley value as the allocation scheme among the players. For coalition symmetric games, i.e., situations where the pooled savings depend on the sum of each player's demand, they introduced a pseudo-polynomial algorithm for its computation.

In a vertical cooperation framework, (Leng & Parlar, 2005) analysed an information-sharing cooperative game involving a supplier, a manufacturer and a retailer. They derived the necessary conditions for stability of each coalition. They also studied the implications of using the Shapley value and the nucleolus as allocation schemes for this type of games.

More recently, (Dror & Hartman, 2007) analysed cost allocation in a multiple product inventory system following an economic order quantity policy to order, where part of the ordering cost is shared and part is specific to each item. They showed that if the part of the ordering cost common to all items is not too small, then the core of the game is non-empty.

(Montrucchio & Scarsini, 2007) considered a newsvendor game with stochastic demand of a single item. They proved that the game is balanced in great generality considering a possibly infinite number of retailers. Under several conditions, they also showed that with a continuum of retailers the core becomes a singleton.

Under vertical cooperation, (Guardiola et al., 2007) analysed a supply chain under decentralised control with a single supplier and n retailers. They proved that the cooperation in this game is stable and proposed a specific allocation rule that is always in the core. This last point is important since the well-known Shapley value does not always belong to the core for this type of games.

(Guardiola et al., 2009) introduced a new class of production-inventory games. Cooperation among agents is given by sharing production processes and warehouses facilities. In this context, the authors proved that the corresponding cooperative game is totally balanced and the set of the Owen-allocations is a point (called the Owen point). Also, the authors showed the relationship between the Owen point, the Shapley value and the nucleolus.

(Özen et al., 2008) conducted a game-theoretical analysis of a supply chain with warehouses, in which retailers have the chance of reallocating their product orders after the demand has been met. In this context, the authors considered a cooperative game between the retailers. They were able to prove that this game has a non-empty core.

(Chen & Zhang, 2009) demonstrated the power of stochastic programming duality approach in studying stochastic inventory games. In fact, their approach is readily applicable to more general models. In this context, as a main result, they showed that stochastic programming provides a way to compute a solution in the core of this kind of games.

Finally, (Özen et al., 2010) considered a simple newsvendor game and investigated the convexity of this type of situations. Whereas it is known that the general newsvendor game is not convex, they focused on the particular family of newsvendor games with independent symmetric unimodal demand distributions. It allowed them to identify several interesting subclasses containing convex games only.

4.4 Further research in supply chain management

We devote this section to suggesting several avenues for further follow-up research in cooperative supply chain games. To this end, we show two interesting contexts related to current and real supply chains. The first is based on (Plambeck & Taylor, 2005) and shows the benefits from collaborating between a pharmaceutical company and a manufacturer. The second context is inspired by the actual Spanish electricity market. We propose to analyse the cooperation between electricity consumers, retailers and the network operator by means of cooperative game theory. In a certain sense, such a framework generalises the approach introduced in (Pettersen et al., 2005) for a single consumer, a single retailer and the network operator in the Nordic electricity market. It is worth mentioning that both contexts are not related to holding costs and inventory problems, a feature that is not usual in the supply chain literature, as we have shown previously.

4.4.1 Contracting manufacturers in the pharmaceutical industry

As pointed out in (Plambeck & Taylor, 2005), firms in the pharmaceutical industry are characterised by long developments cycles and intensive time-to-market pressure. In this industry, any firm that produces its own drug must make a significant capital investment in a plant before the product has completed regulatory trials. Unfortunately, if the drug finally fails, then the plant belonging to the pharmaceutical company (PC) will have little value (Tully, 1994). This drawback is usual in industries where production capacity is low in contrast to their investment power. In this case, contract manufacturing offers the opportunity to outsource production to contract manufacturers (CMs). They are able to pool the total demand from many different pharmaceutical companies and, consequently, achieve high capacity utilization.

Following (Plambeck & Taylor, 2005), we consider two symmetric PCs, j=1, 2, which are developing a new drug. The price per unit when q_j units are sold is $M_j - q_j$. With probability e, the product is successful and $M_j = H_j$ where H_j represents the potential market size. Otherwise, $M_j = L$ with $L < H_j$. On the other hand, each PC should invest in production capacity c at a cost of $k > 0$ per unit before the demand is known. Furthermore, the marginal cost of production is negligible.

Investments by the PCs in innovation (product development) may influence demand through in two ways. On the one hand, increasing the potential market size, H_j. On the other hand, the probability that a drug passes clinical trials influences positively the final success probability. We here consider the first case, i.e., when investment in innovation influences H_j. So, let $f(H_j)$ be the total cost function of innovation of firm j. It is also assumed that this function is increasing at the market size, twice differentiable and convex. Each PC selects a market size H_j that maximizes its total expected profit, V_j.

$$V_j = \max_{H_j} \left\{ \max_{c \geq 0} \left\{ e\left(H_j - c\right)c + \left(1 - e\right) \max_{q_j \in [0,c]} \left\{ \left(L - q_j\right) q_j - kc \right\} \right\} - f\left(H_j\right) \right\}. \tag{16}$$

Consider now that the two PCs pool their production capacity in this game ($c + c = 2c$). In other words, we assume that $\{PC_1, PC_2\}$ is a coalition. In this way, the maximum expected profit that they can achieve is

$$V_{\{1,2\}} = \max_{H_1,H_2}\left\{\max_{c\geq 0}\left\{R\left(c,H_1,H_2\right)-2kc\right\}-f\left(H_1\right)-f\left(H_2\right)\right\}, \tag{17}$$

where

$$\begin{aligned} R\left(c,H_1,H_2\right) = e^2 & \max_{\substack{q_1,q_2\geq 0\\ q_1+q_2\leq 2c}} \left\{\left(H_1-q_1\right)q_1+\left(H_2-q_2\right)q_2\right\}+ \\ e\left(1-e\right) & \sum_{j=1,2}\max_{\substack{q_H,q_L\geq 0\\ q_H+q_L\leq 2c}} \left\{\left(H_j-q_H\right)q_H+\left(L-q_L\right)q_L\right\}+ \\ \left(1-e\right)^2 & 2\max_{q\in[0,c]}\left\{\left(L-q\right)q\right\}. \end{aligned} \tag{18}$$

We now turn to the situation where an independent CM (player number *3*) possesses the capability for producing. We consider that the CM invests in production capacity at a cost of k_{CM} per unit, with $k_{CM} < k$. Therefore, we are considering a situation slightly different of that in (Plambeck & Taylor, 2005).

It is obvious that the CM alone achieves profit zero. This type of firm needs to collaborate with at least one PC to get a strictly positive profit through the production of the final product. Then, the joint profit for the coalition $\{j,3\}$, j=1,2, is equivalent to V_j with k_{CM} instead of k. In the same manner, the profit of the grand coalition would be equivalent to $V_{\{1,2\}}$ with k_{CM} instead of k. Cooperative game theory is the natural way to allocate the value of the grand coalition among all firms. In particular, it could be interesting to analyse stability of cooperation between the pharmaceutical companies and the manufacturer and to look for reasonable and fair distribution of the extra benefits among them.

4.4.2 Supply chain without storage: electricity games

Following the description of the Spanish Electricity Market we propose several games which could be interesting to study. These games have the special feature that the electricity cannot be stored and, therefore, in this context there is not holding or inventory costs. This aspect is not usual in the supply chain literature.

In 1998, the Spanish government liberalised the market for generating electricity and introduced a spot market for electricity. The basic design of this electricity spot market is similar to the previously deregulated UK market and even closer to the Californian electricity market that was deregulated at about the same time. A liberalised electricity market was not new to Spain, as during the 1990s there had been a previous liberalisation of other sectors, such as the media, telephony, oil and gas. In spite of the fact that de-regularisation was a slow process which was not completed until 2009, it was not a process that provided the electricity market with a large number of companies selling energy to small consumers of power. The present situation in Spain continues to be one with few companies on the market which stimulate competition and thereby bring about the expected reduction in prices. The main characteristics of the Spanish electricity sector are the existence of the wholesale Spanish generation market (Spanish pool), and the fact that all consumers are considered to have qualified since 2003. This means that they can choose the electricity company that supplies them with electricity and therefore participate in the pool in an active manner. The electricity production market in Spain is organised around a series of auctions and technical procedures for operating the system: Daily Market, Intradaily

Market, Bilateral Contracts, International Contracts, Technical Constraints, Technical Management, etc. (see, for example, (Sancho et al., 2008)). Since 2006, bilateral contracts and the forward market have become a larger part of the market. On the other hand, generation facilities in Spain operate either under the Spanish ordinary regime or the Spanish special regime. The electricity system must acquire all electricity offered by special regime generators, which consist of small or renewable energy facilities, at tariffs fixed by Royal Decree or Order that vary depending on the type of generation and are generally higher than Spanish market prices. Ordinary regime generators provide electricity at market prices to the Spanish pool and under bilateral contracts to qualified consumers and other suppliers at agreed prices. Suppliers, including last resort suppliers, and consumers can buy electricity in this pool. Foreign companies may also buy and sell in the Spanish pool. The market operator and agency responsible for the market's economic management and bidding process is the Electrical Market Operator (OMEL - www.omel.es). Market participants are undertakings that are authorised to act directly in the electric power market as buyers and sellers of electricity. The following can be market participants:

- Electric power distributors who come to the market to purchase the electricity needed to supply consumers at regulated tariffs or to distributors who are supplied.
- Resellers: They go into the market to purchase power to sell to qualified consumers.
- Qualified consumers: They can purchase power directly in the organised market, through a reseller, by signing a physical bilateral contract with a producer or by continuing temporarily as a regulated tariff consumer.

Transmission companies and regulated distributors must provide network access to all consumers that have chosen to be supplied on the free market. However, these consumers must pay an access tariff to the distribution companies if such access is provided. The electricity transport grid comprises transmission lines, stations, transformers and other electrical equipment with a voltage superior to 220 KV, as well as other facilities, regardless of their voltage, that provide transport or international and extra-peninsular interconnections. Red Eléctrica de España (Spanish Electrical System Operator), REE - www.ree.es, manages most of the transmission network in Spain. It is responsible for the technical management of the Spanish electricity system with regards to developing the high voltage network, in order to guarantee electricity supply and proper coordination between the supply and transmission system, as well as the management of international electricity flows. The system's operator carries out its duties in coordination with the market operator. Liberalised suppliers are free to set a price for their consumers. The main direct activity costs of these entities are the wholesale market price and the regulated access tariffs to be paid to the distribution companies. Electricity generators and liberalised suppliers or qualified consumers may also engage in bilateral contracts without participating in the wholesale market. As from 2009, last resort suppliers, appointed by the Spanish government, supply electricity at a regulated tariff set by the Spanish government to the last resort consumers (low-voltage electricity consumers whose contracted power is less than or equal to 10 KW). Since then, distributors cannot supply electricity to consumers.

All generation facilities that are not governed by the Spanish special regime are governed by the Spanish ordinary regime. Under said ordinary regime, there are four methods of contracting for the sale of electricity and determining a price for the electricity:

- Wholesale energy market or pool. This pool was created on January 1, 1998 and includes a variety of transactions that result from the participation of market agents (including generators, distributors, suppliers and direct consumers) in the daily and intraday market sessions.

- Bilateral contracts. Bilateral contracts are private contracts between market agents, whose terms and conditions are freely negotiated and agreed.
- Auctions for purchase options or primary emissions of energy. Principal market participants are required by law to offer purchase options for a pre-established amount of their power. Some of the remaining market participants are entitled to purchase such options during a certain specified period.
- Energy Auctions for Last Resort Demand. Last resort suppliers in the Iberian Peninsula can acquire electricity in the spot or forward markets to meet last resort demand. However, beginning in 2007, these last resort suppliers were permitted to begin holding energy auctions to purchase electricity at lower prices. Since 2003, all consumers have become qualified consumers. All of them may now choose to acquire electricity under any form of free trading through contracts with suppliers, by going directly to the organised market or through bilateral contracts with producers.

With the coming into force of the Last Resort Supply in 2009, the integral tariff system has been replaced by a last resort tariff system. Last resort tariffs are set on an additive basis and can only be applied to low-voltage electricity consumers whose contracted power is less than or equal to 10 KW. Last resort consumers can choose either to be supplied at last resort tariffs or to be supplied in the liberalised market.

Within the regulatory framework, it is important to point out that there is very low, almost insignificant, participation in the Spanish electricity market by small and medium consumers. To this end, over the last few years, different independent system operators (ISOs) in Europe, Oceania and North America are continuing the development of load response programmes (LRPs) with the objective of changing electricity demand of large power users. Nevertheless, some medium commercial or industrial users may submit offers and bids in new energy markets thanks to lighter requirements for demand reduction with levels of about 100kW (New York ISO or New England ISO). In addition, some ISOs encourage the possibility of demand aggregation through commercial entities (see pilot programmes developed by NYISO since 2002 for small load aggregators (ISO New England Market - www.iso-ne.com) to reach the minimum level for the participation of users. As with these international markets, in the medium term, commercial and aggregating companies will have to offer users in Spain a selling price for power that fits in with the consumption profile of a specific segment of customers (Verdu et al., 2006). They must also offer customers various participation schemes in the demand which will allow the electricity companies to group together sufficient levels of power to be able to buy energy on the electricity market. At the same time, customers signing up to the schemes will receive special offers to reduce or modify their consumptions levels (Valero et al., 2007).

After reading the description of the Spanish Electricity Market it is possible to think that different games could be analysed. For example, in the literature there are many papers analysing from a game theoretical standpoint the electricity auction-market (see, for example, (Aparicio et al., 2008) and (Sancho et al., 2008) and their lists of references). Another interesting problem is the game played by electricity consumers, retailers and the network operator. In (Pettersen et al, 2005) this game for only one electricity consumer, one retailer and one network operator is studied from a non-cooperative point of view. A generalisation of this approach could be to consider a higher number of agents involved in the game. Alternatively, this game could be studied from a cooperative point of view by restricting the possibilities of cooperation in order to respect some level of competition in the market.

Taking into account the possibility of bilateral agreements in the electricity market, the horizontal cooperation among users or consumers could be an interesting problem to be studied from a game theoretical point of view since, at first sight, collaboration among the consumers could be profitable for them because, perhaps, all together could obtain better electricity prices. In this context, we could consider two sides of the electricity market. One side of the market would consist of the suppliers of electricity who should compete for selling electricity. The other side of the market would consist of the consumers who could collaborate in order to get a better position in the market. The analysis of this situation could provide insights on the level of competition among the suppliers and the interests of cooperation among the consumers.

The last game we would like to mention in this part is related to vertical cooperation. At first glance the functioning of the electricity market with respect to small or renewable energy facilities seems appropriate because the market is promoting the use of green energy. However, this could provoke inefficiencies in the system such that a loss of productivity in the firms because of a higher electricity cost. Therefore all agents involved in the electricity market should collaborate in some sense. Of course, this cooperation should not imply a loss of competition in the market but a re-structuring of some aspects of it, for example, the determination of different quotes of electricity production depending on the energy source. Likewise, in the analysis of this problem, the CO_2 market implications or the production of obnoxious residues might also be taken into account. In this situation, perhaps, a cooperative game theoretical approach could be used in order to obtain some insight about the electricity market.

5. Other logistics games

There are a considerable number of papers concerned with other situations related to logistics problems. In this section we show some of these works as an example of the magnitude and relevance of cooperative game theory in this question. In particular, we focus on routing, packing and location games. For each category we will present some approaches trying to illustrate their relationship with logistics. For this reason, we will pay special attention to the modelling stage. In other words, we will try to explain how to go from the logistics problem to cooperative game theory. Also, we will show the main results of each contribution. For a specific revision of the literature on connection and routing problems and cooperative game theory we refer to (Borm et al., 2001).

We start with a couple of problems related to routing (see (Borm et al., 2001), (Hamers et al, 1999) and (Potters et al., 1992)). First, we will study the classical Chinese postman game. Second, we will discuss the well-known travelling salesman game. Both problems are related to the logistics problem of how to design efficient routes to deliver the commodities from the supply nodes to the demand nodes.

In the classical *Chinese postman situation*, a postman must deliver mail to each street of a city. Obviously, she has to start and finish at the post office. Moreover, each street has an associated cost, related to the time that the postman expends in each visit. The aim in this problem is to select the optimal route. To describe mathematically this situation we need a 4-tuple $\langle N, G, v_0, t\rangle$, where N is the set of players (streets), $G = (V, E)$ is a connected undirected

graph with vertex set V and edge set E, $v_0 \in V$ is the post office and t is a nonnegative cost function. We denote a route for coalition $S \subset N$ as $(v_0, e_1, \ldots, e_k, v_0)$, which starts and finishes at the post office and visits each player in S at least once. Finally, $D(S)$ represents the set of all routes for coalition S.
The *Chinese postman game* (N, c) associated with the 4-tuple $\langle N, G, v_0, t\rangle$ is defined from the following cost function for every coalition $S \subset N$.

$$c(S) = \min_{(v_0, e_1, \ldots, e_k, v_0) \in D(S)} \left\{ \sum_{j=1}^{k} t(e_j) - \sum_{i \in S} t(i) \right\}. \tag{19}$$

One result we would like to highlight is that this type of games need not be balanced. For this reason, the Chinese postman game has been studied in the literature under several additional constraints on the underlying graph: efficiency, bridge cluster symmetry, condensation property and so on.
Regarding another routing situation, the *travelling salesman problem* is similar to the Chinese postman problem but in this case there are a set of cities (vertices or nodes) which have to be visited by the salesman and each link connecting two cities has a cost (distance, time, etc.). The objective is to determine a route or tour that visits each city exactly once at minimal cost. The travelling salesman problem can be described formally by means of a triple $\langle N, 0, t\rangle$, where N is the set of player as usual, 0 represents the home location and t is a nonnegative cost function. The costs match the edges linking the vertices in $N \cup \{0\}$. In this case, the characteristic function of the cooperative game, which could be generated from the travelling salesman problem, coincides with the minimal cost of a Hamiltonian circuit in the graph associated with each coalition S. This type of game needs not be balanced, i.e., the core could be empty. Nevertheless, (Potters et al., 1992) showed that the *travelling salesman game* with three players have a non-empty core. Other authors have proved that games with four and five players are balanced as well (see (Borm et al., 2001)).
We now turn to a different class of games: *packing games*. Imagine a set of manufacturers, called A, and a set of transport companies, called B. Each firm $i \in A$ has an item of size a_i, while each individual in B possesses a truck of capacity b_j. The items yield a profit proportional to their size. Nevertheless, it is necessary for each item to be brought to a certain market by means of a truck. Moreover, we assume that each truck can make only one trip to the market. We can define a packing as an assignment of some items in A to the trucks in B such that the total size does not exceed the total truck capacity. The value of a packing coincides with the sum of the sizes of all packed items. In this way, a bin packing problem has as a goal to determine a packing of maximal value. Cooperative game theory tries to share the total profit among the individuals of sets A and B in a reasonable way. (Faigle & Kern, 1993) introduced these games in the literature. They studied the emptiness of the core, showing that (bin) packing games may be not balanced. Due to this fact, (Faigle & Kern, 1993) used a generalisation of the core notion, called the ε-core. The ε-core of a game (N,v) is defined as

$$\varepsilon - \text{core}(v) = \left\{ x \in R^n : x(N) = v(N), x(S) \geq (1-\varepsilon) v(S), \forall S \subseteq N \right\}. \tag{20}$$

Using this concept, (Faigle & Kern, 1993) proved that if $v(N) \geq 0$, then the ε-core is non-empty for a value of ε sufficiently large. Following (Faigle & Kern, 1993), (Kuipers, 1998) showed what is the value of the minimal ε such that the ε-core is nonempty. Also, this author studied, for a specific class of packing games, the minimal ε such that all games in this special class have a nonempty ε-core. Also, for computational purposes, it is worth noting that general bin packing situations are NP-complete problems. Nevertheless, the constraint that all trucks have capacity 1 and that all items are strictly larger than 1/3 makes the problem easier to solve.

For a more recent study of packing games and their applications see (Sanchez-Soriano et al., 2002). There the authors analysed the transport system for university students in the province of *Alacant* (Spain). The question is how to connect different villages and towns in *Alacant* efficiently with the different university campuses. The authors proposed a possible approach to model this situation. They also considered a particular cost sharing rule based on the egalitarian solution.

In a realistic logistics problem, as the previous one, we could combine both the routing problem and the packing problem because in some way they are closely related. In these situations, we would be interested in determining the number of trucks or containers, taking into account their capacities, and their routes to deliver the different possible commodities from the supply nodes to the demand nodes at minimal cost. Of course, a previous logistics problem, which could be considered, is the location of warehouses or factories in order to improve the efficiency of a posterior delivery chain which would be related to the combination of the routing and packing problems.

So, next, we briefly discuss *location games*. (Puerto et al., 2001) introduced a family of cooperative games arising from continuous single facility location problems. In such a situation, there are n users of a certain facility (for example, a hospital), placed in n different points (towns) in R^m, $m \geq 1$. In this structure, the costs depend on the distances from the users to the facility. We seek a location in R^m for the facility that minimises the total transportation cost. (Puerto et al., 2001) showed two sufficient conditions so that their location game has a non-empty core. Also, they studied under which conditions the proportional egalitarian solution provides core allocations for Weber and minimax (continuous) location games. More recently, (Goemans & Skutella, 2004) deeply analysed non-continuous location games. In such a problem, there is a set of F possible locations for the facility/facilities and we have to decide which facility/facilities to build. In addition, each user must be connected to an open facility. Both opening facilities and connecting users have a fix cost. As above, the goal is to minimise the total cost of the system. In this context, (Goemans & Skutella, 2004) established strong links between fair cost allocations and linear programming relaxation. In particular, they proved that a fair cost allocation exists if and only if there is no integrality gap for a corresponding linear programming relaxation. What is much more interesting is that they also showed that it is in general NP-complete to decide whether a fair allocation scheme exists and whether a given cost rule is fair.

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7

Stochastic Game Theory Approach to Robust Synthetic Gene Network Design

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1. Introduction

The development of foundational technologies such as de novo DNA synthesis, milestone experiments such as the computational re-design of enzymes, the opportunity to widely recombine zinc fingers to re-program DNA-binding site specificity and the availability of well-studied model regulatory system for the design of engineering-inspired molecular devices provide a very powerful knowledge and technology basis for building novel biological entities (Heinemann and Panke, 2006). Synthetic biology is to engineer artificial biological systems to investigate natural biological phenomena and for a variety of applications. Synthetic biology will revolutionize how we conceptualize and approach the engineering of biological systems. The vision and applications of this emerging field will influence many other scientific and engineering disciplines, as well as affect various aspects of daily life and society (Andrianantoandro *et al*., 2006). Synthetic biology builds living machines from the off-the-shelf chemical ingredients, utilizing many of the same strategies that electrical engineers employ to make computer chips (Tucker & Zilinskas, 2006). The main goal of the nascent field of synthetic biology is to design and construct biological systems with the desired behavior (Alon, 2003; Alon, 2007; Andrianantoandro *et al*., 2006; Church, 2005; Endy, 2005; Hasty *et al*., 2002; Heinemann & Panke, 2006; Kobayashi *et al*., 2004; Pleiss, 2006; Tucker & Zilinskas, 2006). By a set of powerful techniques for the automated synthesis of DNA molecules and their assembly into genes and microbial genomes, synthetic biology envisions the redesign of natural biological systems for greater efficiency as well as the construction of functional “genetic circuit” and metabolic pathways for practical purposes (Andrianantoandro, *et al*., 2006; Ferber, 2004; Forster & Church, 2007; Gardner, *et al*., 2000; Heinemann & Panke, 2006; Isaacs, *et al*., 2006; Maeda & Sano, 2006; Tucker & Parker, 2000). Synthetic biology is foreseen to have important applications in biotechnology and medicine (Andrianantoandro *et al*., 2006).

Though the engineering of networks of inter-regulating genes, so-called synthetic gene networks, has demonstrated the feasibility of synthetic biology (Gardner *et al*., 2000), the design of gene networks is still a difficult problem and most of the newly designed gene networks cannot work properly. These design failures are mainly due to intrinsic perturbations such as gene expression noises, splicing, mutation, uncertain initial states and disturbances such as changing extra-cellular environments, and interactions with cellular context. Therefore, how to design a robust synthetic gene network, which could tolerate uncertain initial conditions, attenuate the effect of all disturbances and function properly on the host cell, will be an important topic for synthetic biology (Alon, 2003; Alon, 2007; Andrianantoandro *et al*., 2006; Batt *et al*., 2007; Church, 2005; Endy, 2005; Goulian, 2004; Hasty *et al*., 2002; Heinemann & Panke, 2006; Kaznessis, 2006; Kaznessis, 2007; Kitano, 2002; Kitano, 2004; Kobayashi *et al*., 2004; Pleiss, 2006; Salis & Kaznessis, 2006; Tucker & Zilinskas, 2006). Previously, sensitivity analysis has been used for analysis of the dynamic properties of gene networks either in qualitative simulations of coarse-grained models or in extensive numerical simulations of nonlinear differential equation models or stochastic dynamic models (de Jong, 2002; Szallasi *et al*., 2006). For applications in synthetic biology, these approaches are not satisfying. The local sensitivity analysis can provide only a partial description of all possible behaviors of a nonlinear gene network. In particular, it cannot guarantee that a synthetic gene network behaves as expected for all uncertain initial conditions and disturbances. Moreover, obtaining all convergences of states and parameters by extensive numerical simulations quickly becomes computationally intractable when the size of the synthetic network grows (Batt *et al*., 2007).

An approach has recently been developed using semidefinite programming to partition the parameter spaces of polynomial differential equation models into so-called feasible and infeasible regions (Kuepfer *et al*., 2007). Following that, a robustness analysis and tuning approach of synthetic networks was proposed to provide a means to assess the robustness of the expected behavior of a synthetic gene network in spite of parameter variations (Batt *et al*., 2007). This approach has the capability to search for parameter sets for which a given property is satisfied through a publicly available tool called RoVerGeNe. Several gene circuit design networks have been introduced to implement or delete some circuits from an existing gene network so as to modify its structure for improving its robust stability or filtering ability (Chen *et al*., 2008b; Chen & Chen, 2008; Chen & Wu, 2008). However, robust synthetic gene network design is a different topic. It needs to design a complete man-made gene network to be inserted into a host cell. Therefore, the synthetic gene networks should be designed with enough robustness to tolerate uncertain initial conditions and to resist all possible disturbances on the host cell so that they can function properly in a desired steady state. This is a so-called robust regulation design that can achieve a desired steady state of synthetic gene networks despite uncertain initial conditions and disturbances on the host cell. Recently, robust synthetic gene network design has been developed from the robust stabilization method (Chen & Wu, 2009) and minimax method (Chen, et al., 2009).

In this study, a robust regulation design of synthetic gene network is proposed to achieve a desired steady state in spite of uncertain initial conditions, parameter variations and disturbances on the host cell. Because most information of these uncertain factors on the host cell is unavailable, in order to attenuate their detrimental effects, their worst-case effect should be considered by the designer in the regulation design procedure from the worst regulation error perspective. The worst-case effect of all possible initial conditions and disturbances on the regulation error to a desired steady state is minimized for the robust synthetic gene networks, i.e., the proposed robust synthetic gene network is designed from the minimax regulation error perspective. The minimax design scheme is a simple robust synthetic gene network design method because we do not need the precise information of the initial conditions, parameter variations and disturbances on the host cell, which are not easy to measure in the design procedure. This minimax regulation design problem for robust synthetic gene networks could be transformed to an equivalent dynamic game problem (Basar & Olsder, 1999; Chen *et al*., 2002). Dynamic game methods have been widely applied to many fields of robust engineering design problems with external disturbances.

Recently, the application of dynamic game theory has been used for robust model matching control of immune systems under environmental disturbances (Chen *et al.*, 2008a). A robust drug administration (control input) is designed to obtain a prescribed immune response under uncertain initial states and environmental disturbances. In this study, the stochastic game theory will be used for robust synthetic gene network design so that the engineered gene network can work properly under uncertain initial conditions and environmental disturbances on the host cell. The uncertain initial states and disturbances are considered as a player doing his best to deteriorate the regulation performance from the worst-case point of view, while the system parameters to be designed are considered as another player optimizing the regulation performance under the worst-case deterioration of a former player. Since the synthetic gene networks are highly nonlinear, it is not easy to solve the robust synthetic gene network design problem directly by the nonlinear dynamic game method directly. Recently, fuzzy systems have been employed to efficiently approximate nonlinear dynamic systems to solve the nonlinear control problem (Chen *et al.*, 1999; Chen *et al.*, 2000; Hwang, 2004; Li *et al.*, 2004; Lian *et al.*, 2001; Takagi & Sugeno, 1985). A Takagi-Sugeno (T-S) fuzzy model (Takagi & Sugeno, 1985) is proposed to interpolate several linearized genetic networks at different operating points to approximate the nonlinear gene network via some smooth fuzzy membership functions. Then with the help of the fuzzy approximation method, a fuzzy dynamic game scheme (Chen *et al.*, 2002) is developed so that the minimax regulation design of robust synthetic gene networks could be easily solved by the techniques of the linear dynamic game theory, which can be subsequently solved by a constrained optimization scheme via the linear matrix inequality (LMI) technique (Boyd *et al.*, 1994) that can be efficiently solved by the Robust Control Toolbox in Matlab (Balas *et al.*, 2008). Because the fuzzy model can approximate any nonlinear system, the proposed robust regulation design method developed from the fuzzy stochastic game theory can be applied to the robust regulation design problem of any synthetic gene network that can be interpolated by a T-S fuzzy model. For comparison, the conventional optimal regulation design method without considering the effect of disturbances is also proposed for the synthetic gene network. Because the effect of disturbances is not attenuated efficiently, the optimal regulation design method of synthetic gene networks is much influenced by the disturbances on the host cell. Finally, an *in silico* example is given to illustrate the design procedure and to confirm the efficiency and efficacy of the proposed minimax regulation design method for robust synthetic gene networks.

2. Robust synthetic gene network design via stochastic game approach

First, for the convenience of problem description, a simple design example of a four-gene network in (Batt *et al.*, 2007) is provided to give an overview of the design problem of robust synthetic gene networks. A more general design problem of robust synthetic gene networks will be given in the sequel. Let us consider a robust regulation design problem of a cascade loop of transcriptional inhibitions built in *E. coli.* (Hooshangi *et al.*, 2005). The synthetic gene network is represented in Fig. 1. It consists of four genes: *tetR*, *lacI*, *cI* and *eyfp* that code respectively three repressor proteins, TetR, LacI and CI, and the fluorescent potein EYFP (enhanced yellow fluorescent protein) (Batt *et al.*, 2007). aTc (anhydrotetracycline) is the input to the system. The fluorescence of the system, due to the protein EYFP, is the measured output. The protein CI inhibits gene *eyfp*. The protein TetR inhibits gene *lacI*. The protein LacI inhibits gene *cI*. The regulatory dynamic equations of the synthetic transcriptional cascade in Fig. 1 are given as follows (Batt *et al.*, 2007).

$$\begin{aligned}
\dot{x}_{tetR} &= k_{tetR,0} - \gamma_{tetR}x_{tetR} + w_1 \\
\dot{x}_{lacI} &= k_{lacI,0} + k_{lacI}(r_{lacI}(x_{tetR}) + a_{lacI}(u_{aTc}) - r_{lacI}(x_{tetR})a_{lacI}(u_{aTc})) - \gamma_{lacI}x_{lacI} + w_2 \\
\dot{x}_{cI} &= k_{cI,0} + k_{cI}r_{cI}(x_{lacI}) - \gamma_{cI}x_{cI} + w_3 \\
\dot{x}_{eyfp} &= k_{eyfp,0} + k_{eyfp}r_{eyfp}(x_{cI}) - \gamma_{eyfp}x_{eyfp} + w_4
\end{aligned} \tag{1}$$

with the uncertain initial conditions $x_{tetR}(0)$, $x_{lacI}(0)$, $x_{cI}(0)$ and $x_{eyfp}(0)$ in the host cell. $k_{tetR,0}$, $k_{lacI,0}$, $k_{cI,0}$ and $k_{eyfp,0}$ are basal production rates of the corresponding proteins, which are assumed to be given constants. k_{lacI}, k_{cI} and k_{eyfp} are the production rate parameters while γ_{tetR}, γ_{lacI}, γ_{cI} and γ_{eyfp} are decay rate parameters of the corresponding proteins. The regulatory functions r_{lacI}, r_{cI} and r_{eyfp} are the Hill functions for repressors and a_{lacI} for an activator.

The Hill function can be derived from considering the equilibrium binding of the transcription factor to its site on the promoter region. For a repressor, Hill function is an *S*-shaped curve which can be described in the form $r(x) = \dfrac{\beta_r}{1 + (x/K_r)^n}$. β_r is the maximal expression level of promoter. K_r is the repression coefficient. The Hill coefficient n governs the steepness of the input function. For an activator, Hill function can be described in the form $a(x) = \dfrac{\beta_a x^n}{K_a^n + x^n}$. β_a is the maximal expression level of promoter. K_a is the activation coefficient. n determines the steepness of the input function (Alon, 2007). w_1, w_2, w_3 and w_4 are the disturbances of the synthetic gene network, which denote the total of environmental noises, modeling residuals, intrinsic parameter fluctuations in the host cell. Therefore, w_i, i=1~4 are assumed uncertain but bounded disturbances. The synthetic gene network design is to specify k_{lacI}, k_{cI}, k_{eyfp} and γ_{tetR}, γ_{lacI}, γ_{cI}, γ_{eyfp} such that the system states x_{tetR}, x_{lacI}, x_{cI} and x_{eyfp} can approach the desired states x_{d1}, x_{d2}, x_{d3} and x_{d4}, respectively, in spite of uncertain initial conditions and disturbances.

If a synthetic gene network consists of n genes, then equation (1) can be extended to the following n-gene network dynamics.

$$\dot{x} = k_0 + f(x,k,\gamma) + g(u) + w, \; x(0) = x_0 \tag{2}$$

where the state vector x denotes the concentrations of proteins in the synthetic gene network. k_0 denotes the vector of basal production rates of the corresponding proteins. $f(x,k,\gamma)$ denotes the regulation vector of synthetic gene network, which is the function of production rate parameters k and decay rate parameters γ to be designed. $g(u)$ denotes the input function to the synthetic gene network. w denotes the vector of stochastic disturbances on the host cell, whose statistics may be unavailable. The initial condition x_0 is assumed stochastic with unknown covariance. The robust synthetic gene network design is to select parameters k and γ from feasible ranges so that the state vector x can approach a desired state vector x_d in spite of uncertain initial condition $x(0)$ and disturbances w on the host cell. i.e., $x \rightarrow x_d$ at the steady state despite uncertain $x(0)$ and w. This is a robust regulation problem of synthetic gene networks, i.e., the state vector x of synthetic gene networks is robustly regulated to x_d in the host cell.

Let us denote the regulation error as

$$\tilde{x} = x - x_d \tag{3}$$

Then the regulation error dynamic system is given by

$$\dot{\tilde{x}} = f(\tilde{x} + x_d, k, \gamma) + v, \ \tilde{x}(0) = \tilde{x}_0 \tag{4}$$

where $v=k_0+g(u)+w$ denotes the total uncertain disturbance in the regulation error system because these terms always fluctuate in the host cell and are not easily measured correctly. Because of the uncertainty of v and $\tilde{x}(0)$, the minimax regulation design method is an efficient but simple design scheme for robust synthetic gene network. The uncertainty of disturbance v and initial condition $\tilde{x}(0)$ in the following minimax design can be considered as a player maximizing their effects on the regulation error in the following robust design problem of synthetic gene networks (Basar and Olsder, 1999; Chen *et al.*, 2002).

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_{\tilde{x}(0),v} \frac{E\left[\int_0^{t_f} \tilde{x}^T Q \tilde{x} dt\right]}{E\left[\int_0^{t_f} v^T v dt + \tilde{x}^T(0)\tilde{x}(0)\right]} \tag{5}$$

where Q is the weighting matrix. In general, Q is a diagonal weighting matrix with Q=diag([q_{11}, q_{22}, … , q_{nn}]) to denote the punishment on regulation error. If only the last state x_n is required to be regulated to achieve the desired steady state x_{dn}, then we can let q_{nn}=1 and $q_{11}=q_{22}=\ldots=q_{n-1n-1}=0$. $[k_1,k_2]$ and $[\gamma_1,\gamma_2]$ denote the allowable ranges of production rate vector k and decay rate vector γ, respectively. The allowable ranges are determined by the engineering biotechnologies of synthetic biology. k and γ to be designed can be considered as another player minimizing the worst-case effect of $\tilde{x}(0)$ and v on the regulation error. If the disturbances v and initial condition $\tilde{x}(0)$ are deterministic, then the expectation operation E[ ] in (5) could be neglected.

The physical meaning of (5) is that the worst-case effect of uncertain $\tilde{x}(0)$ and v on the regulation error $\tilde{x}$ must be minimized from the mean energy perspective by k and γ, which are chosen from the allowable ranges. Therefore, for uncertain $\tilde{x}(0)$ and v, the robust synthetic gene network design is to solve the minimax problem in (5) subject to the regulation error dynamic system in (4). This is the so-called stochastic game problem in the robust synthetic gene network design (Basar & Olsder, 1999).

In general, it is not easy to solve the nonlinear stochastic game problem in (5) subject to (4) directly. It is always solved by a sub-minimax method. First, let the upper bound g^2 of (5) be (Basar & Olsder, 1999; Chen *et al.*, 2002)

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_{\tilde{x}(0),v} \frac{E\left[\int_0^{t_f} \tilde{x}^T Q \tilde{x} dt\right]}{E\left[\int_0^{t_f} v^T v dt + \tilde{x}^T(0)\tilde{x}(0)\right]} \le g^2 \tag{6}$$

We will first solve the sub-minimax problem in (6) and then decrease the upper bound g^2 as much as possible to approach its minimax solution. In general, the minimax problem in (6) is equivalent to the following minimax problem (Basar & Olsder, 1999; Chen *et al.*, 2002)

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_{v} E\left[\int_0^{t_f} \left(\tilde{x}^T Q \tilde{x} - g^2 v^T v\right) dt\right] \le g^2 E\left[\tilde{x}^T(0)\tilde{x}(0)\right], \quad \forall \tilde{x}(0) \tag{7}$$

where g^2 is to be minimized because it is the upper bound in (6) and should be as small as possible to approach the minimax solution. Let us denote the cost function as

$$J(k,r,v) = E\left[\int_0^{t_f}\left(\tilde{x}^T Q\tilde{x} - g^2 v^T v\right)dt\right] \tag{8}$$

3. Design procedure and result

3.1 Sub-minimax design for robust synthetic gene networks

From the above analysis, the dynamic game problem in (6) or (7) is equivalent to finding the worst-case disturbance v^* which maximizes $J(k,\gamma,v)$ and then the minimax k^* and γ^* which minimize $J(k,\gamma,v^*)$ such that the minimax value $J(k^*,\gamma^*,v^*)$ is less than $g^2 E[\tilde{x}^T(0)\tilde{x}(0)]$, i.e.

$$J\left(k^*,\gamma^*,v^*\right) = \min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} J\left(k,r,v^*\right) = \min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_v J(k,\gamma,v) \le g^2 E\left[\tilde{x}^T(0)\tilde{x}(0)\right], \quad \forall \tilde{x}(0) \tag{9}$$

Hence, if there exist k^*, γ^* and v^* such that the minimax design problem in (9) is solved, then they can satisfy the minimax performance of the robust synthetic gene network design in (6) as well. Therefore, the first step of robust synthetic gene network design is to solve the following dynamic game problem:

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_v J\left(k,\gamma,v\right) \tag{10}$$

subject to the error dynamic equation in (4). Since $J\left(k^*,\gamma^*,v^*\right) \le g^2 E[\tilde{x}^T(0)\tilde{x}(0)]$ according to (9) and g^2 is the upper bound of the game in (6), the sub-minimax has to make g^2 as small as possible, too.

From the above analysis, we obtain the following sub-minimax result for robust synthetic gene network design.

Proposition 1: The sub-minimax synthetic gene network design is equivalent to solving the following constrained optimization for k^* and γ^*,

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} g^2 \tag{11}$$

subject to the following Hamilton-Jacobi inequality (HJI)

$$\begin{aligned}&\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T f(\tilde{x}+x_d,k,\gamma) + \tilde{x}^T Q\tilde{x} + \frac{1}{4g^2}\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right) < 0\\ &E[V(\tilde{x}(0))] \le g^2 E[\tilde{x}^T(0)\tilde{x}(0)]\end{aligned} \tag{12}$$

with $V(\tilde{x}) > 0$ and the worst-case disturbance is given by

$$v^* = \frac{1}{2g^2}\frac{\partial V(\tilde{x})}{\partial \tilde{x}} \tag{13}$$

Proof: see *Appendix A*.

Remark 1:

1. From (6), g^2 is the upper bound of the game. In (11), we minimize the upper bound g^2 to achieve the sub-minimax solution for robust synthetic gene networks.
2. The physical meaning of the constrained minimization in (11) and (12) is that we want to specify k^* and γ^* from the allowable parameter ranges such that the upper bound g^2 is as small as possible until no positive solution $V(\tilde{x}) > 0$ of HJI in (12) exists.

At present, there exists no efficient analytic or numerical method to solve the HJI in (12) for nonlinear stochastic system control or filtering designs (Zhang & Chen, 2006; Zhang *et al.*, 2005).

3.2 Minimax robust synthetic gene networks via fuzzy interpolation method

Because it is very difficult to solve the nonlinear HJI in (12), no simple approach is available for solving the constrained optimization problem in (11) for the minimax robust synthetic gene network design problem. Recently, the Takagi-Sugeno (T-S) fuzzy model has been widely employed (Chen *et al.*, 1999; Chen *et al.*, 2000; Hwang, 2004; Takagi & Sugeno, 1985) to approximate the nonlinear system via interpolating several linearized systems at different operating points so that the nonlinear Nash stochastic problem could be transformed to a fuzzy stochastic game problem (Chen *et al.*, 2002). By using such approach, the HJI in (12) can be replaced by a set of linear matrix inequalities (LMIs). In this situation, the nonlinear stochastic game problem in (10) could be easily solved by the fuzzy dynamic method for the robust design of sub-minimax design problem.
Suppose the nonlinear system in (4) could be approximated by a T-S fuzzy system (Takagi & Sugeno, 1985). The T-S fuzzy model is a piecewise interpolation of several linearized models through fuzzy membership functions. The fuzzy model is described by fuzzy *if-then* rules and will be employed to deal with the nonlinear stochastic game problem for robust synthetic gene network design under uncertain initial conditions and disturbances. The *i*th rule of fuzzy model for nonlinear systems in (4) is of the following form (Chen *et al.*, 1999; Takagi & Sugeno, 1985).

Rule *i*:

If $\tilde{x}_1(t)$ is F_{i1} and … and $\tilde{x}_q(t)$ is F_{iq},

$$\text{then } \dot{\tilde{x}} = \mathbf{A}_i(k,\gamma)\tilde{x} + v,\ i = 1,2,\cdots,L \tag{14}$$

where F_{ij} is the fuzzy set. $\mathbf{A}_i(k,\gamma)$ is constant matrix with the elements of k and γ contained in its entries. q is the number of premise variables and $\tilde{x}_1,\cdots,\tilde{x}_q$ are the premise variables. The fuzzy system is inferred as follows (Chen *et al.*, 1999; Chen *et al.*, 2000; Li *et al.*, 2004; Lian *et al.*, 2001; Takagi and Sugeno, 1985)

$$\dot{\tilde{x}}(t) = \frac{\sum_{i=1}^{L}\mu_i(\tilde{x}(t))[\mathbf{A}_i(k,\gamma)\tilde{x}(t) + v]}{\sum_{i=1}^{L}\mu_i(\tilde{x}(t))}$$

$$= \sum_{i=1}^{L} h_i(\tilde{x}(t))\left[\mathbf{A}_i(k,\gamma)\tilde{x}(t) + v\right],\ \tilde{x}(0) = \tilde{x}_0 \tag{15}$$

where $\mu_i(\tilde{x}(t)) = \prod_{j=1}^{q} F_{ij}(\tilde{x}_j(t))$, $h_i(\tilde{x}(t)) = \frac{\mu_i(\tilde{x}(t))}{\sum_{i=1}^{L}\mu_i(\tilde{x}(t))}$, and $F_{ij}(\tilde{x}_j(t))$ is the grade of membership of $\tilde{x}_j(t)$ in F_{ij}.
We assume

$$\mu_i(\tilde{x}(t)) \geq 0 \text{ and } \sum_{i=1}^{L}\mu_i(\tilde{x}(t)) > 0 \tag{16}$$

Therefore, we get the following fuzzy basis functions

$$h_i(\tilde{x}(t)) \geq 0 \text{ and } \sum_{i=1}^{L} h_i(\tilde{x}(t)) = 1 \tag{17}$$

The T-S fuzzy model in (15) is to interpolate L linear systems to approximate the nonlinear system in (4) via the fuzzy basis functions $h_i(\tilde{x}(t))$. We could specify system parameter $\mathbf{A}_i(k,\gamma)$ easily so that $\sum_{i=1}^{L} h_i(\tilde{x}(t))\mathbf{A}_i(k,\gamma)\tilde{x}$ can approximate $f(\tilde{x}+x_d,k,\gamma)$ in (4) by the fuzzy identification method (Takagi and Sugeno, 1985).
After the nonlinear system in (4) is approximated by the T-S fuzzy system in (15), the nonlinear dynamic game problem in (10) is replaced by solving a dynamic game problem in (6) subject to the fuzzy system (15).
Proposition 2: The sub-minimax robust synthetic gene network design is to solve k^* and γ^* by the following constraint optimization

$$\min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} g^2 \tag{18}$$

subject to

$$\begin{aligned} &P\mathbf{A}_i(k,r) + \mathbf{A}_i^T(k,r)P + Q + \frac{1}{g^2}PP \leq 0,\ i = 1,\cdots,L \\ &P \leq g^2 I,\ P > 0 \end{aligned} \tag{19}$$

and the worst-case disturbance v^* is given by

$$v^* = \frac{1}{g^2}\sum_{i=1}^{L} h_i(\tilde{x})P\tilde{x} \tag{20}$$

Proof: see *Appendix B*.
By the fuzzy approximation, the HJI in (12) can be approximated by a set of algebraic inequalities in (19). By Schur complement (Boyd *et al.*, 1994), the constrained optimization problem in (18)-(19) is equivalent to the following LMI-constrained optimization problem

$$\min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} g^2 \tag{21}$$

subject to

$$\begin{bmatrix} P\mathbf{A}_i(k,\gamma)+\mathbf{A}_i^T(k,\gamma)P+Q & P \\ P & -g^2 I \end{bmatrix} \le 0, \quad i=1,2,\cdots,L \qquad (22)$$
$$P \le g^2 I, \;\; P>0$$

Remark 2:

1. The fuzzy basis functions $h_i(\tilde{x})$ in (15) and (17) can be replaced by other interpolation functions, for example, cubic spline functions.
2. By the fuzzy approximation, the HJI in (12) of nonlinear dynamic game problem can be solved by Robust Control Toolbox in Matlab efficiently (Balas *et al.*, 2008). The constrained optimization in (18) and (19) can be solved by decreasing g^2 until there is no positive definite solution $P > 0$ in (22) with $k^* \in [k_1,k_2]$ and $\gamma^* \in [\gamma_1,\gamma_2]$.
3. In the LMI-constrained optimization in (22) for the robust synthetic gene network design, we do not need the statistics of initial conditions and disturbances on the host cell, which are not easy to be measured. Therefore, the proposed method is simple but robust for synthetic gene networks.

Remark 3:

For comparison, the conventional optimal regulation design is also proposed for synthetic gene networks. If the effect of external disturbances and uncertain initial conditions on the regulation error is not considered as (5) in the design procedure, i.e., only the following optimal regulation design is considered.

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} E\left[\int_0^{t_f} \tilde{x}^T Q \tilde{x} dt\right] \qquad (23)$$

subject to (4)

then we obtain the following sub-optimal regulation design for synthetic gene networks.

Proposition 3: The sub-optimal synthetic gene network design in (23) is to solve the following constrained optimization

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} E\left[V(\tilde{x}(0))\right] \qquad (24)$$

subject to

$$V(\tilde{x})>0, \quad \frac{\partial V(\tilde{x})}{\partial \tilde{x}} f(\tilde{x}+x_d,k,r)+\tilde{x}^T Q\tilde{x}+\frac{1}{2}\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)<0 \qquad (25)$$

Proof: see *Appendix C.*

Because it is not easy to solve the above HJI-constrained optimization for the sub-optimal regulation design in (24) and (25), the fuzzy approximation method is needed to simplify the design procedure. If the nonlinear error dynamic equation in (4) is represented by the fuzzy interpolation system in (15), then the optimal synthetic gene network design in (23) is equivalent to the following optimal regulation design problem.

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} E\left[\int_0^{t_f} \tilde{x}^T Q \tilde{x} dt\right]$$
$$\text{subject to } \dot{\tilde{x}}=\sum_{i=1}^{L} h_i(\tilde{x})\mathbf{A}_i(k,\gamma)\tilde{x}+v \qquad (26)$$

Proposition 4: The sub-optimal regulation design problem in (26) becomes how to solve the following constrained optimization problem

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \mathrm{Tr}\left(PR_0\right) \tag{27}$$

subject to

$$P>0,\ \begin{bmatrix} \mathbf{A}_i^T(k,\gamma)P+P\mathbf{A}_i(k,\gamma)+Q & P \\ P & -2I \end{bmatrix} \le 0,\ \ i=1,2,\cdots,L \tag{28}$$

where R_0 denotes the covariance matrix $E[\tilde{x}(0)\tilde{x}^T(0)]$.
Proof: similar to the proof of ***Proposition 2***.
Since the effect of stochastic disturbances on $\tilde{x}$ is not considered as (5) in the above sub-optimal synthetic gene network design, the synthesized gene networks will be more sensitive to the external disturbances or other uncertain factors. They will be compared with the sub-minimax robust synthetic gene network in the simulation example.
Remark 4:
Since the effect of the disturbance v on the regulation error has not been attenuated efficiently on the design procedure of the sub-optimal regulation in ***Proposition*** 3 and 4, the disturbance will have much effect on the sub-optimal regulation design of synthetic gene network. This property will be discussed and compared with the proposed robust synthetic gene network in the design example in the following section.
According to the analyses above, a design procedure is developed for the proposed robust synthetic gene network.

※ Design Procedure:

1. Give feasible parameter ranges $[k_1,k_2]$ and $[\gamma_1,\gamma_2]$ for production rate parameters k and decay rate parameters γ, respectively, according to the biotechnology ability.
2. Give the desired steady state x_d according to the design purpose and develop a regulation error dynamic (4) for a synthetic gene network.
3. Construct a T-S fuzzy model in (15) to approximate the regulation error dynamic in (4).

Solve the constrained optimization problem from the ranges $k\in[k_1,k_2]$ and $\gamma\in[\gamma_1,\gamma_2]$ in (21) and (22) for the robust synthetic gene network design k^* and γ^*, respectively according to the sub-minimax scheme or solve the constrained optimization problem in (27) and (28) for the sub-optimal regulation design.

3.3 Design example *in silico* for the proposed robust design

Consider the man-made synthetic gene network in the dynamic equations (1) (Batt *et al.*, 2007). The synthetic gene network is shown in Fig. 1. Where $k_{tetR,0}$, $k_{lacI,0}$, $k_{cI,0}$ and $k_{eyfp,0}$ are basal production rates of the corresponding proteins, which are assumed to be 5000, 587, 210 and 3487, respectively (Batt *et al.*, 2007; Hooshangi *et al.*, 2005). k_{lacI}, k_{cI} and k_{eyfp} are the production rate parameters while γ_{tetR}, γ_{lacI}, γ_{cI} and γ_{eyfp} are the decay rate parameters of the corresponding proteins in the host cell (i.e. *E. coli.*). In the robust synthetic gene network design, we should select the parameters k and γ from feasible ranges so that the state of synthetic gene network x_i could approach a desired steady state $x_{d,i}$ for some biotechnical purpose. r_{lacI}, r_{cI} and r_{eyfp} are the decreasing Hill functions for regulations of repressors. a_{lacI} is an increasing function since aTc is an activator. The Hill function is a *S*-shaped curve (Alon, 2007). u_{aTc} is the input to the synthetic gene network system. We assume anhydrotetracycline input concentration to be a constant value 10000 (i.e. u_{aTc} = 10000). For the convenience of simulation, we assume that extrinsic disturbances $w_1 \sim w_4$ are $w_i=[500n_1\ 10000n_2\ 100n_3\ 100000n_4]^T$, where n_i, i=1,2,3,4 are independent Gaussian white noises with zero mean and unit variance.

From the robust synthetic gene network design procedure, we give the feasible parameter ranges of production rate parameters k and decay rate parameters γ as follows (Batt *et al.*, 2007)

$$\begin{array}{cc} k_{lacI} \in [70,7000] & \gamma_{tetR} \in [0.05,5] \\ k_{cI} \in [75,8000] & \gamma_{lacI} \in [0.01314,0.1517] \\ k_{eyfp} \in [30,30000] & \gamma_{cI} \in [0.7617,7.2815] \\ & \gamma_{eyfp} \in [0.007,0.067] \end{array} \tag{29}$$

Then we give the desired steady states of the synthetic gene network are $x_{d,i}=[1000,50000,300,500000]^T$, $i=tetR,\ lacI,\ cI,\ eyfp$. Then the regulation error dynamic equation in (4) is developed for the synthetic gene network. Because it is very difficult to solve the nonlinear HJI in (12), no simple approach is available to solve the constrained optimization problem in (11) for robust parameters k_i^* and γ_i^*. We construct the T-S fuzzy model in (15) to approximate the regulation error dynamic in (4) with the regulation error dynamic system's state variables as the premise variables in the following.

Rule *i*:

If $\tilde{x}_1(t)$ is F_{i1} and $\tilde{x}_2(t)$ is F_{i2} and $\tilde{x}_3(t)$ is F_{i3} and $\tilde{x}_4(t)$ is F_{i4},

then $\dot{\tilde{x}} = \mathbf{A}_i(k,\gamma)\tilde{x} + v,\ i = 1,2,\cdots,L$

where the parameters $\mathbf{A}_i(k,\gamma)$ and the number of fuzzy rules is L=16. To construct the fuzzy model, we need to find the operating points of the regulation error dynamic system. The operating points for $\tilde{x}_1$ are chosen at $\bar{x}_{11} = -40$ and $\bar{x}_{12} = 4040$. Similarly, the operating points of $\tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ are chosen at $\bar{x}_{21} = -38510$, $\bar{x}_{22} = 381$, $\bar{x}_{31} = -16.7$, $\bar{x}_{32} = 1686$, $\bar{x}_{41} = -441590$, and $\bar{x}_{42} = 4372$, respectively. For the convenience of design, triangle-type membership functions are taken for Rule 1 through Rule 16. We create two triangle-type membership functions for each state (see Fig. 2).

In order to simplify the nonlinear stochastic game problem of the robust synthetic gene network, we just solve only the sub-minimax problem in (6) instead. With the help of fuzzy approximation method and LMI technique, we can easily solve the constrained optimization problem in (21) and (22) instead of the nonlinear constrained optimization problem in (11) and (12) for the minimax robust synthetic gene network design. Finally, we obtain the upper bound of the game in (6) g^2 = 0.847536 and a common positive definite symmetric matrix P for (22) as follows

$$P = \begin{bmatrix} 0.45842 & -0.0079 & 0.0143 & -0.00068 \\ -0.0079 & 0.07186 & -0.000557 & 0.00268 \\ 0.0143 & -0.000557 & 0.04847 & 0.000718 \\ -0.00068 & 0.00268 & 0.000718 & 0.0578 \end{bmatrix}$$

with the specified robust production rate parameters $k_{lacI}^{*}=7000$, $k_{cI}^{*}=4037.5$ and $k_{eyfp}^{*}=30000$ and robust decay rate parameters $\gamma_{tetR}^{*}=5$, $\gamma_{lacI}^{*}=0.1517$, $\gamma_{cI}^{*}=4.0216$ and $\gamma_{eyfp}^{*}=0.067$ of the synthetic gene network. With these design parameters, the parameters $\mathbf{A}_i$ of fuzzy model are described in *Appendix D*.

Figure 3 presents the simulation result for robust synthetic gene networks by using Monte Carlo method with 50 rounds and with the uncertain initial values. $x_1(0)$~$x_4(0)$ are assumed normal-distributed random numbers with means 5000, 8000, 2000, 10000 and standard deviations 500, 800, 200, 1000, respectively. As can be seen, the synthetic gene network has robust regulation ability to achieve the desired steady state (black dashed line) in spite of uncertain initial states and the disturbances on the host cell. Obviously, the robust synthetic gene network by the proposed sub-minimax regulation design method has robust stability to the uncertain initial conditions and enough filtering ability to attenuate the disturbances on the host cell and can approach the desired steady states.

For comparison, we solve the sub-optimal regulation design problem in (27) and (28) for the specified production rate parameters $k_{lacI}^{*}=70$, $k_{cI}^{*}=4037.5$ and $k_{eyfp}^{*}=15015$ and decay rate parameters $\gamma_{tetR}^{*}=2.525$, $\gamma_{lacI}^{*}=0.1517$, $\gamma_{cI}^{*}=7.2815$ and $\gamma_{eyfp}^{*}=0.067$ of the synthetic gene network. The simulation result of conventional optimal regulation design is also shown in Fig. 4. As can be seen, the conventional optimal regulation design of the synthetic gene network is more sensitive to the initial conditions and disturbances and cannot achieve the desired steady state under the uncertain initial conditions and disturbances.

Remark 5:

The experimental systems in the above example may not be fully observable. If we want to know whether all state variables can approach to the desired states x_d, several fluorescent proteins (red, green and cyan colour) should be necessary to observe their protein expressions of all state variables in the experimental design.

4. Discussion

Because the initial conditions and disturbances on the host cell are uncertain, to simplify the design problem, a robust synthetic biology design is formulated as a stochastic game problem in this study. The uncertain initial conditions and disturbances due to intrinsic and extrinsic molecular noises on the host cell are considered as a player maximizing the regulation error and the design parameters are considered as another player minimizing the regulation error. In order to avoid solving HJI in the stochastic game theory-based design problem, a T-S fuzzy interpolation method is introduced to simplify the design procedure of robust synthetic gene networks via only solving a set of LMIs, which can be efficiently solved by Robust Control Toolbox in Matlab.

In our study, we can select the weighting matrix Q=diag([q_{11}, q_{22}, q_{33}, q_{44}]) which denotes the punishment on the corresponding tracking error $\tilde{x}$. If we only need to achieve a desired steady state x_{d4} (EYFP), we just assign a value to the fourth diagonal element q_{44} of the weighting matrix Q and set q_{11}=q_{22}= q_{33}=0. The rest of states x_1~x_3 will not approach to the given steady state x_{d1}~x_{d3} because of no any punishment. However, in this case, some infeasible steady states of x_1, x_2, and x_3 may be obtained even an optimal x_4 can be achieved. In this study, the desired steady states of x_1, x_2, and x_3 are given because we can avoid obtaining infeasible steady states in x_1, x_2, and x_3 when an optimal x_4 is achieved. Further, the undesired steady states of x_1, x_2, and x_3 may also have metabolic toxicity on host cell and should be avoided. Since the steady states of x_1~x_3 are not that important, the desired steady states x_{d1}~x_{d3} can be adjusted within feasible ranges, so that the desired steady state x_{d4} can still achieve some optimization as possible. This kind of design can avoid hampering the optimization of x_4 when x_1, x_2, and x_3 achieve some feasible steady states.

In our *in silico* design example, we can design the specified robust production rate parameters k_i^* and decay rate parameters γ_i^* within the feasible parameter ranges to achieve the desired steady states of the synthetic gene network. As for the biological implementation, we could refer to standard biological parts in biological device datasheets to construct the genetic circuits with the fine-tuned production rate parameters k_i^* and decay rate parameters γ_i^*. In this way, synthetic biologists can increase efficiency of gene circuit design through registries of biological parts and standard datasheets, which are developed concerned with proper packing and characterizing of 'modular' biological activities so that these biological parts or devices with some desired characteristics may be efficiently assembled into gene circuits (Canton, *et al.*, 2008).

Quantitative descriptions of devices in the form of standardized, comprehensive datasheets are widely used in many engineering disciplines. A datasheet is intended to allow an engineer to quickly determine whether the behavior of a device will meet the requirements of a system in which a device might be used (Canton, *et al.*, 2008). Such a determination is based on a set of standard characteristics of device behavior, which are the product of engineering theory and experience. In the datasheets of engineering, the characteristics typically reported are common across a wide range of device types, such as sensors, logic elements and actuators. Recently, biological datasheets have been set as standards for characterization, manufacture and sharing of information about modular biological devices for a more efficient, predictable and design-driven genetic engineering science (Arkin, 2008; Canton, *et al.*, 2008). Because datasheets of biological parts or devices are an embodiment of engineering standard for synthetic biology (Canton, *et al.*, 2008), a good device standard should define sufficient information about biological parts or devices to allow the design of gene circuit systems with the optimal parameters. Datasheets contain a formal set of input-output transfer functions, dynamic behaviors, compatibility, requirements and other details about a particular part or device (Arkin, 2008; Canton, *et al.*, 2008). Since parameters k_i are combinations of transcription and translation, they could be measured from the input-output transfer functions and dynamic behaviors of biological parts or devices in biological device datasheets. From properly characterized input-output transfer functions and dynamic behaviors of parts or devices in biological device datasheets, an engineer can estimate the corresponding parameters of biological parts or devices. When the biological parts and devices in datasheets become more complete in future, we can rapidly select from a vast list the parts that will meet our design parameters k_i. Therefore we can ensure that devices selected from datasheets can fit the optimal parameters and systems synthesized from them can satisfy the requirements of design specifications for robust synthetic gene networks.

In order to guarantee the biological feasibility of the calculated optimal parameters, the ranges $[k_1, k_2]$ and $[\gamma_1, \gamma_2]$ of parameters should be determined by the whole parameters of biological parts repositories (http://partsregistry.org/) so that the optimal parameters selected within these ranges to minimize g^2 in equations (21) and (22) have biological meaning, or equivalently from the whole biological parts in biological device datasheets, we can find a set of biological parts whose parameters can minimize the g^2 in equations (21) and (22) to achieve the robust optimal design of synthetic gene network.

In synthetic gene networks, there is much uncertainty about what affects the behavior of biological circuitry and systems. For example, devices will perturb the cellular functions and there are also likely to be parasitic and unpredictable interactions among components as well as with the host. Since k_i is a combination of promoter strength, ribosome binding site and degradation of the transcript, there are some variations or uncertainties on the parameter value k_i. These variations or uncertainties of k_i can be transformed to an equivalent uncertain disturbance w_i in equation (1) from the viewpoint of mathematic model. The proposed robust minimax synthetic biology design method can predict the most robust value of k_i from the perspective of stochastic game. In our robust design method, we don't need the statistics of these parameter uncertainties because the proposed synthetic genetic network not only can achieve the desired steady state but also can tolerate the worst-case effect due to these uncertain parameter variations and external noises on the host cell.

For comparison, a sub-optimal regulation design for synthetic gene network is also developed for synthetic gene network. Because the sub-optimal regulation design cannot efficiently attenuate the effect of uncertain initial conditions and disturbances on the regulation, it is not suitable for robust synthetic gene networks with uncertain initial conditions and disturbances on the host cell. As seen in the example *in silico*, the proposed robust synthetic gene network can function properly in spite of uncertain initial conditions and disturbances on the host cell. Design of more robust and complex genetic circuits is foreseen to have important applications in biotechnology, medicine and biofuel production, and to revolutionize how we conceptualize and approach the engineering of biological systems (Andrianantoandro *et al.*, 2006). Therefore, it has much potential for the robust synthetic gene network design in the near future.

5. Tables and figures

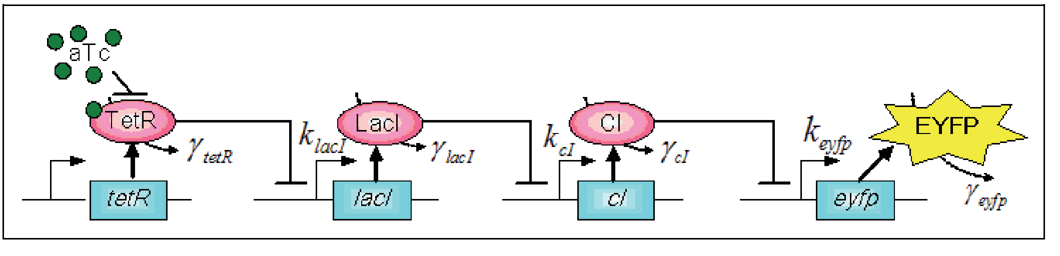

Fig. 1. Synthetic transcription cascade loop *in silico* design example. aTc represses TetR, TetR represses *lacI*, LacI represses *cI*, CI represses *eyfp*. aTc is the system input and the fluorescent protein EYFP is the output.

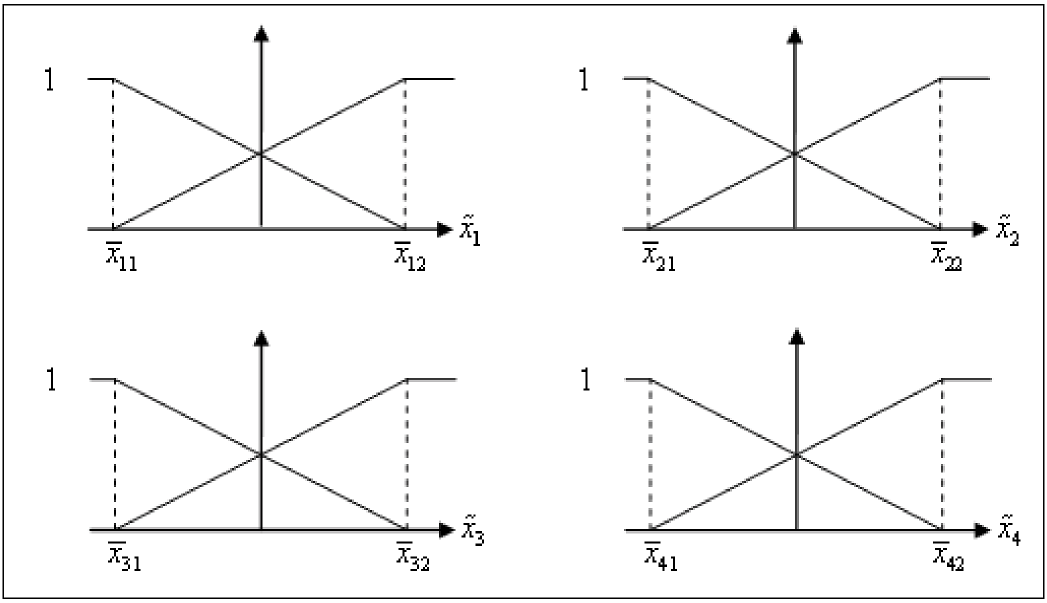

Fig. 2. Membership functions for four states $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ and $\tilde{x}_4$.

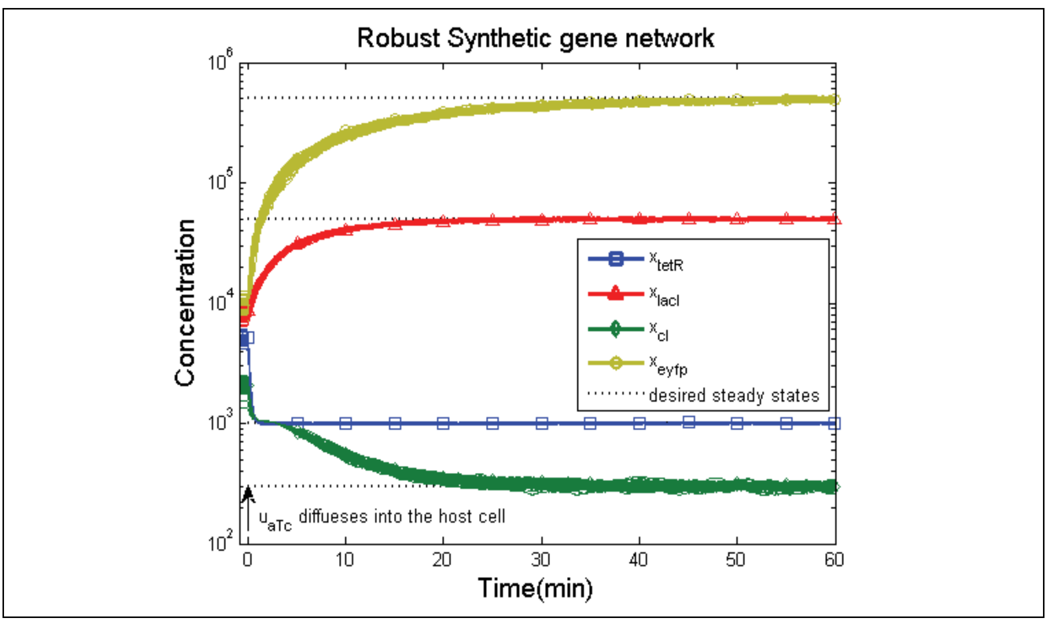

Fig. 3. The robust synthetic gene network design with uncertain initial values and the desired steady states $x_d = [1000, 50000, 300, 500000]^T$. And with specified robust production rate parameters $k^*_{lacI} = 7000$, $k^*_{cI} = 4037.5$ and $k^*_{eyfp} = 30000$ while the specified robust decay rate parameters are $\gamma^*_{tetR} = 5$, $\gamma^*_{lacI} = 0.1517$, $\gamma^*_{cI} = 4.0216$ and $\gamma^*_{eyfp} = 0.067$ of the synthetic gene network. The Monte Carlo simulation method is used with 50 rounds.

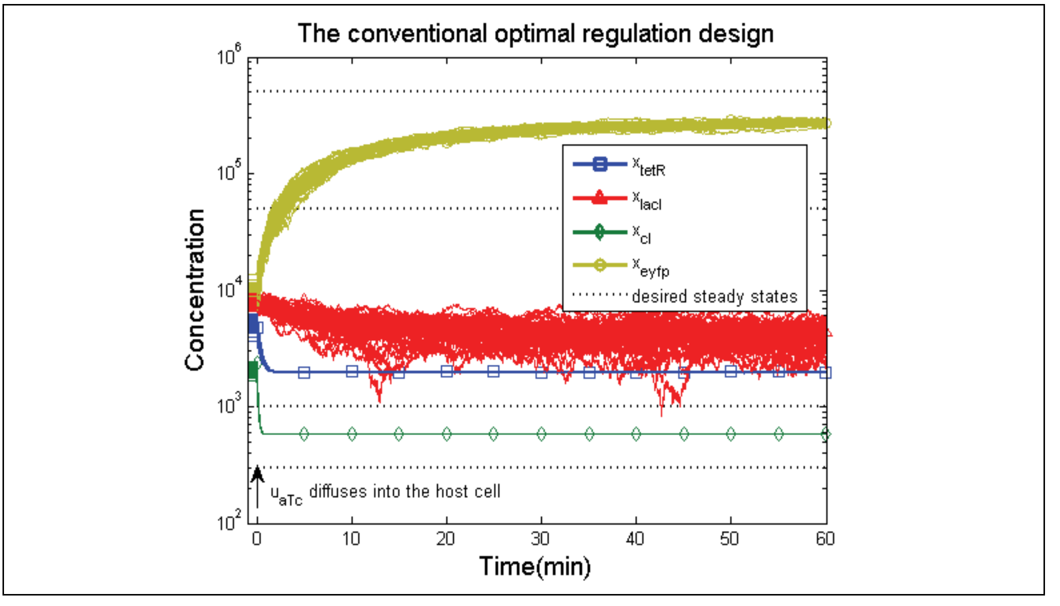

Fig. 4. The conventional optimal regulation design with uncertain initial values and the desired steady states $x_d = [1000, 50000, 300, 500000]^T$. And with specified production rate parameters $k^*_{lacI} = 70$, $k^*_{cI} = 4037.5$ and $k^*_{eyfp} = 15015$ while the specified decay rate parameters are $\gamma^*_{tetR} = 2.525$, $\gamma^*_{lacI} = 0.1517$, $\gamma^*_{cI} = 7.2815$ and $\gamma^*_{eyfp} = 0.067$ of the synthetic gene network. It is seen that the conventional optimal regulation design of the synthetic gene network is sensitive to the initial conditions and disturbances and cannot achieve the desired steady states. The Monte Carlo simulation method is used with 50 rounds.

6. Appendixes

6.1 Appendix A: Proof of proposition 1

Let us consider a Lyapunov energy function $V(\tilde{x}) > 0$, then the cost function in equation (8) is equivalent to

$$J(k,\gamma,v) = E\left[V(\tilde{x}(0)) - V(\tilde{x}(t_f)) + \int_0^{t_f}\left(\tilde{x}^T Q\tilde{x} - g^2 v^T v + \frac{dV(\tilde{x})}{dt}\right)dt\right] \tag{A1}$$

By the chain rule, we get

$$\frac{dV(\tilde{x}(t))}{dt} = \left(\frac{\partial V(\tilde{x}(t))}{\partial \tilde{x}(t)}\right)^T \cdot \frac{d\tilde{x}(t)}{dt} = \left(\frac{\partial V(\tilde{x}(t))}{\partial \tilde{x}(t)}\right)^T \cdot (f(\tilde{x}(t) + x_d(t), k, \gamma) + v(t)) \tag{A2}$$

Substituting (A2) into (A1), we maximize $J(k,\gamma,v)$ by the uncertain disturbance v

$$\max_v J(k,\gamma,v)$$

$$= \max_v E\left[V(\tilde{x}(0)) - V(\tilde{x}(t_f)) + \int_0^{t_f}\left(\tilde{x}^T Q\tilde{x} - g^2 v^T v + \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T f(\tilde{x} + x_d, k, \gamma) + \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T v\right)dt\right]$$

$$= \max_{v} E\left[V(\tilde{x}(0)) - V(\tilde{x}(t_f)) + \int_0^{t_f} \left(\tilde{x}^T Q \tilde{x} + \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T f(\tilde{x}+x_d,k,\gamma) \right.\right.$$

$$\left.\left. - \left(gv - \frac{1}{2g}\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T \left(gv - \frac{1}{2g}\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right) + \frac{1}{4g^2}\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)\right) dt \right]$$

$$= E\left[V(\tilde{x}(0)) - V(\tilde{x}(t_f)) + \int_0^{t_f} \left(\tilde{x}^T Q \tilde{x} + \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T f(\tilde{x}+x_d,k,\gamma) + \frac{1}{4g^2}\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)\right) dt \right] \quad \text{(A3)}$$

with the worst-case disturbance $v^* = \frac{1}{2g^2}\frac{\partial V(\tilde{x})}{\partial \tilde{x}}$.

By the inequality in (12), it is seen that $V(\tilde{x}(0))$ is the upper bound of (A3) i.e., the sub-minimax problem becomes how to solve the following constrained optimization problem

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_{v} J(k,\gamma,v) \le \min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} E\left[V(\tilde{x}(0))\right] \quad \text{(A4)}$$

subject to (12) and $V(\tilde{x}) > 0$.

By the fact in (9), $g^2 E\left[\tilde{x}^T(0)\tilde{x}(0)\right]$ is the upper bound of $\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_{v} J(k,\gamma,v)$. Therefore $E\left[V(\tilde{x}(0))\right]$ in (A4) should be bounded by $g^2 E\left[\tilde{x}^T(0)\tilde{x}(0)\right]$, i.e. $E\left[V(\tilde{x}(0))\right] \le g^2 E\left[\tilde{x}^T(0)\tilde{x}(0)\right]$. Therefore the suboptimal solution is to minimize its upper bound. Hence, the sub-minimax problem in (A4) could be replaced by

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} \max_{v} J(k,\gamma,v) \le \min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} E\left[V(\tilde{x}(0))\right] \le \min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} E\left[g^2\tilde{x}^T(0)\tilde{x}(0)\right] = \min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} g^2 \mathrm{Tr}(R_0) \quad \text{(A5)}$$

where $\mathrm{Tr}(R_0)$ denotes the trace of R_0 and R_0 denotes the covariance of the initial condition $\tilde{x}(0)$ i.e., $R_0 = E\left[\tilde{x}(0)\tilde{x}^T(0)\right]$, which is independent of the choice of k and γ. Therefore, the sub-minimax design problem is equivalent to solving the following constrained optimization

$$\min_{\substack{k\in[k_1,k_2]\\ \gamma\in[\gamma_1,\gamma_2]}} g^2$$

subject to (12) and $V(\tilde{x}) > 0$.

6.2 Appendix B: Proof of proposition 2

We replace error dynamic system in (4) by its fuzzy interpolation system in (15). Then HJI in (12) can be represented by

$$\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T \left(\sum_{i=1}^{L} h_i(\tilde{x})\mathbf{A}_i(k,\gamma)\tilde{x}\right) + \tilde{x}^T Q \tilde{x} + \frac{1}{4g^2}\left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right)^T \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}}\right) < 0 \quad \text{(B1)}$$

Let us choose the Lyapunov function $V(\tilde{x})$ as $V(\tilde{x}) = \tilde{x}^T P \tilde{x}$ for some positive definite symmetric matrix P and substitute it into (B1). Then we get

$$\begin{aligned} &\sum_{i=1}^{L} h_i(\tilde{x}) \left\{ \tilde{x}^T \left(P\mathbf{A}_i(k,r) + \mathbf{A}_i^T(k,r)P + Q + \frac{1}{g^2} PP \right) \tilde{x} \right\} \le 0 \\ &P \le g^2 I \end{aligned} \tag{B2}$$

where the property in (17) is used.

It is seen that the inequalities in (19) implies (B2). Therefore, the sub-minimax design for the fuzzy equivalent system becomes how we solve the constrained optimization in (18) and (19). By substituting $V(\tilde{x}) = \tilde{x}^T P \tilde{x}$ into (13), we get the worst-case disturbances v^* in (20).

6.3 Appendix C: Proof of proposition 3

Again, let us consider a Lyapunov energy function $V(\tilde{x}) > 0$, then the equation (23) is equivalent to

$$\begin{aligned} &\min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} E\left[\int_0^{t_f} \tilde{x}^T Q \tilde{x} dt \right] \\ &= \min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} E\left[V(\tilde{x}(0)) - V(\tilde{x}(t_f)) + \int_0^{t_f} \left(\tilde{x}^T Q \tilde{x} + \frac{dV(\tilde{x})}{dt} \right) dt \right] \\ &= \min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} E\left[V(\tilde{x}(0)) - V(\tilde{x}(t_f)) + \int_0^{t_f} \left(\tilde{x}^T Q \tilde{x} + \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right)^T f(\tilde{x} + x_d, k, \gamma) + \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right)^T v \right) dt \right] \end{aligned}$$

By the fact that $2a^T b \le a^T a + b^T b$ for any two-vectors a and b, we get

$$\begin{aligned} \min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} E\left[\int_0^{t_f} \tilde{x}^T Q \tilde{x} dt \right] = \min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} E\Bigg[& V(\tilde{x}(0)) - V(\tilde{x}(t_f)) + \int_0^{t_f} \Bigg(\tilde{x}^T Q \tilde{x} + \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right)^T f(\tilde{x} + x_d, k, \gamma) \\ & + \frac{1}{2} \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right)^T \left(\frac{\partial V(\tilde{x})}{\partial \tilde{x}} \right) + \frac{1}{2} v^T v \Bigg) dt \Bigg] \end{aligned}$$

By the inequality in (25), we get the sub-optimal regulation problem as follows

$$\min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} E\left[\int_0^{t_f} \tilde{x}^T Q \tilde{x} dt \right] \le \min_{\substack{k\in[k_1,k_2] \\ \gamma\in[\gamma_1,\gamma_2]}} E\left[V(\tilde{x}(0)) + \frac{1}{2} \int_0^{t_f} v^T v dt \right]$$

Since disturbance v is independent of the choice of parameters k and γ, and only the choice of $V(\tilde{x})$ will influence the above minimization, the sub-optimal design becomes how to solve the constrained optimization problem in (24) and (25).

6.4 Appendix D: Parameters of the T-S fuzzy model with the specified kinetic parameters k^* and decay rates γ^*

$$\mathbf{A}_1 = \begin{bmatrix} -1.6879 & -0.060601 & 0.11879 & -0.0092833 \\ 0.38914 & -0.093297 & 0.010249 & -0.0065119 \\ 0.10826 & -0.02841 & -1.4996 & -0.0060343 \\ 0.00097167 & -0.0025457 & 0.0053402 & -0.066832 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -3.5629 & -0.12704 & 0.25074 & -0.0092833 \\ 0.20138 & -0.193 & 0.021476 & -0.0065109 \\ 0.22906 & -0.11458 & -3.1644 & -0.0060447 \\ 0.0014069 & -0.00054073 & 0.0071284 & -0.066833 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} -1.5351 & -0.060529 & 0.11879 & -0.0092832 \\ 0.40408 & -0.092851 & 0.010249 & -0.0065573 \\ -0.18285 & 0.0322 & -1.4996 & 0.0041303 \\ 0.0012516 & -0.0027325 & -0.0017594 & -0.066801 \end{bmatrix}$$

$$\mathbf{A}_4 = \begin{bmatrix} -3.2403 & -0.12689 & 0.25074 & -0.0092832 \\ 0.23298 & -0.19466 & 0.021741 & -0.0065573 \\ -0.38598 & 0.039126 & -3.1671 & 0.0041304 \\ 0.0019632 & 0.00067731 & -0.00013562 & -0.066801 \end{bmatrix}$$

$$\mathbf{A}_5 = \begin{bmatrix} -3.5287 & -0.060601 & 0.24784 & -0.0093278 \\ 0.19497 & -0.093286 & 0.017006 & 0.0019312 \\ 0.22212 & -0.080273 & -3.1614 & -0.0042428 \\ 0.001744 & -0.0025529 & 0.0072707 & -0.067233 \end{bmatrix}$$

$$\mathbf{A}_6 = \begin{bmatrix} -7.4489 & -0.12704 & 0.52318 & -0.0093278 \\ -0.18351 & -0.1982 & 0.095548 & 0.0014778 \\ 0.21344 & -0.11298 & -7.2861 & 0.00040939 \\ -0.012439 & 0.0026832 & -0.025864 & -0.066952 \end{bmatrix}$$

$$\mathbf{A}_7 = \begin{bmatrix} -3.2061 & -0.060529 & 0.24784 & -0.0093277 \\ 0.22649 & -0.092851 & 0.01727 & 0.0018016 \\ -0.38483 & -0.019544 & -3.1642 & 0.0068314 \\ 0.0023517 & -0.0027325 & 6.7334\text{e-}006 & -0.067149 \end{bmatrix}$$

$$\mathbf{A}_8 = \begin{bmatrix} -6.768 & -0.12689 & 0.52318 & -0.0093277 \\ -0.14191 & -0.19465 & -0.023172 & 0.0018026 \\ -0.81178 & -0.012738 & -6.0679 & 0.0068211 \\ 0.0043172 & 0.00067013 & 0.040657 & -0.06715 \end{bmatrix}$$

$$\mathbf{A}_9 = \begin{bmatrix} -1.6879 & 0.12793 & -0.25078 & 0.019598 \\ -0.727 & -0.07319 & -0.026022 & -0.003619 \\ 0.10826 & -0.031432 & -0.80806 & -0.005567 \\ 0.00097182 & -0.0027504 & 0.0047284 & -0.066801 \end{bmatrix}$$

$$\mathbf{A}_{10} = \begin{bmatrix} -3.5629 & 0.26819 & -0.52934 & 0.019598 \\ -0.91465 & -0.15344 & -0.05495 & -0.003619 \\ 0.22793 & -0.094274 & -1.7058 & -0.005567 \\ 0.0013385 & 0.00063963 & 0.0057541 & -0.066801 \end{bmatrix}$$

$$\mathbf{A}_{11} = \begin{bmatrix} -1.5351 & 0.12778 & -0.25078 & 0.019598 \\ -0.71206 & -0.073303 & -0.026022 & -0.0036189 \\ -0.18285 & 0.034225 & -0.80806 & 0.0041294 \\ 0.0012516 & -0.0026058 & -0.0023716 & -0.066797 \end{bmatrix}$$

$$\mathbf{A}_{12} = \begin{bmatrix} -3.2403 & 0.26787 & -0.52934 & 0.019598 \\ -0.88316 & -0.15367 & -0.054951 & -0.0036189 \\ -0.38597 & 0.043382 & -1.7058 & 0.0041294 \\ 0.0019634 & 0.00094337 & -0.0013455 & -0.066797 \end{bmatrix}$$

$$\mathbf{A}_{13} = \begin{bmatrix} -3.5287 & 0.12793 & -0.52322 & 0.019692 \\ -0.92106 & -0.07319 & -0.058507 & 0.0047537 \\ 0.22099 & -0.083177 & -1.7026 & -0.0029125 \\ 0.0016756 & -0.0027503 & 0.0059171 & -0.067149 \end{bmatrix}$$

$$\mathbf{A}_{14} = \begin{bmatrix} -7.4489 & 0.26819 & -1.1045 & 0.019692 \\ -1.3492 & -0.15343 & -0.12363 & 0.0047547 \\ 0.72194 & -0.14614 & -3.593 & -0.0029229 \\ 0.018292 & 0.00063245 & 0.0083449 & -0.06715 \end{bmatrix}$$

$$\mathbf{A}_{15} = \begin{bmatrix} -3.2061 & 0.12778 & -0.52322 & 0.019692 \\ -0.88965 & -0.073303 & -0.058507 & 0.0047076 \\ -0.38483 & -0.01752 & -1.7026 & 0.0073033 \\ 0.0023519 & -0.0026058 & -0.0011826 & -0.067117 \end{bmatrix}$$

$$\mathbf{A}_{16} = \begin{bmatrix} -6.768 & 0.26787 & -1.1045 & 0.019692 \\ -1.2579 & -0.15367 & -0.12336 & 0.0047076 \\ -0.81291 & -0.0083629 & -3.5957 & 0.0073033 \\ 0.0042487 & 0.00094338 & 0.0010809 & -0.067117 \end{bmatrix}$$

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