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Time-Delay Systems

Edited by Dragutin Debeljkovic

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Meet the editor

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Preface

The problem of investigation of time delay systems has been explored over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

So, the title of the book ***Time-Delay Systems*** encompasses broad field of theory and application of many different control applications applied to different classes of aforementioned systems.

It must be admitted that a strong stress, in this monograph, is put on the historical significance of systems stability and in that sense, problems of asymptotic, exponential, non-Lyapunov and technical stability deserved a great attention. Moreover, an evident contribution was given with introductory *chapter* dealing with basic problem of Quasi-polyinomial stability.

Time delay systems can achieve different attributes. Namely, when we speak about singular or descriptor systems, one must have in mind that with some systems we must consider their character of dynamic and static state at the same time. Singular systems (also referred to as degenerate, descriptor, generalized, differential-algebraic systems or semi–state) are systems with dynamics, governed by the mixture of algebraic and - differential equations. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

It must be emphasized that there are lot of systems that show the phenomena of time delay and singularity *simultaneously*, and we call such systems *singular differential systems with time delay*. These systems have many special characteristics. If we want to describe them more exactly, to design them more accurately and to control them more effectively, we must tremendously endeavor to investigate them, but that is obviously very difficult work. When we consider time delay systems in general, within the existing stability criteria, two main ways of approach are adopted. Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria and generally provides simple algebraic conditions. In that sense, the question of their stability deserves great attention.

In the second and third *chapter* authors discuss such systems and some significant consequences, discussing their Lyapunov and non-Lyapunov stability characteristics.

Exponential stability of uncertain switched systems with time-varying delay and actual problems of stabilization and determining of stability characteristics of steady-state regimes are among the central issues in the control theory. Difficulties can be especially met when dealing with the systems containing nonlinearities which are non-analytic function of phase with problems that have been treated in two following *chapters*.

Some of synthesis problems have been discussed in the following *chapters* covering problems such as: static output-feedback stabilization of interval time delay systems, controllers design, decentralized adaptive stabilization for large-scale systems with unknown time-delay and resilient adaptive control of uncertain time-delay systems.

Finally, actual problems with some practical implementation and dealing with sliding mode control, synchronization of multiple time delay systems and T-S fuzzy H_∞ tracking control of input delayed robotic manipulators, were presented in last three *chapters*, including inevitable application of linear matrix inequalities.

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1

Introduction to Stability of Quasipolynomials

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1. Introduction

In this Chapter we shall consider a generalization of Hermite-Biehler Theorem[1] given by Pontryagin in the paper Pontryagin (1955). It should be understood that Pontryagin's generalization is a very relevant formal *tool* for the mathematical analysis of *stability* of quasipolynomials. Thus, from this point of view, the determination of the zeros of a quasipolynomial by means of Pontryagin's Theorem can be considered to be a mathematical method for analysis of stabilization of a class of linear time invariant systems with time delay. Section 2 contains an overview of the representation of entire functions as an infinite product by way of Weierstrass' Theorem—as well as Hadamard's Theorem. Section 3 is devoted to an exposition to the Theory of Quasipolynomials via Pontryagin's Theorem in addition to a generalization of Hermite-Biehler Theorem. Section 4 deals with applications of Pontryagin's Theorem to analysis of stabilization for a class of linear time invariant systems with time delays.

2. Representation of the entire functions by means of infinite products

In this Section we will present the mathematical background with respect to the Theory of Complex Analysis and to provide the necessary tools for studying the Hermite-Biehler Theorem and Pontryagin's Theorems. At the first let us introduce the basic definitions and general results used in the representation of the entire functions as infinite products[2].

2.1 Preliminaries

Definition 1. *(Zeros of analytic functions) Let* $f : \Omega \longrightarrow \mathbb{C}$ *be an analytic function in a region* Ω*—i.e., a nonempty open connected subset of the complex plane. A value* $\alpha \in \Omega$ *is called a zero of* f *with multiplicity (or order)* $m \geq 1$ *if, and only if, there is an analytic function* $g : \Omega \longrightarrow \mathbb{C}$ *such that* $f(z) = (z-\alpha)^m g(z)$*, where* $g(\alpha) \neq 0$*. A zero of order one* $(m = 1)$ *is called a simple zero.*

Definition 2. *(Isolated singularity) Let* $f : \Omega \longrightarrow \mathbb{C}$ *be an analytic function in a region* Ω*. A value* $\beta \in \Omega$ *is called a isolated singularity of* f *if, and only if, there exists* $R > 0$ *such that* f *is analytic in* $\{z \in \mathbb{C}: 0 < |z-\beta| < R\}$ *but not in* $B(\beta, R) = \{z \in \mathbb{C}: |z-\beta| < R\}$*.*

[1] See Levin (1964) for an analytical treatment about the Hermite-Biehler Theorem and a generalization of this theorem to arbitrary entire functions in an alternative way of the Pontryagin's method.
[2] See Ahlfors (1953) and Titchmarsh (1939) for a detailed exposition.

Definition 3. *(Pole) Let Ω be a region. A value $\beta \in \Omega$ is called a pole of analytic function $f : \Omega \longrightarrow \mathbb{C}$ if, and only if, β is a isolated singularity of f and $\lim_{z \longrightarrow \beta} |f(z)| = \infty$.*

Definition 4. *(Pole of order m) Let $\beta \in \Omega$ be a pole of analytic function $f : \Omega \longrightarrow \mathbb{C}$. We say that β is a pole of order $m \geq 1$ of f if, and only if, $f(z) = \dfrac{A_1}{z-\beta} + \dfrac{A_2}{(z-\beta)^2} + \ldots + \dfrac{A_m}{(z-\beta)^m} + g_1(z)$, where g_1 is analytic in $B(\beta, R)$ and $A_1, A_2, \ldots, A_m \in \mathbb{C}$ with $A_m \neq 0$.*

Definition 5. *(Uniform convergence of infinite products) The infinite product*

$$\prod_{n=1}^{+\infty} (1 + f_n(z)) = (1 + f_1(z))(1 + f_2(z)) \ldots (1 + f_n(z)) \ldots \tag{1}$$

where $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of functions of one variable, real or complex, is said to be uniformly convergent if the sequence of partial product ρ_n defined by

$$\rho_n(z) = \prod_{m=1}^{n} (1 + f_m(z)) = (1 + f_1(z))(1 + f_2(z)) \ldots (1 + f_n(z)) \tag{2}$$

converges uniformly in a certain region of values of z to a limit which is never zero.

Theorem 1. *The infinite product (1) is uniformly convergent in any region where the series $\sum_{n=1}^{+\infty} |f_n(z)|$ is uniformly convergent.*

Definition 6. *(Entire function) A function which is analytic in whole complex plane is said to be entire function.*

2.2 Factorization of the entire functions

In this subsection, it will be discussed an important problem in theory of entire functions, namely, the problem of the *decomposition* of an entire function—under the form of an *infinite* product of its zeros—in pursuit of the mathematical basis in order to explain the distribution of the zeros of quasipolynomials.

2.2.1 The problem of factorization of an entire function

Let $P(z) = a_n z^n + \ldots + a_1 z + a_0$ be a polynomial of degree n, $(a_n \neq 0)$. It follows of the Fundamental Theorem of Algebra that $P(z)$ can be decomposed as a *finite* product of the following form: $P(z) = a_n(z - \alpha_1) \ldots (z - \alpha_n)$, where the $\alpha_1, \alpha_2, \ldots, \alpha_n$ are—not necessarily distinct—zeros of $P(z)$. If exactly k_j of the α_j coincide, then the α_j is called a zero of $P(z)$ of *order* k_j [see Definition (1)]. Furthermore, the factorization is *uniquely* determined except for the order of the factors. Remark that we can also find an equivalent form of a polynomial function with a *finite* product of its zeros, more precisely, $P(z) = Cz^m \prod_{j=1}^{N} (1 - \frac{z}{\alpha_j})$, where $C = a_n \prod_{j=1}^{N} (-\alpha_j)$, m is the multiplicity of the zero at the origin, $\alpha_j \neq 0 (j = 1, \ldots, N)$ and $m + N = n$.

We can generalize the problem of factorization of the polynomial function for any entire function expressed likewise as an *infinite* product of its zeros.
Let's supposed that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} (1 - \frac{z}{\alpha_n}) \tag{3}$$

where $g(z)$ is an entire function. Hence, the problem can be established in following way: *the representation (3) should be valid if the infinite product converges uniformly on every compact set* [see Definition (5)].

2.2.2 Weierstrass factorization theorem

The problem characterized above was completely resolved by Weierstrass in 1876. As matter of fact, we have the following definitions and theorems.

Definition 7. *(Elementary factors) We can to take*

$$\begin{aligned} E_0(z) &= 1 - z, \ \textit{and} & (4) \\ E_p(z) &= (1-z)\ \exp(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}), \ \textit{for all} \ \ p = 1, 2, 3, \ldots & (5) \end{aligned}$$

These functions are called elementary factors.

Lemma 1. *If* $|z| \leq 1$ *, then* $|1 - E_p(z)| \leq |z|^{p+1}$*, for* $p = 1, 2, 3, ...$

Theorem 2. *Let* $\{\alpha_n\}_{n \in \mathbb{N}}$ *be a sequence of complex numbers such that* $\alpha_n \neq 0$ *and* $\lim_{n \longrightarrow +\infty} |\alpha_n| = \infty$*. If* $\{p_n\}_{n \in \mathbb{N}}$ *is a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} (\frac{r}{r_n})^{1+p_n} < \infty, \ \textit{where} \ |\alpha_n| = r_n, \tag{6}$$

for every positive r, then the infinite product

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{\alpha_n}) \tag{7}$$

define an entire function f which has a zero at each point α_n*,* $n \in \mathbb{N}$*, and has no other zeros in the complex plane.*

Remark 1. *The condition (6) is always satisfied if* $p_n = n - 1$*. Indeed, for every r, it follows that* $r_n > 2r$ *for all* $n > n_0$*, since* $\lim_{n \longrightarrow +\infty} r_n = \infty$*. Therefore,* $\frac{r}{r_n} < \frac{1}{2}$ *for all* $n > n_0$*, then (6) is valid with respect to* $1 + p_n = n$*.*

Theorem 3. *(Weierstrass Factorization Theorem) Let f be an entire function. Suppose that* $f(0) \neq 0$*, and let* $\alpha_1, \alpha_2, \ldots$ *be the zeros of f, listed according to their multiplicities. Then there exist an entire function g and a sequence* $\{p_n\}_{n \in \mathbb{N}}$ *of nonnegative integers, such that*

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}(\frac{z}{\alpha_n}) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\alpha_n}\right) e^{\left[\frac{z}{\alpha_n} + \frac{1}{2}(\frac{z}{\alpha_n})^2 + \ldots + \frac{1}{n-1}(\frac{z}{\alpha_n})^{n-1}\right]} \tag{8}$$

Notice that, by convention, with respect to $n = 1$ *the first factor of the infinite product should be* $1 - \dfrac{1}{\alpha_1}$.

Remark 2. *If* f *has a zero of multiplicity* m *at* $z = 0$*, the Theorem (3) can be apply to the function* $\dfrac{f(z)}{z^m}$.

Remark 3. *The decomposition (8) is not unique.*

Remark 4. *In the Theorem (3), if the sequence* $\{p_n\}_{n\in\mathbb{N}}$ *of nonnegative integers is constant, i.e.,* $p_n = \rho$ *for all* $n \in \mathbb{N}$*, then the following infinite product:*

$$e^{g(z)} \prod_{n=1}^{\infty} E_\rho\left(\frac{z}{\alpha_n}\right) \tag{9}$$

converges and represents an entire function provided that the series $\dfrac{1}{\rho+1}\displaystyle\sum_{n=1}^{\infty}\left(\frac{R}{|\alpha_n|}\right)^{\rho+1}$ *converges for all* $R > 0$*. Suppose that* ρ *is the smallest integer for which the series* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{|\alpha_n|^{\rho+1}}$ *converges. In this case, the expression (9) is denominated the* ***canonical product*** *associated with the sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *and* ρ *is the* ***genus*** *of the canonical product* [3].

With reference to the Remark (4) we can state:

Hadamard Factorization Theorem. *If* f *is an entire function of finite order* ϑ*, then it admits a factorization of the following manner:* $f(z) = z^m e^{g(z)} \displaystyle\prod_{n=1}^{\infty} E_p\left(\frac{z}{\alpha_n}\right)$*, where* $g(z)$ *is a polynomial function of degree* q*, and* $\max\{p, q\} \leq \vartheta$.

The first example of infinite product representation was given by Euler in 1748, viz., $\sin(\pi z) = \pi z \displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$, where $m = 1$, $p = 1$, $q = 0$ $[g(z) \equiv 0]$, and $\vartheta = 1$.

3. Zeros of quasipolynomials due to Pontryagin's theorem

We know that, under the analytic standpoint and a geometric criterion, results concerning the existence and localization of zeros of entire functions like exponential polynomials have received a considerable attention in the area of research in the automation field. In this section the Pontryagin theory is outlined.

Consider the linear difference-differential equation of differential order n and difference order m defined by

$$\sum_{\mu=0}^{n}\sum_{\nu=0}^{m} a_{\mu\nu} x^{(\mu)}(t+\nu) = 0 \tag{10}$$

[3] See Boas (1954) for analysis of the problem about the connection between the growth of an entire function and the distribution of its zeros.

where m and n are positive integers and $a_{\mu\nu}(\mu = 0, \dots, n, \nu = 0, \dots, m)$ are real numbers. The characteristic function associated to (10) is given by:

$$\delta(z) = P(z, e^z), \tag{11}$$

where $P(z,s) = \sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} z^\mu s^\nu$ is a polynomial in two variables.
Pontryagin's Theorem, in fact, establishes necessary and sufficient conditions such that the real part of all zeros in (11) may be negative. These conditions transform the problem a real variable one.

Definition 8. *(Quasipolynomials)*[4] *We call the quasipolynomials or exponential polynomials the entire functions of the form:*

$$F(z) = \sum_{\xi=0}^{m} f_\xi(z) e^{\lambda_\xi z} = f_0(z)e^{\lambda_0 z} + f_1(z)e^{\lambda_1 z} + \dots + f_m(z)e^{\lambda_m z} \tag{12}$$

where $f_\xi(\xi = 0, \dots, m)$ are polynomial functions with real (or complex) coefficients, and $\lambda_\xi(\xi = 0, \dots, m)$ are real (or complex) numbers. In particular, if the $\lambda_\xi(\xi = 0, \dots, m)$ are commensurable real numbers and $0 = \lambda_0 < \lambda_1, \dots < \lambda_m$, then the quasipolynomial (12) can be written in the form (11) studied by Pontryagin.

Notice that, some trigonometric functions, e.g., sin and cos are quasipolynomials since $\sin(mz) = \frac{1}{2j}e^{jmz} - \frac{1}{2j}e^{-jmz}$ and $\cos(nz) = \frac{1}{2}e^{jnz} + \frac{1}{2}e^{-jnz}$, where $j = \sqrt{-1}$, and $m, n \in \mathbb{N}$.

Remark 5. *If the quasipolynomial $F(z)$ in (12) does not degenerate into a polynomial, then the quasipolynomial $F(z)$ has an **infinite** set of zeros whose unique limit point is infinite. Note that all roots of $F(z)$ are separated from one another by more than some distance $d > 0$, therefore it is possible to determine non-intersecting circles of radius $r < d$ encircling all the roots taken as centers.*

Definition 9. *(Hurwitz Stable) The quasipolynomial $F(z)$ in (12) is said to be a Hurwitz stable if, and only if, all its roots lie in the open left-half of the complex plane.*

Definition 10. *(Interlacing Property) Let $f(\omega)$ and $g(\omega)$ be two real functions of a real variable. The zeros of these two functions interlace (or alternate) if, and only if, we have the following conditions:*

1. *each of the functions has only simple zeros [see Definition1];*
2. *between every two zeros of one of these functions there exists one and only one zero of the other;*
3. *the functions $f(\omega)$ and $g(\omega)$ have no common zeros.*

We cannot refrain from remark that Cebotarev, in 1942, gave the generalization of the Sturm algorithm to quasipolynomials, therefore we have a general principle for solving that problem for arbitrary quasipolynomials. Notwithstanding, it is of interest to note that Chebotarev's result presuppose a generalization of the Hermite-Biehler Theorem to quasipolynomials.

[4] See Pontryagin (1969) for a discussion detailed.

Theorem 4. *(Pontryagin's Theorem) Pontryagin (1955) Let $\delta(z) = P(z, e^z)$ be a quasipolynomial, where $P(z,s)$ is a polynomial function in two variables with real coefficients as defined in (11). Suppose that $a_{nm} \neq 0$. Let $\delta(j\omega)$ be the restriction of the quasipolynomial $\delta(z)$ to imaginary axis. We can express $\delta(j\omega) = f(\omega) + jg(\omega)$, where the real functions (of a real variable) $f(\omega)$ and $g(\omega)$ are the real and imaginary parts of $\delta(j\omega)$, respectively. Let us denote by ω_r and ω_i, respectively, the zeros of the function $f(\omega)$ and $g(\omega)$. If all the zeros of the quasipolynomial $\delta(z)$ lie to the left side of the imaginary axis, then the zeros of the functions $f(\omega)$ and $g(\omega)$ are real, alternating, and*

$$g'(\omega)f(\omega) - g(\omega)f'(\omega) > 0. \tag{13}$$

for each $\omega \in \mathbb{R}$. Reciprocally, if one of the following conditions is satisfied:

1. *All the zeros of the functions $f(\omega)$ and $g(\omega)$ are real and alternate and the inequality (13) is satisfied for at least one value ω;*
2. *All the zeros of the function $f(\omega)$ are real , and for each zero of $f(\omega)$ the inequality (13) is satisfied, that is, $g(\omega_r)f'(\omega_r) < 0$;*
3. *All the zeros of the function $g(\omega)$ are real, and for each zero of $g(\omega)$ the inequality (13) is satisfied, that is, $g'(\omega_i)f(\omega_i) > 0$;*

then all the zeros of the quasipolynomial $\delta(z)$ lie to the left side of the imaginary axis.

Remark 6. *Let us note that the above function $\delta(j\omega)$ in Theorem (4) has, also, the following form:*

$$\delta(j\omega) = \sum_{\mu=0}^{n}\sum_{\nu=0}^{m} a_{\mu\nu}\omega^{\mu}\left[\sum_{\rho=0}^{\nu}(j)^{\mu+\nu-\rho}\frac{\nu!}{\rho!(\nu-\rho)!}(\cos\omega)^{\rho}(\sin\omega)^{\nu-\rho}\right]. \tag{14}$$

Consequently, the functions $f(\omega)$ and $g(\omega)$ can be express as $Q(\omega) = q(\omega, \cos(\omega), \sin(\omega))$, where $q(\omega, u, v)$ is a real polynomial function in three variables with real coefficients.

With respect to the Remark (6), it should be pointed out, the polynomial $q(\omega, u, v)$ may be represented in the form:

$$q(\omega, u, v) = \sum_{\mu=0}^{n}\sum_{\nu=0}^{m}\omega^{\mu}\phi_{\mu}^{(\nu)}(u, v), \tag{15}$$

where $\phi_{\mu}^{(\nu)}(u, v)$ is a real homogeneous polynomial of degree ν in two real variables u and v. The formula $\omega^n\phi_n^{(m)}(u, v)$ is denominated the *principal term* of the polynomial in (15). From (15), we can define $\phi_n^*(u, v)$ as follows

$$\phi_n^*(u, v) = \sum_{\nu=0}^{m}\phi_n^{(\nu)}(u, v). \tag{16}$$

And by substituting $u = \cos(\omega)$ and $v = \sin(\omega)$ in (16) we can express

$$\Phi_n^*(\omega) = \phi_n^*(\cos(\omega), \sin(\omega)). \tag{17}$$

Now, let us consider the above formalization in *complex* field, that is, $\Phi_n^*(z) = \phi_n^*(\cos(z), \sin(z))$, where $z \in \mathbb{C}$.

Theorem 5. *Pontryagin (1955) Let $q(z,u,v)$ be a polynomial function, as given in (15), with three complex variables and real coefficients, in which the principal term is $z^n\phi_n^{(m)}(u,v)$. If ϵ is such that $\Phi_n^*(\epsilon + j\varrho)$ does not take the value zero for every real ϱ, then the function $Q(\omega + j\varrho)$ has exactly $4kn + m$ zeros—for some sufficiently large value of k— for $(\omega,\varrho) \in [-2k\pi + \epsilon, 2k\pi + \epsilon] \times \mathbb{R}$. Hence, in order that the restriction of the function Q to $\mathbb{R}$, denoted by $Q(\omega)$, have only real roots, it is necessary and sufficient that $Q(\omega)$ have exactly $4kn + m$ zeros in the interval $-2k\pi + \epsilon \leq \omega \leq 2k\pi + \epsilon$ for sufficiently large k.*

4. Applications of Pontryagin's theorem to analysis of stabilization for a class of linear time invariant systems with time delay

In this Section we will explain some relevant applications concerning the Hermite-Biehler Theorem and Pontryagin's Theorems in the framework of *Control Theory*. Apropos to the several methodological approaches about the subject of the **Section 3**, we have in technical literature some significant publications, viz., Bellman & Cooke (1963), Bhattacharyya et al. (2009) and Oliveira et al. (2009). These methods constitute a set of analytic tools for mathematical modeling and general criteria for studying of *stability* of the dynamic systems with time delays, that is, for setting a characterization of all stabilizing P, PI or PID controllers for a given plant. It should be pointed out that the definition of the formal concept of *signature*—introduced in the reference Oliveira et al. (2003)—allows to extend results of the polynomial case to quasipolynomial case via property of interlacing in high frequencies of the class of time delay systems considered [5].

The dynamic behavior of many industrial plants may be mathematically modeled by a linear time invariant system with time delay. The problem of stability of linear time invariant systems with time delay make necessary a method for localization of the roots of analytic functions. These systems are described by the linear differential equations with constant coefficients and constant delays of the argument of the following manner

$$\sum_{\mu=0}^{n}\sum_{\nu=0}^{m} a_{\mu\nu}u^{(\mu)}(t-\tau_\nu) = h(t) \tag{18}$$

where the coefficients are denoted by $a_{\mu\nu} \in \mathbb{R}(\mu = 0,\ldots,n, \nu = 0,\ldots,m)$ and the constant delays are symbolized by $\tau_\nu \in \mathbb{R}(\nu = 0,\ldots,m)$ with $0 = \tau_0 < \tau_1,\ldots < \tau_m$.

[5] The Hermite-Biehler Theorem provides necessary and sufficient conditions for Hurwitz stability of real polynomials in terms of an interlacing property. Notice that, if a given real polynomial is not Hurwitz, the Hermite-Biehler Theorem does not provide information on its roots distribution. A generalization of Hermite-Biehler Theorem with respect to real polynomials was first derived in a report by Özgüler & Koçan (1994) in which was given a formula for a signature of polynomial—not necessarily Hurwitz—applicable to real polynomials without zeros on the imaginary axis except possibly a single root at the origin. This formula was employed to solve the constant gain stabilization problem. It may be mentioned that, in Ho et al. (1999), a different formula applicable to arbitrary real polynomials—but without restrictions on root localizations—was derived and used in the problem of stabilizing PID controllers. In addition, as a result of Ho et al. (2000), a generalization of the Hermite-Biehler Theorem for real polynomials—not necessarly Hurwitz—to the polynomials with complex coefficients was derived and, as a consequence of that extension, we have a technical application to a problem of stabilization in area of Control Theory.

We can denominate the equation (18) as an equation with ***delayed*** argument, if the coefficient $a_{n0} \neq 0$ and the remaining coefficients $a_{n\nu} = 0(\nu = 1, \ldots, m)$, that is, $a_{n0}u^{(n)}(t) + \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{m} a_{\mu\nu} u^{(\mu)}(t - \tau_\nu) = h(t)$; analogously, the equation (18) is denominated an equation with ***advanced*** argument, if the coefficient $a_{n0} = 0$ and, if only for one $\nu > 0$, $a_{n\nu} \neq 0$, that is, $a_{n\nu_0}u^{(n)}(t - \tau_{\nu_0}) + \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{m} a_{\mu\nu} u^{(\mu)}(t - \tau_\nu) = h(t)$, for only one $\nu_0 \in \{1, \ldots, m\}$ and, finally, the equation (18) is denominated an equation of ***neutral*** type, if the coefficient $a_{n0} \neq 0$ and, if only for one $\nu > 0$, $a_{n\nu} \neq 0$, that is, $a_{n0}u^{(n)}(t) + a_{n\nu_0}u^{(n)}(t - \tau_{\nu_0}) + \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{m} a_{\mu\nu} u^{(\mu)}(t - \tau_\nu) = h(t)$, for only one $\nu_0 \in \{1, \ldots, m\}$.

Let us consider $h(t) = 0$ in equation (18), we obtain the *homogeneous linear* equation with constant coefficients and constant delays of the argument like

$$\sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} u^{(\mu)}(t - \tau_\nu) = 0. \tag{19}$$

Assuming that $u(t) = e^{zt}$, where z denotes a complex constant, is a particular solution of the equation (19) and, by substituting in (19) we obtain the so-called *characteristic* equation

$$\sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} z^{\mu} e^{-\tau_\nu z} = 0. \tag{20}$$

From the equation (20) we can define the *characteristic quasipolynomial* in the following form

$$\delta^*(z) = \sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} z^{\mu} e^{-\tau_\nu z}. \tag{21}$$

Note that the equation (20) has an infinite set of roots, therefore to every root z_k corresponds a solution $u(t) = e^{z_k t}$ of the equation (19). And, if the sums of infinite series $\sum_{k=0}^{\infty} C_k e^{z_k t}$ of solutions converge and admit $n-$ fold term-by-term differentiation, then those sums are also solutions of the equation (19).
Multiplying the equation (21) by $e^{\tau_m z}$, it follows that

$$\delta(z) = e^{\tau_m z}\delta^*(z) = \sum_{\mu=0}^{n} \sum_{\nu=0}^{m} a_{\mu\nu} z^{\mu} e^{(\tau_m - \tau_\nu)z} = \sum_{\nu=0}^{m} p_\nu(z) e^{(\tau_m - \tau_\nu)z}, \tag{22}$$

where $p_\nu(z) = \sum_{\mu=0}^{n} a_{\mu\nu} z^{\mu} (\nu = 0, \ldots, m)$. For $m \neq 0$, the function (22) belongs to a general class of quasipolynomials [see Definition (8)]. It is evident that $\delta(z) = e^{\tau_m z}\delta^*(z)$ and $\delta^*(z)$ have the same zeros [6]. Thus, from this point of view, the zeros of the function $\delta(z)$ can be obtained from the Theorems (4) and (5).

[6] see El'sgol'ts (1966) for a fully discussion.

Now, consider a special class of quasipolynomials (with *one* delay) given by

$$\delta^*(z) = p_0(z) + e^{-Lz} p_1(z), \tag{23}$$

where $p_0(z) = z^n + \sum_{\mu=0}^{n-1} a_{\mu 0} z^\mu$ with $a_{\mu 0} \in \mathbb{R} (\mu = 0, \ldots, n-1)$, $p_1(z) = \sum_{\mu=0}^{n} a_{\mu 1} z^\mu$ with $a_{\mu 1} \in \mathbb{R} (\mu = 0, \ldots, n)$ and $L > 0$. Multiplying the (23) by e^{Lz}, it follows that

$$\delta(z) = e^{Lz} \delta^*(z) = e^{Lz} p_0(z) + p_1(z). \tag{24}$$

We consider the following *Assumptions*:

Hypothesis 1. $\partial(p_1) < n$ *[retarded type]*

Hypothesis 2. $\partial(p_1) = n$ *and* $0 < |a_{n1}| < 1$ *[neutral type]*

where $\partial(p_1)$ stands for the degree of polynomial p_1. Notice that, Hypothesis (1) implies that $a_{n1} = 0$ and $a_{\mu 1} \neq 0$ for some $\mu = 0, \ldots, n-1$.
Firstly, in what follows, we will state the Lemma (2) and Hypothesis (3) to establish the definition of *signature* of the quasipolynomials.

Lemma 2. *Suppose a quasipolynomial of the form (24) given. Let $f(\omega)$ and $g(\omega)$ be the real and imaginary parts of $\delta(j\omega)$, respectively. Under Hypothesis (1) or (2), there exists $0 < \omega_0 < \infty$ such that in $[\omega_0, \infty)$ the functions $f(\omega)$ and $g(\omega)$ have only real roots and these roots interlace*[7].

Hypothesis 3. *Let $\eta_g + 1$ be the number of zeros of $g(\omega)$ and η_f be the number of zeros of $f(\omega)$ in $(0, \omega_1)$. Suppose that $\omega_1 \in \mathbb{R}^+$, $\eta_g, \eta_f \in \mathbb{N}$ are sufficiently large, such that the zeros of $f(\omega)$ and $g(\omega)$ in $[\omega_0, \infty)$ interlace (with $\omega_0 < \omega_1$). Therefore, if $\eta_f + \eta_g$ is even, then $\omega_0 = \omega_{g_{\eta_g}}$, where $\omega_{g_{\eta_g}}$ denotes the η_g-th (non-null) root of $g(\omega)$, otherwise $\omega_0 = \omega_{f_{\eta_f}}$, where $\omega_{f_{\eta_f}}$ denotes the η_f-th root of $f(\omega)$.*

Note that, the Lemma (2) establishes *only* the condition of *existence* for ω_0 such that $f(\omega)$ and $g(\omega)$ have only real roots and these roots interlace, by another hand the Hypothesis (3) has a *constructive* character, that is, it allows to calculate ω_0.

Definition 11. *(Signature of Quasipolynomials) Let $\delta(z)$ be a given quasipolynomial described as in (24) without real roots in imaginary axis. Under Hypothesis (3), let $0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_{\eta_g}} \leq \omega_0$ and $\omega_{f_1} < \ldots < \omega_{f_{\eta_f}} \leq \omega_0$ be real and distinct zeros of $g(\omega)$ and $f(\omega)$, respectively. Therefore, the signature of δ is defined by*

$$\sigma(\delta) = \begin{cases} \left\{ sgn[f(\omega_{g_0})] + 2\left(\sum_{k=1}^{\eta_g - 1} (-1)^k sgn[f(\omega_{g_k})] \right) + (-1)^{\eta_g} sgn[f(\omega_{g_{\eta_g}})] \right\} (-1)^{\eta_g - 1} sgn[g(\omega^+_{g_{\eta_g - 1}})], \\ \text{if } \eta_f + \eta_g \text{ is even;} \\ \left\{ sgn[f(\omega_{g_0})] + 2\left(\sum_{k=1}^{\eta_g} (-1)^k sgn[f(\omega_{g_k})] \right) \right\} (-1)^{\eta_g} sgn[g(\omega^+_{g_{\eta_g}})], \\ \text{if } \eta_f + \eta_g \text{ is odd;} \end{cases}$$

[7] The proof of Lemma (2) follows from Theorems (4) - (5); indeed, under Hypothesis (2) the roots of $\delta(z)$ go into the left hand complex plane for $|z|$ sufficiently large. A detailed proof can be find in Oliveira et al. (2003) and Oliveira et al. (2009).

where sgn is the standard signum function, $sgn[g(\omega_\lambda^+)]$ *stands for* $\lim_{\omega \longrightarrow \omega_\lambda^+} sgn[g(\omega)]$ *and* $\omega_\lambda, (\lambda = 0, \ldots, g_{\eta_g})$ *is the* λ*-th zero of* $g(\omega)$*.*

Now, by means of the Definition of *Signature* the following Lemma can be established.

Lemma 3. *Consider a Hurwitz stable quasipolynomial* $\delta(z)$ *described as in (24) under Hypothesis (1) or (2). Let* η_f *and* η_g *be given by Hypothesis (3). Then the signature for the quasipolynomial* $\delta(z)$ *is given by* $\sigma(\delta) = \eta_f + \eta_g$*.*

Referring to the feedback system with a *proportional controller* $C(z) = k_p$, the resulted quasipolynomial is given by:

$$\delta(z, k_p) = e^{Lz} p_0(z) + k_p p_1(z) \tag{25}$$

where $p_0(z)$ and $p_1(z)$ are given in (24). In the next Lemma we consider $\delta(z, k_p)$ under Hypothesis (1) or (2). Consequently, we obtain a frequency range signature for the quasipolynomial given by the product $\delta(z, k_p)p_1(-z)$ which is used to establish the subsequent Theorem with respect to the *stabilization* problem.

Lemma 4. *For any stabilizing* k_p*, let* $\eta_g + 1$ *and* η_f *be, respectively, the number of real and distinct zeros of imaginary and real parts of the quasipolynomial* $\delta(j\omega, k_p)$ *given in (25). Suppose* η_g *and* η_f *sufficiently large, it follows that* $\delta(j\omega, k_p)$ *is Hurwitz stable if, and only if, the signature for* $\delta(j\omega, k_p)p_1(-j\omega)$ *in* $[0, \omega_0]$ *with* ω_0 *as in Hypothesis (3), is given by* $\eta_g + \eta_f - \sigma(p_1)$*, where* $\sigma(p_1)$ *stands for the signature of the polynomial* p_1*.*

Definition 12. *(Set of strings) Let* $0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_k} \leq \omega_0$ *be real and distinct zeros of* $g(\omega)$*. Then the set of strings* $\mathcal{A}_k$ *in the range determined by frequency* ω_0 *is defined as*

$$\mathcal{A}_k = \{s_0, \ldots, s_k : s_0 \in \{-1, 0, 1\}; s_l \in \{-1, 1\}; l = 1, \ldots, k\} \tag{26}$$

with s_l *identified as* $sgn[f(\omega_{g_l})]$ *in the Definition (11).*

Theorem 6. *Let* $\delta(z, k_p)$ *be the quasipolynomial given in (25). Consider* $f(\omega, k_p) = f_1(\omega) + k_p f_2(\omega)$ *and* $g(\omega)$ *as the real and imaginary parts of the quasipolynomial* $\delta(j\omega, k_p)p_1(-j\omega)$*, respectively. Suppose there exists a stabilizing* k_p *of the quasipolynomial* $\delta(z, k_p)$*, and by taking* ω_0 *as given in Hypothesis (3) associated to the quasipolynomial* $\delta(z, k_p)$*. Let* $0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_\iota} \leq \omega_0$ *be the real and distinct zeros of* $g(\omega)$ *in* $[0, \omega_0]$*. Assume that the polynomial* $p_1(z)$ *has no zeros at the origin. Then the set of all* k_p*—denoted by* $\mathcal{I}$*—such that* $\delta(z, k_p)$ *is Hurwitz stable may be obtained using the signature of the quasipolynomial* $\delta(z, k_p)p_1(-z)$*. In addition, if* $\mathcal{I}_\iota = (\max_{s_t \in \mathcal{A}_\iota^+}[-\frac{1}{G(j\omega_{g_t})}], \min_{s_t \in \mathcal{A}_\iota^-}[-\frac{1}{G(j\omega_{g_t})}])$*, where* $\frac{1}{G(j\omega)} = \frac{f_1(\omega) - jg(\omega)}{f_2(\omega)}$*,* $\mathcal{A}_\iota$ *is a set of string as in Definition (12) ,* $\mathcal{A}_\iota^+ = \{s_t \in \mathcal{A}_\iota : s_t.sgn[f_2(\omega_{g_t})] = 1\}$ *and* $\mathcal{A}_\iota^- = \{s_t \in \mathcal{A}_\iota : s_t.sgn[f_2(\omega_{g_t})] = -1\}$*, such that* $\max_{s_t \in \mathcal{A}_\iota^+}[-\frac{1}{G(j\omega_{g_t})}] < \min_{s_t \in \mathcal{A}_\iota^-}[-\frac{1}{G(j\omega_{g_t})}]$*, then* $\mathcal{I} = \bigcup \mathcal{I}_\iota$*, with* ι *the number of feasible strings.*

4.1 Stabilization using a PID Controller

In the preceding section we take into account statements *introduced* in Oliveira et al. (2003), namely, Hypothesis (3), Definition (11), Lemma (2), Lemma (3), Lemma (4), and Theorem (6). Now, we shall regard a technical application of these results.

In this subsection we consider the *problem of stabilizing* a first order system with time delay using a PID controller. We will utilize the standard notations of Control Theory, namely, $G(z)$ stands for the plant to be controller and $C(z)$ stands for the PID controller to be designed. Let $G(z)$ be given by

$$G(z) = \frac{k}{1+Tz}e^{-Lz} \tag{27}$$

and $C(z)$ is given by

$$C(z) = k_p + \frac{k_i}{z} + k_d z,$$

where k_p is the proportional gain, k_i is the integral gain, and k_d is the derivative gain.

The *main* problem is to analytically determine the set of controller parameters (k_p, k_i, k_d) for which the closed-loop system is stable. The closed-loop characteristic equation of the system with PID controller is express by means of the quasipolynomial in the following *general* form

$$\delta(j\omega, k_p, k_i, k_d)p_1(-j\omega) = f(\omega, k_i, k_d) + jg(\omega, k_p) \tag{28}$$

where

$$f(\omega, k_i, k_d) = f_1(\omega) + (k_i - k_d\omega^2)f_2(\omega)$$
$$g(\omega, k_p) = g_1(\omega) + k_p g_2(\omega)$$

with

$f_1(\omega) = -\omega[\omega^2 p_0^o(-\omega^2)p_1^o(-\omega^2) + p_0^e(-\omega^2)p_1^e(-\omega^2)]\sin(L\omega) + \omega^2[\omega^2 p_1^o(-\omega^2)p_0^e(-\omega^2) - p_0^o(-\omega^2)p_1^e(-\omega^2)]\cos(L\omega)$

$f_2(\omega) = p_1^e(-\omega^2)p_1^e(-\omega^2) + \omega^2 p_1^o(-\omega^2)p_1^o(-\omega^2)$

$g_1(\omega) = \omega[\omega^2 p_0^o(-\omega^2)p_1^o(-\omega^2) + p_0^e(-\omega^2)p_1^e(-\omega^2)]\cos(L\omega) + \omega^2[\omega^2 p_1^o(-\omega^2)p_0^e(-\omega^2) - p_0^o(-\omega^2)p_1^e(-\omega^2)]\sin(L\omega)$

$g_2(\omega) = \omega f_2(\omega) = \omega[p_1^e(-\omega^2)p_1^e(-\omega^2) + \omega^2 p_1^o(-\omega^2)p_1^o(-\omega^2)]$

where p_0^e and p_0^o stand for the even and odd parts of the decomposition $p_0(\omega) = p_0^e(\omega^2) + \omega p_0^o(\omega^2)$, and analogously for $p_1(\omega) = p_1^e(\omega^2) + \omega p_1^o(\omega^2)$. Notice that for a fixed k_p the polynomial $g(\omega, k_p)$ does not depend on k_i and k_d, therefore we can obtain the stabilizing k_i and k_d values by solving a linear programming problem for each $g(\omega, k_d)$, which is establish in the next Lemma.

Lemma 5. *Consider a stabilizing set (k_p, k_i, k_d) for the quasipolynomial $\delta(j\omega, k_p, k_i, k_d)$ as given in (28). Let $\eta_g + 1$ and η_f be the number of real and distinct zeros, respectively, of the imaginary and real parts of $\delta(j\omega, k_p, k_i, k_d)$ in $[0, \omega_0]$, with a sufficiently large frequency ω_0 as given in the Hypothesis (3). Then, $\delta(j\omega, k_p, k_i, k_d)$ is stable if, and only if, for any stabilizing set (k_p, k_i, k_d) the signature of the* *quasipolynomial* $\delta(z, k_p, k_i, k_d)p_1(-z)$ *determined by the frequency* ω_0 *is given by* $\eta_g + \eta_f - \sigma(p_1)$, *where* $\sigma(p_1)$ *stands for the signature of the polynomial* p_1.

Finally, we make the standing statement to determine the range of stabilizing PID gains.

Theorem 7. *Consider the quasipolynomial* $\delta(j\omega, k_p, k_i, k_d)p_1(-j\omega)$ *as given in (28). Suppose there exists a stabilizing set* (k_p, k_i, k_d) *for a given plant* $G(z)$ *satisfying Hypothesis (1) or (2). Let* η_f, η_g *and* ω_0 *be associated to the quasipolynomial* $\delta(j\omega, k_p, k_i, k_d)$ *be choosen as in Hypothesis (3). For a fixed* k_p, *let* $0 = \omega_{g_0} < \omega_{g_1} < \ldots < \omega_{g_\iota} \leq \omega_0$ *be real and distinct zeros of* $g(\omega, k_p)$ *in the frequency range given by* ω_0. *Then, the* (k_i, k_d) *values—such that the quasipolynomial* $\delta(j\omega, k_p, k_i, k_d)$ *is stable—are obtained by solving the following linear programming problem:*

$$\begin{cases} f_1(\omega_{g_t}) + (k_i - k_d\omega_{g_t}^2)f_2(\omega_{g_t}) > 0, & \text{for } s_t = 1, \\ f_1(\omega_{g_t}) + (k_i - k_d\omega_{g_t}^2)f_2(\omega_{g_t}) < 0, & \text{for } s_t = -1; \end{cases}$$

with $s_t \in \mathcal{A}_\iota (t = 0, 1, \ldots, \iota)$ *and, such that the signature for the quasipolynomial* $\delta(j\omega, k_p, k_i, k_d)p_1(-j\omega)$ *equals* $\eta_g + \eta_f - \sigma(p_1)$, *where* $\sigma(p_1)$ *stands for the signature of the polynomial* p_1.

Now, we shall formulate an *algorithm* for PID controller by way of the above theorem. The algorithm[8] can be state in following form:

Step 1: Adopt a value for the set (k_p, k_i, k_d) to stabilize the given plant $G(z)$. Select η_f and η_g, and choose ω_0 as in the Hypothesis (3).

Step 2: Enter functions $f_1(\omega)$ and $g_1(\omega)$ as given in (28).

Step 3: In the frequency range determined by ω_0 find the zeros of $g(\omega, k_p)$ as defined in (28) for a fixed k_p.

Step 4: Using the Definition(11) for the quasipolynomial $\delta(z, k_p, k_i, k_d)p_1(-z)$, and find the strings $\mathcal{A}_\iota$ that satisfy $\sigma(\delta(z, k_p, k_i, k_d)p_1(-z)) = \eta_g + \eta_f - \sigma(p_1)$.

Step 5: Apply Theorem (7) to obtain the inequalities of the above *linear programming* problem.

5. Conclusion

In view of the following fact concerning the bibliographic references (in this Chapter): all the quasipolynomials have only one delay, it follows that we can express $\delta(z) = P(z, e^z)$ as in (24), where $P(z, s) = p_0(z)s + p_1(z)$ with $\partial(p_0) = 1$, $\partial(p_1) = 0$ and $\partial(p_0) = 2$, $\partial(p_1) = 1$ in Silva et al. (2000), $\partial(p_0) = 2$, $\partial(p_1) = 0$ in Silva et al. (2001), $\partial(p_0) = 2$, $\partial(p_1) = 2$ in Silva et al. (2002), $\partial(p_0) = 2$, $\partial(p_1) = 2$ in Capyrin (1948), $\partial(p_0) = 5$, $\partial(p_1) = 5$ in Capyrin (1953), and $\partial(p_0) = 1$, $\partial(p_1) = 0$ [Hayes' equation] and $\partial(p_0) = 2$, $\partial(p_1) = 0, 1, 2$ [particular cases] in Bellman & Cooke (1963), respectively. Similarly, in the cases studied in Oliveira et al. (2003) and Oliveira et al. (2009)—and described in this Chapter—the Hypothesis (3) and Definition (11) take into account Pontryagin's Theorem. In addition, if we have particularly the following form $F(z) = f_1(z)e^{\lambda_1 z} + f_2(z)e^{\lambda_2 z}$, with $\lambda_1, \lambda_2 \in \mathbb{R}$ (noncommensurable) and $0 < \lambda_1 < \lambda_2$, we can write $F(z) = e^{\lambda_1 z}\delta(z)$, where $\delta(z) = f_1(z) + f_2(z)e^{(\lambda_2 - \lambda_1)z}$ with $\partial(f_2) > \partial(f_1)$, therefore $\delta(z)$ can be studied by Pontryagin's Theorem.

[8] See Oliveira et al. (2009) for an example of PID application with the graphical representation.

It should be observed that, in the state-of-the-art, we do *not* have a *general* mathematical analysis via an extension of Pontryagin's Theorem for the cases in which the quasipolynomials $\delta(z) = P(z, e^z)$ have two or more real (noncommensurable) delays .

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2

Stability of Linear Continuous Singular and Discrete Descriptor Systems over Infinite and Finite Time Interval

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1. Introduction

1.1 Classes of systems to be considered

It should be noticed that in some systems we must consider their character of dynamic and static state at the same time. Singular systems (also referred to as degenerate, descriptor, generalized, differential-algebraic systems or semi-state) are those, the dynamics of which are governed by a mixture of algebraic and differential (difference) equations. Recently many scholars have paid much attention to singular systems and have obtained many good consequences. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a real need for their control.

It is well-known that singular systems have been one of the major research fields of control theory. During the past three decades, singular systems have attracted much attention due to the comprehensive applications in economics as the *Leontief* dynamic model (*Silva & Lima* 2003), in electrical (*Campbell* 1980) and mechanical models (*Muller* 1997), etc. Discussion of singular systems originated in 1974 with the fundamental paper of (*Campbell et al*. 1974) and latter on the anthological paper of (*Luenberger* 1977).

The research activities of the authors in the field of singular systems stability have provided many interesting results, the part of which were documented in the recent references. Still there are many problems in this field to be considered. This chapter gives insight into a detailed preview of the stability problems for particular classes of linear continuous and discrete time delayed systems. Here, we present a number of new results concerning stability properties of this class of systems in the sense of Lyapunov and non-Lyapunov and analyze the relationship between them.

1.2 Stability concepts

Numerous significant contributions have been made in the last sixty years in the area of Lyapunov stabilty for different classes of systems. Listing all contributions in this, always attractive area, at this point would represent a waste of time, since all necessary details and existing results, for so called normal systems, are very well known.

But in practice one is not only interested in system stability (e.g. in sense of Lyapunov), but also in bounds of system trajectories. A system could be stable but completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain sub-sets of state-space, which are *a priori* defined in a given problem.

Besides, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval. These bound properties of system responses, i. e. the solution of system models, are very important from the engineering point of view.

Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of system response. Thus, the analysis of these particular bound properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability concept are concerned.

2. Singular (descriptor) systems

2.1 Continuous singular systems

2.1.1 Continuous singular systems – stability in the sense of Lyapunov

Generally, the time invariant continuous singular control systems can be written, as:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0(t), \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is a generalized state space (co-state, semi-state) vector, $E \in \mathbb{R}^{n \times n}$ is a possibly singular matrix, with $rank\, E = r < n$.

Matrices E and A are of the appropriate dimensions and are defined over the field of real numbers.

System (1) is operatinig in a free regime and no external forces are applied on it. It should be stressed that, in a general case, the initial conditions for an autonomus and a system operating in the forced regime need not be the same.

System models of this form have some important advantages in comparison with models in the *normal form, e.g. when* $E = I$ and an appropriate discussion can be found in (*Debeljkovic et al*. 1996, 2004).

The complex nature of singular systems causes many difficultes *in analytical and numerical treatment* that do not appear when systems represented in the normal form are considered. In this sense questions of existence, solvability, uniqueness, and smothness are presented which must be solved in satisfactory manner. A short and concise, acceptable and understandable explanation of all these questions may be found in the paper of (*Debeljkovic* 2004).

STABILITY DEFINITIONS

Stability plays a central role in the theory of systems and control engineering. There are different kinds of stability problems that arise in the study of dynamic systems, such as *Lyapunov* stability, finite time stability, practical stability, technical stability and BIBO stability. The first part of this section is concerned with the asymptotic stability of the equilibrium points of *linear continuous singular systems.*

As we treat the linear systems this is equivalent to the study of the stability of the systems.
The *Lyapunov* direct method (LDM) is well exposed in a number of very well known references.
Here we present some different and interesting approaches to this problem, mostly based on the contributions of the authors of this paper.

Definition 2.1.1.1 System (1) is *regular* if there exist $s \in \mathbb{C}$, $\det(sE - A) \neq 0$, (*Campbell et al.* 1974).

Definition 2.1.1.2 System (1) with $A = I$ is exponentially stable if one can find two positive constants c_1, c_2 such that $\|\mathbf{x}(t)\| \leq c_2 \cdot e^{-c_1 \cdot t} \|\mathbf{x}(0)\|$ for every solution of (1), (*Pandolfi* 1980).

Definition 2.1.1.3 System (1) will be termed *asymptotically stable* if and only if, for all consistent initial conditions $\mathbf{x}_0$, $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$, (*Owens & Debeljkovic* 1985).

Definition 2.1.1.4 System (1) is *asymptotically stable* if all roots of $\det(sE - A)$, *i.e.* all finite eigenvalues of this matrix pencil, are in the open left-half complex plane, and system under consideration is *impulsive free* if there is no $\mathbf{x}_0$ such that $\mathbf{x}(t)$ exhibits discontinuous behaviour in the free regime, (*Lewis* 1986).

Definition 2.1.1.5 System (1) is called *asymptotically stable* if and only if all finite eigenvalues λ_i, $i = 1, \ldots, n_1$, of the matrix pencil $(\lambda E - A)$ have negative real parts, (*Muller* 1993).

Definition 2.1.1.6 The equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) is said to be *stable* if for every $\varepsilon > 0$, and for any $t_0 \in \Im$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, such that $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon$ holds for all $t \geq t_0$, whenever $\mathbf{x}_0 \in \mathcal{W}_k$ and $\|\mathbf{x}_0\| < \delta$, where $\Im$ denotes time interval such that $\Im = [t_0, +\infty[$, $t_0 \geq 0$, and $\mathcal{W}_k$ is the subspace of consistent intial conditions (*Chen & Liu* 1997).

Definition 2.1.1.7 The equilibrium $\mathbf{x} = \mathbf{0}$ of a system (1) is said to be *unstable* if there exist a $\varepsilon > 0$, and $t_0 \in \Im$, for any $\delta > 0$, such that there exists a $t^* \geq t_0$, for which $\|\mathbf{x}(t^*, t_0, \mathbf{x}_0)\| \geq \varepsilon$ holds, although $\mathbf{x}_0 \in \mathcal{W}_k$ [1] and $\|\mathbf{x}_0\| < \delta$, (*Chen & Liu* 1997).

Definition 2.1.1.8 The equilibrium $\mathbf{x} = \mathbf{0}$ of a system (1) is said to be *attractive* if for every $t_0 \in \Im$, there exists an $\eta = \eta(t_0) > 0$, such that $\lim\limits_{t \to \infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = \mathbf{0}$, whenever $\mathbf{x}_0 \in \mathcal{W}_k$ and $\|\mathbf{x}_0\| < \eta$, (*Chen & Liu* 1997).

Definition 2.1.1.9 The equilibrium $\mathbf{x} = \mathbf{0}$ of a singular system (1) is said to be *asymptotically stable* if it is *stable* and *attractive*, (*Chen & Liu* 1997).

Definition **2.1.1.5** is equivalent to $\lim\limits_{t \to +\infty} \mathbf{x}(t) = \mathbf{0}$.

Lemma 2.1.1.1 The equilibrium $\mathbf{x} = \mathbf{0}$ of a linear singular system (1) is *asymptotically stable* if and only if it is *impulsive-free*, and $\sigma(E, A) \subset \mathbb{C}^-$, (*Chen & Liu* 997).

[1] The solutions of continuous singular system models in this investigation are continuously differentiable functions of time t which satisfy the considered equations of the model. Since for continuous singular systems not all initial values $\mathbf{x}_0$ of $\mathbf{x}(t)$ will generate smooth solution, those that generate such solutions (continuous to the right) we call consistent. Moreover, positive solvability condition guarantees uniqueness and closed form of solutions to (1).

Lemma 2.1.1.2 The equilibrium $\mathbf{x}=\mathbf{0}$ of a system (1) is *asymptotically stable* if and only if it is *impulsive-free, and* $\lim_{t\to\infty}\mathbf{x}(t)=\mathbf{0}$, (*Chen & Liu* 1997).

Due to the system structure and complicated solution, the regularity of the systems is the condition to make the solution to singular control systems exist and be unique.
Moreover if the consistent initial conditions are applied, then the closed form of solutions can be established.

STABILITY THEOREMS

Theorem 2.1.1.1 System (1), with $A=I$, I being the identity matrix, is *exponentially stable* if and only if the eigenvalues of E have non positive real parts, (*Pandolfi* 1980).
Theorem 2.1.1.2 Let $I_{\mathcal{W}_k}$ be the matrix which represents the operator on $\mathbb{R}^n$ which is the identity on $\mathcal{W}_k$ and the zero operator on $\mathcal{W}_k$.
System (1), with $A=I$, is stable if an $(n\times n)$ matrix P exist, which is the solution of the matrix equation:

$$E^TP+PE=-I_{\mathcal{W}_k}, \tag{2}$$

with the following properties:

$$P=P^T, \tag{3}$$

$$P\mathbf{q}=\mathbf{0},\ \mathbf{q}\in\mathcal{V}, \tag{4}$$

$$\mathbf{q}^TP\mathbf{q}>0,\ \mathbf{q}\neq\mathbf{0},\ \mathbf{q}\in\mathcal{W}_k, \tag{5}$$

where:

$$\mathcal{W}_k=\aleph\left(I-EE^D\right) \tag{6}$$

$$\mathcal{V}=\aleph\left(EE^D\right), \tag{7}$$

where $\mathcal{W}_k$ is the subspace of consistent intial conditions, (*Pandolfi* 1980) and $\aleph(\ )$ denotes the kerrnel or null space of the matrix $(\ )$.
Theorem 2.1.1.3 System (1) is *asymptotically stable* if and only if (*Owens & Debeljkovic* 1985):

a. A is invertible.
b. A positive-definite, self-adjoint operator P on $\mathbb{R}^n$ exists, such that:

$$A^TPE+E^TPA=-Q, \tag{8}$$

where Q is self-adjoint and positive in the sense that:

$$\mathbf{x}^T(t)Q\mathbf{x}(t)>0 \text{ for all } \mathbf{x}(t)\in\mathcal{W}_{k^*}\setminus\{\mathbf{0}\}. \tag{9}$$

Theorem 2.1.1.4 System (1) is *asymptotically stable* if and only if (*Owens & Debeljkovic* 1985):

a. A is invertible,

b. there exists a positive-definite, self-adjoin operator P, such that:

$$\mathbf{x}^T(t)\left(A^T PE + E^T PA\right)\mathbf{x}(t) = -\mathbf{x}^T(t) I \mathbf{x}(t), \tag{10}$$

for all $\mathbf{x} \in \mathcal{W}_{k^*}$, where $\mathcal{W}_{k^*}$ denotes the subspace of consistent initial conditions.

2.1.2 Continuous singular systems – stability over finite time interval

Dynamical behaviour of the system (1) is defined over time interval $\Im = \{t : t_0 \le t \le t_0 + T\}$, where quantity T may be either a positive real number or symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously. Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded.
Let index β stand for the set of all allowable states of system and index α for the set of all initial states of the system, such that $\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$.
In general, one may write:

$$\mathcal{S}_\wp = \left\{\mathbf{x} : \left\|\mathbf{x}(t)\right\|_Q < \wp\right\},\ \mathbf{x}(t) \in \mathcal{W}_k \setminus \{0\}, \tag{11}$$

where Q will be assumed to be symmetric, positive definite, real matrix and where $\mathcal{W}_k$ denotes the sub-space of consistent initial conditions generating the smooth solutions.
A short and concise, acceptable and understandable explanation of all these questions can be found in the paper of (*Debeljkovic* 2004). Vector of initial conditions is consistent if there exists continuous, differentiable solution to (1).
A geometric treatment (*Owens & Debeljkovic* 1985) yields $\mathcal{W}_k$ as the limit of the sub-space algorithm:

$$\mathcal{W}_0 = \mathbb{R}^n,\quad \mathcal{W}_{j+1} = A^{-1}\left(E\mathcal{W}_j\right),\ j \ge 0, \tag{12}$$

where $A^{-1}(\cdot)$ denotes inverse image of $(\cdot)$ under the operator A.
Campbell et al. (1974) have shown that sub-space $\mathcal{W}_k$ represents the set of vectors satisfying:

$$\left(I - \hat{E}^D\hat{E}\right)\mathbf{x}_0 = \mathbf{0},\ \text{or}\ \ \mathcal{W}_k = \aleph\left(I - \hat{E}^D\hat{E}\right), \tag{13}$$

where $\hat{E} = (\lambda E - A)^{-1} E$. c is any complex scalar such that:

$$\det(\lambda E - A) \ne 0\ \text{or}\ \mathcal{W}_k \cap \aleph(E) = \{0\}. \tag{14}$$

This condition guarantees the uniqueness of solutions that are generated by $\mathcal{W}_k$ and $(\lambda E - A)$ is invertible for some $\lambda \in \mathbb{R}$. The null space of matrix F is denoted by $\aleph(F)$, range space with $\Re(F)$ and superscript "D" is used to indicate Drazin inverse. Let $\left\|\mathbf{x}(t)\right\|_{(\cdot)}$ be any vector norm (i. g. $\cdot = 1,2,\infty$) and $\left\|(\cdot)\right\|$ the matrix norm induced by this vector.
The matrix measure, for our purposes, is defined as follows:

$$\mu(F) = \frac{1}{2}\max_i \lambda_i\left(F^* + F\right), \tag{15}$$

for any matrix $F \in \mathbb{C}^{n\times n}$. Upper index $*$ denotes transpose conjugate. In case of $F \in \mathbb{R}^{n\times n}$ it follows $F^* = F^T$, where superscript T denotes transpose.
The value of a particular solution at the moment t, which at the moment $t = 0$ passes through the point $\mathbf{x}_0$, is denoted as $\mathbf{x}(t,\mathbf{x}_0)$, in abbreviated notation $\mathbf{x}(t)$.
The set of all points $\mathcal{S}_i$, in the phase space $\mathbb{R}^n$, $\mathcal{S}_i \subseteq \mathbb{R}^n$, which generate smooth solutions can be determined via the *Drazin* inverse technique.

STABILITY DEFINITIONS

Definition 2.1.2.1 System (1) *is finite time stable* w.r.t. $\{\alpha,\beta,Q,\Im\}$, $\alpha < \beta$, iff $\forall \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathcal{W}_k$, satisfying $\|\mathbf{x}_0\|_Q^2 < \alpha$, implies $\|\mathbf{x}(t)\|_Q^2 < \beta$, $\forall t \in \Im$, (*Debeljkovic & Owens* 1985).

Definition 2.1.2.2 System (1) is *finite time instable* w.r.t. $\{\alpha,\beta,Q,\Im\}$, $\alpha < \beta$, iff for $\forall \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathcal{W}_k$, satisfying $\|\mathbf{x}_0\|_Q^2 < \alpha$, exists $t^* \in \Im$ implying $\left\|\mathbf{x}\left(t^*\right)\right\|_Q^2 \geq \beta$, (*Debeljkovic & Owens* 1985).

Preposition 2.1.2.1 If $\varphi(\mathbf{x}) = \mathbf{x}^T(t)M\mathbf{x}(t)$ is quadratic form on $\mathbb{R}^n$ then it follows that there exist numbers $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$, satisfying $-\infty < \lambda_{\min}(M) \leq \lambda_{\max}(M) < +\infty$, such that:

$$\lambda_{\min}(M) \leq \frac{\mathbf{x}^T(t)M\mathbf{x}(t)}{V(\mathbf{x})} \leq \lambda_{\max}(M), \ \forall \mathbf{x} \in \mathcal{W}_k \setminus \{\mathbf{0}\}. \tag{16}$$

If $M = M^T$ and $\mathbf{x}^T(t)M\mathbf{x}(t) > 0$, $\forall \mathbf{x} \in \mathcal{W}_k \setminus \{\mathbf{0}\}$, then $\lambda_{\min}(M) > 0$ and $\lambda_{\max}(M) > 0$, where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ are defined in such way:

$$\begin{aligned} \lambda_{\min}(M) &= \min\left\{\begin{array}{l}\mathbf{x}^T(t)M\mathbf{x}(t), \ \mathbf{x} \in \mathcal{W}_k \setminus \{\mathbf{0}\}, \\ \qquad \mathbf{x}^T(t)E^TPE\mathbf{x}(t) = 1\end{array}\right\}, \\ \lambda_{\max}(M) &= \max\left\{\begin{array}{l}\mathbf{x}^T(t)M\mathbf{x}(t), \ \mathbf{x} \in \mathcal{W}_k \setminus \{\mathbf{0}\}, \\ \qquad \mathbf{x}^T(t)E^TPE\mathbf{x}(t) = 1\end{array}\right\}. \end{aligned} \tag{17}$$

It is convenient to consider, for the purposes of this exposure, the aggregation function for the system (1) in the following manner:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)E^TPE\mathbf{x}(t), \tag{18}$$

with particular choice $P = I$, I being identy matrix.

STABILITY THEOREMS

Theorem 2.1.2.1 The system is *finite stable* with respect to $\{\alpha,\beta,\Im\}$, $\alpha < \beta$, if the following conditiones are satisfied:

(*i*)
$$\beta/\alpha > \frac{\gamma_2(Q)}{\gamma_1(Q)} \tag{19}$$

(*ii*)
$$\ln \beta/\alpha > \Lambda(Q) + \ln \frac{\gamma_2(Q)}{\gamma_1(Q)}, \quad \forall t \in \Im . \tag{20}$$

with $\lambda_{\max}(Q)$ as in Preposition 2.1.2.1, (*Debeljkovic & Owens* 1985).

Preposition 2.1.2.2 There exists matrix $P = P^T > 0$, such that $\gamma_1(Q) = \gamma_2(Q) = 1$, (*Debeljkovic & Owens* 1985).

Corollary 2.1.2.1 If $\beta/\alpha > 1$, there exist choice of P such that

$$\frac{\beta}{\alpha} > \frac{\gamma_2(Q)}{\gamma_1(Q)} . \tag{21}$$

The practical meaning of this result is that condition (i) of *Definition* 2.1.2.1 can be satisfied by initial choice of free parameters of matrix P. Condition (*ii*) depends also on the system data and hence is more complex but it is also natural to ask whether we can choose P such that $\lambda_{\max}(Q) < 0$, (*Debeljkovic & Owens* 1985).

Theorem 2.1.2.2 System (1) is *finite time stable* w.r.t. $\{\alpha, \beta, I, \Im\}$ if the following condition is satisfied

$$\Phi_{CSS}(t) < \sqrt{\frac{\beta}{\alpha}}, \quad \forall t \in \Im , \tag{22}$$

$\Phi_{CSS}(t)$ being the *fundamental matrix* of linear singular system (1), (*Debeljkovic et al.* 1997).

Now we apply matrix mesure approach.

Theorem 2.1.2.3 System (1) is *finite time stable* w.r.t. $\{\alpha, \beta, I, \Im\}$, if the following condition is satisfied (*Debeljkovic et al.* 1997).

$$e^{\mu(\Upsilon)\cdot t} < \frac{\beta}{\alpha}, \quad \forall t \in \Im , \tag{23}$$

where:

$$\Upsilon = \hat{E}^D \hat{A}, \quad \hat{A} = (sE - A)^{-1} A, \quad \hat{E} = (sE - A)^{-1} E . \tag{24}$$

Starting with explicit solution of system (1), derived in (*Campbell* 1980).

$$\mathbf{x}(t) = e^{\hat{E}^D \hat{A}(t - t_0)} \mathbf{x}_0, \qquad \mathbf{x}_0 = \hat{E}\hat{E}^D \mathbf{x}_0 , \tag{25}$$

and differentiating equitation (25), one gets:

$$\dot{\mathbf{x}}(t) = \hat{E}^D \hat{A} e^{\hat{E}^D \hat{A} \cdot t} \cdot \mathbf{x}_0 = \hat{E}^D \hat{A} \mathbf{x}(t) , \tag{26}$$

so only the *regular* singular systems are treated with matrices given in (24).

Theorem 2.1.2.4 For given constant matrix $\hat{E}^D\hat{A}$ any solution of (1) satisfies the following inequality (*Kablar & Debeljkovic* 1998).

$$\|\mathbf{x}(t_0)\| e^{-\mu\left(-\hat{E}^D\hat{A}\right)(t-t_0)} \leq \|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_0)\| e^{\mu\left(\hat{E}^D\hat{A}\right)(t-t_0)}, \quad \forall t \in \Im \tag{27}$$

Theorem 2.1.2.5 In order for the system (1) to *be finite time stable* w.r.t. $\{\alpha,\beta,I,\Im\}$, $\alpha < \beta$, it is *necessary* that the following condition is satisfied:

$$e^{-\mu\left(-\hat{E}^D\hat{A}\right)\cdot(t-t_0)} < \sqrt{\frac{\beta}{\delta}}, \qquad \forall t \in \Im, \tag{28}$$

where $0 < \delta \leq \alpha$, (*Kablar & Debeljkovic* 1998).

Theorem 2.1.2.6 In order for system (1) to be *finite time instable* w.r.t. $\{\alpha,\beta,I,\Im\}$, $\alpha < \beta$, it is *necessary* that there exists $t^* \in \Im$ such that the following condition is satisfied:

$$e^{\mu\left(\hat{E}^D\hat{A}\right)\cdot\left(t^*-t_0\right)} \geq \sqrt{\frac{\beta}{\alpha}}, \quad t^* \in \Im. \tag{29}$$

Theorem 2.1.2.7 System (1) *is finite time instable* w.r.t. $\{\alpha,\beta,I,\Im\}$, $\alpha < \beta$, *if* $\exists \delta$, $0 < \delta \leq \alpha$ and $t^* \in \Im$ such that the following condition is satisfied:

$$e^{-\mu\left(-\hat{E}^D\hat{A}\right)\cdot\left(t^*-t_0\right)} < \sqrt{\frac{\beta}{\delta}}, \qquad t^* \in \Im. \tag{30}$$

Finally, we present *Bellman–Gronwall approach* to derive our results, earlier given in *Theorem* 2.1.2.7.

Lemma 2.1.2.1 Suppose the vector $\mathbf{q}(t,t_0)$ is defined in the following manner (*Debeljkovic & Kablar* 1999):

$$\mathbf{q}(t,t_0) = \Phi(t,t_0)\hat{E}^D\hat{E}\mathbf{v}(t_0). \tag{31}$$

So if:

$$E\mathbf{q}(t,t_0) = E\Phi(t,t_0)\hat{E}^D\hat{E}\mathbf{v}(t_0), \tag{32}$$

then:

$$\|\mathbf{q}(t,t_0)\|^2_{E^TE} \leq \|\mathbf{v}(t_0)\|^2_{E^TE}\, e^{\lambda_{\max}(M)(t-t_0)}, \tag{33}$$

where:

$$\begin{aligned}\lambda_{\max}(M) = \max\{&\mathbf{q}^T(t,t_0)\Xi\mathbf{q}(t,t_0) : \mathbf{q}(t,t_0) \in \mathcal{W}_k \setminus \{0\},\\ &\mathbf{q}^T(t,t_0)E^TE\mathbf{q}(t,t_0) = 1, \quad \Xi = A^TE + E^TA\}\end{aligned} \tag{34}$$

$$\mathbf{v}(t_0) = \mathbf{q}(t_0,t_0). \tag{35}$$

Using this approach the results of *Theorem* 2.1.2.1 can be reformulate in the following manner.

Theorem 2.1.2.8 System (1) is *finite time stable* w.r.t. $\left\{\alpha,\beta,\left\|(\cdot)\right\|_Q^2,\Im\right\}$, $a<\beta$, if the following condition is satisfied:

$$e^{\lambda_{\max}(\Xi)\cdot(t-t_0)}<\frac{\beta}{\alpha},\quad \forall t\in\Im, \tag{36}$$

with $\lambda_{\max}(M)$ given (34), (*Debeljkovic & Kablar* 1999).

2.2 Discrete descriptor system

2.2.1 Discrete descriptor system – stability in sense of Lyapunov

Generally, the time invariant linear discrete descriptor control systems can be written, as:

$$E\mathbf{x}(k+1)=A\mathbf{x}(k),\quad \mathbf{x}(k_0)=\mathbf{x}_0, \tag{37}$$

where $\mathbf{x}(t)\in\mathbb{R}^n$ is a generalized state space (co-state, semi-state) vector, $E\in\mathbb{R}^{n\times n}$ is a possibly singular matrix, with $rank\,E=r<n$. Matrices E and A are of the appropriate dimensions and are defined over the field of real numbers.

NECESSARY CONSIDERATIONS

In the discrete case, the concept of smoothness has little meaning but the idea of consistent initial conditions being these initial conditions $\mathbf{x}_0$, that generate solution sequences $(\mathbf{x}(k):k\geq 0)$ has a physical meaning.

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions $\mathcal{W}_d$,is the subspace sequence:

$$\mathcal{W}_{d,0}=R^n,\ \mathcal{W}_{d,j+1}=A^{-1}\left(E\mathcal{W}_{d,j}\right),\ (j\geq 0). \tag{38}$$

Here $A^{-1}(\cdot)$ denotes the inverse image of $(\cdot)$ under the operator A and we will denote by $\aleph(F)$ and $\Re(F)$ the kernel and range of any operator F, respectively.

Lemma 2.2.1.1 The subspace sequence $\left\{\mathcal{W}_{d,0},\ \mathcal{W}_{d,1},\ \mathcal{W}_{d,2},...\right\}$ is nested in the sense that:

$$\mathcal{W}_{d,0}\supset\mathcal{W}_{d,1}\supset\mathcal{W}_{d,2}\supset\mathcal{W}_{d,3}\supset\cdots. \tag{39}$$

Moreover:

$$\aleph(A)\subset\mathcal{W}_{d,j},\ (j\geq 0), \tag{40}$$

and there exists an integer $k\geq 0$, such that:

$$\mathcal{W}_{d,k+1}=\mathcal{W}_{d,k}, \tag{41}$$

and hence $\mathcal{W}_{d,k+1}=\mathcal{W}_{d,k}$ for $j\geq 1$.

If k^* is the smallest such integer with this property, then:

$$\mathcal{W}_{d,k} \cap \aleph(E) = \{0\}, \ \left(k \geq k^*\right), \tag{42}$$

provided that $(\lambda E - A)$ is invertible for some $\lambda \in R$, (*Owens & Debeljkovic* 1985).

Theorem 2.2.1.1 Under the conditions of *Lemma* 2.2.1.1, $\mathbf{x}_0$ is a consistent initial condition for (37) if $\mathbf{x}_0 \in \mathcal{W}_{d,k^*}$. Moreover $\mathbf{x}_0$ generates a unique solution $\mathbf{x}(t) \in \mathcal{W}_{d,k^*}$, $(k \geq 0)$ that is real - analytic on $\{k : k \geq 0\}$, (*Owens & Debeljkovic* 1985).

Theorem 2.2.1.1 is the geometric counterpart of the algebraic results of *Campbell* (1980). A short and concise, acceptable and understandable explanation of all these questions can be found in the papers of (*Debeljkovic* 2004).

Definition 2.2.1.1 The linear discrete descriptor system (37) is said to be regular if $\det(sE - A)$ is not identically equal to zero, (*Dai* 1989).

Remark 2.2.1.1 Note that the regularity of matrix pair (*E*, *A*) guarantees the existence and uniqueness of solution $\mathbf{x}\,(\cdot)$ for any specified initial condition, and the impulse immunity avoids impulsive behavior at initial time for inconsistent initial conditions. It is clear that, for nontrivial case, det $E \neq 0$, impulse immunity implies regularity.

Definition 2.2.1.2 The linear discrete descriptor system (37) is assumed to be *non-degenerate* (or regular), i.e. $\det(zE - A) \neq 0$. Otherwise, it will be called *degenerate*, (*Syrmos et al.* 1995).

If $(zE - A)$ is non-degenerate, we define the *spectrum* of $(zE - A)$, denoted as $\sigma\{E, A\}$ as those isolated values of z where $\det(zE - A) \neq 0$ fails to hold. The usual spectrum of $(zI - A)$ will be denoted as $\sigma\{A\}$.

Note that owing to (possible) singularity of E, $\sigma\{E, A\}$ may contain finite and infinite values of z.

Definition 2.2.1.3 The linear discrete descriptor system (37) is said to be *causal* if (37) is regular and degree $\det(zE - A) = \operatorname{rank} E$, (*Dai* 1989).

Definition 2.2.1.4 A pair (*E*, *A*) is said to be *admissible* if it is regular, impulse-free and stable, *Hsiung*, *Lee* (1999).

Lemma 2.2.1.2 The linear discrete-time descriptor system (37) is regular, causal and stable if and only if there exists an invertible symmetric matrix $H \in R^{n \times n}$ such that the following two inequalities holds (*Xu & Yang* 1999):

$$E^T H E \geq 0, \tag{43}$$

$$A^T H A - E^T H E < 0. \tag{44}$$

STABILITY DEFINITIONS

Definition 2.2.1.5 Linear discrete descriptor system (37) is said to be stable if and only if (37) is regular and all of its finite poles are within region $\Omega(0,1)$, (*Dai* 1989).

Definition 2.2.1.6 The system in (37) is asymptotically stable if all the finite eigenvalues of the pencil $(zE - A)$ are inside the unit circle, and anticipation free if every admissible $\mathbf{x}(0)$ in (37) admits one-sided solutions, (*Syrmos et al.* 1995).

Definition 2.2.1.7 Linear discrete descriptor system (37) is said to be asymptotically stable if, for all consistent initial conditions $\mathbf{x}_0$, we have that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to +\infty$, (*Owens & Debeljkovic* 1985).

STABILITY THEOREMS

First, we present the fundamental work in the area of stability in the sense of Lyapunov applied to the linear discrete descriptor systems, (*Owens & Debeljkovic* 1985).

Our attention is restricted to the case of singular (i.e. noninvertible) E and the construction of geometric conditions on $\mathbf{x}_0$ for the existence of causal solutions of (37) in terms of the relative subspace structure of matrices E and A. The results are hence a geometric counterpart of the algebraic theory of (*Campbell* 1980) who established the required form of $\mathbf{x}_0$ in terms of the Drazin inverse and the technical trick of replacing E and A by commuting operators.

The ideas in this paper work with E and A directly and commutability is not assumed. The geometric theory of consistency leads to a natural class of positive-definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of a Lyapunov stability theory for linear discrete descriptor systems in the sense that asymptotic stability is equivalent to the existence of symmetric, positive-definite solutions to *a weak form of Lyapunov equation.*

Throughout this exposure it is assumed that $(\lambda E - A)$ is invertible at all but a finite number of points $\lambda \in C$ and hence that if a solution $\mathbf{x}(k), (k \geq 0)$ of $(\mathbf{x}(k) : k = 0,1,...)$ exists for a given choice of $\mathbf{x}_0$, it is unique, (*Campbell* 1980).

The linear discrete descriptor system is said to be stable if (37) is regular and all of its finite poles are within region Ω(0,1), (*Dai* 1989), so careful investigation shows there is no need for the matrix A to be invertible, in comparison with continuous case, see (*Debeljkovic et al.* 2007) so it could be noninvertible.

Theorem 2.2.1.2 The linear discrete descriptor system (37) is asymptotically stable if, and only if, there exists a real number $\lambda^* > 0$ such that, for all λ in the range $0 < |\lambda| < \lambda^*$, there exists a self-adjoint, positive-definite operator H_λ in R^n satisfying:

$$(A - \lambda E)^T H_\lambda (A - \lambda E) - E^T H_\lambda E = -Q_\lambda , \tag{45}$$

for some self-adjoint operator Q_λ satisfying the positivity condition (*Owens & Debeljkovic* 1985):

$$\mathbf{x}^T(t) Q_\lambda \mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_{d,k^*} \setminus \{0\} . \tag{46}$$

Theorem 2.2.1.3 Suppose that matrix A is invertible. Then the linear discrete descriptor system (37) is asymptotically stable if, and only if, there exists a self-adjoint, positive-definite solution H in R^n satisfying

$$A^T H A - E^T H E = -Q , \tag{47}$$

where Q is self-adjoint and positive in the sense that (*Owens & Debeljkovic* 1985):

$$\mathbf{x}^T(t)Q\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_{d,k^*} \setminus \{0\}. \tag{48}$$

Theorem 2.2.1.4 The linear discrete descriptor system (37) is asymptotically stable if and only if there exists a real number $\lambda^* > 0$ such that, for all λ in the range $0 < |\lambda| < \lambda^*$, there exists a self-adjoint, positive-definite operator H_λ in R^n satisfying *Owens, Debeljkovic* (1985):

$$\mathbf{x}^T(t)\Big((A-\lambda E)^T H_\lambda (A-\lambda E) - E^T H_\lambda E\Big)\mathbf{x}(t) = -\mathbf{x}^T(t)\mathbf{x}(t), \quad \forall \mathbf{x}(t) \in \mathcal{W}_{d,k^*}. \tag{49}$$

Corollary 2.2.1.4 If matrix A is invertible, then the linear discrete descriptor system (37) is asymptotically stable if and only if (49) holds for $\lambda = 0$ and some self-adjoint, positive-definite operator H_0, (*Owens & Debeljkovic* 1985).

2.2.2 Discrete descriptor system – stability over infinite time interval

Dynamical behaviour of system (37) is defined over time interval $\mathcal{K} = \{k_0, (k_0 + k_N)\}$, where quantity k_N may be either a positive real number or symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously.

Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded.

Let index β stands for the set of all allowable states of system and index α for the set of all initial states of the system, such that $\forall \mathbf{x}(k_0) = \mathbf{x}_0 \in \mathcal{W}_d$.

Sets are assumed to be open, connected and bounded and defined by (11) in discrete case sense.

Under assumption that *discrete version of* the Preposition 2.1.2.1 is acceptable here, without any limitation, we can give the following *Definitions*.

STABILITY DEFINITIONS

Definition 2.2.2.1 System (37) is finite time stable w.r.t $\{\alpha, \beta, G, \mathcal{K}, \mathcal{W}_d\}$, if and only if a *consistent initial condition*, $\mathbf{x}_0 \in \mathcal{W}_d$, satisfying $\|\mathbf{x}_0\|_G^2 < \alpha$, $G = E^T PE$, implies $\|\mathbf{x}(k)\|_G^2 < \beta$, $\forall k \in \mathcal{K}$. G is chosen to represent physical constraints on the system variables and it is assumed, as before, to satisfy $G = G^T$, $\mathbf{x}^T(k)G\mathbf{x}(k) > 0, \forall \mathbf{x}(k) \in \mathcal{W}_d \setminus \{0\}$, (*Debeljkovic* 1985, 1986), (*Debeljkovic, Owens* 1986), (*Owens, Debeljkovic* 1986).

Definition 2.2.2.2 System (37) is finite time unstable w.r.t respect to $\{K, \alpha, \beta, G, W_q\}$, if and only if there is a *consistent initial condition*, satisfying $\|\mathbf{x}_0\|_G^2 < \alpha$, $G = E^T PE$, and there exists discrete moment $k^* \in K$, such that the next condition is fulfilled $\left\|x\left(k^*\right)\right\|_G^2 > \beta$, for some $k^* \in \mathcal{K}$, (*Debeljkovic & Owens* 1986), (*Owens & Debeljkovic* 1986).

STABILITY THEOREMS

Theorem 2.2.2.1 System (37) is finite time stable w.r.t $\{\alpha, \beta, \mathcal{K}\}$, $\beta > \alpha$, if the following condition is satisfied:

$$\lambda_{\max}^{k}\left(Q\right) < \beta / \alpha, \quad \forall k \in \mathcal{K}, \tag{50}$$

where $\lambda_{\max}^{k}\left(Q\right)$ is defined by:

$$\lambda_{\max}^{k}\left(Q\right) = \max_{\mathbf{x}}\left\{\mathbf{x}^{\mathrm{T}}\left(k\right)A^{\mathrm{T}}PA\mathbf{x}\left(k\right): \ \mathbf{x}\left(k\right) \in \mathcal{W}_{d} \setminus \left\{0\right\}, \ \ \mathbf{x}^{\mathrm{T}}\left(k\right)E^{\mathrm{T}}PE\mathbf{x}\left(k\right) = 1\right\} \tag{51}$$

with matrix $P = P^{T} > 0$, (*Debeljkovic* 1986), (*Debeljkovic & Owens* 1986).

Theorem 2.2.2.2 System (37) is finite time unstable w.r.t $\left\{\alpha, \beta, \mathcal{K}\right\}$, $\beta > \alpha$ if there exists a positive scalar $\gamma \in \left]0, \alpha\right[$ and a discrete moment k^{*}, $\exists\left(k^{*} > k_{0}\right) \in \mathcal{K}$ such that the following condition is satisfied (*Debeljkovic & Owens* 1986):

$$\lambda_{\min}^{k^{*}}\left(Q\right) > \beta / \gamma, \ \text{for some } k^{*} \in \mathcal{K} \tag{52}$$

where $\lambda^{k}\left(Q\right)$ being defined by:

$$\lambda_{\min}^{k}\left(Q\right) = \min_{\mathbf{x}}\left\{\mathbf{x}^{\mathrm{T}}\left(k\right)A^{\mathrm{T}}PA\mathbf{x}\left(k\right): \ \mathbf{x}\left(k\right) \in \mathcal{W}_{d} \setminus \left\{0\right\}, \ \ \mathbf{x}^{\mathrm{T}}\left(k\right)E^{\mathrm{T}}PE\mathbf{x}\left(k\right) = 1\right\}. \tag{53}$$

Theorem 2.2.2.3. System (37) is finite time stable w.r.t $\left\{\alpha, \beta, \mathcal{K}\right\}$, $\beta > \alpha$, if the following condition is satisfied:

$$\left\|\Psi\left(k\right)\right\| < \beta / \alpha, \quad \forall k \in K. \tag{54}$$

where: $\Psi\left(k\right) = \left(\hat{E}^{D}\hat{A}\right)^{k}$ and $\hat{E} = \left(cE - A\right)^{-1}E$, $\hat{A} = \left(cE - A\right)^{-1}A$, (*Debeljkovic* 1986).

3. Conclusion

This chapter considers important stability issues of linear continuous singular and discrete descriptor systems over infinite and finite time interval. Here, we present a number of new results concerning stability properties of this class of systems *in the sense of Lyapunov and non-Lyapunov* and analyze the relationship between them over finite and infinite time interval.

In the first part of the chapter continuous singular systems were considered. Basic stability concepts were introduced, starting with a preview of important stability definitions. Stability in the sense of Lyapunov, as well as the stability over finite time interval were addressed in detail.

Second part of this chapter deals with stability issues for discrete descriptor systems in the sense of Lyapunov and over infinite and finite time interval.

The chapter also represents a comprehensive survey on important stability theorems which apply to studied classes of systems.

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of Lyapunov stability theory even for the *time delay systems* in that sense that asymptotic

stability is equivalent to the existence of symmetric, positive definite solutions to a *weak* form of Lyapunov continuous (discrete) algebraic matrix equation (*Owens*, *Debeljkovic* 1985) respectively, incorporating condition which refers to time delay term.
Time delay systems represent a special and very important class of systems and therefore their investigation deserves special attention. Detailed consideration of time delayed systems, together with important new results of the authors, will be presented in the subsequent chapter, which concerns continuous singular as well as discrete descriptor time delay systems. Presented chapter is therefore a necessary premise as an introduction to the stability issues of continuous singular and discrete descriptor time delay system, which provides consistency and comprehensibility of the presented topics.

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3

Stability of Linear Continuous Singular and Discrete Descriptor Time Delayed Systems

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1. Introduction

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

When the general time delay systems are considered, in the existing stability criteria, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria and generally provides simple algebraic conditions. In that sense the question of their stability deserves great attention. We must emphasize that there are a lot of systems that have the phenomena of time delay and singular characteristics simultaneously. We denote such systems as *the* singular (*descriptor*) differential (*difference*) *systems with time delay*.

These systems have many special properties. If we want to describe them more exactly, to design them more accurately and to control them more effectively, we must pay tremendous endeavor to investigate them, but that is obviously a very difficult work. In recent references authors have discussed such systems and got some consequences. But in the study of such systems, there are still many problems to be considered.

2. Time delay systems

2.1 Continuous time delay systems

2.1.1 Continuous time delay systems – stability in the sense of Lyapunov

The application of *Lyapunov*'s direct method (LDM) is well exposed in a number of very well known references. For the sake of brevity contributions in this field are omitted here. The part of only interesting paper of (*Tissir & Hmamed* 1996), in the context of these investigations, will be presented later.

2.1.2 Continuous time delay systems – stability over finite time interval

A linear, multivariable time-delay system can be represented by differential equation:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \tag{1}$$

and with associated function of initial state:

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \ -\tau \le t \le 0. \tag{2}$$

Equation (1) is referred to as *homogenous,* $\mathbf{x}(t) \in \mathbb{R}^n$ is a state space vector, A_0, A_1, are constant system matrices of appropriate dimensions, and τ is pure time delay, $\tau = const.,(\tau > 0)$.
Dynamical behavior of the system (1) with initial functions (2) is defined over continuous time interval $\Im = \{t_0, \ t_0 + T\}$, where quantity T may be either a positive real number or symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously. It is obvious that $\Im \in \mathbb{R}$. Time invariant sets, used as bounds of system trajectories, satisfy the assumptions stated in the previous chapter (section 2.2).

STABILITY DEFINITIONS

In the context of finite or practical stability for particular class of *nonlinear singularly perturbed multiple time delay systems* various results were, *for the first time*, obtained in *Feng*, *Hunsarg* (1996). It seems that their definitions are very similar to those in *Weiss*, *Infante* (1965, 1967), clearly addopted to time delay systems.
It should be noticed that those definitions are significantly different from definition presented by the autors of this chapter.
In the context of finite time and practical stability for linear continuous time delay systems, various results were first obtained in (*Debeljkovic et al.* 1997.a, 1997.b, 1997.c, 1997.d), (*Nenadic et al.* 1997).
In the paper of (*Debeljkovic et al.* 1997.a) and (*Nenadic et al.* 1997) some basic results of the area of finite time and practical stability were extended to the particular class of linear continuous time delay systems. Stability sufficient conditions dependent on delay, expressed in terms of time delay fundamental system matrix, have been derived. Also, in the circumstances when it is possible to establish the suitable connection between fundamental matrices of linear time delay and non-delay systems, presented results enable an efficient procedure for testing practical as well the finite time stability of time delay system.
Matrix measure approach has been, for the first time applied, in (*Debeljkovic et al.* 1997.b, 1997.c, 1997.d, 1997.e, 1998.a, 1998.b, 1998.d, 1998.d) for the analysis of practical and finite time stability of linear time delayed systems. Based on Coppel's inequality and introducing matrix measure approach one provides a very simple delay – dependent sufficient conditions of practical and finite time stability with no need for time delay fundamental matrix calculation.
In (*Debeljkovic et al.* 1997.c) this problem has been solved for forced time delay system.
Another approach, based on very well known Bellman-Gronwall Lemma, was applied in (*Debeljkovic et al.* 1998.c), to provide new, more efficient sufficient delay-dependent conditions for checking finite and practical stability of continuous systems with state delay.
Collection of all previous results and contributions was presented in paper (*Debeljkovic et al.* 1999) with overall comments and slightly modified Bellman-Gronwall approach.

Finally, modified Bellman-Gronwall principle, has been extended to the particular class of continuous *non-autonomous* time delayed systems operating over the finite time interval, (*Debeljkovic et al.* 2000.a, 2000.b, 2000.c).

Definition 2.1.2.1 Time delay system (1-2) is *stable* with respect to $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$, $\alpha \le \beta$, if for any trajectory $\mathbf{x}(t)$ condition $\|\mathbf{x}_0\| < \alpha$ implies $\|\mathbf{x}(t)\| < \beta \quad \forall t \in [-\Delta, T]$, $\Delta = \tau_{\max}$, (*Feng*, *Hunsarg* 1996).

Definition 2.1.2.2 Time delay system (1-2) is *stable* with respect to $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$, $\gamma < \alpha < \beta$, if for any trajectory $\mathbf{x}(t)$ condition $\|\mathbf{x}_0\| < \alpha$, implies (*Feng*, *Hunsarg* 1996):

i. Stability w.r.t. $\{\alpha, \beta, -\tau, T, \|\mathbf{x}\|\}$,

ii. There exist $t^* \in]0, T[$ such that $\|\mathbf{x}(t)\| < \gamma$ for all $\forall t \in]t^*, T[$.

Definition 2.1.2.3 System (1) satisfying initial condition (2) is finite time stable with respect to $\{\zeta(t), \beta, \Im\}$ if and only if $\|\boldsymbol{\psi}_x(t)\| < \zeta(t)$, implies $\|\mathbf{x}(t)\| < \beta, \ t \in \Im$, $\zeta(t)$ being scalar function with the property $0 < \zeta(t) \le \alpha, \ -\tau \le t \le 0$, $-\tau \le t \le 0$, where α is a real positive number and $\beta \in \mathbb{R}$ and $\beta > \alpha$, (*Debeljkovic et al.* 1997.a, 1997.b, 1997.c, 1997.d), (*Nenadic et al.* 1997).

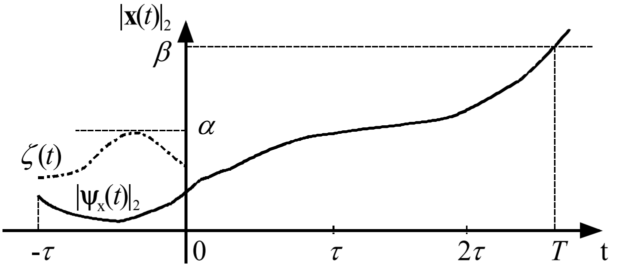

Fig. 2.1 Illustration of preceding definition

Definition 2.1.2.4 System (1) satisfying initial condition (2) is finite time stable with respect to. $\{\zeta(t), \beta, \tau, \Im, \mu(A_0 \ne 0)\}$ iff $\boldsymbol{\psi}_x(t) \in \mathcal{S}_\alpha, \ \forall t \in [-\tau, 0]$, implies $\mathbf{x}(t_0, t, \mathbf{x}_0) \in \mathcal{S}_\beta$, $\forall t \in [0, T]$ (*Debeljkovic et al.* 1997.b, 1997.c).

Definition 2.1.2.5 System (1) satisfying initial condition (2) is finite time stable with respect to $\{\alpha, \beta, \tau, \Im, \mu_2(A_0) \ne 0\}$ iff $\boldsymbol{\psi}_x(t) \in \mathcal{S}_\alpha, \ \forall t \in [-\tau, 0]$, implies $\mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{u}(t)) \in S_\beta$, $\forall t \in \Im$, (*Debeljkovic et al.* 1997.b, 1997.c).

Definition 2.1.2.6 System (1) with initial function (2), is *finite time stable* with respect to $\{t_0, \Im, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, iff $\|\mathbf{x}(t_0)\|^2 = \|\mathbf{x}_0\|^2 < \alpha$, implies $\|\mathbf{x}(t)\|^2 < \beta, \quad \forall t \in \Im$, (*Debeljkovic et al.* 2010).

Definition 2.1.2.7 System (1) with initial function (2), is *attractive practically stable* with respect to $\{t_0, \Im, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, iff $\|\mathbf{x}(t_0)\|_P^2 = \|\mathbf{x}_0\|_P^2 < \alpha$, implies: $\|\mathbf{x}(t)\|_P^2 < \beta, \ \forall t \in \Im$, with property that: $\lim_{k \to \infty} \|\mathbf{x}(t)\|_P^2 \to 0$, (*Debeljkovic et al.* 2010).

STABILITY THEOREMS - Dependent delay stability conditions

Theorem 2.1.2.1 System (1) with the initial function (2) is finite time stable with respect to $\{\alpha, \beta, \tau, \Im\}$ if the following condition is satisfied

$$||\Phi(t)||_2 < \frac{\sqrt{\beta / \alpha}}{1+\tau\|A_1\|_2}, \qquad \forall t \in [0, T] \tag{3}$$

$\|(\cdot)\|$ is Euclidean norm and $\Phi(t)$ is fundamental matrix of system (1), (*Nenadic et al*. 1997), (*Debeljkovic et al*. 1997.a).
When $\tau = 0$ or $\|A_1\| = 0$, the problem is reduced to the case of the ordinary linear systems, (*Angelo* 1974).
Theorem 2.1.2.2 System (1) with initial function (2) is finite time stable w.r.t. $\{\alpha, \beta, \tau, T\}$ if the following condition is satisfied:

$$e^{\mu(A_0)t} < \frac{\sqrt{\beta / \alpha}}{1+\tau\|A_1\|_2}, \quad \forall t \in [0, T], \tag{4}$$

where $\|(\cdot)\|$ denotes Euclidean norm, (*Debeljkovic et al*. 1997.b).
Theorem 2.1.2.3 System (1) with the initial function (2) is finite time stable with respect to $\{\alpha, \beta, \tau, T, \mu_2(A_0) \neq 0\}$ if the following condition is satisfied:

$$e^{\mu(A_0)t} < \frac{\beta / \alpha}{1+\mu_2^{-1}(A_0)\cdot\|A_1\|_2\cdot\left(1-e^{-\mu_2(A_0)\tau}\right)}, \quad \forall t \in [0, T], \tag{5}$$

(*Debeljkovic et al*. 1997.c, 1997.d).
Theorem 2.1.2.4 System (1) with the initial function (2) is finite time stable with respect to $\{\sqrt{\alpha}, \sqrt{\beta}, \tau, T, \mu(A_0) = 0\}$ if the following condition is satisfied:

$$1+\tau\|A_1\|_2 < \sqrt{\beta / \alpha}, \ \forall t \in [0, T], \tag{6}$$

(*Debeljkovic et al.* 1997.d).
Results that will be presented in the sequel enable to check finite time stability of the systems to be considered, namely the system given by (1) and (2), without finding the fundamental matrix or corresponding matrix measure.
Equation (2) can be rewritten in it's general form as:

$$\mathbf{x}(t_0+\vartheta) = \boldsymbol{\psi}_x(\vartheta), \quad \boldsymbol{\psi}_x(\vartheta) \in \mathcal{C}[-\tau, 0], \ -\tau \leq \vartheta \leq 0, \tag{7}$$

where t_0 is the initial time of observation of the system (1) and $\mathcal{C}[-\tau, 0]$ is a *Banach space* of continuous functions over a time interval of length τ, mapping the interval $[(t-\tau), t]$ into $\mathbb{R}^n$ with the norm defined in the following manner:

$$\|\boldsymbol{\psi}\|_{\mathcal{C}} = \max_{-\tau \leq \vartheta \leq 0} \|\boldsymbol{\psi}(\vartheta)\|. \tag{8}$$

It is assumed that the usual smoothness conditions are present so that there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to initial data. Moreover one can write:

$$\mathbf{x}\left(t_0+\vartheta\right)=\boldsymbol{\psi}_x\left(\vartheta\right), \tag{9}$$

as well as:

$$\dot{\mathbf{x}}\left(t_0\right)=\mathbf{f}\left(t_0,\boldsymbol{\psi}_x\left(\vartheta\right)\right). \tag{10}$$

Theorem 2.1.2.5 System given by (1) with initial function (2) is finite time stable w.r.t. $\left\{\alpha,\beta,t_0,\Im\right\}$ if the following condition is satisfied:

$$\left(1+\left(t-t_0\right)\sigma_{\max}\right)^2 e^{2\left(t-t_0\right)\sigma_{\max}}<\frac{\beta}{\alpha},\quad \forall t\in\Im, \tag{11}$$

$\sigma_{\max}\left(\cdot\right)$ being the largest singular value of matrix $\left(\cdot\right)$, namely

$$\sigma_{\max}=\sigma_{\max}\left(A_0\right)+\sigma_{\max}\left(A_1\right). \tag{12}$$

(*Debeljkovic et al.* 1998.c) and (*Lazarevic et al.* 2000).

Remark 2.1.2.1 In the case when in the *Theorem* 2.1.2.5 $A_1=0$, e.g. A_1 is null matrix, we have the result similar to that presented in (*Angelo* 1974).

Before presenting our crucial result, we need some discussion and explanations, as well some additional results.

For the sake of completeness, we present the following result (*Lee & Dianat* 1981).

Lemma 2.1.2.1 Let us consider the system (1) and let $P_1\left(t\right)$ be characteristic matrix of dimension $\left(n\times n\right)$, continuous and differentiable over time interval $\left[0,\ \tau\right]$ and 0 elsewhere, and a set:

$$V\left(\mathbf{x}_t,\tau\right)=\left(\mathbf{x}\left(t\right)+\int_0^h P_1\left(\tau\right)\mathbf{x}\left(t-\tau\right)\mathrm{d}\tau\right)P_0\left(\mathbf{x}\left(t\right)+\int_0^h P_1\left(\tau\right)\mathbf{x}\left(t-\tau\right)\mathrm{d}\tau\right), \tag{13}$$

where $P_0=P_0^*>0$ is Hermitian matrix and $\mathbf{x}_t\left(\vartheta\right)=\mathbf{x}\left(t+\vartheta\right),\ \vartheta\in\left[-\tau,\ 0\right]$.

If:

$$P_0\left(A_0+P_1\left(0\right)\right)+\left(A_0+P_1\left(0\right)\right)^* P_0=-Q, \tag{14}$$

$$\dot{P}_1\left(\kappa\right)=\left(A_0+P_1\left(0\right)\right)P_1\left(\kappa\right),\quad 0\le\kappa\le\tau, \tag{15}$$

where $P_1\left(\tau\right)=A_1$ and $Q=Q^*>0$ is Hermitian matrix, then (*Lee &Dianat* 1981):

$$\dot{V}\left(\mathbf{x}_t,\tau\right)=\frac{d}{dt}V\left(\mathbf{x}_t,\tau\right)<0. \tag{16}$$

Equation (13) defines Lyapunov's function for the system (1) and * denotes conjugate transpose of matrix.
In the paper (*Lee*, *Dianat* 1981) it is emphasized that the key to the success in the construction of a Lyapunov function corresponding to the system (1) is the existence of **at least one solution** $P_1(t)$ of (15) with boundary condition $P_1(\tau) = A_1$.

In other words, it is required that the nonlinear algebraic matrix equation:

$$e^{\left(A_0+P_1(0)\right)\tau} P_1(0) = A_1, \tag{17}$$

has **at least one** solution for $P_1(0)$.

Theorem 2.1.2.6 Let the system be described by (1). If for *any* given positive definite Hermitian matrix Q there exists a positive definite Hermitian matrix P_0, such that:

$$P_0\left(A_0 + P_1(0)\right) + \left(A_0 + P_1(0)\right)^* P + Q = 0, \tag{18}$$

where for $\vartheta \in [0, \tau]$ and $P_1(\vartheta)$ satisfies:

$$\dot{P}_1(\vartheta) = \left(A_0 + P_1(0)\right) P_1(\vartheta), \tag{19}$$

with boundary condition $P_1(\tau) = A_1$ and $P_1(\tau) = 0$ elsewhere, then the system is asymptotically stable, (*Lee*, *Dianat* 1981).
Theorem 2.1.2.7 Let the system be described by (1) and furthermore, let (17) have solution for $P_1(0)$, which is nonsingular. Then, system (1) is asymptotically stable if (19) of *Theorem* 2.1.2.6 is satisfied, (*Lee*, *Dianat* 1981).
Necessary and sufficient conditions for the stability of the system are derived by Lyapunov's direct method through construction of the corresponding "energy" function. This function is known to exist if a solution $P_1(0)$ of the algebraic nonlinear matrix equation $A_1 = \exp\tau\left(A_0 + P_1(0)\right) \cdot P_1(0)$ can be determined.

It is asserted, (*Lee*, *Dianat* 1981), that derivative sign of a Lyapunov function (*Lemma* 2.1.2.1) and thereby asymptotic stability of the system (*Theorem* 2.1.2.6 and *Theorem* 2.1.2.7) can be determined based on the knowledge of **only one or any,** solution of the particular nonlinear matrix equation.
We now demonstrate that *Lemma* 2.1.2.1 should be improved since it does not take into account all possible solutions for (17). The counterexample, based on original approach and supported by the *Lambert* function application, is given in (*Stojanovic & Debeljkovic* 2006), (*Debeljkovic & Stojanovic* 2008).
The final results, that we need in the sequel, should be:
Lemma 2.1.2.2 Suppose that there exist(s) the solution(s) $P_1(0)$ of (19) and let the Lyapunov's function be (13). Then, $\dot{V}(\mathbf{x}_t, \tau) < 0$ **if and only if** for any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (5) holds for **all** solution(s) $P_1(0)$, (*Stojanovic & Debeljkovic* 2006) and (*Debeljkovic & Stojanovic* 2008).

Remark 2.1.2.1 The necessary condition of *Lemma* 2.1.2.2. follows directly from the proof of *Theorem* 2 in (*Lee & Dianat* 1981) and (*Stojanovic & Debeljkovic* 2006).
Theorem 2.1.2.8 Suppose that there exist(s) the solution(s) of $P_1(0)$ of (17). Then, the system (1) is asymptotically stable if for any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (14) holds for **all** solutions $P_1(0)$ of (17), (*Stojanovic & Debeljkovic* 2006) and (*Debeljkovic & Stojanovic* 2008).
Remark 2.1.2.2 Statements *Lemma* 2.1.2.2. and *Theorems* 2.1.2.7 and *Theorems* 2.1.2.8 require that corresponding conditions are fulfilled for any solution $P_1(0)$ of (17) .

These matrix conditions are analogous to the following known scalar condition of asymptotic stability.
System (1) *is asymptotically stable* iff *the condition* $\mathrm{Re}(s) < 0$ holds for **all** solutions s of :

$$f(s) = \det\left(sI - A_0 - e^{-s\tau} A_1\right) = 0 \,. \tag{20}$$

Now, we can present our main result, concerning practical stability of system (1).
Theorem 2.1.2.9 System (1) with initial function (2), is *attractive practically stable* with respect to $\left\{t_0, \Im, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if there exist a positive real number q, $q > 1$, such that:

$$\left\|\mathbf{x}(t+\tau)\right\|_{P_0} \le \sup_{\vartheta\in[-\tau,0]} \left\|\mathbf{x}(t+\vartheta)\right\|_{P_0} < q\left\|\mathbf{x}(t)\right\|_{P_0}, \ q > 1, \ t \ge t_0 \ , \ \forall t \in \Im \ , \ \forall \mathbf{x}(t) \in \mathcal{S}_\beta, \tag{21}$$

and if for any matrix $Q = Q^* > 0$ there exists matrix $P_0 = P_0^* > 0$ such that (14) holds for **all** solutions $P_1(0)$ of (17) and if the following conditions are satisfied (*Debeljkovic et al.* 2011.b):

$$e^{\bar{\lambda}_{\max}(\Upsilon)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \Im \,, \tag{22}$$

where:

$$\bar{\lambda}_{\max}\left(\Upsilon\right) = \bar{\lambda}_{\max}\left(\mathbf{x}^T(t)\left(P_0 A_1 P_0^{-1} A_1^T P_0 + q^2 P_0\right)\mathbf{x}(t) : \ \mathbf{x}^T(t) P_0 \mathbf{x}(t) = 1\right), \tag{23}$$

Proof. Define tentative aggregation function, as:

$$\begin{aligned} V(\mathbf{x}_t,\tau) &= \mathbf{x}^T(t)P_0\mathbf{x}(t) + \int_0^\tau\int_0^\tau \mathbf{x}^T(t-\nu)P_1^T(\nu)P_0P_1(\eta)\mathbf{x}(t-\eta)\,d\nu\, d\eta \\ &+ \mathbf{x}^T(t)P_0\int_0^\tau P_1(\eta)\mathbf{x}(t-\eta)\,d\eta + \int_0^\tau \mathbf{x}^T(t-\eta)P_1(\eta)\,d\eta \end{aligned} \tag{24}$$

The total derivative $\dot{V}(t,\mathbf{x}(t))$ along the trajectories of the system, yields[1]

[1] Under conditions of Lemma 2.1.2.1.

$$\dot{V}(\mathbf{x}_t,\tau)=\left(\mathbf{x}(t)+\int_0^{\tau}P_1(\eta)\mathbf{x}(t-\eta)d\eta\right)^T\times(-Q)\times\left(\mathbf{x}(t)+\int_0^{\tau}P_1(\eta)\mathbf{x}(t-\eta)d\eta\right), \tag{25}$$

and since, $(-Q)$ is negative definite and obviously $\dot{V}(\mathbf{x}_t,\tau)<0$, time delay system (1) possesses atractivity property.
Furthermore, it is obvious that

$$\begin{aligned}\frac{dV(\mathbf{x}_t,\tau)}{dt}&=\frac{d\left(\mathbf{x}^T(t)P_0\mathbf{x}(t)\right)}{dt}+\frac{d}{dt}\left(\int_0^{\tau}\int_0^{\tau}\mathbf{x}^T(t-\nu)P_1^T(\nu)P_0P_1(\eta)\mathbf{x}(t-\eta)d\nu d\eta\right.\\&\left.+\mathbf{x}^T(t)P_0\int_0^{\tau}P_1(\eta)\mathbf{x}(t-\eta)d\eta+\int_0^{\tau}\mathbf{x}^T(t-\eta)P_1(\eta)d\eta\right)\end{aligned} \tag{26}$$

so, the standard procedure, leads to:

$$\frac{d}{dt}\left(\mathbf{x}^T(t)P_0\mathbf{x}(t)\right)=\mathbf{x}^T(t)\left(A_0^TP_0+P_0A_0\right)\mathbf{x}(t)+2\mathbf{x}^T(t)P_0A_1\mathbf{x}(t-\tau),\quad \text{or} \tag{27}$$

$$\frac{d}{dt}\left(\mathbf{x}^T(t)P_0\,\mathbf{x}(t)\right)=\mathbf{x}^T(t)\left(A_0^TP_0+P_0A_0+Q\right)\mathbf{x}(t)+2\mathbf{x}^T(t)P_0A_1\mathbf{x}(t-\tau)-\mathbf{x}^T(t)Q\mathbf{x}(t) \tag{28}$$

From the fact that the time delay system under consideration, upon the statement of the *Theorem*, is asymptotically stable [2], follows:

$$\frac{d}{dt}\left(\mathbf{x}^T(t)P_0\,\mathbf{x}(t)\right)=-\mathbf{x}^T(t)Q\mathbf{x}(t)+2\mathbf{x}^T(t)P_0A_1\mathbf{x}(t-\tau), \tag{29}$$

and using very well known inequality [3], with particular choice:

$$\mathbf{x}^T(t)\Gamma\mathbf{x}^T(t)=\mathbf{x}^T(t)P_0\mathbf{x}^T(t)>0,\ \forall t\in\Im, \tag{30}$$

and the fact that:

$$\mathbf{x}^T(t)Q\mathbf{x}(t)>0,\ \forall t\in\Im, \tag{31}$$

is positive definite quadratic form, one can get:

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{x}^T(t)P_0\,\mathbf{x}(t)\right)&=2\mathbf{x}^T(t)P_0A_1\mathbf{x}(t-\tau)\\&\le\mathbf{x}^T(t)P_0A_1P_0^{-1}A_1^TP_0\,\mathbf{x}(t)+\mathbf{x}^T(t-\tau)P_0\mathbf{x}(t-\tau)\end{aligned} \tag{32}$$

and using (21), (*Su & Huang* 1992), (*Xu &Liu* 1994) and (*Mao* 1997), clearly (32) reduces to:

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[2] Clarify *Theorem 2.1.2.8.*

[3] $2\mathbf{u}^T(t)\mathbf{v}(t-\tau)\le\mathbf{u}^T(t)\Gamma^{-1}\mathbf{u}(t)+\mathbf{v}^T(t-\tau)\Gamma\mathbf{v}(t-\tau),\ \Gamma=\Gamma^T>0$.

$$\frac{d}{dt}\left(\mathbf{x}^T(t)P_0\mathbf{x}(t)\right) < \mathbf{x}^T(t)\left(P_0A_1P_0^{-1}A_1^TP_0 + q^2P\right)\mathbf{x}(t), \tag{33}$$

or, using (22), one can get:

$$\frac{d}{dt}\left(\mathbf{x}^T(t)P_0\mathbf{x}(t)\right) < \bar{\lambda}_{\max}\left(\bar{\Upsilon}\right)\mathbf{x}^T(t)P_0\mathbf{x}(t), \tag{34}$$

or:

$$\int_{t_0}^{t}\frac{d\left(\mathbf{x}^T(t)P_0\mathbf{x}(t)\right)}{\mathbf{x}^T(t)P_0\mathbf{x}(t)} < \int_{t_0}^{t}\bar{\lambda}_{\max}\left(\bar{\Upsilon}\right)dt, \tag{35}$$

and:

$$\mathbf{x}^T(t)P_0\mathbf{x}(t) < \mathbf{x}^T(t_0)P_0\mathbf{x}(t_0)e^{\bar{\lambda}_{\max}\left(\bar{\Upsilon}\right)(t-t_0)}. \tag{36}$$

Finally, if one applies the first condition, given in *Definition* 2.1.2.7 , and then:

$$\mathbf{x}^T(t)P_0\mathbf{x}(t) < \alpha\cdot e^{\bar{\lambda}_{\max}\left(\bar{\Upsilon}\right)(t-t_0)}, \tag{37}$$

and by applying the basic condition (22) of the *Theorem* 2.1.2.9, one can get

$$\mathbf{x}^T(t)P_0\mathbf{x}(t) < \alpha\cdot\frac{\beta}{\alpha} < \beta, \quad \forall t\in\Im. \quad Q.E.D. \tag{38}$$

STABILITY THEOREMS - Independent delay stability conditions

Theorem 2.1.2.10 Time delayed system (1), is *finite time stable* w.r.t. $\left\{t_0,\Im,\alpha,\beta,\left\|(\cdot)\right\|^2\right\}$, $\alpha<\beta$, if there exist a positive real number q, $q>1$, such that:

$$\left\|\mathbf{x}(t+\tau)\right\| \le \sup_{\vartheta\in[-\tau,0]}\left\|\mathbf{x}(t+\vartheta)\right\| < q\left\|\mathbf{x}(t)\right\|,\ q>1,\ t\ge t_0,\forall t\in\Im,\ \forall\mathbf{x}(t)\in\mathcal{S}_\beta, \tag{39}$$

if the following condition is satisfied (*Debeljkovic et al.* 2010):

$$e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t\in\Im, \tag{40}$$

where:

$$\lambda_{\max}(\Pi) = \lambda_{\max}\left(A_0^T + A_0 + A_1A_1^T + q^2I\right). \tag{41}$$

Proof. Define tentative aggregation function as:

$$V\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta. \tag{42}$$

The total derivative $\dot{V}\left(t,\mathbf{x}(t)\right)$ along the trajectories of the system, yields:

$$\begin{aligned}\dot{V}\left(t,\mathbf{x}(t)\right) &= \frac{d}{dt}\left(\mathbf{x}^T(t)\mathbf{x}(t)\right) + \frac{d}{dt}\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta \\ &= \mathbf{x}^T(t)\left(A_0^T + A_0\right)\mathbf{x}(t) + 2\mathbf{x}^T(t)A_1\mathbf{x}(t-\tau) + \mathbf{x}^T(t)\mathbf{x}(t) + \mathbf{x}^T(t-\tau)\mathbf{x}(t-\tau).\end{aligned} \tag{43}$$

From (43), it is obvious:

$$\frac{d}{dt}\left(\mathbf{x}^T(t)\mathbf{x}(t)\right) = \mathbf{x}^T(t)\left(A_0^T + A_0\right)\mathbf{x}(t) + 2\mathbf{x}^T(t)A_1\mathbf{x}(t-\tau), \tag{44}$$

and based on the previous inequality and with the particular choice:

$$\mathbf{x}^T(t)\Gamma\mathbf{x}(t) = \mathbf{x}^T(t)\mathbf{x}(t) > 0,\ \forall t \in \Im, \quad \text{so that} \tag{45}$$

$$\frac{d}{dt}\left(\mathbf{x}^T(t)\mathbf{x}(t)\right) \le \mathbf{x}^T(t)\left(A_0^T + A_0\right)\mathbf{x}(t) + \mathbf{x}^T(t)A_1A_1^T\mathbf{x}(t) + \mathbf{x}^T(t-\tau)I\mathbf{x}(t-\tau), \tag{46}$$

Based on (39), (*Su & Huang* 1992), (*Xu & Liu* 1994) and (*Mao* 1997), it is clear that (46) reduces to:

$$\frac{d}{dt}\left(\mathbf{x}^T(t)\mathbf{x}(t)\right) < \mathbf{x}^T(t)\left(A_0^T + A_0 + A_1A_1^T + q^2I\right)\mathbf{x}(t) < \lambda_{\max}(\Pi)\mathbf{x}^T(t)\mathbf{x}(t), \tag{47}$$

where matrix Π is defined by (41). From (47) one can get:

$$\int_{t_0}^{t}\frac{d\left(\mathbf{x}^T(t)\mathbf{x}(t)\right)}{\mathbf{x}^T(t)\mathbf{x}(t)} < \int_{t_0}^{t}\lambda_{\max}(\Pi)dt, \quad \text{and:} \tag{48}$$

$$\mathbf{x}^T(t)\mathbf{x}(t) < \mathbf{x}^T(t_0)\mathbf{x}(t_0)e^{\lambda_{\max}(\Pi)(t-t_0)} < \alpha\cdot e^{\lambda_{\max}(\Pi)(t-t_0)} < \alpha\cdot\frac{\beta}{\alpha} < \beta, \quad \forall t \in \Im. \tag{49}$$

under the identical technique from the previous proof of *Theorem* 2.1.2.9. *Q.E.D.*

2.2 Discrete time delay systems

2.2.1 Discrete time delay systems – stability in the sense of Lyapunov

ASYMPTOTIC STABILITY-APPROACH BASED ON THE RESULTS OF TISSIR AND HMAMED[4]

In particular case we are concerned with a linear, autonomous, multivariable discrete time delay system in the form:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1), \tag{50}$$

[4] (*Tissir & Hmamed* 1996).

The equation (50) is referred to as homogenous or the unforced state equation, $\mathbf{x}(k)$ is the state vector, A_0 and A_1 are constant system matrices of appropriate dimensions.
Theorem 2.2.1.1. System (50) is asymptotically stable if:

$$\|A_0\| + \|A_1\| < 1, \tag{51}$$

holds, (*Mori et al.* 1981).
Theorem 2.2.1.2. System (50) is asymptotically stable, independent of delay, if:

$$\|A_1\| < \frac{\sigma_{\min}\left(Q^{-\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}A_0^T P\right)}, \tag{52}$$

where P is the solution of the *discrete Lyapunov matrix equation*:

$$A_0^T P A_0 - P = -\left(2Q + A_1^T P A_1\right), \tag{53}$$

where $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, (*Debeljkovic et al.* 2004.a, 2004.b, 2004.d, 2005.a).
Theorem 2.2.1.3 Suppose the matrix $\left(Q - A_1^T P A_1\right)$ is regular.
System (50) is asymptotically stable, independent of delay, if:

$$\|A_1\| < \frac{\sigma_{\min}\left(\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}A_0^T P\right)}, \tag{54}$$

where P is the solution of the *discrete Lyapunov matrix equation*:

$$A_0^T P A_0 - P = -2Q, \tag{55}$$

where $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, (*Debeljkovic et al.* 2004.c, 2004.d, 2005.a, 2005.b).

ASYMPTOTIC STABILTY- LYAPUNOV BASED APPROACH

A linear, autonomous, multivariable linear discrete time-delay system can be represented by the difference equation:

$$\mathbf{x}(k+1) = \sum_{j=0}^{N} A_j \mathbf{x}\left(k - h_j\right), \quad \mathbf{x}(\vartheta) = \boldsymbol{\psi}(\vartheta), \quad \vartheta \in \{-h_N, -h_N + 1, \ldots, 0\} \triangleq \Delta, \tag{56}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n \times n}$, $0 = h_0 < h_1 < h_2 < \ldots < h_N$ - are integers and represent the systems time delays. Let $V(\mathbf{x}(k))$: $\mathbb{R}^n \to \mathbb{R}$, so that $V(\mathbf{x}(k))$ is bounded for, and for which $\|\mathbf{x}(k)\|$ is also bounded.

Lemma 2.2.1.1 For any two matrices of the same dimenssions F and G and for some pozitive constant ε the following statement is true (*Wang & Mau* 1997):

$$(F+G)^T(F+G)\le(1+\varepsilon)F^TF+\left(1+\varepsilon^{-1}\right)G^TG\,. \tag{57}$$

Theorem 2.2.1.4 Suppose that A_0 is not null matrix. If for any given matrix $Q=Q^T>0$ there exists matrix $P=P^T>0$ such that the following matrix equation is fulfilled:

$$(1+\varepsilon_{\min})A_0^TPA_0+\left(1+\varepsilon_{\min}^{-1}\right)A_1^TPA_1-P=-Q\,, \tag{58}$$

where:

$$\varepsilon_{\min}=\frac{\|A_1\|_2}{\|A_0\|_2}\,, \tag{59}$$

then, system (56) is asymptotically stable, (*Stojanovic & Debeljkovic* 2005.b).

Corollary 2.2.1.1 If for any given matrix $Q=Q^T>0$ there exists matrix $P=P^T>0$ being the solution of the following Lyapunov matrix equation:

$$A_0^TPA_0-P=-\frac{\varepsilon_{\min}}{1+\varepsilon_{\min}}Q\,, \tag{60}$$

where $\varepsilon_{\min}$ is defined by (59) and if the following condition is satisfied:

$$\sigma_{\max}(A_0)+\sigma_{\max}(A_1)<\frac{\lambda_{\min}(Q-P)}{\sigma_{\max}(A_0)\lambda_{\max}(P)}\,, \tag{61}$$

then, system (59) is asymptotically stable, (*Stojanovic & Debeljkovic* 2005.b).

Corollary 2.2.1.2 If for any given matrix $Q=Q^T>0$ there exists matrix $P=P^T>0$ being solution of the following matrix equation:

$$(1+\varepsilon_{\min})A_0^TPA_0-P=-\varepsilon_{\min}Q\,, \tag{62}$$

where $\varepsilon_{\min}$ is defined by (59), and if the following condition is satisfied, too:

$$\sigma_{\max}(A_0)+\sigma_{\max}(A_1)<\frac{\lambda_{\min}(Q)}{\sigma_{\max}(A_0)\lambda_{\max}(P)}\,, \tag{63}$$

then, system (56) is asymptotically stable, (*Stojanovic & Debeljkovic* 2005.b).

Theorem 2.2.1.5 If for any given matrix $Q=Q^T>0$ there exists matrix $P=P^T>0$ such that the following matrix equation is fulfilled:

$$2A_0^TPA_0+2A_1^TPA_1-P=-Q\,, \tag{64}$$

then, system (56) is asymptotically stable, (*Stojanovic & Debeljkovic* 2006.a).

Corollary 2.2.1.3 System (56) is asymptotically stable, independent of delay, if :

$$\sigma_{\max}^2\left(A_1\right)<\frac{\lambda_{\min}\left(2Q-P\right)}{2\sigma_{\max}^2\left(P^{\frac{1}{2}}\right)}, \tag{65}$$

where, for any given matrix $Q=Q^T>0$ there exists matrix $P=P^T>0$ being the solution of the following *Lyapunov matrix equation* (*Stojanovic & Debeljkovic* 2006.a):

$$A_0^T P A_0 - P = -Q\,. \tag{66}$$

Corollary 2.2.1.4 System (56) is asymptotically stable, independent of delay, if:

$$\sigma_{\max}^2\left(A_1\right)<\frac{\lambda_{\min}\left(Q\right)}{2\sigma_{\max}^2\left(P^{\frac{1}{2}}\right)}, \tag{67}$$

where, for any given matrix $Q=Q^T>0$ there exists matrix $P=P^T>0$ being the solution of the following *Lyapunov matrix equation* (*Stojanovic & Debeljkovic* 2006.a):

$$2A_0^T P A_0 - P = -Q\,. \tag{68}$$

2.2.2 Discrete time delay systems – Stability over finite time interval

As far as we know the only result, considering and investigating the problem of non-Lyapunov analysis of linear discrete time delay systems, is one that has been mentioned in the introduction, e.g. (*Debeljkovic & Aleksendric* 2003), where this problem has been considered for the first time.

Investigating the system stability throughout the discrete fundamental matrix is very cumbersome, so there is a need to find some more efficient expressions that should be based on calculation appropriate eigenvalues or norm of appropriate systems matrices as it has been done in continuous case.

SYSTEM DESCRIPTION

Consider a linear discrete system with state delay, described by:

$$\mathbf{x}\left(k+1\right)=A_0\,\mathbf{x}\left(k\right)+A_1\,\mathbf{x}\left(k-1\right), \tag{69}$$

with known vector valued function of initial conditions:

$$\mathbf{x}\left(k_0\right)=\boldsymbol{\psi}\left(k_0\right),\ \ -1\le k_0\le 0\,, \tag{70}$$

where $\mathbf{x}\left(k\right)\in\mathbb{R}^n$ is a state vector and with constant matrices A_0 and A_1 of appropriate dimensions. Time delay is constant and equals one. For some other purposes, the state delay equation can be represented in the following way:

$$\mathbf{x}\left(k+1\right)=A_0\,\mathbf{x}\left(k\right)+\sum_{j=1}^{M}A_j\,\mathbf{x}\left(k-h_j\right), \tag{71}$$

$$\mathbf{x}(\vartheta) = \boldsymbol{\psi}(\vartheta), \ \ \vartheta \in \{-h, -h+1, \ldots, 0\}, \tag{72}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n \times n}$, $j = 1,2$, h – is integer representing system time delay and $\boldsymbol{\psi}(\cdot)$ is a priori known vector function of initial conditions, as well.

STABILITY DEFINITIONS

Definition 2.2.2.1 System, given by (69), is *attractive practically stable* with respect to $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, iff $\|\mathbf{x}(k_0)\|^2_{A_0^T P A_0} = \|\mathbf{x}_0\|^2_{A_0^T P A_0} < \alpha$, implies:

$$\|\mathbf{x}(k)\|^2_{A_0^T P A_0} < \beta, \quad \forall k \in \mathcal{K}_N$$

with property that $\lim_{k \to \infty} \|\mathbf{x}(k)\|^2_{A_0^T P A_0} \to 0$, (*Nestorovic et al*. 2011).

Definition 2.2.2.2 System, given by (69), is *practically stable* with respect to $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, if and only if: $\|\mathbf{x}_0\|^2 < \alpha$, implies $\|\mathbf{x}(k)\|^2 < \beta, \ \ \forall k \in \mathcal{K}_N$.

Definition 2.2.2.3 System given by (69), is *attractive practically unstable* with respect $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if for $\|\mathbf{x}_0\|^2_{A_0^T P A_0} < \alpha$, there exist a moment: $k = k^* \in \mathcal{K}_N$, so that the next condition is fulfilled $\left\|\mathbf{x}\left(k^*\right)\right\|^2_{A_0^T P A_0} \geq \beta$ with property that $\lim_{k \to \infty} \|\mathbf{x}(k)\|^2_{A_0^T P A_0} \to 0$, (*Nestorovic et al*. 2011).

Definition 2.2.2.4 System given by (69), is *practically unstable* with respect $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if for $\|\mathbf{x}_0\|^2 < \alpha$ there exist a moment: $k = k^* \in \mathcal{K}_N$, such that the next condition is fulfilled $\left\|\mathbf{x}\left(k^*\right)\right\|^2 \geq \beta$ for some $k = k^* \in \mathcal{K}_N$.

Definition 2.2.2.5 Linear discrete time delay system (69) is *finite time stable* with respect to $\{\alpha, \beta, k_0, k_N, \|(\cdot)\|\}$, $\alpha \leq \beta$, if and only if for every trajectory $\mathbf{x}(k)$ satisfying initial function, (70) such that $\|\mathbf{x}(k)\| < \alpha, \ \ k = 0, -1, -2, \cdots, -N$ imply $\|\mathbf{x}(k)\|^2 < \beta, \ \ k \in \mathcal{K}_N$, (*Aleksendric* 2002), (*Aleksendric & Debeljkovic* 2002), (*Debeljkovic & Aleksendric* 2003).
This Definition is analogous to that presented, for the first time, in (*Debeljković et al*. 1997.a, 1997.b, 1997.c, 1997.d) and (*Nenadic et al*. 1997).

SOME PREVIOUS RESULTS

Theorem 2.2.2.1 Linear discrete time delay system (69), is *finite time stable* with respect to $\left\{\alpha, \beta, M, N, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}_+$, if the following sufficient condition is fulfilled:

$$\|\Phi(k)\| < \frac{\beta}{\alpha} \cdot \frac{1}{1 + \sum_{j=1}^{M} \|A_j\|}, \ \forall k = 0, 1, \cdots, N, \tag{73}$$

$\Phi(k)$ being fundamental matrix, (*Aleksendric* 2002), (*Aleksendric & Debeljkovic* 2002), (*Debeljkovic & Aleksendric* 2003).
This result is analogous to that, for the first time derived, in (*Debeljkovic et al*. 1997.a) for continuous time delay systems.
Remark 2.2.2.1 The matrix measure is widely used when continuous time delay system are investigated, (*Coppel* 1965), (*Desoer & Vidysagar* 1975). The nature of discrete time delay enables one to use this approach as well as Bellman's principle, so the problem must be attack from the point of view which is based only on norms.

STABILITY THEOREMS: PRACTICAL AND FINITE TIME STABILITY

Theorem 2.2.2.2 System given by (71), with $\det A_1 \neq 0$, is *attractive practically stable* with respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if there exist $P = P^T > 0$, being the solution of:

$$2A_0^T P A_0 - P = -Q\,, \tag{74}$$

where $Q = Q^T > 0$ and if the following conditions are satisfied (*Nestorovic et al*. 2011):

$$\left\| A_1 \right\| < \sigma_{\min}\left(\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}} \right) \sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}} A_0^T P \right), \tag{75}$$

$$\bar{\lambda}_{\max}^{\frac{1}{2}k}(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in K_N\,, \tag{76}$$

where:

$$\bar{\lambda}_{\max}(\ ) = \max\left\{ \mathbf{x}^T(k) A_1^T P A_1 \mathbf{x}(k) : \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) = 1 \right\}. \tag{77}$$

Proof. Let us use a functional, as a possible aggregation function, for the system to be considered:

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k) P \mathbf{x}(k) + \mathbf{x}^T(k-1) Q \mathbf{x}(k-1)\,, \tag{78}$$

with matrices $P = P^T > 0$ and $Q = Q^T > 0$.
Clearly, using the equation of motion of (69), we have:

$$\Delta V\left(\mathbf{x}(k)\right) = V\left(\mathbf{x}(k+1)\right) - V\left(\mathbf{x}(k)\right)\,, \tag{79}$$

or:

$$\begin{aligned} \Delta V\left(\mathbf{x}(k)\right) &= \mathbf{x}^T(k+1) P \mathbf{x}(k+1) - \mathbf{x}^T(k) P \mathbf{x}(k) \\ &\quad + \mathbf{x}^T(k) Q \mathbf{x}(k) - \mathbf{x}^T(k-1) Q \mathbf{x}(k-1) \\ &= \mathbf{x}^T(k)\left(A_0^T P A_0 + Q - P\right)\mathbf{x}(k) + 2\mathbf{x}^T(k) A_0^T P A_1 \mathbf{x}(k-1) \\ &\quad - \mathbf{x}^T(k-1)\left(Q - A_1^T P A_1\right)\mathbf{x}(k-1) \end{aligned} \tag{80}$$

It has been shown, (*Debeljković et al*. 2004, 2008), that if:

$$2A_0^T P A_0 - P = -Q\ , \tag{81}$$

where $P = P^T > 0$ and $Q = Q^T > 0$, then for:

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k)P\mathbf{x}(k) + \mathbf{x}^T(k-1)Q\mathbf{x}(k-1)\ , \tag{82}$$

the backward difference along the trajectories of the systems is:

$$\begin{aligned}\Delta V\left(\mathbf{x}(k)\right) &= V\left(\mathbf{x}(k+1)\right) - V\left(\mathbf{x}(k)\right)\\ &= \mathbf{x}^T(k)\left(A_0^T P A_0 - P + Q\right)\mathbf{x}(k) + \mathbf{x}^T(k-1)\left(A_1^T P A_1 - Q\right)\mathbf{x}(k-1)\\ &+\mathbf{x}^T(k)A_0^T P A_1\mathbf{x}(k-1) + \mathbf{x}^T(k-1)A_1^T P A_0\mathbf{x}(k)\end{aligned} \tag{83}$$

or:

$$\begin{aligned}\Delta V\left(\mathbf{x}(k)\right) &= \mathbf{x}^T(k)\left(2A_0^T P A_0 - P + Q\right)\mathbf{x}(k)\\ &+\mathbf{x}^T(k-1)\left(2A_1^T P A_1 - Q\right)\mathbf{x}(k-1) + \mathbf{x}^T(k)A_0^T P A_1\mathbf{x}(k-1)\\ &+\mathbf{x}^T(k-1)A_1^T P A_0\mathbf{x}(k) - \mathbf{x}^T(k)A_0^T P A_0\mathbf{x}(k) - \mathbf{x}^T(k-1)A_1^T P A_1\mathbf{x}(k-1)\end{aligned} \tag{84}$$

and since we have to take into account (80), one can get:

$$\begin{aligned}\Delta V\left(\mathbf{x}(k)\right) &= \mathbf{x}^T(k-1)\left(2A_1^T P A_1 - Q\right)\mathbf{x}(k-1)\\ &-\left[A_0\mathbf{x}(k) - A_1\mathbf{x}(k-1)\right]^T P\left[A_0\mathbf{x}(k) - A_1\mathbf{x}(k-1)\right].\end{aligned} \tag{85}$$

Since the matrix $P = P^T > 0$, it is more than obvious, that:

$$\Delta V\left(\mathbf{x}(k)\right) < \mathbf{x}^T(k-1)\left(2A_1^T P A_1 - Q\right)\mathbf{x}(k-1)\,. \tag{86}$$

Combining the right sides of (80) and (86), yields:

$$\begin{aligned}\Delta V\left(\mathbf{x}(k)\right) &= \mathbf{x}^T(k)\left(A_0^T P A_0 + Q - P\right)\mathbf{x}(k) + 2\mathbf{x}^T(k)A_0^T P A_1\mathbf{x}(k-1)\\ &< \mathbf{x}^T(k-1)\left(A_1^T P A_1\right)\mathbf{x}(k-1)\end{aligned} \tag{87}$$

Using the very well known inequality, with particular choice:

$$\Gamma = \frac{1}{2}\left(A_1^T P A_1\right), \tag{88}$$

it can be obtained:

$$\begin{aligned}&\mathbf{x}^T(k)\left(A_0^T P A_0 + Q - P + A_0^T P A_1\left(\frac{1}{2}A_1^T P A_1\right)^{-1} A_1^T P A_0\right)\mathbf{x}(k)\\ &+\frac{1}{2}\mathbf{x}^T(k-1)\left(A_1^T P A_1\right)\mathbf{x}(k-1) < \mathbf{x}^T(k-1)\left(A_1^T P A_1\right)\mathbf{x}(k-1)\end{aligned} \tag{89}$$

$$\mathbf{x}^T(k)\left(2A_0^TPA_0+Q-P+A_0^TPA_0\right)\mathbf{x}(k)<\frac{1}{2}\mathbf{x}^T(k-1)\left(A_1^TPA_1\right)\mathbf{x}(k-1). \tag{90}$$

Since:
$$2A_0^TPA_0+Q-P=0, \tag{91}$$

it is finally obtained:

$$\mathbf{x}^T(k)A_0^TPA_0\,\mathbf{x}(k)<\frac{1}{2}\mathbf{x}^T(k-1)\left(A_1^TPA_1\right)\mathbf{x}(k-1), \tag{92}$$

or:
$$\mathbf{x}^T(k)A_0^TPA_0\,\mathbf{x}(k)<\frac{1}{2}\bar{\lambda}_{\max}(\ )\mathbf{x}^T(k-1)A_0^TPA_0\,\mathbf{x}(k-1), \tag{93}$$

where:

$$\bar{\lambda}_{\max}(\ )=\max\left\{\mathbf{x}^T(k)A_1^TPA_1\mathbf{x}(k):\left(2A_0^TPA_0-P\right)=-Q,\ \mathbf{x}^T(k)A_0^TPA_0\,\mathbf{x}(k)=1\right\}. \tag{94}$$

Since this manipulation is independent of k, it can be written:

$$\mathbf{x}^T(k+1)A_0^TPA_0\,\mathbf{x}(k+1)<\frac{1}{2}\bar{\lambda}_{\max}(\ )\mathbf{x}^T(k)A_0^TPA_0\,\mathbf{x}(k), \tag{95}$$

or:

$$\begin{aligned}\ln\mathbf{x}^T(k+1)A_0^TPA_0\,\mathbf{x}(k+1)&<\ln\frac{1}{2}\bar{\lambda}_{\max}(\ )\mathbf{x}^T(k)A_0^TPA_0\,\mathbf{x}(k)\\&<\ln\frac{1}{2}\bar{\lambda}_{\max}(\ )+\ln\mathbf{x}^T(k)A_0^TPA_0\,\mathbf{x}(k)\end{aligned} \tag{96}$$

and:

$$\ln\mathbf{x}^T(k+1)A_0^TPA_0\,\mathbf{x}(k+1)-\ln\mathbf{x}^T(k)A_0^TPA_0\,\mathbf{x}(k)<\ln\bar{\lambda}_{\max}^{\frac{1}{2}}(\ ). \tag{97}$$

It can be shown that:

$$\begin{aligned}&\sum_{j=k_0}^{k_0+k-1}\left(\ln\mathbf{x}^T(j+1)\mathbf{x}(j+1)-\ln\mathbf{x}^T(j)\mathbf{x}(j)\right)=\\&=\ln\mathbf{x}^T(k_0+1)\mathbf{x}(k_0+1)+\ln\mathbf{x}^T(k_0+2)\mathbf{x}(k_0+2)+\ldots+\\&+\ln\mathbf{x}^T(k_0+k-2+1)\mathbf{x}(k_0+k-2+1)+\ln\mathbf{x}^T(k_0+k-1+1)\mathbf{x}(k_0+k-1+1)\\&-\left(\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0)+\ln\mathbf{x}^T(k_0+1)\mathbf{x}(k_0+1)+\ldots+\ln\mathbf{x}^T(k_0+k-1)\mathbf{x}(k_0+k-1)\right)\\&=\ln\mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k)-\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0)\end{aligned} \tag{98}$$

If the summing $\sum_{j=k_0}^{k_0+k-1}$ is applied to both sides of (97) for $\forall k\in\mathcal{K}_N$, one can obtain:

$$\sum_{j=k_0}^{k_0+k-1} \ln \mathbf{x}^T(k+1) A_0^T P A_0 \mathbf{x}(k+1) - \ln \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k)$$
$$\leq \sum_{j=k_0}^{k_0+k-1} \ln \lambda_{\max}^{\frac{1}{2}}(\ ) \leq \ln \prod_{j=k_0}^{k_0+k-1} \lambda_{\max}^{\frac{1}{2}}(\ ) \tag{99}$$

so that, for (99), it seems to be:

$$\ln \mathbf{x}^T(k_0+k) A_0^T P A_0 \mathbf{x}(k_0+k) - \ln \mathbf{x}^T(k_0) A_0^T P A_0 \mathbf{x}(k_0)$$
$$< \ln \prod_{j=k_0}^{k_0+k-1} \bar{\lambda}_{\max}^{\frac{1}{2}}(\ ) < \ln \bar{\lambda}_{\max}^{\frac{1}{2}k}(\ ), \quad \forall k \in \mathcal{K}_N \tag{100}$$

as well as:

$$\ln \mathbf{x}^T(k_0+k) A_0^T P A_0 \mathbf{x}(k_0+k) \leq \ln \prod_{j=k_0}^{k_0+k-1} \bar{\lambda}_{\max}^{\frac{1}{2}}(\ )$$
$$\leq \ln \bar{\lambda}_{\max}^{\frac{1}{2}k}(\ ) + \ln \mathbf{x}^T(k_0) A_0^T P A_0 \mathbf{x}(k_0) \quad \forall k \in \mathcal{K}_N \tag{101}$$

Taking into account fact that $\|\mathbf{x}_0\|_{A_0^T P A_0}^2 < \alpha$ and basic condition of *Theorem* 2.2.2.2, (76), one can get:

$$\ln \mathbf{x}^T(k_0+k) A_0^T P A_0 \mathbf{x}(k_0+k) < \ln \bar{\lambda}_{\max}^{\frac{1}{2}k}(\ ) + \ln \mathbf{x}^T(k_0) A_0^T P A_0 \mathbf{x}(k_0)$$
$$< \ln \alpha \cdot \bar{\lambda}_{\max}^{\frac{1}{2}k}(\ ) < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in \mathcal{K}_N. \quad Q.E.D. \tag{102}$$

Remark 2.2.2.2 Assumption $\det A_1 \neq 0$ do not reduce the generality of this result, since this condition is not crucial when discrete time systems are considered.

Remark 2.2.2.3 Lyapunov asymptotic stability and finite time stability are independent concepts: a system that is finite time stable may not be Lyapunov asymptotically stable, conversely Lyapunov asymptotically stable system could not be finite time stable if, during the transients, its motion exceeds the pre-specified bounds (β). Attraction property is guaranteed by (74) and (75), (*Debeljković et al.* 2004) and system motion within pre-specified boundaries is well provided by (76).

Remark 2.2.2.4 For the numerical treatment of this problem $\bar{\lambda}_{\max}(\ )$ can be calculated in the following way (*Kalman*, *Bertram* 1960):

$$\bar{\lambda}_{\max}(\ ) = \max_{\mathbf{x}}\{\ \} = \bar{\lambda}_{\max}\left(A_1^T P A_1 \left(A_0^T P A_0 \right)^{-1} \right). \tag{103}$$

Remark 2.2.2.5 These results are in some sense analogous to those given in (*Amato et al.* 2003), although results presented there are derived for continuous time varying systems.

Now we proceed to develop delay independent criteria, for finite time stability of system under consideration, not to be necessarily asymptotic stable, e.g. so we reduce previous demand that basic system matrix A_0 should be discrete stable matrix.

Theorem 2.2.2.3 Suppose the matrix $\left(I - A_1^T A_1\right) > 0$. System given by (69), is *finite time stable* with respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if there exist a positive real number p, $p > 1$, such that:

$$\left\|\mathbf{x}(k-1)\right\|^2 < p^2 \left\|\mathbf{x}(k)\right\|^2, \ \forall k \in \mathcal{K}_N, \ \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \tag{104}$$

and if the following condition is satisfied (*Nestorovic et al*. 2011):

$$\lambda_{\max}^k(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \tag{105}$$

where:

$$\lambda_{\max}(\ ) = \lambda_{\max}\left(A_0^T\left(I - A_1^T A_1\right)A_0 + p^2 I\right). \tag{106}$$

Proof. Now we consider, again, system given by (69). Define:

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k)\mathbf{x}(k) + \mathbf{x}^T(k-1)\mathbf{x}(k-1), \tag{107}$$

as a tentative Lyapunov-like function for the system, given by (69).
Then, the $\Delta V\left(\mathbf{x}(k)\right)$ along the trajectory, is obtained as:

$$\begin{aligned}\Delta V\left(\mathbf{x}(k)\right) = V\left(\mathbf{x}(k+1)\right) - V\left(\mathbf{x}(k)\right) &= \mathbf{x}^T(k+1)\mathbf{x}(k+1) - \mathbf{x}^T(k-1)\mathbf{x}(k-1) \\ &= \mathbf{x}^T(k)A_0^T A_0\mathbf{x}(k) + 2\mathbf{x}^T(k)A_0^T A_1\mathbf{x}(k-1) \\ &+ \mathbf{x}^T(k-1)A_1^T A_1\mathbf{x}(k-1) - \mathbf{x}^T(k-1)\mathbf{x}(k-1)\end{aligned} \tag{108}$$

From (108), one can get:

$$\begin{aligned}\mathbf{x}^T(k+1)\mathbf{x}(k+1) &= \mathbf{x}^T(k)A_0^T A_0\mathbf{x}(k) \\ +2\mathbf{x}^T(k)A_0^T A_1\mathbf{x}(k-1) &+ \mathbf{x}^T(k-1)A_1^T A_1\mathbf{x}(k-1)\end{aligned} \tag{109}$$

Using the very well known inequality, with choice:

$$\Gamma = \left(I - A_1^T A_1\right) > 0, \tag{110}$$

I being the identity matrix, it can be obtained:

$$\begin{aligned}\mathbf{x}^T(k+1)\mathbf{x}(k+1) &\le \mathbf{x}^T(k)A_0^T A_0\mathbf{x}(k) + \\ \mathbf{x}^T(k)A_1\left(I - A_1^T A_1\right)^{-1} A_1^T\mathbf{x}(k) &+ \mathbf{x}^T(k-1)\mathbf{x}(k-1)\end{aligned} \tag{111}$$

and using assumption (104), it is clear that (111) reduces to:

$$\begin{aligned}\mathbf{x}^T(k+1)\mathbf{x}(k+1) &< \mathbf{x}^T(k)A_0^T\left(\left(I-A_1A_1^T\right)^{-1}+p^2I\right)A_0\,\mathbf{x}(k)\\ &< \lambda_{\max}\left(A_0,A_1,p\right)\mathbf{x}^T(k)\mathbf{x}(k)\end{aligned} \tag{112}$$

where:
$$\lambda_{\max}\left(A_0,A_1,p\right)=\lambda_{\max}\left(A_0^T\left(I-A_1A_1^T\right)^{-1}A_0+p^2I\right) \tag{113}$$

with obvious property, that gives the natural sense to this problem: $\lambda_{\max}\left(A_0,A_1,p\right)\geq 0$ when $\left(I-A_1A_1^T\right)\geq 0$.

Following the procedure from the previous section, it can be written:

$$\ln\mathbf{x}^T(k+1)\mathbf{x}(k+1)-\ln\mathbf{x}^T(k)\mathbf{x}(k)<\ln\lambda_{\max}(\ ). \tag{114}$$

By applying the sum $\sum_{j=k_0}^{k_0+k-1}$ on both sides of (112) for $\forall k\in\mathcal{K}_N$, one can obtain:

$$\ln\mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k)\leq\ln\prod_{j=k_0}^{k_0+k-1}\lambda_{\max}(\ )\leq\ln\lambda^{k}_{\max}(\ )+\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0),\ \forall k\in\mathcal{K}_N \tag{115}$$

Taking into account the fact that $\left\|\mathbf{x}_0\right\|^2<\alpha$ and condition of *Theorem* 2.2.2.3, (105), one can get:

$$\begin{aligned}\ln\mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) &< \ln\lambda_{\max}^k\left(A_0,A_1,p\right)+\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0)\\ &< \ln\alpha\cdot\lambda_{\max}^k\left(A_0,A_1,p\right)<\ln\alpha\cdot\frac{\beta}{\alpha}<\ln\beta,\ \ \forall k\in K_N\end{aligned} \tag{116}$$

Remark 2.2.2.6 In the case when A_1 is null matrix and $p=0$ result, given by (106), reduces to that given in (*Debeljkovic* 2001) earlier developed for ordinary discrete time systems.

Theorem 2.2.2.4 Suppose the matrix $\left(I-A_1^TA_1\right)>0$. System, given by (69), is *practically unstable* with respect to $\left\{k_0,\mathcal{K}_N,\alpha,\beta,\left\|(\cdot)\right\|^2\right\}$, $\alpha<\beta$, if there exist a positive real number p, $p>1$, such that:

$$\left\|\mathbf{x}(k-1)\right\|^2<p^2\left\|\mathbf{x}(k)\right\|^2,\ \forall k\in\mathcal{K}_N,\ \forall\mathbf{x}(k)\in\mathcal{S}_\beta, \tag{117}$$

and if there exist: real, positive number δ, $\delta\in\left]0,\alpha\right[$ and time instant $k,\ k=k^*:\exists!\left(k^*>k_0\right)\in\mathcal{K}_N$ for which the next condition is fulfilled:

$$\lambda_{\min}^{k^*}>\frac{\beta}{\delta},\quad k^*\in\mathcal{K}_N. \tag{118}$$

Proof. Let:

$$V\left(\mathbf{x}(k)\right)=\mathbf{x}^T(k)\mathbf{x}(k)+\mathbf{x}^T(k-1)\mathbf{x}(k-1) \tag{119}$$

Then following the identical procedure as in the previous *Theorem*, one can get:

$$\ln\mathbf{x}^T(k+1)\mathbf{x}(k+1)-\ln\mathbf{x}^T(k)\mathbf{x}(k)>\ln\lambda_{\min}(\ ), \tag{120}$$

where:

$$\lambda_{\min}\left(A_0,A_1,p\right)=\lambda_{\min}\left(A_0^T\left(I-A_1A_1^T\right)^{-1}A_0+p^2I\right). \tag{121}$$

If we apply the summing $\sum_{j=k_0}^{k_0+k-1}$ on both sides of (120) for $\forall k\in\mathcal{K}_N$, one can obtain:

$$\ln\mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k)\geq\ln\prod_{j=k_0}^{k_0+k-1}\lambda_{\min}(\ )\geq\ln\lambda_{\min}^{k}(\ )+\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0),\ \forall k\in K_N. \tag{122}$$

It is clear that for any $\mathbf{x}_0$ follows: $\delta<\left\|\mathbf{x}_0\right\|^2<\alpha$ and for some $k^*\in\mathcal{K}_N$ and with (118), one can get:

$$\begin{aligned}&\ln\mathbf{x}^T\left(k_0+k^*\right)\mathbf{x}\left(k_0+k^*\right)>\ln\lambda_{\min}^{k^*}\left(A_0,A_1,p\right)+\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0)\\&>\ln\delta\cdot\lambda_{\min}^{k^*}\left(A_0,A_1,p\right)>\ln\delta\cdot\frac{\beta}{\delta}>\ln\beta,\ \exists!\,k^*\in\mathcal{K}_N.\ \ Q.E.D.\end{aligned} \tag{123}$$

3. Singular and descriptive time delay systems

Singular and descriptive systems represent very important classes of systems. Their stability was considered in detail in the previous chapter. Time delay phenomena, which often occur in real systems, may introduce instability, which must not be neglected. Therefore a special attention is paid to stability of singular and descriptive time delay systems, which are considered in detail in this section.

3.1 Continuous singular time delayed systems

3.1.1 Continuous singular time delayed systems – Stability in the sense of Lyapunov

Consider a linear continuous singular system with state delay, described by:

$$E\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau), \tag{124}$$

with known compatible vector valued function of initial conditions:

$$\mathbf{x}(t)=\boldsymbol{\psi}(t),\ -\tau\leq t\leq 0, \tag{125}$$

where A_0 and A_1 are constant matrices of appropriate dimensions.
Time delay is constant, e.g. $\tau\in\mathbb{R}_+$. Moreover we shall assume that $rank\,E=r<n$.

Definition 3.1.1.1 The matrix pair (E, A_0) is regular if $\det(sE - A_0)$ is not identically zero, (*Xu et al*. 2002.a).

Definition 3.1.1.2 The matrix pair (E, A_0) is impulse free if degree $\det(sE - A) = \operatorname{rank} E$, (*Xu et al*. 2002.a).

The linear continuous singular time delay system (124) may have an impulsive solution, however, the regularity and the absence of impulses of the matrix pair (E, A_0) ensure the existence and uniqueness of an impulse free solution to the system under consideration, which is defined in the following *Lemma*.

Lemma 3.1.1.1 Suppose that the matrix pair (E, A_0) is *regular* and *impulsive free* and *unique* on $\left[0, \infty\right[$, (*Xu et al.* 2002).

Necessity for system stability investigation makes need for establishing a proper stability definition. So one can has:

Definition 3.1.1.3 Linear continuous singular time delay system (124) is said to be *regular* and *impulsive free* if the matrix pair (E, A_0) is regular and impulsive free, (*Xu et al.* 2002.a).

STABILITY DEFINITIONS

Definition 3.1.1.4 If $\forall t_0 \in T$ and $\forall \varepsilon > 0$, there always exists $\delta(t_0, \varepsilon)$, such that $\forall \psi \in \mathbb{S}_\delta(0, \delta) \cap \mathbb{S}(t_0, t^*)$, the solution $\mathbf{x}(t, t_0, \boldsymbol{\psi})$ to (124) satisfies that $\left\| \mathbf{q}(t, \mathbf{x}(t)) \right\| \le \varepsilon$, $\forall t \in (t_0, t^*)$, then the zero solution to (124) is said to be stable on $\left\{ \mathbf{q}(t, \mathbf{x}(t)), T \right\}$, where $T = \left[0, +t^*\right]$, $0 < t^* \le +\infty$ and $\mathbb{S}_\delta(0, \delta) = \left\{ \boldsymbol{\psi} \in \mathcal{C}\left(\left[-\tau, 0\right], \mathbb{R}^n\right), \left\| \boldsymbol{\psi} \right\| < \delta, \delta > 0 \right\}$. $\mathbb{S}_*(t_0, t^*)$ is a set of all consistency initial functions and for $\forall \boldsymbol{\psi} \in \mathbb{S}_*(t_0, t^*)$, there exists a continuous solution to (122) in $\left[t_0 - \tau, t^*\right)$ through $(t_0, \boldsymbol{\psi})$ at least, (*Li & Liu* 1997, 1998).

Definition 3.1.1.5 If δ is only related to ε and has nothing to do with t_0, then the zero solution is said to be uniformly stable on $\left\{ \mathbf{q}(t, \mathbf{x}(t)), T \right\}$, (*Li & Liu* 1997, 1998).

Definition 3.1.1.6 Linear continuous singular time delay system (124) is said to be stable if for any $\varepsilon > 0$ there exist a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial conditions $\boldsymbol{\psi}(t)$, satisfying condition: $\sup_{-\tau \le t \le 0} \left\| \boldsymbol{\psi}(t) \right\| \le \delta(\varepsilon)$, the solution $\mathbf{x}(t)$ of system (2) satisfies $\left\| \mathbf{x}(t) \right\| \le \varepsilon, \forall t \ge 0$.

Moreover if $\lim_{t \to \infty} \left\| \mathbf{x}(t) \right\| \to 0$, system is said to be *asymptotically stable*, (*Xu et al.* 2002.a).

STABILITY THEOREMS

Theorem 3.1.1.1 Suppose that the matrix pair (E, A_0) is *regular* with system matrix A_0 being nonsingular., e.i. $\det A_0 \ne 0$. System (124) is *asymptotically stable*, independent of delay, if there exist a symmetric positive definite matrix $P = P^T > 0$, being the solution of Lyapunov matrix equation

$$A_0^T PE + E^T PA_0 = -2\left(S+Q\right), \tag{126}$$

with matrices $Q = Q^T > 0$ and $S = S^T$, such that:

$$\mathbf{x}^T\left(t\right)\left(S+Q\right)\mathbf{x}\left(t\right) > 0, \quad \forall \mathbf{x}\left(t\right) \in \mathcal{W}_{k^*} \setminus \left\{0\right\}, \tag{127}$$

is positive definite quadratic form on $\mathcal{W}_{k^*} \setminus \left\{0\right\}$, $\mathcal{W}_{k^*}$ being the subspace of consistent initial conditions, and if the following condition is satisfied:

$$\left\| A_1 \right\| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E^T P\right), \tag{128}$$

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively, (*Debeljkovic et al.* 2003, 2004.c, 2006, 2007).
Proof. Let us consider the functional:

$$V\left(\mathbf{x}\left(t\right)\right) = \mathbf{x}^T\left(t\right)E^T P E\mathbf{x}\left(t\right) + \int\limits_{t-\tau}^{t} \mathbf{x}^T\left(\vartheta\right)Q\mathbf{x}\left(\vartheta\right)d\kappa \, . \tag{129}$$

Note that (*Owens, Debeljković* 1985) indicates that:

$$V\left(\mathbf{x}\left(t\right)\right) = \mathbf{x}^T\left(t\right)E^T PE\mathbf{x}\left(t\right), \tag{130}$$

is *positive quadratic form* on $\mathcal{W}_{k^*}$, and it is obvious that all smooth solutions $\mathbf{x}\left(t\right)$ evolve in $\mathcal{W}_{k^*}$, so $V\left(\mathbf{x}\left(t\right)\right)$ can be used as a *Lyapunov function* for the system under consideration, (*Owens*, *Debeljkovic* 1985). It will be shown that the same argument can be used to declare the same property of another quadratic form present in (129).
Clearly, using the equation of motion of (124), we have:

$$\begin{aligned}\dot{V}\left(\mathbf{x}\left(t\right)\right) &= \mathbf{x}^T\left(t\right)\left(A_0^T PE + E^T PA_0 + Q\right)\mathbf{x}\left(t\right)\\ &+2\mathbf{x}^T\left(t\right)\left(E^T PA_1\right)\mathbf{x}\left(t-\tau\right) - \mathbf{x}^T\left(t-\tau\right)Q\mathbf{x}\left(t-\tau\right)\end{aligned} \tag{131}$$

and after some manipulations, to the following expression is obtained:

$$\begin{aligned}\dot{V}\left(\mathbf{x}\left(t\right)\right) &= \mathbf{x}^T\left(A_0^T PE + E^T PA_0 + 2Q + 2S\right)\mathbf{x} + 2\mathbf{x}^T\left(t\right)\left(E^T PA_1\right)\mathbf{x}\left(t-\tau\right)\\ &-\mathbf{x}^T\left(t\right)Q\mathbf{x}\left(t\right) - 2\mathbf{x}^T\left(t\right)S\mathbf{x}\left(t\right) - \mathbf{x}^T\left(t-\tau\right)Q\mathbf{x}\left(t-\tau\right)\end{aligned} \tag{132}$$

From (126) and the fact that the choice of matrix S , can be done, such that:

$$\mathbf{x}^T\left(t\right)S\mathbf{x}\left(t\right) \ge 0, \ \forall \mathbf{x}\left(t\right) \in \mathcal{W}_{k^*} \setminus \left\{0\right\}, \tag{133}$$

one obtains the following result:

$$\dot{V}\left(\mathbf{x}(t)\right)\leq 2\mathbf{x}^T(t)\left(E^TPA_1\right)\mathbf{x}(t-\tau)-\mathbf{x}^T(t)Q\mathbf{x}(t)-\mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau), \tag{134}$$

and based on well known inequality:

$$\begin{aligned}2\mathbf{x}^T(t)E^TPA_1\mathbf{x}(t-\tau)&=2\mathbf{x}^T(t)E^TPA_1Q^{-\frac{1}{2}}Q^{\frac{1}{2}}\mathbf{x}(t-\tau)\\ &\leq \mathbf{x}^T(t)E^TPA_1Q^{-1}A_1^TPE^T\mathbf{x}(t)+\mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau)\end{aligned} \tag{135}$$

and by substituting into (134), it yields:

$$\dot{V}\left(\mathbf{x}(t)\right)\leq -\mathbf{x}^T(t)Q\mathbf{x}(t)+\mathbf{x}^T(t)E^TPA_1Q^{-1}A_1^TPE\mathbf{x}(t)\leq -\mathbf{x}^T(t)Q^{\frac{1}{2}}\Gamma Q^{\frac{1}{2}}\mathbf{x}(t)(t), \tag{136}$$

with matrix Γ defined by:

$$\Gamma=\left(I-Q^{-\frac{1}{2}}E^TPA_1Q^{-\frac{1}{2}}Q^{-\frac{1}{2}}A_1^TPEQ^{-\frac{1}{2}}\right) \tag{137}$$

$\dot{V}\left(\mathbf{x}(t)\right)$ is negative definite, if:

$$1-\lambda_{\max}\left(Q^{-\frac{1}{2}}E^TPA_1Q^{-\frac{1}{2}}Q^{-\frac{1}{2}-1}A_1^TPEQ^{-\frac{1}{2}}\right)>0, \tag{138}$$

which is satisfied, if:

$$1-\sigma_{\max}^2\left(Q^{-\frac{1}{2}}E^TPA_1Q^{-\frac{1}{2}}\right)>0. \tag{139}$$

Using the properties of the singular matrix values, (*Amir - Moez* 1956), the condition (139), holds if:

$$1-\sigma_{\max}^2\left(Q^{-\frac{1}{2}}E^TP\right)\sigma_{\max}^2\left(A_1Q^{-\frac{1}{2}}\right)>0, \tag{140}$$

which is satisfied if:

$$1-\sigma_{\min}^{-2}\left(Q^{\frac{1}{2}}\right)\left(\left\|A_1\right\|^2\sigma_{\max}^2\left(Q^{-\frac{1}{2}}E^TP\right)\right)>0.\ Q.E.D. \tag{141}$$

Remark 3.1.1.1 (126-127) are, in modified form, taken from (*Owens & Debeljkovic* 1985).
Remark 3.1.1.2 If the system under consideration is just ordinary time delay, e.g. $E=I$, we have result identical to that presented in (*Tissir & Hmamed* 1996).
Remark 3.1.1.3 Let us discuss first the case when the time delay *is absent.*
Then the *singular* (weak) Lyapunov matrix (126) is natural generalization of classical Lyapunov theory. In particular:

a. If E is *nonsingular matrix*, then the system is asymptotically stable if and only if $A = E^{-1}A_0$ *Hurwitz* matrix. (126) can be written in the form:

$$A^T E^T PE + E^T PEA = -(Q+S), \tag{142}$$

with matrix Q being symmetric and positive definite, in whole state space, since then $\mathcal{W}_{k^*} = \Re\left(E^{k^*}\right) = \mathbb{R}^n$. In this circumstances $E^T PE$ is a Lyapunov function for the system.

b. The matrix A_0 by necessity is nonsingular and hence the system has the form:

$$E_0 \dot{\mathbf{x}}(t) = \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{143}$$

Then for this system to be stable (143) must hold also, and has familiar Lyapunov structure:

$$E_0^T P + PE_0 = -Q, \tag{144}$$

where Q is symmetric matrix but only required to be positive definite on $\mathcal{W}_{k^*}$.

Remark 3.1.1.4 There is no need for the system, under consideration, to posses properties given in *Definition* 3.1.1.2, since this is obviously guaranteed by demand that all smooth solutions $\mathbf{x}(t)$ evolve in $\mathcal{W}_{k^*}$.

Remark 3.1.1.5 Idea and approach is based upon the papers of (*Owens & Debeljkovic* 1985) and (*Tissir & Hmamed* 1996).

Theorem 3.1.1.2 Suppose that the system matrix A_0 is nonsingular., e.i. $\det A_0 \neq 0$. Then we can consider system (124) with known compatible vector valued function of initial conditions and we shall assume that $rank\, E_0 = r < n$.

Matrix E_0 is defined in the following way $E_0 = A_0^{-1}E$. System (124) is *asymptotically stable*, independent of delay, if :

$$\left\| A_1 \right\| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E_0^T P\right), \tag{145}$$

and if there exist $(n \times n)$ matrix P, being the solution of Lyapunov matrix:

$$E_0^T P + PE_0 = -2I_{\mathcal{W}_k}, \tag{146}$$

with the properties given by (3)–(7).

Moreover matrix P is symmetric and positive definite on the subspace of consistent initial conditions. Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively (*Debeljkovic et al.* 2005.b, 2005.c, 2006.a).

For the sake of brevity the proof is here omitted and is completely identical to that of preceding *Theorem.*

Remark 3.1.1.6 Basic idea and approach is based upon the paper of (*Pandolfi* 1980) and (*Tissir*, *Hmamed* 1996).

3.1.2 Continuous singular time delayed systems – stability over finite time interval

Let us consider the case when the subspace of consistent initial conditions for *singular time delay* and *singular nondelay system* coincide.

STABILITY DEFINITIONS

Definition 3.1.2.1 Regular and impulsive free singular time delayed system (124), is *finite time stable* with respect to $\left\{t_0,\Im,\mathcal{S}_\alpha,\mathcal{S}_\beta\right\}$, if and only if $\forall \mathbf{x}_0 \in \mathcal{W}_k^*$ satisfying $\left\|\mathbf{x}(t_0)\right\|_{E^TE}^2 = \left\|\mathbf{x}_0\right\|_{E^TE}^2 < \alpha$, implies $\left\|\mathbf{x}(t)\right\|_{E^TE}^2 < \beta,\ \forall t\in\Im$.

Definition 3.1.2.2 . Regular and impulsive free singular time delayed system (124), is *attractive practically stable* with respect to $\left\{t_0,\Im,\mathcal{S}_\alpha,\mathcal{S}_\beta\right\}$, if and only if $\forall \mathbf{x}_0 \in \mathcal{W}_k^*$ satisfying $\left\|\mathbf{x}(t_0)\right\|_{G=E^TPE}^2 = \left\|\mathbf{x}_0\right\|_{G=E^TPE}^2 < \alpha$ implies: $\left\|\mathbf{x}(t)\right\|_{G=E^TPE}^2 < \beta,\ \forall t\in\Im$, with property that $\lim\limits_{k\to\infty}\left\|\mathbf{x}(t)\right\|_{G=E^TPE}^2 \to 0$, $\mathcal{W}_k^*$ being the subspace of consistent initial conditions, (*Debeljkovic et al.* 2011.b).

Remark 3.1.2.1 The singularity of matrix E will ensure that solutions to (6) exist for only special choice of $\mathbf{x}_0$.

In (*Owens, Debeljković* 1985) the subspace of $\mathcal{W}_k^*$ of consistent initial conditions is shown to be the limit of the nested subspace algorithm (12)–(14).

STABILITY THEOREMS

Theorem 3.1.2.1 Suppose that $\left(I-E^TE\right)>0$. Singular time delayed system (124), is *finite time stable* with respect to $\left\{t_0,\Im,\alpha,\beta,\left\|(\cdot)\right\|^2\right\}$, $\alpha<\beta$, if there exist a positive real number q, $q>1$, such that:

$$\left\|\mathbf{x}(t+\vartheta)\right\|^2 < q^2\left\|\mathbf{x}(t)\right\|^2,\ \vartheta\in[-\tau,0],\ \forall t\in\Im,\ \mathbf{x}(t)\in\mathcal{W}_k^*,\ \forall\mathbf{x}(t)\in\mathcal{S}_\beta, \tag{147}$$

and if the following condition is satisfied:

$$e^{\bar{\lambda}_{\max}(\Xi)(t-t_0)} < \frac{\beta}{\alpha},\quad \forall t\in\Im, \tag{148}$$

where:

$$\begin{aligned}\bar{\lambda}_{\max}(\Xi) = \bar{\lambda}_{\max}\{&\mathbf{x}^T(t)(A_0^TE+E^TA_0+E^TA_1(I-E^TE)^{-1}A_1^TE\\ &+q^2I)\mathbf{x}(t),\mathbf{x}(t)\in\mathcal{W}_k^*,\ \mathbf{x}^T(t)E^TE\mathbf{x}(t)=1\}.\end{aligned} \tag{149}$$

Proof. Define tentative aggregation function as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)E^TE\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)\,d\vartheta. \tag{150}$$

Let $\mathbf{x}_0$ be an arbitrary consistent initial condition and $\mathbf{x}(t)$ resulting system trajectory. The total derivative $\dot{V}(t,\mathbf{x}(t))$ along the trajectories of the system, yields:

$$\begin{aligned}\dot{V}(t,\mathbf{x}(t)) &= \frac{d}{dt}\left(\mathbf{x}^T(t)E^TE\mathbf{x}(t)\right)+\frac{d}{dt}\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta \\ &= \mathbf{x}^T(t)\left(A_0^TE+E^TA_0\right)\mathbf{x}(t)+2\mathbf{x}^T(t)E^TA_1\mathbf{x}(t-\tau)+\mathbf{x}^T(t)\mathbf{x}(t)-\mathbf{x}^T(t-\tau)\mathbf{x}(t-\tau)\end{aligned} \tag{151}$$

From (148) it is obvious:

$$\frac{d}{dt}\left(\mathbf{x}^T(t)E^TE\mathbf{x}(t)\right)=\mathbf{x}^T(t)\left(A_0^TE+E^TA_0\right)\mathbf{x}(t)+2\mathbf{x}^T(t)E^TA_1\mathbf{x}(t-\tau), \tag{152}$$

and based on well known inequality and with the particular choice:

$$\mathbf{x}^T(t)\Gamma\mathbf{x}(t)=\mathbf{x}^T(t)\left(I-E^TE\right)\mathbf{x}(t)>0,\ \forall\mathbf{x}(t)\in\mathcal{W}_k^*\setminus\{0\}, \tag{153}$$

so:

$$\begin{aligned}&\frac{d}{dt}\left(\mathbf{x}^T(t)E^TE\mathbf{x}(t)\right)\le\mathbf{x}^T(t)\left(A_0^TE+E^TA_0\right)\mathbf{x}(t)\\ &+\mathbf{x}^T(t)E^TA_1\left(I-E^TE\right)^{-1}A_1^TE\mathbf{x}(t)+\mathbf{x}^T(t-\tau)\left(I-E^TE\right)\mathbf{x}(t-\tau).\end{aligned} \tag{154}$$

Moreover, since:

$$\left\|\mathbf{x}(t-\tau)\right\|_{E^TE}^2\ge 0,\ \ \forall\mathbf{x}(t)\in\mathcal{W}_k^*\setminus\{0\}, \tag{155}$$

and using assumption (147), it is clear that (154) reduces to:

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{x}^T(t)E^TE\mathbf{x}(t)\right)&<\mathbf{x}^T(t)\left(A_0^TE+E^TA_0+E^TA_1\left(E^TE-I\right)^{-1}A_1^TE+q^2I\right)\mathbf{x}(t)\\ &<\bar{\lambda}_{\max}(\Xi)\mathbf{x}^T(t)E^TE\mathbf{x}(t)\end{aligned} \tag{156}$$

Remark 3.1.2. 2 Note that *Lemma* 2.2.1.1 and *Theorem* 2.2.1.1 indicates that:

$$V(\mathbf{x}(t))=\mathbf{x}^T(t)E^TE\mathbf{x}(t), \tag{157}$$

is *positive quadratic form* on $\mathcal{W}_k^*$, and it is obvious that all smooth solutions $\mathbf{x}(t)$ evolve in $\mathcal{W}_k^*$, so $V(\mathbf{x}(t))$ can be used as a *Lyapunov function* for the system under consideration, (*Owens*, *Debeljkovic* 1985).

Using (149) one can get (*Debeljkovic et al.* 2011.b):

$$\int_{t_0}^{t}\frac{d\left(\mathbf{x}^T(t)E^TE\mathbf{x}(t)\right)}{\mathbf{x}^T(t)E^TE\mathbf{x}(t)}<\int_{t_0}^{t}\bar{\lambda}_{\max}(\Xi)dt, \tag{158}$$

and:

$$\begin{aligned}&\mathbf{x}^T(t)E^TE\mathbf{x}(t)<\mathbf{x}^T(t_0)E^TE\mathbf{x}(t_0)e^{\bar{\lambda}_{\max}(\Xi)(t-t_0)}\\&<\alpha\cdot e^{\bar{\lambda}_{\max}(\Xi)(t-t_0)}<\alpha\cdot\frac{\beta}{\alpha}<\beta,\forall t\in\Im.\quad Q.E.D.\end{aligned}\tag{159}$$

Remark 3.1.2.3 In the case on non-delay system, e.g. $A_1\equiv 0$, (148) reduces to basic result, (*Debeljkovic*, *Owens* 1985).

Theorem 3.1.2.2 Suppose that $\left(Q-E^TE\right)>0$. Singular time delayed system (124), with system matrix A_0 being nonsingular, is *attractive practically stable* with respect to $\left\{t_0,\Im,\alpha,\beta,\left\|(\cdot)\right\|^2_{G=E^TPE}\right\}$, $\alpha<\beta$, if there exist matrix $P=P^T>0$, being solution of:

$$A_0^TPE+E^TPA_0=-Q,\tag{160}$$

with matrices $Q=Q^T>0\wedge S=S^T$, such that:

$$\mathbf{x}^T(t)(S+Q)\mathbf{x}(t)>0,\ \forall\mathbf{x}(t)\in\mathcal{W}_k^*\backslash\{0\},\tag{161}$$

is positive definite quadratic form on $\mathcal{W}_k^*\backslash\{0\}$, $\mathcal{W}_k^*$ being the subspace of consistent initial conditions, if there exist a positive real number q, $q>1$, such that:

$$\left\|\mathbf{x}(t-\tau)\right\|_Q^2<q^2\left\|\mathbf{x}(t)\right\|_Q^2,\ \forall t\in\Im,\ \forall\mathbf{x}(t)\in\mathcal{S}_\beta,\ \forall\mathbf{x}(t)\in\mathcal{W}_k^*\backslash\{0\},\tag{162}$$

and if the following conditions are satisfied (*Debeljkovic et al.* 2011.b):

$$\left\|A_1\right\|<\sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}A_0^TP\right),\tag{163}$$

and:

$$e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)}<\frac{\beta}{\alpha},\quad\forall t\in\Im,\tag{164}$$

where:

$$\begin{aligned}\bar{\lambda}_{\max}(\Psi)=\max\{&\mathbf{x}^T(t)(E^TPA_1(Q-E^TPE)^{-1}A_1^TPE+q^2Q)\mathbf{x}(t),\\&\mathbf{x}(t)\in\mathcal{W}_k^*,\ \mathbf{x}^T(t)E^TPE\mathbf{x}(t)=1\}.\end{aligned}\tag{165}$$

Proof. Define tentative aggregation function as:

$$V(\mathbf{x}(t))=\mathbf{x}^T(t)E^TPE\mathbf{x}(t)+\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)Q\mathbf{x}(\vartheta)d\vartheta.\tag{166}$$

The total derivative $\dot{V}(t,\mathbf{x}(t))$ along the trajectories of the system, yields:

$$\begin{aligned}\dot{V}\big(t,\mathbf{x}(t)\big) &= \frac{d}{dt}\Big(\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\Big) + \frac{d}{dt}\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)Q\mathbf{x}(\vartheta)d\vartheta \\ &= \mathbf{x}^T(t)\Big(A_0^T PE + E^T PA_0\Big)\mathbf{x}(t) + 2\mathbf{x}^T(t)E^T PA_1\mathbf{x}(t-\tau) \\ &\quad + \mathbf{x}^T(t)Q\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau).\end{aligned} \tag{167}$$

From (162), it is obvious:

$$\frac{d}{dt}\Big(\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\Big) = \mathbf{x}^T(t)\Big(A_0^T PE + E^T PA_0\Big)\mathbf{x}(t) + 2\mathbf{x}^T(t)E^T PA_1\mathbf{x}(t-\tau), \tag{168}$$

or:

$$\begin{aligned}\frac{d}{dt}\Big(\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\Big) &= \mathbf{x}^T(t)\Big(A_0^T PE + E^T PA_0 + Q + S\Big)\mathbf{x}(t) \\ &\quad + 2\mathbf{x}^T(t)E^T PA_1\mathbf{x}(t-\tau) - \mathbf{x}^T(t)(Q+S)\mathbf{x}(t).\end{aligned} \tag{169}$$

From (160), it follows:

$$\frac{d}{dt}\Big(\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\Big) = -\mathbf{x}^T(t)(Q+S)Q\mathbf{x}(t) + 2\mathbf{x}^T(t)E^T PA_1\mathbf{x}(t-\tau), \tag{170}$$

as well, using before mentioned inequality, with particular choice:

$$\mathbf{x}^T(t)\Gamma\mathbf{x}^T(t) = \mathbf{x}^T(t)\Big(Q - E^T PE\Big)\mathbf{x}^T(t) > 0, \quad \forall\mathbf{x}(t) \in \mathcal{W}_k^*\backslash\{0\}, \tag{171}$$

and fact that:

$$\mathbf{x}^T(t)(Q+S)\mathbf{x}(t) > 0, \quad \forall\mathbf{x}(t) \in \mathcal{W}_k^*\backslash\{0\}, \tag{172}$$

is positive definite quadratic form on $\mathcal{W}_k^*\backslash\{0\}$, one can get :

$$\begin{aligned}\frac{d}{dt}\Big(\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\Big) &= 2\mathbf{x}^T(t)E^T PA_1\mathbf{x}(t-\tau) \\ \le \mathbf{x}^T(t)E^T PA_1\Big(Q - E^T PE\Big)^{-1}A_1^T PE\mathbf{x}(t) &+ \mathbf{x}^T(t-\tau)\Big(Q - E^T PE\Big)\mathbf{x}(t-\tau)\end{aligned} \tag{173}$$

Moreover, since:

$$\big\|\mathbf{x}(t-\tau)\big\|_{E^T PE}^2 \ge 0, \quad \forall\mathbf{x}(t) \in \mathcal{W}_k^*\backslash\{0\}, \tag{174}$$

and using assumption (162) it is clear that (173), reduces to:

$$\frac{d}{dt}\Big(\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\Big) < \mathbf{x}^T(t)\bigg(E^T PA_1\Big(E^T PE - Q\Big)^{-1}A_1^T PE + q^2 Q\bigg)\mathbf{x}(t), \tag{175}$$

or using (169), one can get:

$$\begin{aligned}\frac{d}{dt}\Big(\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\Big) &< \mathbf{x}^T(t)\bigg(E^T PA_1\Big(E^T PE - Q\Big)^{-1}A_1^T PE + q^2 Q\bigg)\mathbf{x}(t) \\ &< \bar{\lambda}_{\max}(\Psi)\mathbf{x}^T(t)E^T PE\mathbf{x}(t)\end{aligned} \tag{176}$$

or finally:

$$\mathbf{x}^T(t)E^T P E\mathbf{x}(t) < \mathbf{x}^T(t_0)E^T P E\mathbf{x}(t_0)e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)} < \alpha \cdot e^{\bar{\lambda}_{\max}(\Psi)(t-t_0)} < \alpha \cdot \frac{\beta}{\alpha} < \beta, \forall t \in \mathfrak{I}. \ Q.E.D. \tag{177}$$

3.2 Discrete descriptor time delayed systems

3.2.1 Discrete descriptor time delayed systems – Stability in the sense of Lyapunov

Consider a linear discrete descriptor system with state delay, described by:

$$E\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1), \tag{178}$$

$$\mathbf{x}(k_0) = \boldsymbol{\varphi}(k_0), \quad -1 \le k_0 \le 0, \tag{179}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is a state vector. The matrix $E \in \mathbb{R}^{n\times n}$ is a necessarily singular matrix, with property $rank\, E = r < n$ and with matrices A_0 and A_1 of appropriate dimensions.

For a (DDTDS), (178), we present the following definitions taken from, (*Xu et al*. 2002.b).

Definition 3.2.1.1 The (DDTDS) is said to be *regular* if $\det\left(z^2E - zA_0 - A_1\right)$, is not identically zero.

Definition 3.2.1.2 The (DDTDS) is said to be *causal* if it is *regular* and $\deg\left(z^n \det\left(zE - A_0 - z^{-1}A_1\right)\right) = n + rang\, E$.

Definition 3.2.1.3 The (DDTDS) is said to be *stable* if it is *regular* and $\rho\left(E, A_0, A_1\right) \subset D(0,1)$, where $\rho\left(E, A_0, A_1\right) = \left\{z \mid \det\left(z^2E - zA_0 - A_1\right) = 0\right\}$.

Definition 3.2.1.4 The (DDTDS) is said to be *admissible* if it is *regular*, *causal* and *stable*.

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Definition 3.2.1.5 System (178) is E-stable if for any $\varepsilon > 0$, there always exists a positive δ such that $\|E\mathbf{x}(k)\| < \varepsilon$, when $\|E\mathbf{x}_0\| < \delta$, (*Liang* 2000).

Definition 3.2.1.6 System (178) is E- asymptotically stable if (178) is E- stable and $\lim_{k\to+\infty} E\mathbf{x}(k) \to \mathbf{0}$, (*Liang* 2000).

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Theorem 3.2.1.1 Suppose that system (173) is *regular* and *causal* with system matrix A_0 being nonsingular, i.e. $\det A_0 \ne 0$. System (178) is *asymptotically stable*, independent of delay, if

$$\|A_1\| < \frac{\sigma_{\min}\left(Q^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}A_0^T P\right)}, \tag{180}$$

and if there exist a symmetric positive definite matrix P on the whole state space, being the solution of *discrete Lyapunov matrix equation* :

$$A_0^T P A_0 - E^T P E = -2\left(S + Q\right), \tag{181}$$

with matrices $Q = Q^T > 0$ and $S = S^T$, such that:

$$\mathbf{x}^T\left(k\right)\left(S + Q\right)\mathbf{x}\left(k\right) > 0, \ \forall \mathbf{x}\left(k\right) \in \mathcal{W}_{d,k^*} \setminus \left\{0\right\}, \tag{182}$$

is positive definite quadratic form on $\mathcal{W}_{d,k^*} \setminus \{0\}$, $\mathcal{W}_{d,k^*}$ being the subspace of consistent initial conditions. Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively, (*Debeljkovic et al.* 2004).

Remark 3.2.1.1 (181 - 182) are, in modify form, taken from (*Owens*, *Debeljkovic* 1985).

Remark 3.2.1.2 If the system under consideration is just ordinary time delay, e.g. $E = I$, we have result identical to that presented in *Debeljkovic et al.* (2004.a – 2004.d, 2005.a, 2005.b).

Remark 3.2.1.3 Idea and approach is based upon the papers of (*Owens*, *Debeljkovic* 1985) and (*Tissir*, *Hmamed* 1996).

Theorem 3.2.1.2 Suppose that system (178) is *regular* and *causal.* Moreover, suppose matrix $\left(Q_\lambda - A_1^T P_\lambda A_1\right)$ is regular, with $Q_\lambda = Q_\lambda^T > 0$.

System (178) is *asymptotically stable*, independent of delay, if:

$$\left\| A_1 \right\| < \frac{\sigma_{\min}\left(\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q_\lambda^{-\frac{1}{2}}\left(A_0 - \lambda E\right)^T P_\lambda\right)}, \tag{183}$$

and if there exist real positive scalar $\lambda^* > 0$ such that for all λ within the range $0 < \left|\lambda\right| < \lambda^*$ there exist symmetric positive definite matrix P_λ, being the solution of *discrete Lyapunov matrix equation*:

$$\left(A_0 - \lambda E\right)^T P_\lambda \left(A_0 - \lambda E\right) - E^T P_\lambda E = -2\left(S_\lambda + Q_\lambda\right) \tag{184}$$

with matrix $S_\lambda = S_\lambda^T$, such that:

$$\mathbf{x}^T\left(k\right)\left(S_\lambda + Q_\lambda\right)\mathbf{x}\left(k\right) > 0, \quad \forall \mathbf{x}\left(k\right) \in \mathcal{W}_{d,k^*} \setminus \left\{0\right\} \tag{185}$$

is positive definite quadratic form on $\mathcal{W}_{d,k^*} \setminus \{0\}$, $\mathcal{W}_{d,k^*}$ being the subspace of consistent initial conditions for both *time delay* and *non-time delay* discrete descriptor system. Such conditions we call *compatible consistent initial conditions*. Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$ respectively, (*Debeljkovic et al.* 2007).

3.2.2 Discrete descriptor time delayed systems – stability over finite time interval

To the best knowledge of the authors, there is not any paper treating the problem of finite time stability for *discrete descriptor* time delay systems. Only one paper has been written in context of practical and finite time stability for *continuous* singular time delay systems, see (*Yang et al.* 2006).

Definition 3.2.2.1 *Causal* system, given by (178), is *finite time stable* with respect to $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, if and only if $\forall \mathbf{x}_0 \in \mathcal{W}_{d,k^*}$ satisfying $\|\mathbf{x}_0\|^2_{E^TE} < \alpha$, implies: $\|\mathbf{x}(k)\|^2_{E^TE} < \beta, \quad \forall k \in \mathcal{K}_N$.

Definition 3.2.2.2 *Causal* system given by (178), is *practically unstable* with respect $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if and only if $\exists \mathbf{x}_0 \in \mathcal{W}_{d,k^*}$ such that $\|\mathbf{x}_0\|^2_{E^TE} < \alpha$, there exist some $k^* \in K_N$, such that the following condition is fulfilled $\left\|\mathbf{x}\left(k^*\right)\right\|^2_{E^TE} \geq \beta$, for some $k^* \in \mathcal{K}_N$.

Definition 3.2.2.3 *Causal* system, given by (178), is *attractive practically stable* with respect to $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, if and only if $\forall \mathbf{x}_0 \in \mathcal{W}_{d,k^*}$ satisfying $\|\mathbf{x}(k_0)\|^2_{G=E^TPE} = \|\mathbf{x}_0\|^2_{G=E^TPE} < \alpha$, implies $\|\mathbf{x}(k)\|^2_{G=E^TPE} < \beta, \quad \forall k \in \mathcal{K}_N$, with property that $\lim_{k\to\infty} \|\mathbf{x}(k)\|^2_{G=E^TPE} \to 0$, (*Nestorovic & Debeljkovic* 2011).

Remark 3.2.2.1 We shall also need the following *Definitions* of the smallest and the largest eigenvalues, respectively, of the matrix $R = R^T$, with respect to subspace of consistent initial conditions $\mathcal{W}_{d,k^*}$ and matrix G.

Proposition 3.2.2.1 If $\mathbf{x}^T(t)R\mathbf{x}(t)$ is quadratic form on $\mathbb{R}^n$, then it follows that there exist numbers $\lambda_{\min}(R)$ and $\lambda_{\max}(R)$ satisfying: $-\infty \leq \lambda_{\min}(R) \leq \lambda_{\max}(R) \leq +\infty$, such that:

$$\lambda_{\min}(\Xi) \leq \frac{\mathbf{x}^T(k)R\mathbf{x}(k)}{\mathbf{x}^T(k)G\mathbf{x}(k)} \leq \lambda_{\max}(\Xi), \ \forall \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\}, \tag{186}$$

with matrix $R = R^T$ and corresponding eigenvalues:

$$\lambda_{\min}\left(R, G, \mathcal{W}_{d,k^*}\right) = \min\left\{\mathbf{x}^T(k)R\mathbf{x}(k): \ \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\}, \ \mathbf{x}^T(k)G\mathbf{x}(k) = 1\right\}, \tag{187}$$

$$\lambda_{\max}\left(R, G, \mathcal{W}_{d,k^*}\right) = \max\left\{\mathbf{x}^T(k)R\mathbf{x}(k): \ \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\}, \ \mathbf{x}^T(k)G\mathbf{x}(k) = 1\right\}. \tag{188}$$

Note that $\lambda_{\min} > 0$ if $R = R^T > 0$.

Let us consider the case when the subspace of consistent initial conditions for *discrete descriptor time delay* and *discrete descriptor nondelay system* coincide.

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Theorem 3.2.2.1 Suppose matrix $\left(A_1^TA_1 - E^TE\right) > 0$. *Causal* system given by (178), is *finite time stable* with respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there exist a positive real number p, $p > 1$, such that:

$$\left\| \mathbf{x}(k-1) \right\|_{A_1^T A_1}^2 < p^2 \left\| \mathbf{x}(k) \right\|_{A_1^T A_1}^2, \ \forall k \in \mathcal{K}_N, \ \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \ \forall \mathbf{x}(k) \in \mathcal{W}_{d,k}^* \backslash \{0\} \tag{189}$$

and if the following condition is satisfied (*Nestorovic & Debeljkovic* 2011):

$$\bar{\lambda}_{\max}^k(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \tag{190}$$

where:

$$\begin{aligned} \bar{\lambda}_{\max}(\ ) = \bar{\lambda}_{\max}\{ & \mathbf{x}^T(k) A_0^T (I - A_1 (A_1^T A_1 - E^T E)^{-1} A_1^T \\ & + p^2 A_1^T A_1) A_0 \mathbf{x}(k), \ \mathbf{x}(k) \in \mathcal{W}_{d,k^*}, \ \mathbf{x}^T(k) E^T E \mathbf{x}(k) = 1 \}. \end{aligned} \tag{191}$$

Proof. Define:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbf{x}(k) + \mathbf{x}^T(k-1)\mathbf{x}(k-1). \tag{192}$$

Let $\mathbf{x}_0$ be an arbitrary consistent initial condition and $\mathbf{x}(k)$ the resulting system trajectory. The backward difference $\Delta V(\mathbf{x}(k))$ along the trajectories of the system, yields:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) = & \mathbf{x}^T(k)\left(A_0^T A_0 - E^T E + I\right)\mathbf{x}(k) \\ & + 2\mathbf{x}^T(k)\left(A_0^T A_1\right)\mathbf{x}(k-1) + \mathbf{x}^T(k-1)\left(A_1^T A_1 - I\right)\mathbf{x}(k-1) \end{aligned} \tag{193}$$

From (192) one can get:

$$\begin{aligned} \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) = & \mathbf{x}^T(k)\left(A_0^T A_0\right)\mathbf{x}(k) \\ & + 2\mathbf{x}^T(k)\left(A_0^T A_1\right)\mathbf{x}(k-1) + \mathbf{x}^T(k-1)\left(A_1^T A_1\right)\mathbf{x}(k-1) \end{aligned} \tag{194}$$

Using the very well known inequality, with particular choice:

$$\mathbf{x}^T(k)\Gamma\mathbf{x}(k) = \mathbf{x}^T(k)\left(A_1^T A_1 - E^T E\right)\mathbf{x}(k) \geq 0, \ \mathbf{x}(k) \in \mathcal{W}_{d,k^*}, \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \forall k \in \mathcal{K}_N, \tag{195}$$

it can be obtained:

$$\begin{aligned} \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) \leq & \mathbf{x}^T(k) A_0^T A_0 \mathbf{x}(k) \\ - \mathbf{x}^T(k) A_0^T A_1 \left(A_1^T A_1 - E^T E\right)^{-1} & A_1^T A_0 \mathbf{x}(k) + \mathbf{x}^T(k-1)\left(2A_1^T A_1 - E^T E\right)\mathbf{x}(k-1) \end{aligned} \tag{196}$$

Moreover, since:

$$\left\| \mathbf{x}(k-1) \right\|_{E^T E}^2 \geq 0, \ \forall k \in \mathcal{K}_N, \ \forall \mathbf{x}(k) \in \mathcal{W}_{d,k}^* \backslash \{0\} \tag{197}$$

and using assumption (189) it is clear that (196), reduces to:

$$\begin{aligned} \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) < & \mathbf{x}^T(k) A_0^T \left(I - A_1 \left(A_1^T A_1 - E^T E\right)^{-1} A_1^T + 2p^2 I \right) A_0 \mathbf{x}(k) \\ & < \bar{\lambda}_{\max}(\ ) \mathbf{x}^T(k) E^T E \mathbf{x}(k) \end{aligned} \tag{198}$$

$$\text{where:}\quad \bar{\lambda}_{\max}(\;) = \{\mathbf{x}^T(k)A_0^T\left(I - A_1\left(A_1^T A_1 - E^T E\right)^{-1} A_1^T + 2p^2 I\right)A_0\mathbf{x}(k),\\ \mathbf{x}(k)\in\mathcal{W}^*_{dis},\ \mathbf{x}^T(k)E^T E\mathbf{x}(k) = 1\}. \tag{199}$$

Following the procedure from the previous section, it can be written:

$$\ln\mathbf{x}^T(k+1)E^T E\mathbf{x}(k+1) - \ln\mathbf{x}^T(k)E^T E\mathbf{x}(k) < \ln\bar{\lambda}_{\max}(\;). \tag{200}$$

By applying the summing $\sum_{j=k_0}^{k_0+k-1}$ on both sides of (200) for $\forall k\in\mathcal{K}_N$, one can obtain:

$$\ln\mathbf{x}^T(k_0+k)E^T E\mathbf{x}(k_0+k) \le \ln\prod_{j=k_0}^{k_0+k-1}\bar{\lambda}_{\max}(\;)\\ \le \ln\bar{\lambda}^k_{\max}(\;) + \ln\mathbf{x}^T(k_0)E^T E\mathbf{x}(k_0),\quad \forall k\in\mathcal{K}_N \tag{201}$$

Taking into account the fact that $\|\mathbf{x}_0\|^2_{E^T E} < \alpha$ and the condition of *Theorem* 3.2.2.1, eq. (190), one can get:

$$\ln\mathbf{x}^T(k_0+k)E^T E\mathbf{x}(k_0+k) < \ln\bar{\lambda}^k_{\max}(\;) + \ln\mathbf{x}^T(k_0)E^T E\mathbf{x}(k_0)\\ < \ln\alpha\cdot\bar{\lambda}^k_{\max}(\;) < \ln\alpha\cdot\frac{\beta}{\alpha} < \ln\beta,\ \forall k\in\mathcal{K}_N.\qquad Q.E.D. \tag{202}$$

Theorem 3.2.2.2 Suppose matrix $\left(A_1^T A_1 - E^T E\right) > 0$. *Causal* system (178), is *finite time unstable* with respect to $\left\{k_0,\mathcal{K}_N,\alpha,\beta,\|(\cdot)\|^2\right\}, \alpha<\beta$, if there exist a positive real number p, $p>1$, such that:

$$\|\mathbf{x}(k-1)\|^2_{A_1^T A_1} < p^2\|\mathbf{x}(k)\|^2_{A_1^T A_1},\ \forall k\in\mathbf{K}_N,\ \forall\mathbf{x}(k)\in\mathbf{S}_\beta,\ \forall\mathbf{x}(k)\in\mathcal{W}^*_{d,k}\backslash\{0\} \tag{203}$$

and if for $\forall\mathbf{x}_0\in\mathcal{W}_{d,k^*}$ and $\|\mathbf{x}_0\|^2_{G=E^T E} < \alpha$ there exist: real, positive number δ, $\delta\in\left]0,\alpha\right[$ and time instant k, $k=k^*$: $\exists!\left(k^*>k_0\right)\in\mathcal{K}_N$, for which the next condition is fulfilled (*Nestorovic & Debeljkovic* 2011):

$$\bar{\lambda}^{k^*}_{\min}(\;) > \frac{\beta}{\delta},\quad k^*\in\mathcal{K}_N \tag{204}$$

where:

$$\bar{\lambda}_{\min}(\;) = \bar{\lambda}_{\min}\{\mathbf{x}^T(k)A_0^T\left(I - A_1\left(A_1^T A_1 - E^T E\right)^{-1} A_1^T + 2\wp(k)I\right)A_0\mathbf{x}(k),\\ \mathbf{x}(k)\in\mathcal{W}^d_{k^*},\ \mathbf{x}^T(k)E^T E\mathbf{x}(k) = 1\}. \tag{205}$$

Proof. Following the identical procedure as in the previous *Theorem*, with the same aggregation function, one can get:

$$\begin{aligned}\ln \mathbf{x}^T\left(k_0+k^*\right)E^T E\mathbf{x}\left(k_0+k^*\right) &> \ln \bar{\lambda}_{\min}^{k^*}(\ ) + \ln \mathbf{x}^T\left(k_0\right)E^T E\mathbf{x}\left(k_0\right) \\ &> \ln\delta\cdot\bar{\lambda}_{\min}^{k^*}(\ ) > \ln\delta\cdot\frac{\beta}{\delta} > \ln\beta,\ \text{for some } k^*\in\mathcal{K}_N,\end{aligned} \tag{206}$$

where $\bar{\lambda}_{\min}(\ )$ is given by (187). *Q.E.D.*

Theorem 3.2.2.3 Suppose matrix $\left(A_1^T P A_1 - E^T P E\right) \geq 0$. *Causal* system given by (178), with $\det A_0 \neq 0$, is *attractive practically stable* with respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there exists a matrix $P = P^T > 0$, being the solution of:

$$A_0^T P A_0 - E^T P E = -2(Q+S), \tag{207}$$

with matrices $Q = Q^T > 0$ and $S = S^T$, such that:

$$\mathbf{x}^T(k)(Q+S)\mathbf{x}(k) > 0,\ \forall \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\} \tag{208}$$

is positive definite quadratic form on $\mathcal{W}_{d,k^*} \setminus \{0\}$, p real number, $p > 1$, such that:

$$\left\|\mathbf{x}(k-1)\right\|_{A_1^T P A_1}^2 < p^2 \left\|\mathbf{x}(k)\right\|_{A_1^T P A_1}^2,\ \forall k\in\mathcal{K}_N,\ \forall \mathbf{x}(k)\in \mathcal{S}_\beta,\ \forall \mathbf{x}(k)\in\mathcal{W}_{k,d}^*\setminus\{0\} \tag{209}$$

and if the following conditions are satisfied (*Nestorovic & Debeljkovic* 2011):

$$\left\|A_1\right\| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E^T P\right), \tag{210}$$

and

$$\bar{\lambda}_{\max}^{k}(\ ) < \frac{\beta}{\alpha},\ \ \forall k\in\mathcal{K}_N, \tag{211}$$

where:

$$\begin{aligned}\bar{\lambda}_{\max}(\ ) = \max\{&\mathbf{x}^T(k)A_0^T P^{\frac{1}{2}}\left(I - A_1\left(A_1^T P A_1 - E^T P E\right)^{-1}A_1^T + p^2 I\right)P^{\frac{1}{2}}A_0\mathbf{x}(k): \\ &\mathbf{x}(k)\in\mathcal{W}_{d,k^*},\ \mathbf{x}^T(k)E^T P E\mathbf{x}(k) = 1\}.\end{aligned} \tag{212}$$

Proof. Let us consider the functional:

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k)E^T P E\mathbf{x}(k) + \mathbf{x}^T(k-1)Q\mathbf{x}(k-1) \tag{213}$$

with matrices $P = P^T > 0$ and $Q = Q^T > 0$.

Remark 3.2.2.2 (208 – 209) are, in modified form, taken from (*Owens*, *Debeljkovic* 1985).

For given (213), general backward difference is:

$$\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) = \mathbf{x}^T(k+1)E^T PE\mathbf{x}(k+1)$$
$$+\mathbf{x}^T(k)Q\mathbf{x}(k) - \mathbf{x}^T(k)E^T PE\mathbf{x}(k) - \mathbf{x}^T(k-1)Q\mathbf{x}(k-1). \tag{214}$$

Clearly, using the equation of motion (178), we have:

$$\Delta V(\mathbf{x}(k)) = \mathbf{x}^T(k)\left(A_0^T PA_0 - E^T PE + Q\right)\mathbf{x}(k)$$
$$+2\mathbf{x}^T(k)\left(A_0^T PA_1\right)\mathbf{x}(k-1) - \mathbf{x}^T(k-1)\left(Q - A_1^T PA_1\right)\mathbf{x}(k-1), \tag{215}$$

or

$$\Delta V(\mathbf{x}(k)) = \mathbf{x}^T(k)\left(A_0^T PA_0 - E^T PE + 2Q + 2S\right)\mathbf{x}(k) - \mathbf{x}^T(k)Q\mathbf{x}(k)$$
$$-2\mathbf{x}^T(k)S\mathbf{x}(k) + 2\mathbf{x}^T(k)\left(A_0^T PA_1\right)\mathbf{x}(k-1) - \mathbf{x}^T(k-1)\left(Q - A_1^T PA_1\right)\mathbf{x}(k-1). \tag{216}$$

Using (208) and (209) yields:

$$\mathbf{x}^T(k+1)E^T PE\mathbf{x}(k+1) = \mathbf{x}^T(k)A_0^T PA_0\mathbf{x}(k)$$
$$+2\mathbf{x}^T(k)A_0^T PA_1\mathbf{x}(k-1) + \mathbf{x}^T(k-1)A_1^T PA_1\mathbf{x}(k-1). \tag{217}$$

Using the very well known inequality, with particular choice:

$$\mathbf{x}^T(k)\Gamma\mathbf{x}(k) = \mathbf{x}^T(k)\left(A_1^T PA_1 - E^T PE\right)\mathbf{x}(k) \geq 0,$$
$$\mathbf{x}(k) \in \mathcal{W}_{k^*}^d,\ \forall \mathbf{x}(k) \in \mathcal{S}_\beta,\ \forall k \in \mathcal{K}_N \tag{218}$$

one can get:

$$\mathbf{x}^T(k+1)E^T PE\mathbf{x}(k+1) \leq \mathbf{x}^T(k)A_0^T PA_0\mathbf{x}(k) - \mathbf{x}^T(k)A_0^T PA_1(A_1^T PA_1 -$$
$$-E^T PE)^{-1}A_1^T PA_0\mathbf{x}(k) + \mathbf{x}^T(k-1)(2A_1^T PA_1 - E^T PE)\mathbf{x}(k-1). \tag{219}$$

Moreover, since:

$$\left\|\mathbf{x}(k-1)\right\|_{E^T PE}^2 \geq 0,\ \ \forall k \in \mathcal{K}_N,\ \ \forall \mathbf{x}(k) \in \mathcal{W}_{d,k}^{*} \setminus \{0\} \tag{220}$$

and using assumption (209) it is clear that (219), reduces to:

$$\mathbf{x}^T(k+1)E^T PE\mathbf{x}(k+1) \leq \mathbf{x}^T(k)A_0^T P^{\frac{1}{2}}\left(I - A_1\left(A_1^T PA_1 - E^T PE\right)^{-1}A_1^T + 2p^2 I\right)P^{\frac{1}{2}}A_0\mathbf{x}(k) \tag{221}$$

Using very well known the property of quadratic form, one can get:

$$\mathbf{x}^T(k+1)E^T PE\mathbf{x}(k+1) \leq \overline{\lambda}_{\max}(\ )\mathbf{x}^T(k)E^T PE\mathbf{x}(k) \tag{222}$$

where:

$$\overline{\lambda}_{\max}(\ ) = \{\mathbf{x}^T(k)A_0^T P^{\frac{1}{2}}(I - A_1(A_1^T PA_1 - E^T PE)^{-1}A_1^T + 2p^2 I)P^{\frac{1}{2}}A_0\mathbf{x}(k),$$
$$\mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\},\ \mathbf{x}^T(k)E^T PE\mathbf{x}(k) = 1\} \tag{223}$$

Then following the identical procedure as in the *Theorem* 3.2.2.1, one can get:

$$\ln \mathbf{x}^T(k+1)E^T PE\mathbf{x}(k+1) - \ln \mathbf{x}^T(k)E^T PE\mathbf{x}(k) < \ln \bar{\lambda}_{\max}(\;) \tag{224}$$

where $\bar{\lambda}_{\max}(\;)$ is given by (223).

If the summing $\sum_{j=k_0}^{k_0+k-1}$ is applied to both sides of (224) for $\forall k \in \mathcal{K}_N$, one can obtain:

$$\begin{aligned} &\ln \mathbf{x}^T(k_0+k)E^T PE\mathbf{x}(k_0+k) \le \ln \prod_{j=k_0}^{k_0+k-1} \bar{\lambda}_{\max}(\;) \\ &\le \ln \bar{\lambda}^{k}_{\max}(\;) + \ln \mathbf{x}^T(k_0)E^T PE\mathbf{x}(k_0), \quad \forall k \in \mathcal{K}_N \end{aligned} \tag{225}$$

Taking into account the fact that $\|\mathbf{x}_0\|^2_{E^T PE} < \alpha$ and the basic condition of *Theorem* 3.2.2.3, (211), one can get:

$$\begin{aligned} &\ln \mathbf{x}^T(k_0+k)E^T PE\mathbf{x}(k_0+k) < \ln \bar{\lambda}^{k}_{\max}(\;) + \ln \mathbf{x}^T(k_0)E^T PE\mathbf{x}(k_0) \\ &< \ln \alpha \cdot \bar{\lambda}^{k}_{\max}(\;) < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \;\; \forall k \in \mathcal{K}_N. \quad Q.E.D. \end{aligned} \tag{226}$$

4. Conclusion

The first part of this chapter is devoted to the stability of particular classes of linear continuous and discrete time delayed systems. Here, we present a number of new results concerning stability properties of this class of systems *in the sense of Lyapunov and non-Lyapunov* and analyze the relationship between them. Some open question can arise when particular choice of parameters p and q is needed, see (*Su & Huang* 1992), (*Xu & Liu 1994*) and (*Su* 1994).

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of Lyapunov stability theory even for linear continuous singular time delayed systems (LCSTDS) and linear discrete descriptor time delayed systems (LDDTDS) in that sense that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a *weak* form of Lyapunov continuous (discrete) algebraic matrix equation (*Owens*, *Debeljkovic* 1985) respectively, incorporating condition which refers to time delay term.

To assure *asymptotical stability* for (LCSTDS) it is not only enough to have the eigenvalues of the matrix pair (E, A) in the left half complex plane or within the unit circle, respectively, but also to provide an impulse-free motion and some other certain conditions to be fulfilled for the systems under consideration. The idea and the approach, in this exposure, are based upon the papers by (*Owens*, *Debeljkovic*1985) and (*Tissir*, *Hmamed* 1996).

Some different approaches have been shown in order to construct *Lyapunov* stability theory for a particular class of autonomous (LCSTDS) and (LDDTDS).

The second part of the chapter is concerned with the stability of particular classes of (LCSTDS) and (LDDTDS). There, we present a number of *new results* concerning stability properties of this class of systems *in the sense of non-Lyapunov* (*finite time stability, practical stability, attractive practical stability, etc.*) and analyse the relationship between them.

And finally this chapter extends some of the basic results in the area of non-Lyapunov to *linear*, *continuous singular time invariant time-delay systems* (LCSTDS) and (LDDTDS). In that sense the part of this result is hence a geometric counterpart of the algebraic theory of *Campbell* (1980) charged with appropriate criteria to cover the need for system stability in the presence of actual time delay term. To assure *practical stability* for (LCSTDS) it is not enough only to have the eigenvalues of matrix pair (E, A) somewhere in the complex plane, but also to provide an impulse-free motion and certain conditions to be fulfilled for the system under consideration.

Some different approaches have been shown in order to construct *non-Lyapunov* stability theory for a particular class of autonomous (LDDTDS). The geometric description of consistent initial conditions that generate tractable solutions to such problems and the construction of non-Lyapunov stability theory to bound rates of decay of such solutions are also investigated. Result are based on existing Lyapunov-like functions and their properties on sub-space of consistent initial functions (conditions). In particular, these functions need not to have: *a*) Properties of positivity in the whole state space and *b*) negative derivatives along the system trajectories.

And finally a quite new approach leads to the sufficient delay–independent criteria for finite and attractive practical stability of (LCSTDS) and (LDDTDS).

Stability issues, as well as time delay and singularity phenomena play a significant role in modeling of real systems. A need for their consideration arises from growing interest and extensive application possibilities in different areas such as large-scale systems, flexible light-weight structures and their vibration and noise control, optimization of smart structures (*Nestorovic* et al. 2005, 2006, 2008) etc. Development of reliable models plays a crucial role especially in early development phases, which enables performance testing, design review, optimization and controller design (*Nestorovic & Trajkov* 2010.a). Assumptions introduced along with model development, especially e.g. reduction of large numerical models of smart structures require consideration of many important questions from the control theory point of view, whereby the stability and singularity phenomena count among some of the most important. Therefore they represent the focus of the authors' ongoing and further research activities (*Debeljkovic* et al. 2011.b, *Nestorovic & Trajkov* 2010.b).

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4

Exponential Stability of Uncertain Switched System with Time-Varying Delay

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1. Introduction

During the past decades, many researchers have investigated stability of switched systems; due to its potential for real world application such as transportation systems, computer systems, communication systems, control of mechanical systems, etc. A switched systems is composed of a family of continuous time (Alan & Lib, 2008; Alan & Lib, 2009, Alan et al., 2008; Hien et al., 2009; Hien & Phat, 2009; Kim et al., 2006; Li et al., 2009; Niamsup, 2008; Li et al., 2009; Lien et al., 2009; Lib et al., 2008) or discrete time systems (Wu et al., 2004) and a switching condition determining at any time instant which subsystem is activated.

In recent years, the stability of systems with time delay has received considerable attention. Switched system in which all subsystems are stable was studied in (Lien et al., 2009) and switched system in which subsystems are both stable and unstable was studied in (Alan & Lib, 2008; Alan & Lib, 2009, Alan et al., 2008). The commonly used approach to stability analysis of switched systems is Lyapunov theory and some important preliminaries results have been applied to obtain sufficient conditions for stability of switched systems. A single Lyapunov function approach is used in (Alan & Lib, 2008) and a multiple Lyapunov functions approach is used in (Hien et al., 2009; Kim et al., 2006; Li et al., 2009; Lien et al., 2009; Lib et al., 2008) and the references therein. The asymptotical stability of the linear with time delay and uncertainties has been considered in (Lien et al., 2009). In (L.V.Hien et al., 2009), the authors investigated the exponential stability and stabilization of switched linear systems with time varying delay and uncertainties by using the strictly complete systems of matrices approach. The strictly complete of the matrices has been also used for the switching condition, see (Hien et al., 2009; Huang et al., 2005; Niamsup, 2008; Lib et al., 2008; Wu et al., 2004). In this paper, stability analysis for switched linear and nonlinear systems with uncertainties and time-varying delay are studied. We obtain the new conditions for exponential stability of switched system in which subsystems consist of stable and unstable subsystems. The stability conditions are derived in terms of linear matrix inequality (LMI) by using a new Lyapunov

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function. The free weighting matrices and Newton-Leibniz formula are applied. As a results, the obtained stability conditions are less conservative comparing to some previous existing results in the literatures. In particular, comparing to (Alan & Lib, 2008), our results give a much less conservative results, namely, for stable subsystems, the condition that state matrices are Hurwitz stable is not required. Moreover, advantages of the paper are that the delay is time-varying and switched system may have uncertainties. The paper is organized as follows. In section 1, problem formulation and introduction is addressed. In section 2, we give some notations, definitions and the preliminary results that will be used in this paper. Switching design for the exponential stability of the switched system is presented in Section 3. In section 4, numerical examples are given to illustrate the theoretical results. The paper ends with conclusions and cited references.

2. Preliminaries

The following notations will be used throughout this paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times n}$ denotes the space of all matrices of $n \times n$-dimensions. A^T denotes the transpose of A. I denotes the identity matrix. $\lambda(A), \lambda_M(A), \lambda_m(A)$ denote the set of all eigenvalues of A, the maximum eigenvalue of A, and the minimum eigenvalue of A, respectively. For all real symmetric matrix X, the notation $X > 0 (X \geq 0, X < 0, X \leq 0)$ means that X is positive definite (positive semidefinite, negative definite, negative semidefinite, respectively.) For a vector x, $\|x_t\| = \sup_{s\in[-h_M,0]} \|x(t+s)\|$ with $\|x\|$ being the Euclidean norm of vector x.

The switched system under the consideration is described by

$$\begin{aligned} \dot{x}(t) &= [A_\sigma + \Delta A_\sigma(t)]x(t) + [B_\sigma + \Delta B_\sigma(t)]x(t-h(t)) \\ &\quad + f_\sigma(t, x(t), x(t-h(t))),\ t > 0, \\ x(t) &= \phi(t),\ t \in [-h_M, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector. $\sigma(\cdot) : \mathbb{R}^n \to S = \{1, 2, ..., N\}$ is the switching function. Let $i \in S = S_u \cup S_s$ such that $S_u = \{1, 2, ..., r\}$ and $S_s = \{r+1, r+2, ..., N\}$ be the set of the unstable and stable modes, respectively. N denotes the number of subsystems. $A_i, B_i \in \mathbb{R}^{n\times n}$ are given constant matrices. $\Delta A_i(t), \Delta B_i(t)$ are uncertain matrices satisfying the following conditions:

$$\Delta A_i(t) = E_{1i}F_{1i}(t)H_{1i},\ \Delta B_i(t) = E_{2i}F_{2i}(t)H_{2i}, \tag{2}$$

where $E_{ji}, H_{ji}, j = 1, 2, i = 1, 2, ..., N$ are given constant matrices with appropriate dimensions. $F_{ji}(t)$ are unknown, real matrices satisfying:

$$F_{ji}^T(t)F_{ji}(t) \leq I,\ j = 1, 2,\ i = 1, 2, ..., N,\ \forall t \geq 0, \tag{3}$$

where I is the identity matrix of appropriate dimension.
The nonlinear perturbation $f_i(t, x(t), x(t-h(t)))$, $i = 1, 2, ..., N$ satisfies the following condition:

$$\| f_i(t, x(t), x(t-h(t))) \| \leq \gamma_i \| x(t) \| + \delta_i \| x(t-h(t)) \| \tag{4}$$

for some $\gamma_i,\ \delta_i > 0$. The time-varying delay function $h(t)$ is assumed to satisfy one of the following conditions:

(i) when $\Delta A_i(t) = 0$ and $\Delta B_i(t) = 0$ and $f_i(t, x(t), x(t-h(t))) = 0$

$$0 \le h_m \le h(t) \le h_M,\ \dot{h}(t) \le \mu,\ t \ge 0,$$

(ii) when $\Delta A_i(t) \ne 0$ or $\Delta B_i(t) \ne 0$ or $f_i(t, x(t), x(t-h(t))) \ne 0$

$$0 \le h_m \le h(t) \le h_M,\ \dot{h}(t) \le \mu < 1,\ t \ge 0,$$

where h_m, h_M and μ are given constants.

Definition 2.1 (Hien et al., 2009) Given $\beta > 0$. The system (1) is $\beta-$exponentially stable if there exists a switching function $\sigma(\cdot)$ and positive number γ such that any solution $x(t,\phi)$ of the system satisfies

$$\| x(t,\phi) \| \le \gamma e^{-\beta t} \| \phi \|,\ \forall t \in \mathbb{R}^+,$$

for all the uncertainties.

Lemma 2.1 (Hien et al., 2009) For any $x, y \in \mathbb{R}^n$, matrices W, E, F, H with $W > 0, F^T F \le I$, and scalar $\varepsilon > 0$, one has

(1.) $EFH + H^T F^T E^T \le \varepsilon^{-1} E E^T + \varepsilon H^T H,$

(2.) $2x^T y \le x^T W^{-1} x + y^T W y.$

Lemma 2.2 (Alan & Lib, 2008) Let $u : [t_0, \infty] \to \mathbb{R}$ satisfy the following delay differential inequality:

$$\dot{u}(t) \le \alpha u(t) + \beta \sup_{\theta \in [t-\tau, t]} u(\theta),\ t \ge t_0.$$

Assume that $\alpha + \beta > 0$. Then, there exist positive constant ξ and k such that

$$u(t) \le k e^{\xi(t-t_0)},\ t \ge t_0,$$

where $\xi = \alpha + \beta$ and $k = \sup_{\theta \in [t_0-\tau, t_0]} u(\theta)$.

Lemma 2.3 (Alan & Lib, 2008) Let the following differential inequality:

$$\dot{u} \le -\alpha u(t) + \beta \sup_{\theta \in [t-\tau, t]} u(\theta),\ t \ge t_0,$$

hold. If $\alpha > \beta > 0$, then there exist positive k and ζ such that

$$u(t) \le k e^{-\zeta(t-t_0)},\ t \ge t_0,$$

where $\zeta = \alpha - \beta$ and $k = \sup_{\theta \in [t_0-\tau, t_0]} u(\theta)$.

Lemma 2.4 (Schur Complement Lemma) (Boyd et al., 1985) Given constant symmetric Q, S and $R \in \mathbb{R}^{n \times n}$ where $R > 0, Q = Q^T$ and $R = R^T$ we have

$$\begin{bmatrix} Q & S \\ S^T & -R \end{bmatrix} < 0 \Leftrightarrow Q + S R^{-1} S^T < 0.$$

3. Main results

In this section, we establish exponential stability of uncertain switched system with time-varying delay. For simplicity of later presentation, we use the following notations:

$\lambda^+ = \max_i \{\xi_i, \forall i \in S_u\}$, ξ_i denotes the growth rates of the unstable modes.

$\lambda^- = \min_i \{\zeta_i, \forall i \in S_s\}$, ζ_i denotes the decay rates of the stable modes.

$T^{+}(t_0, t)$ denotes the total activation times of the unstable modes over $[t_0, t)$.
$T^{-}(t_0, t)$ denotes the total activation times of the stable modes over $[t_0, t)$.
$N(t)$ denotes the number of times the system is switched on $[t_0, t)$.
$l(t)$ denotes the number of times the unstable subsystems are activated on $[t_0, t)$.
$N(t) - l(t)$ denotes the number of times the stable subsystems are activated on $[t_0, t)$.

$$\psi = \frac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}}.$$

$$\alpha_1 = \min_i\{\lambda_m(P_i)\}.$$

$$\begin{aligned}\alpha_2 = &\max_i\{\lambda_M(P_i)\} + h_M \max_i\{\lambda_M(Q_i)\} + \frac{h_M^2}{2}\max_i\{\lambda_M(R_i)\}\\ &+ h_M^2 \max_i\{\lambda_M(\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i}\end{bmatrix})\}\\ &+ 2h_M^2 \max_i\{\lambda_M(A_i^T T_i A_i), \lambda_M(A_i^T T_i B_i), \lambda_M(B_i^T T_i A_i), \lambda_M(B_i^T T_i B_i)\},\end{aligned}$$

$$\begin{aligned}\alpha_3 = &\max_i\{\lambda_M(P_i)\} + h_M \max_i\{\lambda_M(Q_i)\} + \frac{h_M^2}{2}\max_i\{\lambda_M(R_i)\}\\ &+ h_M^2 \max_i\{\lambda_M(\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i}\end{bmatrix})\}.\end{aligned}$$

$$\Omega_{1,i} = \begin{bmatrix} \Phi_{11,i} & \Phi_{12,i} \\ * & \Phi_{13,i}\end{bmatrix},$$

$$\Phi_{11,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + h_M A_i^T T_i A_i,$$
$$\Phi_{12,i} = B_i^T P_i + h_M S_{12,i} + h_M A_i^T T_i B_i,$$
$$\Phi_{13,i} = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + h_M B_i^T T_i B_i.$$

$$\Omega_{2,i} = \begin{bmatrix} \Phi_{21,i} & \Phi_{22,i} \\ * & \Phi_{23,i}\end{bmatrix},$$
$$\Phi_{21,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + h_M A_i^T T_i A_i + h_M X_{11,i} + Y_i + Y_i^T,$$
$$\Phi_{22,i} = B_i^T P_i + h_M S_{12,i} + h_M A_i^T T_i B_i + h_M X_{12,i} - Y_i + Z_i^T,$$
$$\Phi_{23,i} = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + h_M B_i^T T_i B_i + h_M X_{22,i} - Z_i - Z_i^T.$$

$$\Omega_{3,i} = \begin{bmatrix} X_{11,i} & X_{12,i} & Y_i \\ * & X_{22,i} & Z_i \\ * & * & \frac{T_i}{2}\end{bmatrix}.$$

$$\Xi_i = \begin{bmatrix} \Phi_{31,i} & \Phi_{32,i} \\ * & \Phi_{33,i}\end{bmatrix},$$
$$\Phi_{31,i} = A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{1i}^{-1} H_{1i}^T H_{1i} + \varepsilon_{1i} P_i E_{1i}^T E_{1i} P_i + \varepsilon_{2i} P_i E_{2i}^T E_{2i} P_i,$$
$$\Phi_{32,i} = B_i^T P_i + h_M S_{12,i},$$
$$\Phi_{33,i} = -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} + \varepsilon_{2i}^{-1} H_{2i}^T H_{2i}.$$

$$\Theta_i = \begin{bmatrix} \Phi_{41,i} & \Phi_{42,i} \\ * & \Upsilon_{43,i}\end{bmatrix},$$
$$\begin{aligned}\Phi_{41,i} = &A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} + \varepsilon_{3i}^{-1}\gamma_i I + \varepsilon_{3i} P_i P_i + \varepsilon_{4i}^{-1} H_{4i}^T H_{4i}\\ &+ \varepsilon_{4i} P_i E_{4i}^T E_{4i} P_i + \varepsilon_{6i} P_i E_{5i}^T E_{5i} P_i,\end{aligned}$$
$$\Phi_{42,i} = B_i^T P_i + h_M S_{12,i},$$

$\Phi_{43,i} = -(1-\mu)e^{-2\beta h_M}Q_i + h_M S_{22,i} + \varepsilon_{3i}^{-1}\delta_i I + \varepsilon_{5i}^{-1}H_{5i}^T H_{5i}.$

3.1 Exponential stability of linear switched system with time-varying delay

In this section, we deal with the problem for exponential stability of the zero solution of system (1) without the uncertainties and nonlinear perturbation ($\Delta A_i(t) = \Delta B_i(t) = 0,\ f_i(t, x(t), x(t-h(t))) = 0$).

Theorem 3.1 *The zero solution of system (1) with $\Delta A_i(t) = \Delta B_i(t) = 0$ and $f_i(t, x(t), x(t-h(t))) = 0$ is exponentially stable if there exist symmetric positive definite matrices P_i, Q_i, R_i, $\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}$, T_i and appropriate dimension matrices Y_i, Z_i such that the following conditions hold:*

A1. (i) For $i \in S_u$,

$$\Omega_{1,i} > 0. \tag{5}$$

(ii) For $i \in S_s$,

$$\Omega_{2,i} < 0 \ \ and \ \ \Omega_{3,i} \geq 0. \tag{6}$$

A2. Assume that, for any t_0 the switching law guarantees that

$$\inf_{t \geq t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{7}$$

where $\lambda^* \in (0, \lambda^-)$. Furthermore, there exists $0 < \nu < \lambda^*$ such that
(i) If the subsystem $i \in S_u$ is activated in time intervals $[t_{i_k-1}, t_{i_k}), k = 1, 2, ...,$ then

$$ln\ \psi - \nu(t_{i_k} - t_{i_k-1}) \leq 0,\ k = 1, 2, ..., l(t). \tag{8}$$

(ii) If the subsystem $j \in S_s$ is activated in time intervals $[t_{j_k-1}, t_{j_k}), k = 1, 2, ...,$ then

$$ln\ \psi + \zeta_j h_M - \nu(t_{j_k} - t_{j_k-1}) \leq 0,\ k = 1, 2, ..., N(t) - 1. \tag{9}$$

Proof. Consider the following Lyapunov functional:

$$V_i(x_t) = V_{1,i}(x(t)) + V_{2,i}(x_t) + V_{3,i}(x_t) + V_{4,i}(x_t) + V_{5,i}(x_t)$$

where $x_t \in C([-h_M, 0], \mathbb{R}^n),\ x_t(s) = x(t+s),\ s \in [-h_M, 0]$ and

$V_{1,i}(x(t)) = x^T(t)P_i x(t),$

$V_{2,i}(x_t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)Q_i x(s)ds,$

$V_{3,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)}x^T(\xi)R_i x(\xi)d\xi ds,$

$V_{4,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix} d\xi ds,$

$V_{5,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} \dot{x}^T(\xi)T_i\dot{x}(\xi)d\xi ds.$

It is easy to verify that

$$\alpha_1 \parallel x(t) \parallel^2 \leq V_i(x_t) \leq \alpha_2 \parallel x_t \parallel^2,\ t \geq 0. \tag{10}$$

We have

$$\begin{aligned} V_{1,i}(x(t)) &\le \max_i\{\lambda_M(P_i)\} \parallel x(t) \parallel^2 \\ &= \frac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}} \min_j\{\lambda_m(P_j)\}x^T(t)x(t) \\ &\le \frac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}} x^T(t)P_jx(t) \\ &= \frac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}} V_{1,j}(x(t)). \end{aligned}$$

Let $\psi = \dfrac{\max_i\{\lambda_M(P_i)\}}{\min_j\{\lambda_m(P_j)\}}$. Obviously $\psi \ge 1$ and we get

$$V_i(x_t) \le \psi V_j(x_t),\ \forall i, j \in S. \tag{11}$$

Taking derivative of $V_{1,i}(x(t))$ along trajectories of any subsystem ith we have

$$\begin{aligned} \dot{V}_{1,i}(x(t)) &= \dot{x}^T(t)P_ix(t) + x^T(t)P_i\dot{x}(t) \\ &= \sum_{i=1}^{N} \lambda_i(t)[x^T(t)A_i^TP_ix(t) + x^T(t-h(t))B_i^TP_ix(t) \\ &\quad + x^T(t)P_iA_ix(t) + x^T(t)P_iB_ix(t-h(t))]. \end{aligned}$$

Next, by taking derivative of $V_{2,i}(x_t), V_{3,i}(x_t), V_{4,i}(x_t)$ and $V_{5,i}(x_t)$, respectively, along the system trajectories yields

$$\begin{aligned} \dot{V}_{2,i}(x_t) &= x^T(t)Q_ix(t) - (1-\dot{h}(t))e^{-2\beta h(t)}x^T(t-h(t))Q_ix(t-h(t)) - 2\beta V_{2,i}(x_t) \\ &\le x^T(t)Q_ix(t) - (1-\mu)e^{-2\beta h(t)}x^T(t-h(t))Q_ix(t-h(t)) - 2\beta V_{2,i}(x_t), \end{aligned}$$

$$\begin{aligned} \dot{V}_{3,i}(x_t) &= \int_{-h(t)}^{0}[x^T(t)R_ix(t) - e^{2\beta s}x^T(t+s)R_ix(t+s)]ds - 2\beta V_{3,i}(x_t) \\ &\le h_Mx^T(t)R_ix(t) - \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_ix(s)ds - 2\beta V_{3,i}(x_t), \end{aligned}$$

$$\begin{aligned}
\dot{V}_{4,i}(x_t) &= \int_{-h(t)}^{0} [\begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix} \\
&\quad - e^{2\beta s} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}]ds \\
&\quad - e^{2\beta s} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}]ds \\
&\quad - 2\beta V_{4,i}(x_t) \\
&\le h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\
&\quad - \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds \\
&\quad - e^{2\beta s} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t+s) \\ x(t+s-h(t+s)) \end{bmatrix}]ds \\
&\quad - 2\beta V_{4,i}(x_t) \\
&\le h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\
&\quad - \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds \\
&\quad - 2\beta V_{4,i}(x_t),
\end{aligned}$$

$$\begin{aligned}
\dot{V}_{5,i}(x_t) &= \int_{-h(t)}^{0} [\dot{x}^T(t) T_i \dot{x}(t) - \dot{x}^T(t+s) T_i \dot{x}(t+s)] ds \\
&\le h_M \dot{x}^T(t) T_i \dot{x}(t) - \int_{t-h(t)}^{t} \dot{x}^T(s) T_i \dot{x}(s) ds \\
&= h_M \dot{x}^T(t) T_i \dot{x}(t) - \frac{1}{2} \int_{t-h(t)}^{t} \dot{x}^T(s) T_i \dot{x}(s) ds - \frac{1}{2} \int_{t-h(t)}^{t} \dot{x}^T(s) T_i \dot{x}(s) ds.
\end{aligned}$$

Then, the derivative of $V_i(x_t)$ along the any trajectory of solution of (1) is estimated by

$$\begin{aligned}
\dot{V}_i(x_t) &\le \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Omega_{1,i}^{\star} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t) \\
&\quad - \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) R_i x(s) ds - 2\beta V_{3,i}(x_t) \\
&\quad - \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds \\
&\quad - 2\beta V_{4,i}(x_t) + h_M \dot{x}(t)^T T_i \dot{x}(t) - \frac{1}{2} \int_{t-h(t)}^{t} \dot{x}^T(s) T_i \dot{x}(s) ds \\
&\quad - \frac{1}{2} \int_{t-h(t)}^{t} \dot{x}^T(s) T_i \dot{x}(s) ds, \qquad (12)
\end{aligned}$$

where

$$\Omega_{1,i}^{\star} = \begin{bmatrix} A_i^T P_i + P_i A_i + Q_i + h_M R_i + h_M S_{11,i} & B_i^T P_i + h_M S_{12,i} \\ * & -(1-\mu)e^{-2\beta h_M} Q_i + h_M S_{22,i} \end{bmatrix}$$

Since

$$\begin{aligned} \int_{-h(t)}^{0} \int_{t+s}^{t} e^{2\beta(\xi - t)} x^T(\xi) R_i x(\xi) d\xi ds &\leq \int_{-h(t)}^{0} \int_{t-h(t)}^{t} e^{2\beta(\xi - t)} x^T(\xi) R_i x(\xi) d\xi ds \\ &\leq h_M \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) R_i x(s) ds, \end{aligned}$$

we have

$$\begin{aligned} -\int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) R_i x(s) ds &\leq -\frac{1}{h_M} \int_{-h(t)}^{0} \int_{t+s}^{t} e^{2\beta(\xi - t)} x^T(\xi) R_i x(\xi) d\xi ds \\ &= -\frac{1}{h_M} V_{3,i}(x_t). \end{aligned} \tag{13}$$

Similarly, we have

$$-\int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds \leq -\frac{1}{h_M} V_{4,i}(x_t), \tag{14}$$

and

$$-\frac{1}{2} \int_{t-h(t)}^{t} \dot{x}(s) T_i \dot{x}(s) ds \leq -\frac{1}{2h_M} V_{5,i}(x_t). \tag{15}$$

From (12), (13), (14) and (15), we obtain

$$\begin{aligned} \dot{V}_i(x_t) \leq & \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Omega_{1,i} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t) \\ & -(2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t)) - \frac{1}{2h_M} V_{5,i}(x_t) \\ & -\frac{1}{2} \int_{t-h(t)}^{t} \dot{x}(s) T_i \dot{x}(s) ds. \end{aligned} \tag{16}$$

For $i \in S_u$, we have

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Omega_{1,i} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}.$$

By (5), (16) and Lemma 2.2, there exists $\xi_i > 0$ such that

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{\xi_i (t - t_0)},\ t \geq t_0. \tag{17}$$

where $\xi_i = \frac{2\max\limits_i\{\lambda_M(\Omega_{1,i})\}}{\min\limits_i\{\lambda_m(P_i)\}}$.

For $i \in S_s$, we have that when $X_i = \begin{bmatrix} X_{11,i} & X_{12,i} \\ * & X_{22,i} \end{bmatrix} \geq 0$, the following holds:

$$h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T X_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T X_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} ds \geq 0. \quad (18)$$

Using the Newton-Leibniz formula, (Wu et al., 2004), we can write

$$x(t-h(t)) = x(t) - \int_{t-h(t)}^{t} \dot{x}(s)ds.$$

Then, for any appropriate dimension matrices Y_i and Z_i, we have

$$2[x^T(t)Y_i + x^T(t-h(t))Z_i][x(t) - \int_{t-h(t)}^{t} \dot{x}(s)ds - x(t-h(t))] = 0.$$

It follows that

$$2x^T(t)Y_i x(t) - 2x^T(t)Y_i \int_{t-h(t)}^{t} \dot{x}(s)ds - 2x^T(t)Y_i x(t-h(t)) + 2x^T(t-h(t))Z_i x(t) - 2x^T(t-h(t))Z_i \int_{t-h(t)}^{t} \dot{x}(s)ds - 2x^T(t-h(t))Z_i x(t-h(t)) = 0. \quad (19)$$

From (16) with (18) and (19), we have

$$\begin{aligned} \dot{V}_i(x_t) \leq & \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Omega_{2,i} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t) \\ & -(2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t)) - \frac{1}{2h_M} V_{5,i}(x_t) \\ & - \int_{t-h(t)}^{t} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix}^T \Omega_{3,i} \begin{bmatrix} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{bmatrix} ds. \end{aligned} \quad (20)$$

By (6), (20) and Lemma 2.3, there exist $\zeta_i > 0$ such that

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{-\zeta_i(t-t_0)},\ t \geq t_0. \quad (21)$$

where $\zeta_i = \min\{\frac{\min\limits_i\{\lambda_m(-\Omega_{2,i})\}}{\max\limits_i\{\lambda_M(P_i)\}}, 2\beta, \frac{1}{2h_M}\}$.

Let $N(t)$ denotes the number of times the system is switched on $[t_0, t)$ such that $\lim\limits_{t\to+\infty} N(t) = +\infty$. Suppose that $\sigma(t_0) = i_0, \sigma(t_1) = i_1, ...$ and $\sigma(t) = i$.

Let $l(t)$ denotes the number of times the unstable subsystems are activated on $[t_0, t)$ and $N(t) - l(t)$ denotes the number of times the stable subsystems are activated on $[t_0, t)$. Suppose that $t_0 < t_1 < t_2 < ...$ and $\lim_{n\to+\infty} t_n = +\infty$.

From (11), (17) and (21), suppose that the j th subsystem of unstable mode is activated on the interval $[t_l, t_{l+1})$,

- if the i th subsystem of unstable mode is activated on the interval $[t_{l-1}, t_l)$, then
$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{\xi_i(t_l - t_{l-1})} e^{\xi_j(t-t_l)},\ t \in [t_l, t_{l+1})$.
- if the i th subsystem of stable mode is activated on the interval $[t_{l-1}, t_l)$, then
$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{-\zeta_i(t_l - t_{l-1})} e^{\xi_j(t-t_l)},\ t \in [t_l, t_{l+1})$.

Suppose that the j th subsystem of stable mode is activated on the interval $[t_l, t_{l+1})$,

- if the i th subsystem of unstable mode is activated on the interval $[t_{l-1}, t_l)$, then
$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{\xi_i(t_l - t_{l-1})} e^{-\zeta_j(t-t_l)},\ t \in [t_l, t_{l+1})$.
- if the i th subsystem of stable mode is activated on the interval $[t_{l-1}, t_l)$, then
$V_j(x_t) \leq \psi \parallel V_i(x_{t_{l-1}}) \parallel e^{-\zeta_i(t_l - t_{l-1})} e^{-\zeta_j(t-t_l)},\ t \in [t_l, t_{l+1})$.

In general, we get

$$
\begin{aligned}
V_i(x_t) &\leq \prod_{m=1}^{l(t)} \psi e^{\xi_{i_m}(t_m - t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} e^{-\zeta_{i_n}(t_n - t_{n-1})} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\zeta_i(t - t_{N(t)-1})} \\
&\leq \prod_{m=1}^{l(t)} \psi e^{\lambda^+(t_m - t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} e^{-\lambda^-(t_n - t_{n-1})} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^-(t - t_{N(t)-1})},
\end{aligned}
$$

$t \geq t_0$. Using (7), we have

$$
V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^*(t-t_0)},\ t \geq t_0.
$$

By (8) and (9), we get

$$
V_i(x_t) \leq \parallel V_{i_0}(x_{t_0}) \parallel e^{-(\lambda^* - \nu)(t-t_0)},\ t \geq t_0.
$$

Thus, by (10), we have

$$
\parallel x(t) \parallel \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \parallel x_{t_0} \parallel e^{-\frac{1}{2}(\lambda^* - \nu)(t-t_0)},\ t \geq t_0,
$$

which concludes the proof of the Theorem 3.1. □

3.2 Robust exponential stability of linear switched system with time-varying delay

In this section, we give conditions for robust exponential stability of the zero solution of system (1) without nonlinear perturbation, namely $f_i(t, x(t), x(t-h(t))) = 0$. The following is the main result.

Theorem 3.2 *The zero solution of system (1) with $f_i(t, x(t), x(t-h(t))) = 0$ is robustly exponentially stable if there exist positive real numbers $\varepsilon_{1i}, \varepsilon_{2i}$, positive definite matrices P_i, Q_i, R_i and $\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}$ such that the following conditions hold:*

A1. (i) For $i \in S_u$,

$$
\Xi_i > 0. \tag{22}
$$

(ii) For $i \in S_s$,

$$\Xi_i < 0. \tag{23}$$

A2. Assume that, for any t_0 *the switching law guarantees that*

$$\inf_{t \geq t_0} \frac{T^-(t_0,t)}{T^+(t_0,t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{24}$$

where $\lambda^* \in (0, \lambda^-)$*. Furthermore, there exists* $0 < \nu < \lambda^*$ *such that*
(i) If the subsystem $i \in S_u$ *is activated in time intervals* $[t_{i_k-1}, t_{i_k}), k = 1, 2, ...,$ *then*

$$ln\ \psi - \nu(t_{i_k} - t_{i_k-1}) \leq 0,\ k = 1, 2, ..., l(t). \tag{25}$$

(ii) If the subsystem $j \in S_s$ *is activated in time intervals* $[t_{j_k-1}, t_{j_k}), k = 1, 2, ...,$ *then*

$$ln\ \psi + \zeta_j h_M - \nu(t_{j_k} - t_{j_k-1}) \leq 0,\ k = 1, 2, ..., N(t) - 1. \tag{26}$$

Proof. Consider the following Lyapunov functional:

$$V_i(x_t) = V_{1,i}(x(t)) + V_{2,i}(x_t) + V_{3,i}(x_t) + V_{4,i}(x_t)$$

where $x_t \in C([-h_M, 0], \mathbb{R}^n), x_t(s) = x(t+s), s \in [-h_M, 0]$, and $V_{1,i}(x(t)) = x^T(t)P_i x(t)$,

$$V_{2,i}(x_t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) Q_i x(s) ds,$$

$$V_{3,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)} x^T(\xi) R_i x(\xi) d\xi ds,$$

$$V_{4,i}(x_t) = \int_{-h(t)}^{0}\int_{t+s}^{t} e^{2\beta(\xi-t)} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix} d\xi ds.$$

It is easy to verify that

$$\alpha_1 \parallel x(t) \parallel^2 \leq V_i(x_t) \leq \alpha_3 \parallel x_t \parallel^2,\ t \geq 0. \tag{27}$$

Similar to (11), we have

$$V_i(x_t) \leq \psi V_j(x_t),\ \forall i, j \in S. \tag{28}$$

Taking derivative of $V_{1,i}(x(t))$ along trajectories of any subsystem ith, we have

$$\begin{aligned}
\dot{V}_{1,i}(x(t)) &= \dot{x}^T(t)P_i x(t) + x^T(t)P_i\dot{x}(t) \\
&= \sum_{i=1}^{N} \lambda_i(t)[x^T(t)A_i^T P_i x(t) + x^T(t)\Delta A_i^T(t) P_i x(t) + x^T(t-h(t))B_i^T P_i x(t) \\
&\quad + x^T(t-h(t))\Delta B_i^T(t) P_i x(t) + x^T(t) P_i A_i x(t) + x^T(t) P_i \Delta A_i(t) x(t) \\
&\quad + x^T(t) P_i B_i x(t-h(t)) + x^T(t) P_i \Delta B_i(t) x(t-h(t))].
\end{aligned}$$

Applying Lemma 2.1 and from (2) and (3), we get

$$\begin{aligned}
2x^T(t)\Delta A_i^T(t) P_i x(t) &\leq \varepsilon_{1i}^{-1} x^T(t) H_{1i}^T H_{1i} x(t) + \varepsilon_{1i} x^T(t) P_i E_{1i}^T E_{1i} P_i x(t), \\
2x^T(t-h(t))\Delta B_i^T(t) P_i x(t) &\leq \varepsilon_{2i}^{-1} x^T(t-h(t)) H_{2i}^T H_{2i} x(t-h(t)) + \varepsilon_{2i} x^T(t) P_i E_{2i}^T E_{2i} P_i x(t).
\end{aligned}$$

Next, by taking derivative of $V_{2,i}(x_t), V_{3,i}(x_t)$ and $V_{4,i}(x_t)$, respectively, along the system trajectories yields

$$\dot{V}_{2,i}(x_t) \le x^T(t)Q_i x(t) - (1-\mu)e^{-2\beta h(t)}x^T(t-h(t))Q_i x(t-h(t)) - 2\beta V_{2,i}(x_t),$$

$$\dot{V}_{3,i}(x_t) \le h_M x^T(t)R_i x(t) - \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_i x(s)ds - 2\beta V_{3,i}(x_t),$$

$$\begin{aligned}\dot{V}_{4,i}(x_t) \le\ & h_M \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \\ & - \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds \\ & -2\beta V_{4,i}(x_t).\end{aligned}$$

Therefore, the estimation of derivative of $V_i(x_t)$ along any trajectory of solution of (1) is given by

$$\begin{aligned}\dot{V}_i(x_t) \le\ & \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Xi_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t) \\ & - \int_{t-h(t)}^{t} e^{2\beta(s-t)}x^T(s)R_i x(s)ds - 2\beta V_{3,i}(x_t) \\ & - \int_{t-h(t)}^{t} e^{2\beta(s-t)} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-h(s)) \end{bmatrix} ds \\ & -2\beta V_{4,i}(x_t). \end{aligned} \tag{29}$$

For $i \in S_u$, we have

$$\dot{V}_i(x_t) \le \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Xi_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}.$$

Similar to Theorem 3.1, from (22) and (29), we get

$$V_i(x_t) \le \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{\xi_i(t-t_0)},\ t \ge t_0, \tag{30}$$

where $\xi_i = \dfrac{2\max\limits_i\{\lambda_M(\Xi_i)\}}{\min\limits_i\{\lambda_m(P_i)\}}$.

For $i \in S_s$, from (13), (14) and (29), we have

$$\begin{aligned}\dot{V}_i(x_t) \le\ & \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Xi_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t) \\ & -(2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t)) \end{aligned} \tag{31}$$

Similar to Theorem 3.1, from (23) and (31), we get

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{-\zeta_i(t-t_0)},\ t \geq t_0. \tag{32}$$

where $\zeta_i = \min\{\frac{\min_i\{\lambda_m(-\Xi_i)\}}{\max_i\{\lambda_M(P_i)\}}, 2\beta\}$.

In general, from (28), (30) and (32), with the same argument as in the proof of Theorem 3.1, we get

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi e^{\lambda^+(t_m - t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} e^{-\lambda^-(t_n - t_{n-1})} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^-(t - t_{N(t)-1})},$$

$t \geq t_0$. Using (24), we have

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^*(t-t_0)},\ t \geq t_0.$$

By (25) and (26), we get

$$V_i(x_t) \leq \parallel V_{i_0}(x_{t_0}) \parallel e^{-(\lambda^* - \nu)(t-t_0)},\ t \geq t_0.$$

Thus, by (27), we have

$$\parallel x(t) \parallel \leq \sqrt{\frac{\alpha_3}{\alpha_1}} \parallel x_{t_0} \parallel e^{-\frac{1}{2}(\lambda^* - \nu)(t-t_0)},\ t \geq t_0,$$

which concludes the proof of the Theorem 3.2. □

3.3 Robust exponential stability of switched system with time-varying delay and nonlinear perturbation

In this section, we deal with the problem for robust exponential stability of the zero solution of system (1).

Theorem 3.3 *The zero solution of system (1) is robust exponentially stable if there exist positive real numbers* $\varepsilon_{3i}, \varepsilon_{4i}, \varepsilon_{5i}$, *positive definite matrices* P_i, Q_i, R_i *and* $\begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix}$ *such that the following conditions hold:*

*A*1. (i) *For* $i \in S_u$,

$$\Theta_i > 0. \tag{33}$$

(ii) *For* $i \in S_s$,

$$\Theta_i < 0. \tag{34}$$

*A*2. *Assume that, for any* t_0 *the switching law guarantees that*

$$\inf_{t \geq t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \geq \frac{\lambda^+ + \lambda^*}{\lambda^- - \lambda^*} \tag{35}$$

where $\lambda^* \in (0, \lambda^-)$. *Furthermore, there exists* $0 < \nu < \lambda^*$ *such that*
(i) *If the subsystem* $i \in S_u$ *is activated in time intervals* $[t_{i_k-1}, t_{i_k}), k = 1, 2, ...,$ *then*

$$ln\ \psi - \nu(t_{i_k} - t_{i_k-1}) \leq 0,\ k = 1, 2, ..., l(t). \tag{36}$$

(ii) *If the subsystem* $j \in S_s$ *is activated in time intervals* $[t_{j_k-1}, t_{j_k}), k = 1, 2, ...,$ *then*

$$ln\ \psi + \zeta_j h_M - \nu(t_{j_k} - t_{j_k-1}) \leq 0,\ k = 1, 2, ..., N(t) - 1. \tag{37}$$

Proof. Consider the following Lyapunov functional:

$$V_i(x_t) = V_{1,i}(x(t)) + V_{2,i}(x_t) + V_{3,i}(x_t) + V_{4,i}(x_t)$$

where $x_t \in C([-h_M, 0], \mathbb{R}^n), x_t(s) = x(t+s), s \in [-h_M, 0]$ and
$V_{1,i}(x(t)) = x^T(t)P_i x(t)$,
$V_{2,i}(x_t) = \int_{t-h(t)}^{t} e^{2\beta(s-t)} x^T(s) Q_i x(s) ds$,
$V_{3,i}(x_t) = \int_{-h(t)}^{0} \int_{t+s}^{t} e^{2\beta(\xi-t)} x^T(\xi) R_i x(\xi) d\xi ds$,
$V_{4,i}(x_t) = \int_{-h(t)}^{0} \int_{t+s}^{t} e^{2\beta(\xi-t)} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix}^T \begin{bmatrix} S_{11,i} & S_{12,i} \\ S_{12,i}^T & S_{22,i} \end{bmatrix} \begin{bmatrix} x(\xi) \\ x(\xi - h(\xi)) \end{bmatrix} d\xi ds.$
It is easy to verify that

$$\alpha_1 \parallel x(t) \parallel^2 \leq V_i(x_t) \leq \alpha_3 \parallel x_t \parallel^2,\ t \geq 0. \tag{38}$$

Similar to (11), we have

$$V_i(x_t) \leq \psi V_j(x_t),\ \forall i, j \in S. \tag{39}$$

Taking derivative of $V_{1,i}(x(t))$ along trajectories of any subsystem ith we have

$$\begin{aligned}
\dot{V}_{1,i}(x(t)) &= \dot{x}^T(t)P_i x(t) + x^T(t)P_i \dot{x}(t) \\
&= \sum_{i=1}^{N} \lambda_i(t)[x^T(t)A_i^T P_i x(t) + x^T(t)\Delta A_i^T(t)P_i x(t) + x^T(t-h(t))B_i^T P_i x(t) \\
&\quad + x^T(t-h(t))\Delta B_i^T(t)P_i x(t) + f_i^T(t, x(t), x(t-h(t)))P_i x(t) + x^T(t)P_i A_i x(t) \\
&\quad + x^T(t)P_i \Delta A_i(t)x(t) + x^T(t)P_i B_i x(t-h(t)) + x^T(t)P_i \Delta B_i(t)x(t-h(t)) \\
&\quad + x^T(t)P_i f_i(t, x(t), x(t-h(t)))].
\end{aligned}$$

From lemma 2.1, we have

$$\begin{aligned}
2f_i^T(t, x(t), x(t-h(t)))P_i x(t) &\leq f_i^T(t, x(t), x(t-h(t)))W_i^{-1} f_i(t, x(t), x(t-h(t))) \\
&\quad + x^T(t)P_i W_i P_i x(t).
\end{aligned}$$

By choosing $W_i = \varepsilon_{3i} I_i$ and from (4), we have

$$\begin{aligned}
2f_i^T(t, x(t), x(t-h(t)))P_i x(t) &\leq \varepsilon_{3i}^{-1} f_i^T(t, x(t), x(t-h(t))) f_i(t, x(t), x(t-h(t))) \\
&\quad + \varepsilon_{3i} x^T(t)P_i P_i x(t) \\
&\leq \varepsilon_{3i}^{-1}[\gamma_i x^T(t)x(t) + \delta_i x^T(t-h(t))x(t-h(t))] \\
&\quad + \varepsilon_{3i} x^T(t)P_i P_i x(t).
\end{aligned}$$

Applying Lemma 2.1 and from (2) and (3), we get

$$2x^T(t)\Delta A_i^T(t)P_ix(t) \leq \varepsilon_{4i}^{-1}x^T(t)H_{4i}^TH_{4i}x(t) + \varepsilon_{4i}x^T(t)P_iE_{4i}^TE_{4i}P_ix(t),$$
$$2x^T(t-h(t))\Delta B_i^T(t)P_ix(t) \leq \varepsilon_{5i}^{-1}x^T(t-h(t))H_{5i}^TH_{5i}x(t-h(t)) + \varepsilon_{5i}x^T(t)P_iE_{5i}^TE_{5i}P_ix(t).$$

Next, by taking derivative of $V_{2,i}(x_t), V_{3,i}(x_t)$ and $V_{4,i}(x_t)$, respectively, along the system trajectories yields

$$\begin{aligned}\dot{V}_{2,i}(x_t) &\leq x^T(t)Q_ix(t) - (1-\mu)e^{-2\beta h(t)}x^T(t-h(t))Q_ix(t-h(t)) - 2\beta V_{2,i}(x_t),\\ \dot{V}_{3,i}(x_t) &\leq h_Mx^T(t)R_ix(t) - \int_{t-h(t)}^t e^{2\beta(s-t)}x^T(s)R_ix(s)ds - 2\beta V_{3,i}(x_t),\\ \dot{V}_{4,i}(x_t) &\leq h_M\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}^T\begin{bmatrix}S_{11,i} & S_{12,i}\\S_{12,i}^T & S_{22,i}\end{bmatrix}\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}\\ &\quad - \int_{t-h(t)}^t e^{2\beta(s-t)}\begin{bmatrix}x(s)\\x(s-h(s))\end{bmatrix}^T\begin{bmatrix}S_{11,i} & S_{12,i}\\S_{12,i}^T & S_{22,i}\end{bmatrix}\begin{bmatrix}x(s)\\x(s-h(s))\end{bmatrix}ds\\ &\quad -2\beta V_{4,i}(x_t).\end{aligned}$$

Then, the derivative of $V_i(x_t)$ along any trajectory of solution of (1) is estimated by

$$\begin{aligned}\dot{V}_i(x_t) \leq \sum_{i=1}^N \lambda_i(t)&\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}^T\Theta_i\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix} - 2\beta V_{2,i}(x_t)\\ &- \int_{t-h(t)}^t e^{2\beta(s-t)}x^T(s)R_ix(s)ds - 2\beta V_{3,i}(x_t)\\ &- \int_{t-h(t)}^t e^{2\beta(s-t)}\begin{bmatrix}x(s)\\x(s-h(s))\end{bmatrix}^T\begin{bmatrix}S_{11,i} & S_{12,i}\\S_{12,i}^T & S_{22,i}\end{bmatrix}\begin{bmatrix}x(s)\\x(s-h(s))\end{bmatrix}ds\\ &-2\beta V_{4,i}(x_t).\end{aligned} \tag{40}$$

For $i \in S_u$, it follows from (40) that

$$\dot{V}_i(x_t) \leq \sum_{i=1}^N \lambda_i(t)\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}^T\Theta_i\begin{bmatrix}x(t)\\x(t-h(t))\end{bmatrix}. \tag{41}$$

Similar to Theorem 3.1, from (33) and (41), we get

$$V_i(x_t) \leq \sum_{i=1}^N \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{\xi_i(t-t_0)},\ t \geq t_0. \tag{42}$$

where $\xi_i = \dfrac{2\max\limits_i\{\lambda_M(\Theta_i)\}}{\min\limits_i\{\lambda_m(P_i)\}}$.

For $i \in S_s$, from (13), (14) and (40), we have

$$\dot{V}_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix}^T \Theta_i \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} - 2\beta V_{2,i}(x_t)$$
$$-(2\beta + \frac{1}{h_M})(V_{3,i}(x_t) + V_{4,i}(x_t)). \quad (43)$$

Similar to Theorem 3.1, from (34) and (43), we get

$$V_i(x_t) \leq \sum_{i=1}^{N} \lambda_i(t) \parallel V_i(x_{t_0}) \parallel e^{-\zeta_i(t-t_0)},\ t \geq t_0. \quad (44)$$

where $\zeta_i = \min\{\frac{\min\limits_i\{\lambda_m(-\Theta_i)\}}{\max\limits_i\{\lambda_M(P_i)\}}, 2\beta\}$.

In general, from (39), (42) and (44), with the same argument as in the proof of Theorem 3.1, we get

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi e^{\lambda^+(t_m - t_{m-1})} \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} e^{-\lambda^-(t_n - t_{n-1})} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^-(t - t_{N(t)-1})},$$

$t \geq t_0$. Using (35), we have

$$V_i(x_t) \leq \prod_{m=1}^{l(t)} \psi \times \prod_{n=l(t)+1}^{N(t)-1} \psi e^{\zeta_{i_n} h_M} \times \parallel V_{i_0}(x_{t_0}) \parallel e^{-\lambda^*(t-t_0)},\ t \geq t_0.$$

By (36) and (37), we get

$$V_i(x_t) \leq \parallel V_{i_0}(x_{t_0}) \parallel e^{-(\lambda^* - \nu)(t-t_0)},\ t \geq t_0.$$

Thus, by (38), we have

$$\parallel x(t) \parallel \leq \sqrt{\frac{\alpha_3}{\alpha_1}} \parallel x_{t_0} \parallel e^{-\frac{1}{2}(\lambda^* - \nu)(t-t_0)},\ t \geq t_0,$$

which concludes the proof of the Theorem 3.3. □

4. Numerical examples

Example 4.1 Consider linear switched system (1) with time-varying delay but without matrix uncertainties and without nonlinear perturbations. Let $N = 2$, $S_u = \{1\}, S_s = \{2\}$. Let the delay function be $h(t) = 0.51 \sin^2 t$. We have $h_M = 0.51$, $\mu = 1.02, \lambda(A_1 + B_1) = 0.0046, -0.0399, \lambda(A_2) = -0.2156, 0.0007$. Let $\beta = 0.5$.

Since one of the eigenvalues of $A_1 + B_1$ is negative and one of eigenvalues of A_2 is positive, we can't use results in (Alan & Lib, 2008) to consider stability of switched system (1). By using the LMI toolbox in Matlab, we have matrix solutions of (5) for unstable subsystems and (6) for stable subsystems as the following:

For unstable subsystems, we get

$P_1 = \begin{bmatrix} 41.6819 & 0.0001 \\ 0.0001 & 41.5691 \end{bmatrix}, Q_1 = \begin{bmatrix} 24.7813 & -0.0002 \\ -0.0002 & 24.7848 \end{bmatrix}, R_1 = \begin{bmatrix} 33.1027 & -0.0001 \\ -0.0001 & 33.1044 \end{bmatrix},$
$S_{11,1} = \begin{bmatrix} 33.1027 & -0.0001 \\ -0.0001 & 33.1044 \end{bmatrix}, S_{12,1} = \begin{bmatrix} -0.0372 & -0.0023 \\ -0.0023 & 0.7075 \end{bmatrix}, S_{22,1} = \begin{bmatrix} 50.0412 & 0.0001 \\ 0.0001 & 50.0115 \end{bmatrix},$
$T_1 = \begin{bmatrix} 41.7637 & -0.0001 \\ -0.0001 & 41.7920 \end{bmatrix}.$
For stable subsystems, we get
$P_2 = \begin{bmatrix} 71.8776 & 2.3932 \\ 2.3932 & 110.8889 \end{bmatrix}, Q_2 = \begin{bmatrix} 7.2590 & -0.3265 \\ -0.3265 & 0.8745 \end{bmatrix}, R_2 = \begin{bmatrix} 10.4001 & -0.4667 \\ -0.4667 & 1.2806 \end{bmatrix},$
$S_{11,2} = \begin{bmatrix} 12.7990 & -0.4854 \\ -0.4854 & 3.5031 \end{bmatrix}, S_{12,2} = \begin{bmatrix} -3.1787 & 0.0240 \\ 0.0240 & -2.8307 \end{bmatrix}, S_{22,2} = \begin{bmatrix} 4.6346 & -0.0289 \\ -0.0289 & 4.0835 \end{bmatrix},$
$T_2 = \begin{bmatrix} 16.9964 & 0.0394 \\ 0.0394 & 17.7152 \end{bmatrix}, X_{11,2} = \begin{bmatrix} 17.2639 & -0.1536 \\ -0.1536 & 14.2310 \end{bmatrix}, X_{12,2} = \begin{bmatrix} -9.6485 & -0.1466 \\ -0.1466 & -12.5573 \end{bmatrix},$
$X_{22,2} = \begin{bmatrix} 16.9716 & -0.1635 \\ -0.1635 & 13.8095 \end{bmatrix}, Y_2 = \begin{bmatrix} -3.4666 & -0.1525 \\ -0.1525 & -6.3485 \end{bmatrix}, Z_2 = \begin{bmatrix} 6.8776 & -0.0574 \\ -0.0574 & 5.7924 \end{bmatrix}.$
By straight forward calculation, the growth rate is $\lambda^+ = \xi = 2.8291$, the decay rate is $\lambda^- = \zeta = 0.0063$, $\lambda(\Omega_{1,1}) = 25.8187, 25.8188, 58.7463, 58.8011$, $\lambda(\Omega_{2,2}) = -10.1108, -3.7678, -2.0403, -0.7032$ and $\lambda(\Omega_{3,2}) = 1.4217, 4.2448, 5.4006, 9.1514, 29.3526, 30.0607$. Thus, we may take $\lambda^* = 0.0001$ and $\nu = 0.00001$. Thus, from inequality (7), we have $T^- \geq 456.3226\ T^+$. By choosing $T^+ = 0.1$, we get $T^- \geq 45.63226$. We choose the following switching rules:
(i) for $t \in [0, 0.1) \cup [50, 50.1) \cup [100, 100.1) \cup [150, 150.1) \cup ...$, subsystem $i = 1$ is activated.
(ii) for $t \in [0.1, 50) \cup [50.1, 100) \cup [100.1, 150) \cup [150.1, 200) \cup ...$, subsystem $i = 2$ is activated.
Then, by Theorem 3.1, the switching system (1) is exponentially stable. Moreover, the solution $x(t)$ of the system satisfies

$$\| x(t) \| \leq 11.8915 e^{-0.000045t},\ t \in [0, \infty).$$

The trajectories of solution of switched system switching between the subsystems $i = 1$ and $i = 2$ are shown in Figure 1, Figure 2 and Figure 3, respectively.

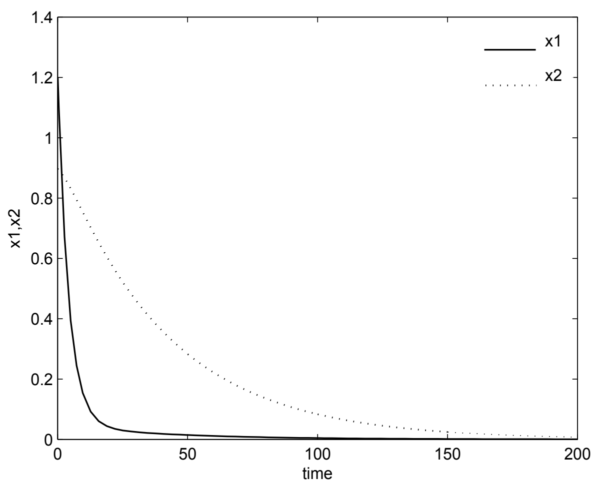

Fig. 1. The trajectories of solution of linear switched system.

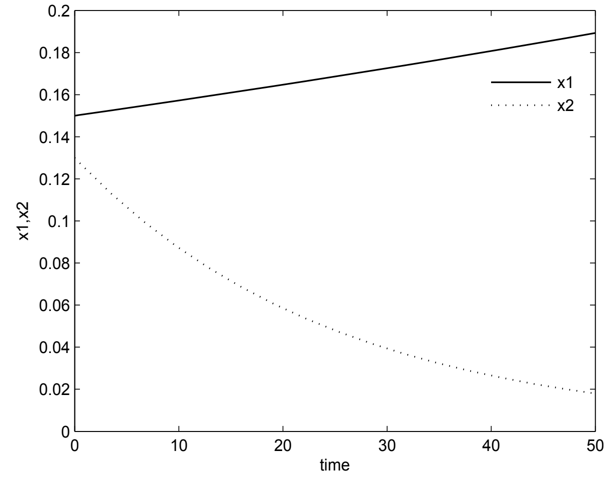

Fig. 2. The trajectories of solution of subsystem $i = 1$.

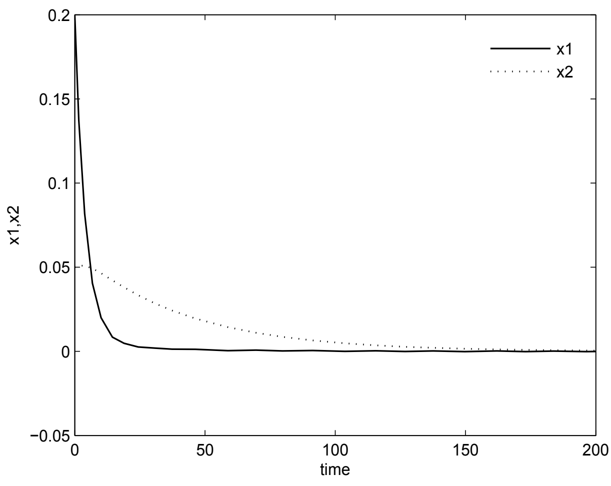

Fig. 3. The trajectories of solution of subsystem $i = 2$.

Example 4.2 Consider uncertain switched system (1) with time-varying delay and nonlinear perturbation. Let $N = 2$, $S_u = \{1\}, S_s = \{2\}$ where

$$A_1 = \begin{bmatrix} 0.1130 & 0.00013 \\ 0.00015 & -0.0033 \end{bmatrix}, B_1 = \begin{bmatrix} 0.0002 & 0.0012 \\ 0.0014 & -0.5002 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -5.5200 & 1.0002 \\ 1.0003 & -6.5500 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0245 & 0.0001 \\ 0.0001 & 0.0237 \end{bmatrix},$$

$$E_{1i} = E_{2i} = \begin{bmatrix} 0.2000 & 0.0000 \\ 0.0000 & 0.2000 \end{bmatrix}, H_{1i} = H_{2i} = \begin{bmatrix} 0.1000 & 0.0000 \\ 0.0000 & 0.1000 \end{bmatrix}, i = 1, 2,$$

$$F_{1i} = F_{2i} = \begin{bmatrix} \sin t & 0 \\ 0 & \sin t \end{bmatrix}, i = 1, 2,$$

$$f_1(t,x(t),x(t-h(t))) = \begin{bmatrix} 0.1x_1(t)\sin(x_1(t)) \\ 0.1x_2(t-h(t))\cos(x_2(t)) \end{bmatrix},$$
$$f_2(t,x(t),x(t-h(t))) = \begin{bmatrix} 0.5x_1(t)\sin(x_1(t)) \\ 0.5x_2(t-h(t))\cos(x_2(t)) \end{bmatrix}.$$

From

$$\begin{aligned} \| f_1(t,x(t),x(t-h(t))) \|^2 &= [0.1x_1(t)\sin(x_1(t))]^2 + [0.1x_2(t-h(t))\cos(x_2(t))]^2 \\ &\le 0.01x_1^2(t) + 0.01x_2^2(t-h(t)) \\ &\le 0.01 \| x(t) \|^2 + 0.01 \| x(t-h(t)) \|^2 \\ &\le 0.01[\| x(t) \| + \| x(t-h(t)) \|]^2, \end{aligned}$$

we obtain

$$\| f_1(t,x(t),x(t-h(t))) \| \le 0.1 \| x(t) \| + 0.1 \| x(t-h(t)) \| .$$

The delay function is chosen as $h(t) = 0.25\sin^2 t$. From

$$\begin{aligned} \| f_2(t,x(t),x(t-h(t))) \|^2 &= [0.5x_1(t)\sin(x_1(t))]^2 + [0.5x_2(t-h(t))\cos(x_2(t))]^2 \\ &\le 0.25x_1^2(t) + 0.25x_2^2(t-h(t)) \\ &\le 0.25 \| x(t) \|^2 + 0.25 \| x(t-h(t)) \|^2 \\ &\le 0.25[\| x(t) \| + \| x(t-h(t)) \|]^2, \end{aligned}$$

we obtain

$$\| f_2(t,x(t),x(t-h(t))) \| \le 0.5 \| x(t) \| + 0.5 \| x(t-h(t)) \| .$$

We may take $h_M = 0.25$, and from (4), we take $\gamma_1 = 0.1, \delta_1 = 0.1, \gamma_2 = 0.5, \delta_2 = 0.5$. Note that $\lambda(A_1) = 0.11300016, -0.00330016$. Let $\beta = 0.5, \mu = 0.5$. Since one of the eigenvalues of A_1 is negative, we can't use results in (Alan & Lib, 2008) to consider stability of switched system (1). From Lemma 2.4 , we have the matrix solutions of (33) for unstable subsystems and of (34) for stable subsystems by using the LMI toolbox in Matlab as the following:

For unstable subsystems, we get

$\varepsilon_{31} = 0.8901,\ \varepsilon_{41} = 0.8901,\ \varepsilon_{51} = 0.8901,$

$$P_1 = \begin{bmatrix} 0.2745 & -0.0000 \\ -0.0000 & 0.2818 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.4818 & -0.0000 \\ -0.0000 & 0.5097 \end{bmatrix}, R_1 = \begin{bmatrix} 0.8649 & -0.0000 \\ -0.0000 & 0.8729 \end{bmatrix},$$
$$S_{11,1} = \begin{bmatrix} 0.8649 & -0.0000 \\ -0.0000 & 0.8729 \end{bmatrix}, S_{12,1} = 10^{-4} \times \begin{bmatrix} -0.1291 & -0.8517 \\ -0.8517 & 0.1326 \end{bmatrix},$$
$$S_{22,1} = \begin{bmatrix} 1.0877 & -0.0000 \\ -0.0000 & 1.0902 \end{bmatrix}.$$

For stable subsystems, we get

$\varepsilon_{32} = 2.0180,\ \varepsilon_{42} = 2.0180,\ \varepsilon_{52} = 2.0180,$

$$P_2 = \begin{bmatrix} 0.2741 & 0.0407 \\ 0.0407 & 0.2323 \end{bmatrix}, Q_2 = \begin{bmatrix} 1.3330 & -0.0069 \\ -0.0069 & 1.3330 \end{bmatrix}, R_2 = \begin{bmatrix} 1.0210 & -0.0002 \\ -0.0002 & 1.0210 \end{bmatrix},$$
$$S_{11,2} = \begin{bmatrix} 1.0210 & -0.0002 \\ -0.0002 & 1.0210 \end{bmatrix}, S_{12,2} = \begin{bmatrix} -0.0016 & -0.0002 \\ -0.0002 & -0.0016 \end{bmatrix},$$
$$S_{22,2} = \begin{bmatrix} 0.8236 & -0.0006 \\ -0.0006 & 0.8236 \end{bmatrix}.$$

By straight forward calculation, the growth rate is $\lambda^+ = \xi = 8.5413$, the decay

rate is $\lambda^- = \zeta = 0.1967$, $\lambda(\Theta_1) = 0.1976, 0.2079, 1.1443, 1.1723$ and $\lambda(\Theta_2) = -0.7682, -0.6494, -0.0646, -0.0588$. Thus, we may take $\lambda^* = 0.0001$ and $\nu = 0.00001$.
Thus, from inequality (35), we have $T^- \geq 43.4456\ T^+$. By choosing $T^+ = 0.1$, we get $T^- \geq 4.34456$. We choose the following switching rules:
(i) for $t \in [0, 0.1) \cup [5.0, 5.1) \cup [10.0, 10.1) \cup [15.0, 15.1) \cup ...$, system $i = 1$ is activated.
(ii) for $t \in [0.1, 5.0) \cup [5.1, 10.0) \cup [10.1, 15.0) \cup [15.1, 20.0) \cup ...$, system $i = 2$ is activated.
Then, by theorem 3.3.1, the switched system (1) is exponentially stable. Moreover, the solution $x(t)$ of the system satisfies

$$\| x(t) \| \leq 1.8770 e^{-0.000045t},\ t \in [0, \infty).$$

The trajectories of solution of switched system switching between the subsystems $i = 1$ and $i = 2$ are shown in Figure 4, Figure 5 and Figure 6, respectively.

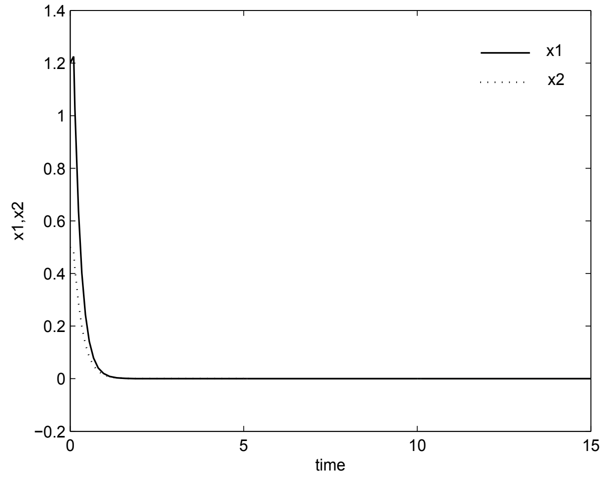

Fig. 4. The trajectories of solution of switched system with nonlinear perturbations

5. Conclusion

In this paper, we have studied the exponential stability of uncertain switched system with time varying delay and nonlinear perturbations. We allow switched system to contain stable and unstable subsystems. By using a new Lyapunov functional, we obtain the conditions for robust exponential stability for switched system in terms of linear matrix inequalities (LMIs) which may be solved by various algorithms. Numerical examples are given to illustrate the effectiveness of our theoretical results.

6. Acknowledgments

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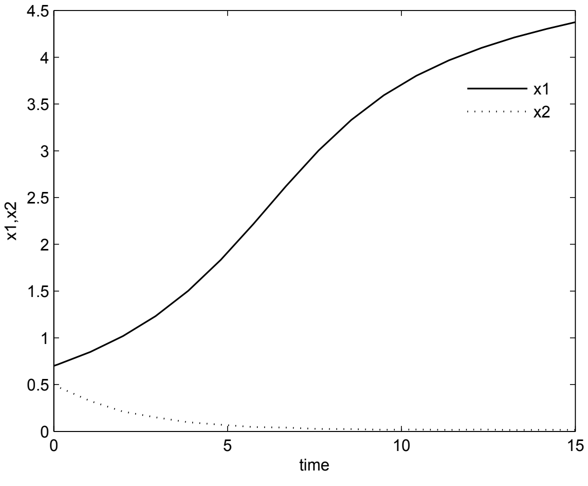

Fig. 5. The trajectories of solution of system $i = 1$

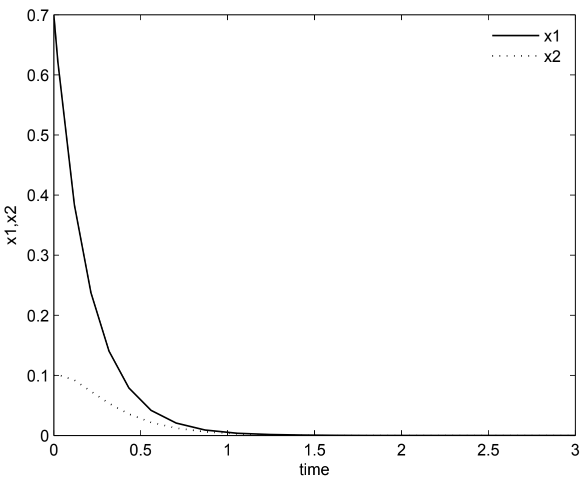

Fig. 6. The trajectories of solution of system $i = 2$

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5

On Stable Periodic Solutions of One Time Delay System Containing Some Nonideal Relay Nonlinearities

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1. Introduction

Problems of stabilization and determining of stablility characteristics of steady-state regimes are among the central in a control theory. Especial difficulties can be met when dealing with the systems containing nonlinearities which are nonanalytic function of phase. Different models describing nonlinear effects in real control systems (e.g. servomechanisms, such as servo drives, autopilots, stabilizers etc.) are just concern this type, numerous works are devoted to the analysis of problem of stable oscillations presence in such systems.
Time delays appear in control systems frequently and are important due to significant impact on them. They affect substantially on stability properties and configuration of steady state solutions. An accurate simultaneous account of nonlinear effects and time delays allows to receive adequate models of real control systems.
This work contains some results concerning to a stability problem for periodic solutions of nonlinear controlled system containing time delay. It corresponds further development of an article: Kamachkin & Stepanov (2009). Main results obtained below might generally be put in connection with classical results of V.I. Zubov's control theory school (see Zubov (1999), Zubov & Zubov (1996)) and based generally on work Zubov & Zubov (1996).
Note that all examples presented here are purely illustrative; some examples concerning to similar systems can be found in Petrov & Gordeev (1979), Varigonda & Georgiou (2001).

2. Models under consideration

Consider a system

$$\dot{x} = Ax + cu(t-\tau), \tag{1}$$

here $x = x(t) \in \mathbb{E}^n$, $t \geq t_0 \geq \tau$, A is real $n \times n$ matrix, $c \in \mathbb{E}^n$, vector $x(t)$, $t \in [t_0 - \tau,\, t_0]$, is considered to be known. Quantity $\tau > 0$ describes time delay of actuator or observer. Control statement u is defined in the following way:

$$u(t-\tau) = f\left(\sigma(t-\tau)\right), \quad \sigma(t-\tau) = \gamma' x(t-\tau), \quad \gamma \in \mathbb{E}^n, \quad \|\gamma\| \neq 0;$$

nonlinearity f can, for example, describe a nonideal two-position relay with hysteresis:

$$f(\sigma) = \begin{cases} m_1, & \sigma < l_2, \\ m_2, & \sigma > l_1, \end{cases} \tag{2}$$

here $l_1 < l_2$, $m_1 < m_2$; and $f(\sigma(t)) = f_- = f(\sigma(t-0))$ if $\sigma \in [l_1; l_2]$.
In addition to the nonlinearity (2) a three-position relay with hysteresis will be considered:

$$f(\sigma) = \begin{cases} 0, & \begin{cases} |\sigma| \le l_0, \\ |\sigma| \in (l_0;\ l], & f_- = 0; \end{cases} \\ m_1, & \begin{cases} \sigma \in [-l;\ -l_0), & f_- = m_1, \\ \sigma < -l; \end{cases} \\ m_2, & \begin{cases} \sigma \in (l_0;\ l], & f_- = m_2, \\ \sigma > l; \end{cases} \end{cases} \tag{3}$$

(here $m_1 < m < m_2$, $0 < l_0 < l$);
Suppose that hysteresis loops for the nonlinearities are walked around in counterclockwise direction.

3. Stability of periodic solutions

Denote $x(t - t_0, x_0, u)$ solution of the system (1) for unchanging control law u and initial conditions (t_0, x_0).
Let the system (1), (3) has a periodic solution with four switching points $\hat{s}_i$ such as

$$\hat{s}_1 = x\left(T_4, \hat{s}_4, m_2\right), \quad \hat{s}_2 = x\left(T_1, \hat{s}_1, 0\right), \quad \hat{s}_3 = x\left(T_2, \hat{s}_2, m_1\right), \quad \hat{s}_4 = x\left(T_3, \hat{s}_3, 0\right).$$

Let s_i, $i = \overline{1,4}$ are points of this solution (preceding to the corresponding $\hat{s}_i$) such as

$$\gamma' s_1 = l_0, \quad \gamma' s_2 = -l, \quad \gamma' s_3 = -l_0, \quad \gamma' s_4 = l,$$

(let us name them Ťpre-switching pointsŤ, for example), and

$$\hat{s}_1 = x\left(\tau, s_1, m_2\right), \quad \hat{s}_2 = x\left(\tau, s_2, 0\right), \quad \hat{s}_3 = x\left(\tau, s_3, m_1\right), \quad \hat{s}_4 = x\left(\tau, s_4, 0\right),$$

or

$$\hat{s}_{i+1} = x\left(T_i, \hat{s}_i, u_i\right), \quad \hat{s}_i = x\left(\tau, s_i, u_{i-1}\right),$$

where

$$u_1 = 0, \quad u_2 = m_1, \quad u_3 = 0, \quad u_4 = m_2$$

(hereafter suppose that indices are cyclic, i.e. for $i = \overline{1,m}$ one have $i + 1 = 1$ if $i = m$ and $i - 1 = m$ if $i = 1$).
Denote

$$v_i = A s_{i+1} + c u_i, \quad k_i = \gamma' v_i.$$

Theorem 1. *Let $k_i \neq 0$ and $\|M\| < 1$, where*

$$M = \prod_{i=4}^{1} M_i, \quad M_i = \left(I - k_i^{-1} v_i \gamma'\right) e^{AT_i},$$

then the periodic solution under consideration is orbitally asymptotically stable.

Proof As

$$s_{i+1} = e^{A(T_i-\tau)}\hat{s}_i + \int_0^{T_i-\tau} e^{A(T_i-\tau-t)}cu_i\,dt, \quad \hat{s}_i = e^{A\tau}s_i + \int_0^{\tau} e^{A(\tau-t)}cu_{i-1}\,dt,$$

then the expression for s_{i+1} can be written in a following form:

$$\begin{aligned} s_{i+1} &= e^{AT_i}s_i + e^{AT_i}\int_0^{\tau} e^{-At}cu_{i-1}dt + \int_0^{T_i-\tau} e^{A(T_i-\tau-t)}cu_i dt = \\ &= e^{AT_i}\left(s_i + \int_0^{\tau} e^{-At}cu_{i-1}\,dt + \int_{\tau}^{T_i} e^{-At}cu_i\,dt\right). \end{aligned}$$

So,

$$(s_{i+1})'_{s_i} = e^{AT_i}, \quad (s_{i+1})'_{T_i} = As_{i+1} + cu_i = v_i,$$

and

$$\begin{aligned} d\left(\gamma' s_{i+1}\right) = 0 = \gamma' e^{AT_i}ds_i + \gamma' v_i dT_i, \quad dT_i = -k_i^{-1}\gamma' e^{AT_i}ds_i, \\ ds_{i+1} = e^{AT_i}ds_i - v_i k_i^{-1}\gamma' e^{AT_i}ds_i = \left(I - k_i^{-1}v_i\gamma'\right)e^{AT_i}\,ds_i = M_i ds_i. \end{aligned}$$

Denote ds_1^k the successive deviations of pre-switching points of some diverged solution from s_1. In such a case

$$ds_1^{k+1} = \prod_{i=4}^{1} M_i ds_1^k.$$

The system under consideration causes continuous contracting mapping of some neighbourhood of the point s_1 lying on hyperplane $s = l_0$, to itself. Use of fixed point principle (Nelepin (2002)) completes the proof. ∎

Example 1. *Let* $\tau = 0.3$,

$$A = \begin{pmatrix} -0.1 & -0.1 & 0 \\ 0.1 & -0.1 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0.2 \\ 0 \\ -1 \end{pmatrix},$$

$$m_{1,2} = \mp 1, \quad l_0 = 0.1, \quad l = 0.5.$$

System (1), (3) has periodic solution with four switching points; the pre-switching points are:

$$s_1 \approx \begin{pmatrix} 0.468349 \\ 0.497302 \\ -0.006307 \end{pmatrix}, \quad s_2 \approx \begin{pmatrix} 0.005176 \\ -0.000633 \\ 0.501036 \end{pmatrix}, \quad s_3 = -s_1, \quad s_4 = -s_2;$$

and

$$T_1 \approx 53.6354, \quad T_2 \approx 0.7973, \quad T_3 = T_1, \quad T_4 = T_2.$$

As $\|M\| \approx 0.0078 < 1$, *then the periodic solution is orbitally asymptotically stable.*

Similarly, the system (1), (3) may have a periodic solution with a pair of switching points $\hat{s}_{1,2}$ and a pair of pre-switching points $s_{1,2}$ such as

$$\hat{s}_1 = x\left(T_2, \hat{s}_2, m_2\right), \quad \hat{s}_2 = x\left(T_1, \hat{s}_1, 0\right),$$
$$\hat{s}_1 = x\left(\tau, s_1, m_2\right), \quad \gamma' s_1 = l_0, \quad \hat{s}_2 = x\left(\tau, s_2, 0\right), \quad \gamma' s_1 = l.$$

for some positive constants $T_{1,2}$. This solution will be orbitally asymptotically stable if

$$k_1 = \gamma' v_{1,2} \neq 0, \quad \text{where} \quad v_i = As_j + cu_i, \quad i \neq j, \quad u_1 = 0, \quad u_2 = m_2,$$

and

$$\|M\| = \left\| \left(I - k_2^{-1} v_2 \gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \gamma'\right) e^{AT_1} \right\| < 1$$

(the proof is similar to the previous one).

Example 2. *Let* $\tau = 0.5$,

$$A = \begin{pmatrix} -0.1 & -0.2 & 0 \\ 0.2 & -0.1 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0.1 \\ 0 \\ -1 \end{pmatrix},$$
$$l_0 = 0.75, \quad l = 1, \quad m_{1,2} = \mp 1.$$

Then the system (1), (3) has a periodic solution with pre-switching points

$$s_1 = \begin{pmatrix} 0.2727 \\ 0.2886 \\ -0.7227 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad T_1 = 149.6021, \quad T_2 = 0.7847,$$
$$\|M\| \approx 0.9286 < 1,$$

and the solution is orbitally asymptotically stable.

4. Some extensions (bilinear system, multiple control etc.)

Consider a bilinear system

$$\dot{x} = Ax + (Cx + c)\, u(t - \tau), \tag{4}$$

In case of piecewise constant nonlinearity it is easy to obtain sufficient conditions for orbital asymptotical stability of periodic solutions of this system.
Denote $x_i(t - t_0, x_0)$, $i = \overline{1,4}$ solution of the system

$$\dot{x} = A_i x + c_i,$$

where (t_0, x_0) are initial conditions and

$$A_1 = A_3 = A, \quad A_2 = A + Cm_1, \quad A_4 = A + Cm_2, \quad c_1 = c_3 = \mathbf{0}, \quad c_2 = cm_1, \quad c_4 = cm_2.$$

Lef the control u is given by (3) and the system (4), (3) has a periodic solution with four control switching points (see the Theorem 1) $\hat{s}_i$ and "pre-switching" points s_i such as

$$\hat{s}_{i+1} = x_i\left(T_i, \hat{s}_i\right), \quad \gamma' s_1 = l_0, \quad \gamma' s_2 = -l, \quad \gamma' s_3 = -l_0, \quad \gamma' s_4 = l.$$

Denote

$$v_i = A_i s_{i+1} + c_i, \quad k_i = \gamma' v_i, \quad i = \overline{1,4}.$$

Theorem 2. *If $k_i \neq 0$ and*

$$\|M\| = \left\| \prod_{i=4}^{1} \left(I - k_i^{-1} v_i \gamma' \right) e^{A_i T_i + (A_{i-1} - A_i)\tau} \right\| < 1,$$

then the periodic solution under consideration is orbitally asymptotically stable.

Proof As

$$s_{i+1} = x_i\left(T_i - \tau, \hat{s}_i\right) = x_i\left(T_i - \tau, x_{i-1}\left(\tau, s_i\right)\right) =$$
$$= e^{A_i(T_i - \tau)} \left(e^{A_{i-1}\tau} s_i + \int_0^\tau e^{A_{i-1}(\tau - t)} c_{i-1} dt \right) + \int_0^{T_i - \tau} e^{A_i(T_i - \tau - t)} c_i dt =$$
$$= e^{A_i T_i + (A_{i-1} - A_i)\tau} s_i + e^{A_i(T_i - \tau)} \int_0^\tau e^{A_{i-1}(\tau - t)} c_{i-1} dt + + \int_0^{T_i - \tau} e^{A_i(T_i - \tau - t)} c_i dt,$$

then

$$\left(s_{i+1}\right)'_{s_i} = e^{A_i T_i + (A_{i-1} - A_i)\tau}, \quad \left(s_{i+1}\right)'_{T_i} = A_i s_{i+1} + c_i.$$

So, as $d\left(\gamma' s_{i+1}\right) = 0$,

$$\gamma' e^{A_i T_i + (A_{i-1} - A_i)\tau} ds_i = -k_i dT_i, \quad ds_{i+1} = \left(I - k_i^{-1} v_i \gamma' \right) e^{A_i T_i + (A_{i-1} - A_i)\tau} ds_i,$$

and $ds_1^{k+1} = M ds_1^k$. Use of fixed point principle completes the proof. ■

Example 3. *Let, for example, $\tau = 0.3$,*

$$A = \begin{pmatrix} -0.1 & -0.05 & 0 \\ 0.1 & -0.05 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0.05 & 0 \\ 0.05 & -0.1 & 0.05 \\ 0 & -0.05 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$
$$\gamma' = \begin{pmatrix} -0.2 & 0.5 & -1 \end{pmatrix}, \quad l_0 = 0.1, \quad l = 0.5, \quad m_{1,2} = \mp 1.$$

In such a case the system (4), (3) has periodic solution with pre-switching points

$$s_1 \approx \begin{pmatrix} 0.6819 \\ 0.5383 \\ 0.0328 \end{pmatrix}, \quad s_2 \approx \begin{pmatrix} -0.0534 \\ -0.0073 \\ 0.5070 \end{pmatrix}, \quad s_3 \approx \begin{pmatrix} -0.6096 \\ -0.6396 \\ -0.0979 \end{pmatrix}, \quad s_4 \approx \begin{pmatrix} 0.1127 \\ -0.0664 \\ -0.5557 \end{pmatrix},$$
$$T_1 \approx 42.2723, \quad T_2 \approx 0.8977, \quad T_3 \approx 33.5405, \quad T_4 \approx 0.8969.$$

One can verify that $k_i \neq 0$, and

$$\|M\| \approx 0.8223 < 1.$$

So, the solution under consideration is orbitally asymptotically stable.

Note that if matrices $A_{1,2} = A + Cm_{1,2}$ are Hurwitz, and

$$-\gamma' A_2^{-1} c m_2 < l_1, \quad -\gamma' A_1^{-1} c m_1 > l_2,$$

then the system (4), (2) has at least one periodic solution.
By the analogy with the system (1), a system with multiple controls can be observed:

$$\dot{x} = Ax + c_1 u_1\left(\sigma_1(t - \tau_1)\right) + c_2 u_2\left(\sigma_2(t - \tau_2)\right). \tag{5}$$

Suppose for simplicity that u_i are simple hysteresis nonlinearities given by (2):

$$u_i(\sigma) = u(\sigma) = \begin{cases} m_1, & \sigma_i < l_2, \\ m_2, & \sigma_i > l_1, \end{cases} \quad \sigma_i = \gamma_i' x, \quad i = 1,2.$$

Denote $x\left(t - t_0, x_0, u_1, u_2\right)$ solution of the system (5) for unchanging control laws $u_{1,2}$ and initial conditions (t_0, x_0). Let the system has periodic solution with four switching $(\hat{s}_i)$ and pre-switching (s_i) points such as

$$\hat{s}_1 = x(T_4, \hat{s}_4, m_2, m_2), \quad \hat{s}_2 = x(T_1, \hat{s}_1, m_1, m_2), \quad \hat{s}_3 = x(T_2, \hat{s}_2, m_1, m_1), \quad \hat{s}_4 = x(T_3, \hat{s}_3, m_2, m_1),$$
$$\hat{s}_1 = x(\tau, s_1, m_2, m_2), \quad \hat{s}_2 = x(\tau, s_2, m_1, m_2), \quad \hat{s}_3 = x(\tau, s_3, m_1, m_1), \quad \hat{s}_4 = x(\tau, s_4, m_2, m_1),$$
$$\gamma_1' s_1 = -l_1, \quad \gamma_2' s_2 = -l_2, \quad \gamma_1' s_3 = l_1, \quad \gamma_2' s_4 = l_2.$$

Denote

$$p_1 = c_1 m_1 + c_2 m_2, \quad p_2 = c_1 m_1 + c_2 m_1, \quad p_3 = c_1 m_2 + c_2 m_1, \quad p_4 = c_1 m_2 + c_2 m_2,$$
$$v_i = A s_{i+1} + p_i, \quad i = \overline{1,4}, \quad k_1 = \gamma_2' v_1, \quad k_2 = \gamma_1' v_2, \quad k_3 = \gamma_2' v_3, \quad k_4 = \gamma_1' v_4,$$
$$M_1 = \left(I - k_1^{-1} v_1 \gamma_2'\right) e^{AT_1}, \quad M_2 = \left(I - k_2^{-1} v_2 \gamma_1'\right) e^{AT_2},$$
$$M_3 = \left(I - k_3^{-1} v_3 \gamma_2'\right) e^{AT_3}, \quad M_4 = \left(I - k_4^{-1} v_4 \gamma_1'\right) e^{AT_4}.$$

It is easy to verify that the solution under consideration is orbitally asymptotically stable if $k_i \neq 0$ and

$$\left\| \prod_{i=4}^{1} M_i \right\| < 1.$$

Example 4. *Consider a trivial case:*

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix}.$$

So the system can be rewritten as a pair of independent equations

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + u\left(\alpha_1 x(t - \tau_1)\right), \\ \dot{x}_2 = \lambda_2 x_2 + u\left(\alpha_2 x(t - \tau_2)\right); \end{cases}$$

or

$$\begin{cases} \dot{\sigma}_1 = \lambda_1 \sigma_1 + \alpha_1 u\left(\sigma_1(t - \tau_1)\right), \\ \dot{\sigma}_2 = \lambda_2 \sigma_2 + \alpha_2 u\left(\sigma_2(t - \tau_2)\right). \end{cases}$$

Let, for example, $\lambda_1 > 0$, $\lambda_2 < 0$, $l_1 = -l_2 = -l$, $m_1 = -m_2 = -m$, $\tau_1 = \tau_2 = \tau$. *Denote*

$$\hat{l}_i = e^{\lambda_i \tau} l - \alpha_i \lambda_i^{-1}\left(e^{\lambda_i \tau} - 1\right) m, \quad i = 1,2.$$

Between switchings σ *looks as follows:*

$$\sigma_i(t) = e^{\lambda_i t} \sigma_i(0) + \alpha_i \lambda_i^{-1}\left(e^{\lambda_i t} - 1\right) u, \quad i = 1,2.$$

Suppose t_1 is a positive constant such as

$$\sigma_1(0) = -\hat{l}_1, \quad \sigma_1(0.5t_1) = \hat{l}_1, \quad u = -m;$$

i.e.

$$\frac{\alpha_1 m}{\lambda_1} - \hat{l}_1 = \left(\frac{\alpha_1 m}{\lambda_1} + \hat{l}_1\right) e^{0.5\lambda_1 t_1}, \quad t_1 = \frac{2}{\lambda_1} \ln \frac{\alpha_1 m - \lambda_1 \hat{l}_1}{\alpha_1 m + \lambda_1 \hat{l}_1}.$$

Similarly,

$$t_2 = \frac{2}{\lambda_2} \ln \frac{\alpha_2 m - \lambda_2 \hat{l}_2}{\alpha_2 m + \lambda_2 \hat{l}_2}.$$

If t_i are commensurable quantities (i.e. t_1/t_2 is rational number) then the system has a periodic solution with the period $T = LCM\,(t_1, t_2)$.
This example also demonstrates that there can exist an almost periodic solution of the system (5) (as a superposition of two periodic solutions with incommensurable periods) if $t_1/t_2 \in \mathcal{I}$.
Let, for example,

$$\tau = 0.1, \quad \lambda_1 = -\lambda_2 = \lambda = 0.1, \quad l = m = 1.$$

Let us choose parameters $\alpha_{1,2}$ in such a way that $t_1 = t_2$. It is easy to verify that the latest equality holds true if

$$\frac{\alpha_1 - \lambda \hat{l}_1}{\alpha_1 + \lambda \hat{l}_1} = \frac{\alpha_2 - \lambda \hat{l}_2}{\alpha_2 + \lambda \hat{l}_2}, \quad or \quad \frac{\alpha_1}{\alpha_2} = \frac{\hat{l}_1}{\hat{l}_2}$$

So,

$$\alpha_2 = \frac{\alpha_1 \lambda l}{(\lambda l - \alpha_1 m)\, e^{2\lambda\tau} + 2\alpha_1 m e^{\lambda\tau} - \alpha_1 m}.$$

Let $\alpha_1 = -1$, then

$$\alpha_2 \approx -0.979229,$$

then we can calculate $\hat{l}_{1,2}$:

$$\hat{l}_1 \approx 1.110552, \quad \hat{l}_2 \approx 1.087485.$$

And, finally,

$$t_1 = t_2 \approx 4.460606.$$

The system under consideration has a T-periodic solution, $T = t_i$. Let $s_1' = \begin{pmatrix}1 & 0\end{pmatrix}$, then

$$s_2' \approx \begin{pmatrix}0.19809 & 1.02122\end{pmatrix}, \quad s_3 = -s_1, \quad s_4 = -s_4,$$
$$T_1 = T_3 \approx 1.07715, \quad T_2 = T_4 \approx 1.15315;$$

and

$$ds_1^{k+1} = M ds_1^k, \quad M = \begin{pmatrix} 0 & 0 \\ 1.1362... & 1 \end{pmatrix}.$$

So, as $s_{1,1} = 1$, then $ds_{1,1} = 0$,

$$ds_{1,2}^{k+1} = ds_{1,2}^k,$$

and the periodic solution under consideration cannot be asymptotically stable (of course this fact can be established from other general considerations).
It is obvious that the system under consideration may have periodic solutions with greater amount of switching points (depending of $LCM\,(t_1, t_2)$ value).

Similar computations can be observed in case of nonlinearity (3).

5. Stability in case of multiple delays

In more general case the system under consideration can also contain several nonlinearities or several positive delays τ_i ($i = \overline{1,k}$) in control loop:

$$\dot{x}(t) = Ax(t) + c\, f\left(\sum_{i=1}^{k} \gamma_i' x(t-\tau_i)\right), \quad \gamma_i \in \mathbb{E}^n, \quad \|\gamma_i\| \neq 0. \tag{6}$$

Let, for example, $k = 2$, $\tau_1 = 0$, $\tau_2 = \tau$, denote $\hat{\gamma} = \gamma_1$, $\gamma = \gamma_2$, i.e.

$$\dot{x}(t) = Ax(t) + cf\left(\hat{\sigma}(t) + \sigma(t-\tau)\right), \quad \hat{\sigma} = \hat{\gamma}'x, \quad \sigma = \gamma'x. \tag{7}$$

Consider one simple particular case. Let f is given by the (2) and the system (7), (2) has a periodic solution with two switching points $\hat{s}_{1,2}$ such as

$$\hat{s}_1 = x(T_2, \hat{s}_2, m_2), \quad \hat{s}_2 = x(T_1, \hat{s}_1, m_1),$$
$$\hat{\gamma}'\hat{s}_1 + \gamma' s_1 = l_1, \quad \hat{\gamma}'\hat{s}_2 + \gamma' s_2 = l_2.$$

Here

$$\hat{s}_2 = e^{A\tau} s_2 + \int_0^\tau e^{A(\tau-t)} cm_1 dt, \quad \hat{s}_1 = e^{A\tau} s_1 + \int_0^\tau e^{A(\tau-t)} cm_2 dt.$$

Denote

$$\Gamma = \left(e^{A\tau}\right)' \hat{\gamma} + \gamma, \qquad \hat{l}_1 = l_1 - \hat{\gamma}' \int_0^\tau e^{A(\tau-t)} cm_2 dt, \qquad \hat{l}_2 = l_2 - \hat{\gamma}' \int_0^\tau e^{A(\tau-t)} cm_1 dt.$$

then

$$\Gamma' s_1 = \hat{l}_1, \quad \Gamma' s_2 = \hat{l}_2.$$

Theorem 3. *Let*

$$v_1 = As_2 + cm_1, \quad v_2 = As_1 + cm_2, \quad k_{1,2} = \Gamma' v_{1,2} \neq 0,$$

and

$$\left\| \left(I - k_2^{-1} v_2 \Gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \Gamma'\right) e^{AT_1} \right\| < 1,$$

then the periodic solution under consideration is orbitally asymptotically stable.

Proof The proof is similar to the previous proofs. As $d\left(\Gamma' s_i\right) = 0$, then

$$ds_{i+1} = \left(I - k_I^{-1} v_i \Gamma'\right) e^{AT_i} ds_i = M_i ds_i.$$

So, $ds_1^{k+1} = M_2 M_1 ds_1^k$, and use of fixed point principle completes the proof. ■

Note that here we can obtain sufficient conditions for the orbital stability in the alternative way. Suppose

$$\Gamma = \hat{\gamma} + \left(e^{-A\tau}\right)' \gamma, \quad \hat{l}_1 = l_1 + \gamma' \int_0^\tau e^{-At} cm_2 dt, \quad \hat{l}_2 = l_2 + \gamma' \int_0^\tau e^{-At} cm_1 dt,$$
$$v_1 = A\hat{s}_2 + cm_1, \quad v_2 = A\hat{s}_1 + cm_2, \quad k_{1,2} = \Gamma' v_{1,2},$$

in such a case

$$\Gamma' \hat{s}_i = \hat{l}_i, \quad i = 1, 2,$$

and the periodic solution will be orbitally asymptotically stable if $k_{1,2} \neq 0$ and

$$\left\| \left(I - k_2^{-1} v_2 \Gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \Gamma'\right) e^{AT_1} \right\| < 1.$$

All the above statements we can reformulate in a similar way, defining the above vector Γ, considering the switching points instead of pre-switching and re-defining threshold values l_i (or l_0, l in case of (3)).

Let us return to the system (6). In general case we can repeate the previous derivations. Let it has a periodic solution with two control switching points $\hat{s}_{1,2}$, such as

$$\sum_{i=1}^{k} \gamma_i' s_{1,i} = l_1, \quad \sum_{i=1}^{k} \gamma_i' s_{2,i} = l_2,$$

where

$$\hat{s}_1 = x\left(\tau_i, s_{1,i}, m_2\right), \quad \hat{s}_2 = x\left(\tau_i, s_{2,i}, m_1\right), \quad i = \overline{1,k}.$$

Then

$$\sum_{i=1}^{k} \gamma_i' \left(e^{-A\tau_i}\hat{s}_1 - \int_0^{\tau_i} e^{-At} c m_2 dt\right) = l_1, \quad \sum_{i=1}^{k} \gamma_i' \left(e^{-A\tau_i}\hat{s}_2 - \int_0^{\tau_i} e^{-At} c m_1 dt\right) = l_2,$$

and

$$\Gamma \hat{s}_j = \hat{l}_j, \quad j = 1, 2,$$

here

$$\Gamma = \sum_{i=1}^{k} \left(e^{-A\tau_i}\right)' \gamma_i, \quad \hat{l}_1 = l_1 + \sum_{i=1}^{k} \gamma_i' \int_0^{\tau_i} e^{-At} c m_2 dt, \quad \hat{l}_2 = l_2 + \sum_{i=1}^{k} \gamma_i' \int_0^{\tau_i} e^{-At} c m_1 dt.$$

So the considered periodic solution will be orbitally asymptotically stable if $k_{1,2} \neq 0$ and

$$\left\| \left(I - k_2^{-1} v_2 \Gamma'\right) e^{AT_2} \left(I - k_1^{-1} v_1 \Gamma'\right) e^{AT_1} \right\| < 1,$$

where

$$v_1 = A\hat{s}_2 + c m_1, \quad v_2 = A\hat{s}_1 + c m_2, \quad k_{1,2} = \Gamma' v_{1,2}.$$

Of course the system considered can have periodic solutions with amount of control switching points larger then two. Consider an example:

Example 5. *Consider the system (6), (2). Let* $\tau_1 = 0.013$, $\tau_2 = 0.015$,

$$A = \begin{pmatrix} -0.25 & -1. & -0.25 \\ 0.75 & 1. & 0.75 \\ 0.25 & -7. & -3.75 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0.536 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 \\ -1.108 \\ -0.567 \end{pmatrix},$$

$$m_{1,2} = \mp 1, \quad l_1 = -0.1, \quad l_2 = 0.5.$$

System (1), (2) has periodic solution with six switching points:

$$\hat{s}_1 \approx \begin{pmatrix} 0.69484 \\ -0.64902 \\ 2.12876 \end{pmatrix}, \quad \hat{s}_2 \approx \begin{pmatrix} 0.06226 \\ -1.91945 \\ 2.92801 \end{pmatrix}, \quad \hat{s}_3 \approx \begin{pmatrix} 0.72238 \\ -1.05935 \\ 2.95759 \end{pmatrix},$$

$$\hat{s}_4 \approx \begin{pmatrix} 0.51706 \\ -1.95858 \\ 3.43423 \end{pmatrix}, \quad \hat{s}_5 \approx \begin{pmatrix} 1.08072 \\ -0.87355 \\ 2.93260 \end{pmatrix}, \quad \hat{s}_6 \approx \begin{pmatrix} 0.11909 \\ -1.44650 \\ 2.05635 \end{pmatrix},$$

$$T_1 \approx 1.8724, \quad T_2 \approx 0.4018, \quad T_3 \approx 6.8301, \quad T_4 \approx 0.4019, \quad T_5 \approx 1.6087, \quad T_6 \approx 0.4084.$$

Let

$$\Gamma = \left(e^{-A\tau_1}\right)' \gamma_1 + \left(e^{-A\tau_2}\right)' \gamma_2 \approx \left(0.552607 \ -1.144496 \ -0.584956\right),$$
$$\hat{l}_1 = l_1 + \gamma_1' \int_0^{\tau_1} e^{-At} c m_2 dt + \gamma_2' \int_0^{\tau_2} e^{-At} c m_2 dt \approx -0.118450,$$
$$\hat{l}_2 = l_2 + \gamma_1' \int_0^{\tau_1} e^{-At} c m_1 dt + \gamma_2' \int_0^{\tau_2} e^{-At} c m_1 dt \approx 0.518450,$$

then

$$\Gamma' \hat{s}_1 = \Gamma' \hat{s}_3 = \hat{l}_1, \quad \Gamma' \hat{s}_2 = \Gamma' \hat{s}_4 = \hat{l}_1.$$

Denote

$$u_{2k+1} = m_1, \quad u_{2k} = m_2.$$

One can verify that

$$k_i = \Gamma' \left(A\hat{s}_{i+1} + cu_i\right) \neq 0, \quad i = \overline{1,6}.$$

Let

$$M_i = \left(I - k_i^{-1} \left(As_{i+1} + cu_i\right) \Gamma'\right) e^{AT_i},$$

in such a case

$$\|M\| = \left\| \prod_{i=6}^{1} M_i \right\| \approx 0.13771 < 1$$

and the periodic solution under consideration is asymptotically orbitally stable.

Let us obtain similar results for the system (4). Suppose for simplicity that

$$\dot{x} = Ax + (Cx + c) f\left(\hat{\sigma}(t) + \sigma(t - \tau)\right), \quad \hat{\sigma} = \hat{\gamma}' x, \quad \sigma = \gamma' x. \tag{8}$$

Let f is given by the (2). Denote

$$A_i = A + Cm_i, \quad c_i = cm_i, \quad i = 1, 2, \quad x_i(T, x_0) = e^{A_i T} x_0 + \int_0^T e^{A_i(T-t)} c_i dt.$$

Let the system (8), (2) has a periodic solution with two switching points $\hat{s}_{1,2}$ such as

$$\hat{s}_1 = x_2(T_2, \hat{s}_2), \quad \hat{s}_2 = x_1(T_1, \hat{s}_1),$$
$$\hat{\gamma}' \hat{s}_1 + \gamma' s_1 = l_1, \quad \hat{\gamma}' \hat{s}_2 + \gamma' s_2 = l_2,$$

here

$$\hat{s}_1 = e^{A_2\tau} s_1 + \int_0^\tau e^{A_2(\tau-t)} c_2 dt, \quad \hat{s}_2 = e^{A_1\tau} s_2 + \int_0^\tau e^{A_1(\tau-t)} c_1 dt.$$

So,

$$\hat{\gamma}' e^{A_2\tau} s_1 + \hat{\gamma}' \int_0^\tau e^{A_2(\tau-t)} c_2 dt + \gamma' s_1 = l_1, \quad \hat{\gamma}' e^{A_1\tau} s_2 + \hat{\gamma}' \int_0^\tau e^{A_1(\tau-t)} c_1 dt + \gamma' s_2 = l_2,$$

or

$$\Gamma_1' s_1 = \hat{l}_1, \quad \Gamma_2' s_2 = \hat{l}_2,$$

where

$$\Gamma_1 = \left(e^{A_2\tau}\right)'\hat{\gamma} + \gamma, \quad \Gamma_2 = \left(e^{A_1\tau}\right)'\hat{\gamma} + \gamma,$$

$$\hat{l}_1 = l_1 - \hat{\gamma}'\int_0^\tau e^{A_2(\tau-t)}c_2 dt, \quad \hat{l}_2 = l_2 - \hat{\gamma}'\int_0^\tau e^{A_1(\tau-t)}c_1 dt.$$

Let

$$v_1 = A_1 s_2 + c_1, \quad v_2 = A_2 s_1 + c_2, \quad k_1 = \Gamma_2' v_1, \quad k_2 = \Gamma_1' v_2.$$

Theorem 4. *If* $k_{1,2} \neq 0$ *and*

$$\left\| \left(I - k_2^{-1} v_2 \Gamma_1'\right) e^{A_2T_2 + (A_1-A_2)\tau} \left(I - k_1^{-1} v_1 \Gamma_2'\right) e^{A_1T_1 + (A_2-A_1)\tau} \right\| < 1,$$

where

$$A_i = A + Cm_i, \quad c_i = cm_i, \quad i = 1, 2.$$

Then the considered periodic solution is orbitally asymptotically stable.

Proof As

$$s_2 = x_1\left(T_1 - \tau, x_2(\tau, s_1)\right) =$$

$$= e^{A_1T_1 + (A_2-A_1)\tau}s_1 + e^{A_1(T_1-\tau)}\int_0^\tau e^{A_2(\tau-t)}c_2 dt + \int_0^{T_1-\tau} e^{A_1(T_1-\tau-t)}c_1 dt,$$

$$(s_2)'_{s_1} = e^{A_1T_1 + (A_2-A_1)\tau}, \quad (s_2)'_{T_1} = A_1 s_2 + c_1 = v_1,$$

then

$$0 = d\left(\Gamma_2' s_2\right) = \Gamma_2' e^{A_1T_1 + (A_2-A_1)\tau} ds_1 + k_1 dT_1,$$

$$dT_1 = -k_1^{-1}\Gamma_2' e^{A_1T_1 + (A_2-A_1)\tau} ds_1, \quad \text{and} \quad ds_2 = \left(I - k_1^{-1} v_1 \Gamma_2'\right) e^{A_1T_1 + (A_2-A_1)\tau} ds_1.$$

Similarly,

$$ds_2 = \left(I - k_2^{-1} v_2 \Gamma_1'\right) e^{A_2T_2 + (A_1-A_2)\tau} ds_2.$$

In order to finalize the proof one can use the fixed point principle for s_1. ■

In case of the system (8), (3) the sufficient conditions for orbital stability will change slightly. Let the system has periodic solution with four control switching points $\hat{s}_i$, $i = \overline{1,4}$, where

$$\hat{s}_{i+1} = x_i\left(T_1, \hat{s}_i\right).$$

Let s_i, $i = \overline{1,4}$, are points on the trajectory of the solution such as

$$\hat{s}_i = x_{i-1}\left(s_i, \tau\right),$$

and

$$\hat{\gamma}'\hat{s}_i + \gamma' s_i = l_i, \quad l_1 = l_0, \quad l_2 = -l, \quad l_3 = -l_0, \quad l_4 = l.$$

In such a case

$$\hat{\gamma}' e^{A_{i-1}\tau} s_i + \hat{\gamma}'\int_0^\tau e^{A_{i-1}(\tau-t)}c_{i-1} dt + \gamma' s_i = \hat{l}_i,$$

or

$$\Gamma_i s_i = \hat{l}_i, \quad i = \overline{1,4}, \quad \Gamma_i = \left(e^{A_{i-1}\tau}\right)'\hat{\gamma} + \gamma, \quad \hat{l}_i = l_i - \hat{\gamma}'\int_0^\tau e^{A_{i-1}(\tau-t)}c_{i-1} dt.$$

Denote

$$v_i = A_i s_{i+1} + c_i, \quad k_i = \Gamma'_{i+1} v_i, \quad M_i = \left(I - k_i^{-1} v_i \Gamma'_{i+1} \right) e^{A_i T_i + (A_{i-1} - A_i)\tau}$$

Theorem 5. *Let $k_i \neq 0$, $i = \overline{1,4}$, and*

$$\left\| \prod_{i=4}^{1} M_i \right\| < 1, \tag{9}$$

then the periodic solution is orbitally asymptotically stable.

Let us skip the proof, it is similar to the above one.

Example 6. *Let A, c, $l_{1,2}$, $m_{1,2}$ are the same as in the example 5,*

$$C = \begin{pmatrix} -0.01 & 0 & 0 \\ 0 & 0.005 & 0 \\ -0.01 & 0.01 & 0.005 \end{pmatrix},$$

and

$$\dot{x} = Ax + (Cx + c) f \left(-0.565 x_3(t) - 1.11 x_2(t - 0.015) + 0.54 x_1(t - 0.1) \right),$$

where f is given by the (2). I.e.

$$\tau_1 = 0, \quad \tau_2 = 0.015, \quad \tau_3 = 0.1,$$
$$\gamma'_1 = \begin{pmatrix} 0 & 0 & -0.565 \end{pmatrix}, \quad \gamma'_2 = \begin{pmatrix} 0 & -1.11 & 0 \end{pmatrix}, \quad \gamma'_3 = \begin{pmatrix} 0.54 & 0 & 0 \end{pmatrix}.$$

In such a case the system has a periodic solution with four switching points

$$\hat{s}'_1 \approx \begin{pmatrix} 1.1250 & -1.0662 & 3.3411 \end{pmatrix}, \quad \hat{s}'_2 \approx \begin{pmatrix} 0.1806 & -1.3848 & 2.0040 \end{pmatrix},$$
$$\hat{s}'_3 \approx \begin{pmatrix} 0.7081 & -0.6317 & 2.0672 \end{pmatrix}, \quad \hat{s}'_4 \approx \begin{pmatrix} 0.5502 & -2.1717 & 3.9062 \end{pmatrix},$$
$$T_1 \approx 1.5668, \quad T_2 \approx 0.3846, \quad T_3 \approx 4.4353, \quad T_4 \approx 0.3890.$$

Denote

$$A_{1,2} = A + Cm_{1,2},$$
$$\Gamma_1 = \gamma_1 + \left(e^{-A_2 \tau_2} \right)' \gamma_2 + \left(e^{-A_2 \tau_3} \right)' \gamma_3 \approx \begin{pmatrix} 0.564337 & -1.035933 & -0.538052 \end{pmatrix}',$$
$$\Gamma_2 = \gamma_1 + \left(e^{-A_1 \tau_2} \right)' \gamma_2 + \left(e^{-A_1 \tau_3} \right)' \gamma_3 \approx \begin{pmatrix} 0.563215 & -1.036110 & -0.538057 \end{pmatrix}',$$

$$\hat{l}_1 = l_1 + \gamma'_2 \int_0^{\tau_2} e^{-A_2 t} c m_2 dt + \gamma'_3 \int_0^{\tau_3} e^{-A_2 t} c m_2 dt \approx -0.058212,$$
$$\hat{l}_2 = l_2 + \gamma'_2 \int_0^{\tau_2} e^{-A_1 t} c m_1 dt + \gamma'_3 \int_0^{\tau_3} e^{-A_1 t} c m_1 dt \approx 0.458270.$$

Then

$$\Gamma'_1 \hat{s}_1 = \Gamma'_1 \hat{s}_3 = \hat{l}_1, \quad \Gamma'_2 \hat{s}_2 = \Gamma'_2 \hat{s}_4 = \hat{l}_2.$$

Let

$$v_1 = A_1 \hat{s}_2 + c m_1, \quad v_2 = A_2 \hat{s}_3 + c m_2, \quad v_3 = A_1 \hat{s}_4 + c m_1, \quad v_4 = A_2 \hat{s}_1 + c m_2,$$

One can easy verify that

$$k_1 = \Gamma_2' v_1 \neq 0, \quad k_2 = \Gamma_1' v_2 \neq 0, \quad k_3 = \Gamma_2' v_3 \neq 0, \quad k_4 = \Gamma_1' v_4 \neq 0.$$

Denote

$$M_1 = \left(I - k_1^{-1} v_1 \Gamma_2'\right) e^{A_1 T_1}, \quad M_2 = \left(I - k_2^{-1} v_2 \Gamma_1'\right) e^{A_2 T_2},$$
$$M_3 = \left(I - k_3^{-1} v_3 \Gamma_2'\right) e^{A_1 T_3}, \quad M_4 = \left(I - k_4^{-1} v_4 \Gamma_1'\right) e^{A_2 T_4}.$$

and

$$\|M\| = \left\| \prod_{i=1}^{4} M_i \right\| \approx 0.3033 < 1.$$

So, as $ds_1^{k+1} = M ds_1^k$, the periodic solution under consideration is orbitally asymptotically stable.
Similar results can be obtained in case of nonlinearity (3).

6. Perturbed system

Consider a system:

$$\dot{x} = Ax + c\big(\varphi(t) + u(t-\tau)\big), \tag{10}$$

where $\varphi(t)$ is scalar T_φ-periodic continuous function of time. Let f is given by (3).
Consider a special case of the previous system (see Nelepin (2002), Kamachkin & Shamberov (1995)). Let $n = 2$,

$$\ddot{y} + g_1 \dot{y} + g_2 y = u(t-\tau) + \varphi(t), \tag{11}$$

here $y(t) \in \mathbb{R}$ is sought-for time variable, $g_{1,2}$ are real constants, $\sigma = \alpha_1 y + \alpha_2 \dot{y}$, $\alpha_{1,2}$ are real constants. Let us rewrite system (11) in vector form. Denote $z' = \begin{pmatrix} y & \dot{y} \end{pmatrix}$, in that case

$$\begin{aligned} \dot{z} &= Pz + q\left(\varphi(t) + u(t-\tau)\right), \\ u(t-\tau) &= f\left(\sigma(t-\tau)\right), \quad \sigma = \alpha' z, \end{aligned} \tag{12}$$

where

$$P = \begin{pmatrix} 0 & 1 \\ -g_2 & -g_1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Suppose that characteristic determinant $D(s) = \det(P - sI)$ has real simple roots $\lambda_{1,2}$, and vectors q, Pq are linearly independent. In that case system (12) may be reduced to the form (10), where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

by means of nonsingular linear transformation

$$z = Tx, \qquad T = \begin{pmatrix} \frac{N_1(\lambda_1)}{D'(\lambda_1)} & \frac{N_1(\lambda_2)}{D'(\lambda_2)} \\ \frac{N_2(\lambda_1)}{D'(\lambda_1)} & \frac{N_2(\lambda_2)}{D'(\lambda_2)} \end{pmatrix}, \quad D'(\lambda_j) = \frac{d}{ds} D(s) \Big|_{s=\lambda_j}, \quad N_j(s) = \sum_{i=1}^{2} q_i D_{ij}(s), \tag{13}$$

$D_{ij}(s)$ is algebraic supplement for element lying in the intersection of i-th row and j-th column of determinant $D(s)$.

Note that

$$\sigma = \gamma' x, \quad \gamma = T' \alpha.$$

Furthermore, since

$$\gamma_i = -\left(D'\left(\lambda_i\right)\right)^{-1} \sum_{j=1}^{2} \alpha_j N_j\left(\lambda_i\right), \quad i = 1, 2.$$

then

$$\gamma_1 = \left(\lambda_1 - \lambda_2\right)^{-1}\left(\alpha_1 + \alpha_2\lambda_1\right), \quad \gamma_2 = \left(\lambda_2 - \lambda_1\right)^{-1}\left(\alpha_1 + \alpha_2\lambda_2\right).$$

Transformation (13) leads to the following system:

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + f\left(\sigma(t-\tau)\right) + \varphi(t), \\ \dot{x}_2 = \lambda_2 x_2 + f\left(\sigma(t-\tau)\right) + \varphi(t). \end{cases} \tag{14}$$

If, for example,

$$\alpha_1 = -\lambda_1\alpha_2,$$

then

$$\gamma_1 = 0, \quad \gamma_2 = \alpha_2, \quad \sigma = \gamma_2 x_2.$$

Function f in that case is independent of variable x_1, and

$$\dot{\sigma} = \lambda_2\sigma + \gamma_2\left(f(\gamma_2\, x_2(t-\tau)) + \varphi(t)\right).$$

Solution of the latest equation when $f = u$ (where $u = m_1, m_2$ or 0) has the following form:

$$\sigma\left(t,\, t_0,\, \sigma_0,\, u\right) = e^{\lambda_2(t-t_0)}\sigma_0 + \gamma_2\, e^{\lambda_2 t}\int_{t_0}^{t} e^{-\lambda_2 s}\Big(u + \varphi(s)\Big)\, ds.$$

Let us trace out necessary conditions for existing of periodic solution of the system (10), (3) having four switching points $\hat{s}_i$:

$$\begin{aligned} \sigma_2 &= \sigma\left(t_1, t_0+\tau, \hat{\sigma}_1, 0\right), & \hat{\sigma}_2 &= \sigma\left(t_1+\tau, t_1, \sigma_2, 0\right), \\ \sigma_3 &= \sigma\left(t_2, t_1+\tau, \hat{\sigma}_2, m_1\right), & \hat{\sigma}_3 &= \sigma\left(t_2+\tau, t_2, \sigma_3, m_1\right), \\ \sigma_4 &= \sigma\left(t_3, t_2+\tau, \hat{\sigma}_3, 0\right), & \hat{\sigma}_4 &= \sigma\left(t_3+\tau, t_3, \sigma_4, 0\right), \\ \sigma_1 &= \sigma\left(t_4, t_3+\tau, \hat{\sigma}_4, m_2\right), & \hat{\sigma}_1 &= \sigma\left(t_4+\tau, t_4, \sigma_1, m_2\right), \end{aligned}$$

for some positive T_i, $i = \overline{1,4}$, and $t_i = t_{i-1} + T_i$. Denote $u_1 = 0, \quad u_2 = m_1, \quad u_3 = 0, \quad u_4 = m_2$, then

$$\begin{aligned} \sigma_{i+1} &= \sigma\left(t_i,\, t_{i-1}+\tau,\, \sigma\left(t_{i-1}+\tau,\, t_{i-1},\, \sigma_i,\, u_{i-1}\right),\, u_i\right) = \\ &= e^{\lambda_2(T_i-\tau)}\left(e^{\lambda_2\tau}\sigma_i + \gamma_2 e^{\lambda_2(t_{i-1}+\tau)}\int_{t_{i-1}}^{t_{i-1}+\tau} e^{-\lambda_2 t}\left(u_{i-1} + \varphi(t)\right)\, dt\right) + \\ &\quad + \gamma_2 e^{\lambda_2 t_i}\int_{t_{i-1}+\tau}^{t_i} e^{-\lambda_2 t}\left(u_i + \varphi(t)\right)\, dt = e^{\lambda_2 T_i}\sigma_i + K_i, \end{aligned}$$

where

$$K_i = \gamma_2 e^{\lambda_2 t_i}\left(\int_{t_{i-1}}^{t_i} e^{-\lambda_2 t}\varphi(t)\, dt + \int_{t_{i-1}}^{t_{i-1}+\tau} e^{-\lambda_2 t}u_{i-1}\, dt + \int_{t_{i-1}+\tau}^{t_i} e^{-\lambda_2 t}u_i\, dt\right).$$

So,

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & e^{\lambda_2 T_4}\\ e^{\lambda_2 T_1} & 0 & 0 & 0\\ 0 & e^{\lambda_2 T_2} & 0 & 0\\ 0 & 0 & e^{\lambda_2 T_3} & 0 \end{pmatrix} \begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix} + \begin{pmatrix}K_1\\ K_2\\ K_3\\ K_4\end{pmatrix}$$

and

$$\begin{aligned}
\sigma_1 &= \left(1 - e^{\lambda_2 T}\right)\left(K_2 e^{\lambda_2(T_2+T_3+T_4)} + K_3 e^{\lambda_2(T_3+T_4)} + K_4 e^{\lambda_2 T_4} + K_1\right) = l_0,\\
\sigma_2 &= \left(1 - e^{\lambda_2 T}\right)\left(K_3 e^{\lambda_2(T_1+T_3+T_4)} + K_4 e^{\lambda_2(T_1+T_4)} + K_1 e^{\lambda_2 T_1} + K_2\right) = -l,\\
\sigma_3 &= \left(1 - e^{\lambda_2 T}\right)\left(K_4 e^{\lambda_2(T_1+T_2+T_4)} + K_1 e^{\lambda_2(T_1+T_2)} + K_2 e^{\lambda_2 T_2} + K_3\right) = -l_0,\\
\sigma_4 &= \left(1 - e^{\lambda_2 T}\right)\left(K_1 e^{\lambda_2(T_1+T_2+T_3)} + K_2 e^{\lambda_2(T_2+T_3)} + K_3 e^{\lambda_3 T_3} + K_4\right) = l,
\end{aligned}$$

here $T = T_1 + T_2 + T_3 + T_4$ is a period of the solution (let it is multiple of T_φ). Consider the latest system as a system of linear equations with respect to γ_2, m (for example), i.e.

$$\sigma_1 = \Psi_1(m,\gamma_2) = l_0, \quad \sigma_2 = \Psi_2(m,\gamma_2) = -l, \quad \sigma_3 = \Psi_3(m,\gamma_2) = -l_0, \quad \sigma_4 = \Psi_4(m,\gamma_2) = l.$$

Suppose $\Psi_i \equiv -\Psi_{i+2}$ (it can be if the solution is origin-symmetric).
Denote

$$\begin{aligned}
\hat{\psi}_i(t) &= \sigma\left(t_i + t,\, t_i,\, \sigma_i,\, u_{i-1}\right), \quad t \in [0,\, \tau),\\
\psi_i(t) &= \sigma\left(t_i + \tau + t,\, t_i + \tau,\, \hat{\sigma}_1,\, u_i\right), \quad t \in [0,\, T_i - \tau)
\end{aligned}$$

Following result may be formulated.

Theorem 6. *Let the system*

$$\begin{cases}\Psi_1(m,\gamma_2) = l_0,\\ \Psi_2(m,\gamma_2) = -l.\end{cases}$$

has a solution such as for given $\gamma = \left(0,\, \gamma_2\right)'$ and m conditions

$$\begin{cases}
\hat{\psi}_1(t) > -l, & t \in [0,\, \tau),\\
\psi_1(t) > -l, & t \in [0,\, T_1 - \tau),\\
\hat{\psi}_2(t) > -l_0, & t \in [0,\, \tau),\\
\psi_2(t) > -l_0, & t \in [0,\, T_2 - \tau),\\
\hat{\psi}_3(t) < l, & t \in [0,\, \tau),\\
\psi_3(t) < l, & t \in [0,\, T_3 - \tau),\\
\hat{\psi}_4(t) > l_0, & t \in [0,\, \tau),\\
\psi_4(t) > l_0, & t \in [0,\, T_4 - \tau)
\end{cases} \tag{15}$$

are satisfied. In that case system (14) has a stable T-periodic solution with switching points $\hat{s}_i$, if $\lambda_1 < 0$ and

$$T\, T_\varphi^{-1} \in \mathbb{N}.$$

Proof In order to prove the theorem it is enough to note that under above-listed conditions system (14) settles self-mapping of switching lines $\sigma = l_i$. Moreover, for any $x^{(i)}$ lying on switching line,

$$x_1^{(i+1)} = e^{\lambda_1 T} x_1^{(i)} + \Theta, \quad \Theta \in \mathbb{R},$$

and in general case ($\Theta \neq 0$) the latter difference equation has stable solution only if $\lambda_1 < 0$. ■
In order to pass onto variables z_i it is enough to effect linear transform (13).
Note that conditions (15) may be readily verified using mathematical symbolic packages.
Of course the statement Theorem 6 is just an outline. Further investigation of the system (11) requires specification of φ function, detailed computations are quite laborious.
On the analogy with the previous section a case of multiple delays can be observed.

7. Conclusion

The above results suppose further development. Investigation of stable modes of the forced system (10) is an individual complex task (systems with several delays may also be considered). Results similar to obtained in the last part can be outlined for periodic solutions of the system (10) having a quite complicated configuration (large amount of control switching point etc.).
Stabilization problem (i.e. how to choose setup variables of a system in order to put its steady state solution in a prescribed neighbourhood of the origin) was not discussed. This problem was elucidated in Zubov (1999), Zubov & Zubov (1996) for a bit different systems.

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6

Design of Controllers for Time Delay Systems: Integrating and Unstable Systems

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1. Introduction

The presence of a time delay is a common property of many technological processes. In addition, a part of time delay systems can be unstable or have integrating properties. Typical examples of such processes are e.g. pumps, liquid storing tanks, distillation columns or some types of chemical reactors.

Plants with a time delay often cannot be controlled by usual controllers designed without consideration of the dead-time. There are various ways to control such systems. A number of methods utilise PI or PID controllers in the classical feedback closed-loop structure, e.g. (Park et al., 1998; Zhang and Xu, 1999; Wang and Cluett, 1997; Silva et al., 2005). Other methods employ ideas of the IMC (Tan et al., 2003) or robust control (Prokop and Corriou, 1997). Control results of a good quality can be achieved by modified Smith predictor methods, e.g. (Åström et al., 1994; De Paor, 1985; Liu et al., 2005; Majhi and Atherton, 1999; and Matausek and Micic, 1996).

Principles of the methods used in this work and design procedures in the 1DOF and 2DOF control system structures can be found in papers of authors of this article (Dostál et al., 2001; Dostál et al., 2002). The control system structure with two feedback controllers is considered (Dostál et al., 2007; Dostál et al., 2008). The procedure of obtaining controllers is based on the time delay first order Padé approximation and on the polynomial approach (Kučera, 1993). For tuning of the controller parameters, the pole assignment method exploiting the LQ control technique is used (Hunt et al., 1993). The resulting proper and stable controllers obtained via polynomial Diophantine equations and spectral factorization techniques ensure asymptotic tracking of step references as well as step disturbances attenuation. Structures of developed controllers together with analytically derived formulas for computation of their parameters are presented for five typical plant types of integrating and unstable time delay systems: an integrating time delay system (ITDS), an unstable first order time delay system (UFOTDS), an unstable second order time delay system (USOTDS), a stable first order plus integrating time delay system (SFOPITDS) and an unstable plus integrating time delay system (UFOPITDS). Presented simulation results document usefulness of the proposed method providing stable control responses of a good quality also for a higher ratio between the time delay and unstable time constants of the controlled system.

2. Approximate transfer functions

The transfer functions in the sequence ITDS, UFOTDS, USOTDS, SFOPITDS and UFOPITDS have these forms:

$$G_1(s) = \frac{K}{s} e^{-\tau_d s} \tag{1}$$

$$G_2(s) = \frac{K}{\tau s - 1} e^{-\tau_d s} \tag{2}$$

$$G_3(s) = \frac{K}{(\tau_1 s - 1)(\tau_2 s + 1)} e^{-\tau_d s} \tag{3}$$

$$G_{4,5}(s) = \frac{K}{s(\tau s \pm 1)} e^{-\tau_d s} . \tag{4}$$

Using the first order Padé approximation, the time delay term in (1) – (4) is approximated by

$$e^{-\tau_d s} \approx \frac{2 - \tau_d s}{2 + \tau_d s} . \tag{5}$$

Then, the approximate transfer functions take forms

$$G_{A1}(s) = \frac{K(2 - \tau_d s)}{s(2 + \tau_d s)} = \frac{b_0 - b_1 s}{s^2 + a_1 s} \tag{6}$$

where $b_0 = \dfrac{2K}{\tau_d}$, $b_1 = K$ and $a_1 = \dfrac{2}{\tau_d}$ for the ITDS,

$$G_{A2}(s) = \frac{K(2 - \tau_d s)}{(\tau s - 1)(2 + \tau_d s)} = \frac{b_0 - b_1 s}{s^2 + a_1 s + a_0} \tag{7}$$

with $b_0 = \dfrac{2K}{\tau \tau_d}$, $b_1 = \dfrac{K}{\tau}$, $a_0 = -\dfrac{2}{\tau \tau_d}$, $a_1 = \dfrac{2\tau - \tau_d}{\tau \tau_d}$ and $\tau_d \neq 2\tau$ for the UFOTDS,

$$G_{A3}(s) = \frac{K(2 - \tau_d s)}{(\tau_1 s - 1)(\tau_2 s + 1)(2 + \tau_d s)} = \frac{b_0 - b_1 s}{s^3 + a_2 s^2 + a_1 s - a_0} \tag{8}$$

where

$b_0 = \dfrac{2K}{\tau_1 \tau_2 \tau_d}$, $b_1 = \dfrac{K}{\tau_1 \tau_2}$, $a_0 = \dfrac{2}{\tau_1 \tau_2 \tau_d}$, $a_1 = \dfrac{2(\tau_1 - \tau_2) - \tau_d}{\tau_1 \tau_2 \tau_d}$,

$a_2 = \dfrac{2\tau_1\tau_2 + \tau_1\tau_d - \tau_2\tau_d}{\tau_1 \tau_2 \tau_d}$ and $\tau_d \neq 2\tau_1$ for the USOTDS, and,

$$G_{A\,4,5}(s) = \frac{K(2 - \tau_d s)}{s(\tau s \pm 1)(2 + \tau_d s)} = \frac{b_0 - b_1 s}{s^3 + a_2 s^2 + a_1 s} \tag{9}$$

where $b_0 = \dfrac{2K}{\tau\tau_d}$, $b_1 = \dfrac{K}{\tau}$, $a_1 = \pm\dfrac{2}{\tau\tau_d}$, $a_2 = \dfrac{2\tau \pm \tau_d}{\tau\tau_d}$ and $\tau_d \neq 2\tau$ for the SFOPITDS and UFOPTDS, respectively.
All approximate transfer functions (6) – (9) are strictly proper transfer functions

$$G_A(s) = \frac{b(s)}{a(s)} \tag{10}$$

where b and a are coprime polynomials in s that fulfill the inequality $\deg b < \deg a$.
The polynomial $a(s)$ in their denominators can be expressed as a product of the stable and unstable part

$$a(s) = a^+(s)a^-(s) \tag{11}$$

so that for ITDS, UFOTDS, USOTDS and SFOPITDS the equality

$$\deg a^+ = \deg a - 1 \tag{12}$$

is fulfilled.

3. Control system description

The control system with two feedback controllers is depicted in Fig. 1. In the scheme, w is the reference, v is the load disturbance, e is the tracking error, u_0 is the controller output, y is the controlled output, u is the control input and G_A represents one of the approximate transfer functions (6) – (9) in the general form (10).
Remark: Here, the approximate transfer function G_A is used only for a controller derivation. For control simulations, the models G_1 – G_5 are utilized.
Both w and v are considered to be step functions with Laplace transforms

$$W(s) = \frac{w_0}{s}, \quad V(s) = \frac{v_0}{s}. \tag{13}$$

The transfer functions of controllers are assumed as

$$Q(s) = \frac{\tilde{q}(s)}{\tilde{p}(s)}, \quad R(s) = \frac{r(s)}{\tilde{p}(s)} \tag{14}$$

where $\tilde{q}, r$ and $\tilde{p}$ are polynomials in s.

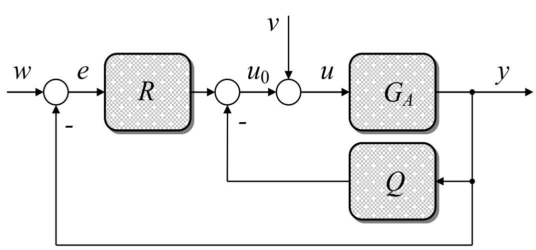

Fig. 1. The control system.

4. Application of the polynomial method

The controller design described in this section follows the polynomial approach. General requirements on the control system are formulated as its internal properness and strong stability (in addition to the control system stability, also the controller stability is required), asymptotic tracking of the reference and load disturbance attenuation. The procedure to derive admissible controllers can be performed as follows:
Transforms of basic signals in the closed-loop system from Fig.1 take following forms (for simplification, the argument *s* is in some equations omitted)

$$Y(s) = \frac{b}{d}\left[r\,W(s) + \tilde{p}\,V(s) \right] \tag{15}$$

$$E(s) = \frac{1}{d}\left[(a\tilde{p} + b\tilde{q})W(s) - b\tilde{p}\,V(s) \right] \tag{16}$$

$$U(s) = \frac{a}{d}\left[r\,W(s) + \tilde{p}\,V(s) \right] \tag{17}$$

where

$$d(s) = a(s)\,\tilde{p}(s) + b(s)\left[r(s) + \tilde{q}(s) \right] \tag{18}$$

is the characteristic polynomial with roots as poles of the closed-loop.
Establishing the polynomial t as

$$t(s) = r(s) + \tilde{q}(s) \tag{19}$$

and substituting (19) into (18), the condition of the control system stability is ensured when polynomials $\tilde{p}$ and t are given by a solution of the polynomial Diophantine equation

$$a(s)\tilde{p}(s) + b(s)\,t(s) = d(s) \tag{20}$$

with a stable polynomial d on the right side.
With regard to transforms (13), the asymptotic tracking and load disturbance attenuation are provided by divisibility of both terms $a\tilde{p} + b\tilde{q}$ and $\tilde{p}$ in (16) by s. This condition is fulfilled for polynomials $\tilde{p}$ and $\tilde{q}$ having forms

$$\tilde{p}(s) = s\,p(s)\,, \quad \tilde{q}(s) = s\,q(s)\,. \tag{21}$$

Subsequently, the transfer functions (14) take forms

$$Q(s) = \frac{q(s)}{p(s)}\,, \quad R(s) = \frac{r(s)}{s\,p(s)} \tag{22}$$

and, a stable polynomial $p(s)$ in their denominators ensures the stability of controllers (the strong stability of the control system).
The control system satisfies the condition of internal properness when the transfer functions of all its components are proper. Consequently, the degrees of polynomials q and r must fulfil these inequalities

$$\deg q \le \deg p\,,\quad \deg r \le \deg p + 1\,. \tag{23}$$

Now, the polynomial t can be rewritten to the form

$$t(s) = r(s) + s\,q(s)\,. \tag{24}$$

Taking into account solvability of (20) and conditions (23), the degrees of polynomials in (19) and (20) can be easily derived as

$$\deg t = \deg r = \deg a\,,\ \deg q = \deg a - 1\,,\ \deg p \ge \deg a - 1\,,\ \deg d \ge 2\deg a\,. \tag{25}$$

Denoting deg a = n, polynomials t, r and q have forms

$$t(s) = \sum_{i=0}^{n} t_i s^i\ ,\ r(s) = \sum_{i=0}^{n} r_i s^i\ ,\ q(s) = \sum_{i=1}^{n} q_i s^{i-1} \tag{26}$$

and, relations among their coefficients are

$$r_0 = t_0\,,\quad r_i + q_i = t_i \ \text{ for } \ i = 1, \dots, n \tag{27}$$

Since by a solution of the polynomial equation (20) only coefficients t_i can be calculated, unknown coefficients r_i and q_i can be obtained by a choice of selectable coefficients $\gamma_i \in \langle 0,1 \rangle$ such that

$$r_i = \gamma_i t_i\,,\quad q_i = (1-\gamma_i)t_i \ \text{ for } \ i = 1, \dots, n\,. \tag{28}$$

The coefficients γ_i divide a weight between numerators of transfer functions Q and R.

Remark: If $\gamma_i = 1$ for all i, the control system in Fig. 1 reduces to the 1DOF control configuration (Q = 0). If $\gamma_i = 0$ for all i, and, both reference and load disturbance are step functions, the control system corresponds to the 2DOF control configuration.

The controller parameters then result from solutions of the polynomial equation (20) and depend upon coefficients of the polynomial d. The next problem here is to find a stable polynomial d that enables to obtain acceptable stabilizing and stable controllers.

5. Pole assignment

The polynomial d is considered as a product of two stable polynomials g and m in the form

$$d(s) = g(s)\,m(s) \tag{29}$$

where the polynomial g is a monic form of the polynomial g' obtained by the spectral factorization

$$\left[s\,a(s)\right]^* \varphi \left[s\,a(s)\right] + b^*(s)b(s) = g'^*(s)\,g'(s) \tag{30}$$

where $\varphi > 0$ is the weighting coefficient.

Remark: In the LQ control theory, the polynomial g' results from minimization of the quadratic cost function

$$J = \int_0^{\infty} \left\{ e^2(t) + \varphi \dot{u}^2(t) \right\} dt \tag{31}$$

where $e(t)$ is the tracking error and $\dot{u}(t)$ is the control input derivative.
The second polynomial m ensuring properness of controllers is given as

$$m(s) = a^{+}(s) = s + \frac{2}{\tau_d} \tag{32}$$

for both ITDS and UFOTDS,

$$m(s) = a^{+}(s) = \left(s + \frac{2}{\tau_d} \right)\left(s + \frac{1}{\tau_2} \right) \tag{33}$$

for the USOTDS, and,

$$m(s) = \left(s + \frac{2}{\tau_d} \right)\left(s + \frac{1}{\tau} \right). \tag{34}$$

for both UFOPITDS and SFOPITDS.
The coefficients of the polynomial *d* include only a single selectable parameter φ and all other coefficients are given by parameters of polynomials *b* and *a*. Consequently, the closed loop poles location can be affected by a single selectable parameter. As known, the closed loop poles location determines both step reference and step load disturbance responses. However, with respect to the transform (13), it may be expected that weighting coefficients γ influence only step reference responses.
Then, the monic polynomial g and derived formulas for their parameters have forms

$$g(s) = s^3 + g_2 s^2 + g_1 s + g_0 \tag{35}$$

for both ITDS and UFOTDS, where

$$g_0 = \frac{2K}{\tau_d}\sqrt{\frac{1}{\varphi}}, \quad g_1 = \sqrt{\frac{1}{\varphi}\left(\frac{4K}{\tau_d} g_2 + K^2 \right)}, \quad g_2 = \sqrt{\frac{2}{\sqrt{\varphi}} g_1 + \frac{4}{\tau_d^2}} \tag{36}$$

for the ITDS, and,

$$g_0 = \frac{2K}{\tau\tau_d}\frac{1}{\sqrt{\varphi}}, \quad g_1 = \frac{1}{\tau\tau_d}\sqrt{4\left(K\tau\tau_d \frac{1}{\varphi} g_2 + 1 \right) + K^2\tau_d^2 \frac{1}{\varphi}},$$
$$g_2 = \frac{1}{\tau\tau_d}\sqrt{2\tau^2\tau_d^2\sqrt{\frac{1}{\varphi}} g_1 + 4\tau^2 + \tau_d^2} \tag{37}$$

for the UFOTDS, and,

$$g(s) = s^4 + g_3 s^3 + g_2 s^2 + g_1 s + g_0 \tag{38}$$

for USOTDS, SFOPITDS and UFOPITDS, where

$$g_0 = \frac{2K}{\tau_1\tau_2\tau_d}\frac{1}{\sqrt{\varphi}}, \quad g_1 = \sqrt{\left(\frac{4K}{\tau_1\tau_2\tau_d}g_2 + \frac{K^2}{\tau_1^2\tau_2^2}\right)\frac{1}{\varphi} + \frac{4}{\tau_1^2\tau_2^2\tau_d^2}}$$
$$g_2 = \sqrt{\frac{2}{\varphi}g_1g_3 - \frac{4K}{\tau_1\tau_2\tau_d}\sqrt{\frac{1}{\varphi}} + \frac{4(\tau_1^2+\tau_2^2)+\tau_d^2}{\tau_1^2\tau_2^2\tau_d^2}}, \quad g_3 = \sqrt{\frac{2}{\sqrt{\varphi}}g_2 + \frac{4}{\tau_d^2} + \frac{1}{\tau_1^2} + \frac{1}{\tau_2^2}} \tag{39}$$

for the USOTDS, and,

$$g_0 = \frac{2K}{\tau\tau_d}\frac{1}{\sqrt{\varphi}}, \quad g_1 = \sqrt{\frac{1}{\varphi}\left(\frac{4K}{\tau\tau_d}g_2 + \frac{K^2}{\tau^2}\right)},$$
$$g_2 = \sqrt{2g_1g_3 + \frac{4}{\tau\tau_d}\left(\frac{1}{\tau\tau_d} - K\frac{1}{\sqrt{\varphi}}\right)}, \quad g_3 = \sqrt{\frac{2}{\sqrt{\varphi}}g_2 + \frac{4}{\tau_d^2} + \frac{1}{\tau^2}} \tag{40}$$

for both SFOPITDS ans UFOPITDS.
The transfer functions of controllers are

$$Q(s) = \frac{q_2 s + q_1}{s + p_0}, \quad R(s) = \frac{r_2 s^2 + r_1 s + r_0}{s(s+p_0)} \tag{41}$$

for both ITDS and UFOTDS, and,

$$Q(s) = \frac{q_3 s^2 + q_2 s + q_1}{s^2 + p_1 s + p_0}, \quad R(s) = \frac{r_3 s^3 + r_2 s^2 + r_1 s + r_0}{s(s^2 + p_1 s + p_0)} \tag{42}$$

for the USOTDS, SFOPITDS and UFOPITDS.

6. Controller parameters

For the sake of limited space, formulas derived from (20) for all considered systems together with conditions of the controllers' stability are introduced in the form of tables. Parameters r_i and q_i in (41) and (42) can then be calculated from t_i according to (28).

$p_0 = g_2 + \frac{\tau_d}{4}(2g_1 + \tau_d g_0)$, $t_0 = \frac{1}{K}g_0$ $t_1 = \frac{1}{K}(g_1 + \tau_d g_0)$, $t_2 = \frac{\tau_d}{4K}(2g_1 + \tau_d g_0)$
$p_0 > 0$ for all τ_d

Table 1. Controller parameters for the ITDS

$p_0 = \dfrac{\tau\left[2g_2+\tau_d\left(g_1+\frac{\tau_d}{2}g_0\right)\right]+2}{2\tau-\tau_d}$ $t_0=\dfrac{\tau}{K}g_0,\ t_1=\dfrac{1}{K}\left[p_0+\tau\left(g_1+\tau_d g_0\right)\right],\ t_2=\dfrac{1}{K}\left[\tau\left(p_0-g_2\right)-1\right]$
$p_0>0$ for $\tau_d<2\tau$

Table 2. Controller parameters for the UFOTDS

$p_0=\dfrac{2g_3+\tau_1\left[2g_2+\tau_d\left(g_1+\frac{\tau_d}{2}g_0\right)\right]+\frac{2}{\tau_1}}{2\tau_1-\tau_d},\ p_1=g_3+\dfrac{1}{\tau_1}$ $t_0=\dfrac{\tau_1}{K}g_0,\ t_1=\dfrac{1}{K}\left[p_0+\tau_1\left(g_1+(\tau_2+\tau_d)g_0\right)\right]$ $t_2=\dfrac{1}{K}\left[\left[\left(\dfrac{4\tau_1\tau_2}{\tau_d}+\tau_1-\tau_2\right)p_0-\left(\dfrac{4\tau_2}{\tau_d}+1\right)\left(g_3+\tau_1 g_2+\dfrac{1}{\tau_1}\right)-\tau_1\tau_2 g_1\right]\right]$ $t_3=\dfrac{\tau_2}{K}\left[\tau_1(p_0-g_2)-g_3-\dfrac{1}{\tau_1}\right]$
$p_1>0$ for all τ_d, $p_0>0$ for $\tau_d<2\tau_1$

Table 3. Controller parameters for the USOTDS

$p_0=g_2+\dfrac{\tau_d}{4}(2g_1+\tau_d g_0),\ p_1=g_3+\dfrac{1}{\tau}$ $t_0=\dfrac{1}{K}g_0,\ t_1=\dfrac{1}{K}\left[g_1+(\tau+\tau_d)g_0\right],\ t_2=\dfrac{1}{4K}\left[(2\tau+\tau_d)(2g_1+\tau_d g_0)+2\tau\tau_d g_0\right]$ $t_3=\dfrac{\tau\tau_d}{4K}\left[2g_1+\tau_d g_0\right]$
p_1, $p_0>0$ for all τ_d

Table 4. Controller parameters for the SFOPITDS

$p_0=\dfrac{4g_3+(2\tau+\tau_d)\left(g_2+\frac{\tau_d}{2}g_1+\frac{\tau_d^2}{4}g_0\right)+\frac{4}{\tau}}{2\tau-\tau_d},\ p_1=g_3+\dfrac{2}{\tau}$ $t_0=\dfrac{1}{K}g_0,\ t_1=\dfrac{1}{K}\left[g_1+(\tau+\tau_d)g_0\right],$ $t_2=\dfrac{1}{K}\left[\left(\dfrac{4\tau}{\tau_d}-1\right)p_0-\dfrac{8}{\tau_d}g_3-\left(\dfrac{4\tau}{\tau_d}+1\right)g_2-\tau g_1-\dfrac{8}{\tau\tau_d}\right],\ t_3=\dfrac{1}{K}\left[\tau(p_0-g_2)-2g_3-\dfrac{2}{\tau}\right]$
$p_1>0$ for all τ_d, $p_0>0$ for $\tau_d<2\tau$

Table 5. Controller parameters for the UFOPITDS

7. Simulation results

The simulations were performed by MATLAB-Simulink tools. For all simulations, the unit step reference w was introduced at the time $t = 0$ and the step load disturbance v after settling of the step reference responses.

7.1 ITDS

In the transfer function (1), let $K = 1$. The responses in Fig. 2 for $\tau_d = 5$ show the effect of φ upon the control quality. An increasing value φ improves control stability, and, by choosing its value higher, aperiodic responses can be obtained. Simulation results shown in Fig. 3 demonstrate the influence of parameters γ on the control responses. Their smaller values accelerate step reference responses but they do not affect load disturbance responses. Higher values of γ can lead to overshoots and oscillations. The effect of parameters γ on the control

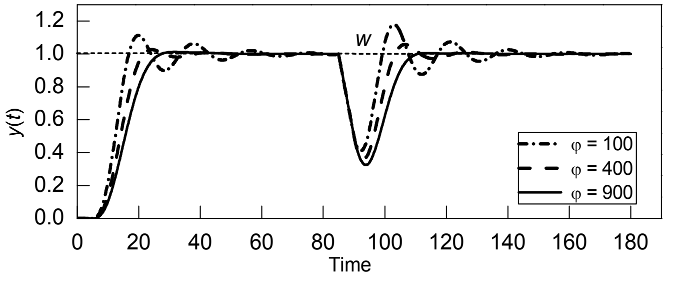

Fig. 2. ITDS: controlled output responses ($\tau_d = 5$, $v = -0.1$, $\gamma_1 = \gamma_2 = 0$)

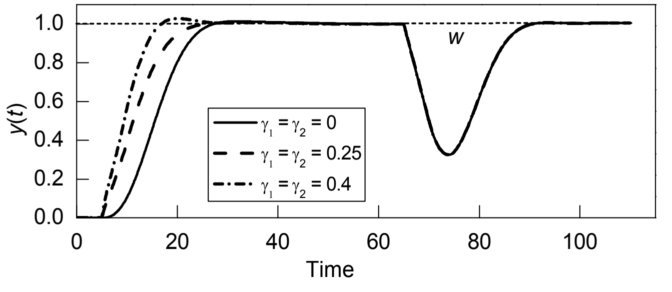

Fig. 3. ITDS: controlled output response ($\tau_d = 5$, $v = -0.1$, $\varphi = 900$).

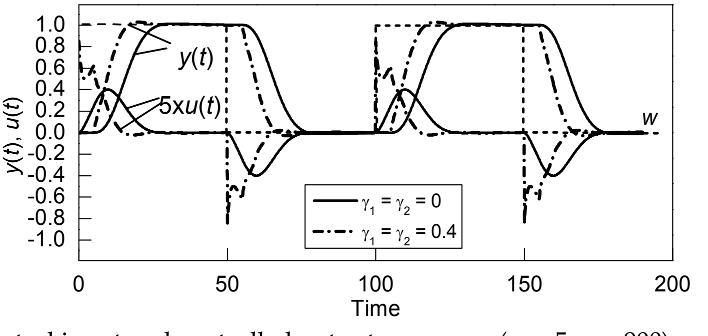

Fig. 4. ITDS: Control input and controlled output responses ($\tau_d = 5$, $\varphi = 900$)

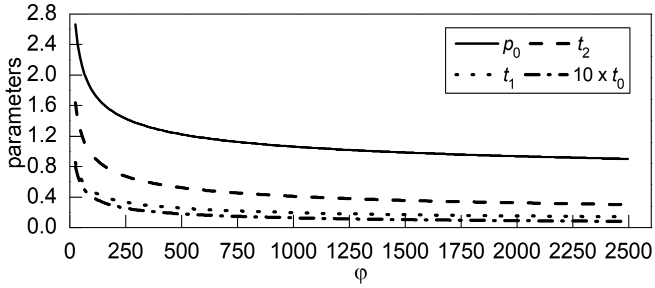

Fig. 5. ITDS: Controller parameters' dependence on φ (τ_d = 5)

input can be seen in Fig.4. Their higher values result in greater control inputs and their changes. This fact can be important in control of realistic processes. Dependence of the controller parameters on φ for τ_d = 5 is shown in Fig. 5.

7.2 UFOTDS

In this case, the parameters in (2) have been chosen as K = 4, τ = 4. The effect of φ on the control responses is similar to the ITDS, as shown in Fig. 6. The control responses for limiting values $\gamma_1 = \gamma_2 = 1$ and $\gamma_1 = \gamma_2 = 0$ (corresponding to the 1DOF and 2DOF structure) are in Fig. 7. The responses document unsuitability of the 1DOF structure application. The control response for τ_d = 4 is shown in Fig. 8. The presented response without any overshoots documents usefulness of the proposed method also for relatively high values of τ_d. The responses in Fig. 9 demonstrate robustness of the proposed method against changes of τ_d.

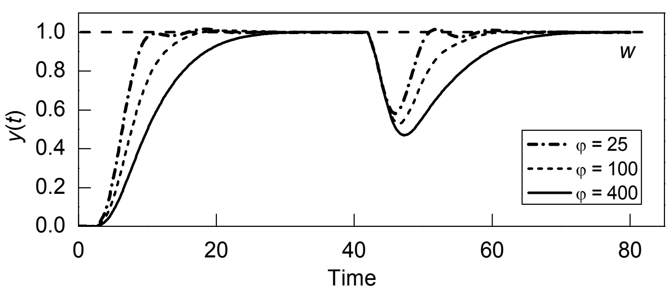

Fig. 6. UFOTDS: controlled output responses (τ_d = 2, v = - 0.1, $\gamma_1 = \gamma_2 = 0$)

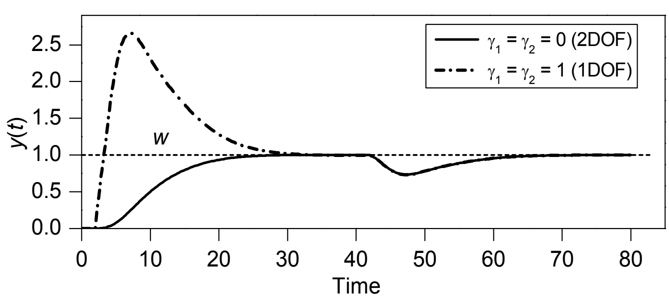

Fig. 7. UFOTDS: controlled output responses (τ_d = 2, v = - 0.05, φ = 400)

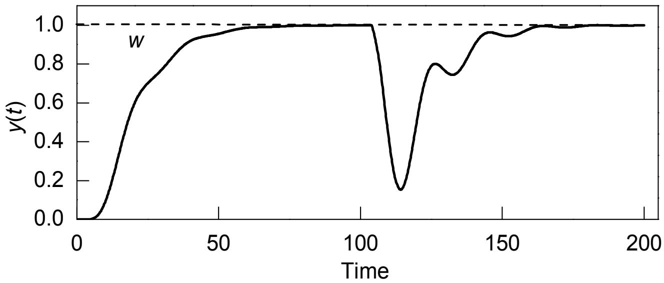

Fig. 8. UFOTDS: controlled output response (τ_d = 4, v = - 0.05, φ = 2500, $\gamma_1 = \gamma_2$ = 0)

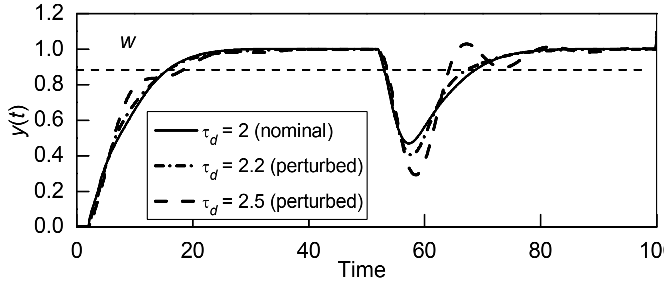

Fig. 9. Robustness against a change of τ_d (v = - 0.1, φ = 400, $\gamma_1 = \gamma_2$ = 0)

The controller parameters were computed for a nominal model with τ_d = 2 and subsequently used for perturbed models with the +10% and +25% estimation error in the τ_d.

7.3 USOTDS

In this case, the parameters in (3) were selected to be K = 1, τ_1 = 4, τ_2 = 2. Analogous to controlling the UFOTDS, the responses in Fig. 10 prove applicability of the proposed method also for an USOTDS with a relatively high ratio between the time delay and an unstable time constant (τ_d / τ_1 = 1). The responses in Fig. 11 demonstrate the possibility of extensive control acceleration, and, also high sensitivity of the control responses to the selection of parameters γ.

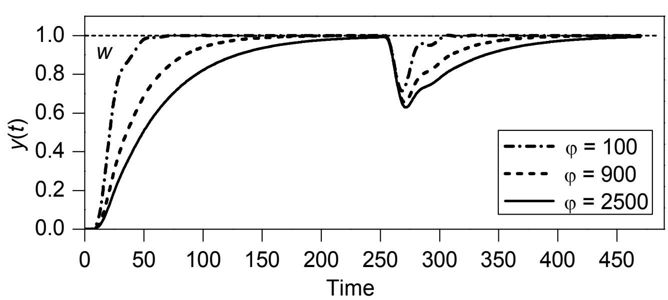

Fig. 10. USOTDS: controlled output responses (τ_d = 4, v = - 0.05, $\gamma_1 = \gamma_2 = \gamma_3$ = 0)

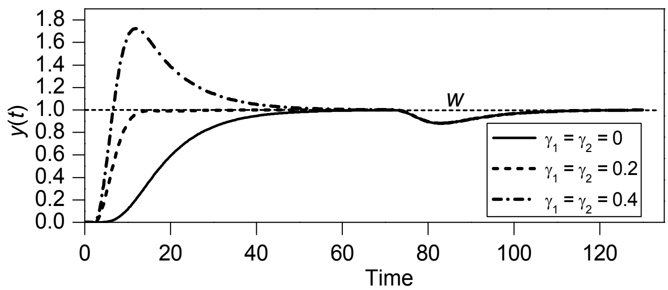

Fig. 11. USOTDS: controlled output responses (τ_d = 2, v = - 0.05, φ = 100, γ_3 = 0)

7.4 SFOPITDS

For this model, the parameters in (4) have been chosen as $K = 1$, $\tau = 4$, $\tau_d = 4$. A suitable selection of parameters φ and γ provides control responses of a good quality, as illustrated in Figs. 12 and 13.

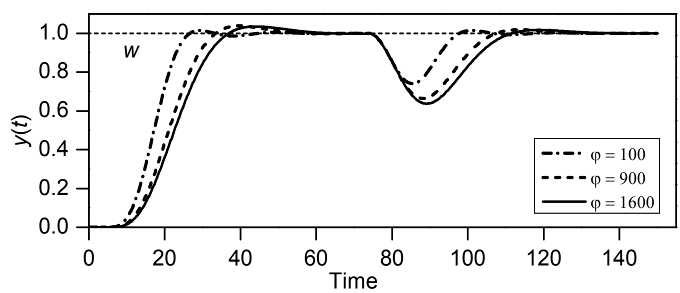

Fig. 12. SFOPITDS: controlled output responses (τ_d = 4, v = - 0.05, $\gamma_1 = \gamma_2 = \gamma_3 = 0$)

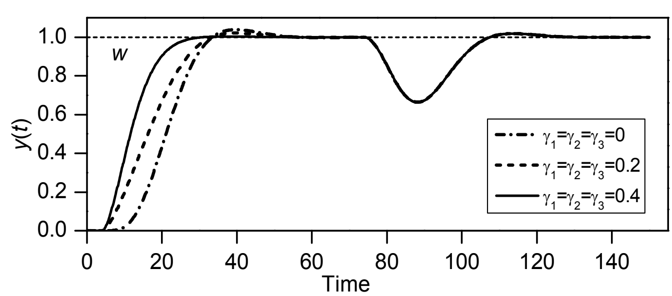

Fig. 13. SFOPITDS: controlled output responses (τ_d = 4, v = - 0.05, φ = 900).

7.5 UFOPITDS

Here, the model parameters in (4) have been chosen the same as for the SFOPITDS. With regard to the presence of both integrating and unstable parts, the UFOPITDS belongs to hardly controllable systems. However, the control responses in Fig. 14 document usefulness of the proposed method also for such systems. Obviously, for higher values τ_d also higher values of φ have to be chosen. Moreover, for this system, only the 2DOF structure with zero parameters γ should be used as follows from Fig. 15.

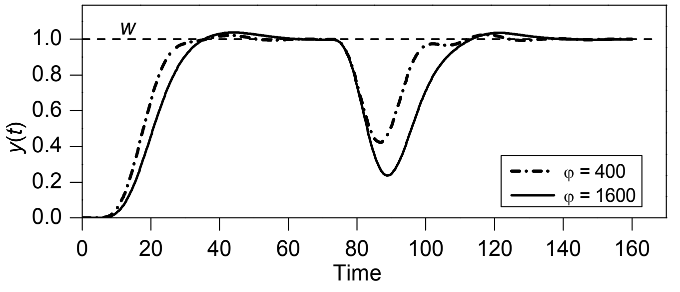

Fig. 14. UFOPITDS: controlled output responses (τ_d = 3, v = - 0.025, $\gamma_1 = \gamma_2 = \gamma_3 = 0$)

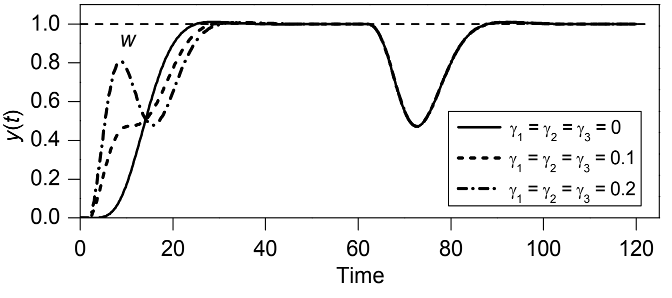

Fig. 15. UFOPITDS: controlled output responses (τ_d = 2, v = - 0.05, φ = 100)

8. Conclusions

The problem of control design for integrating and unstable time delay systems has been solved and analysed. The proposed method is based on the Padé time delay approximation. The controller design uses the polynomial synthesis and results of the LQ control theory. The presented procedure provides satisfactory control responses for the tracking of a step reference as well as for the step load disturbance attenuation. The procedure enables tuning of the controller parameters by two types of selectable parameters. Using derived formulas, the controller parameters can be automatically computed. As a consequence, the method could also be used for adaptive control.

9. Acknowledgment

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7

Decentralized Adaptive Stabilization for Large-Scale Systems with Unknown Time-Delay

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1. Introduction

In dealing with a large-scale system, one usually does not have adequate knowledge of the plant parameters and interactions among subsystems. The decentralized adaptive technique, designed independently for local subsystems and using locally available signals for feedback propose, is an appropriate strategy to be employed. In the context of decentralized adaptive control, a number of results have been obtained, see for examples Ioannou (1986); Narendra & Oleng (2002); Ortega (1996); Wen (1994). Since backstepping technique was proposed, it has been widely used to design adaptive controllers for uncertain systems Krstic et al. (1995). This technique has a number of advantages over the conventional approaches such as providing a promising way to improve the transient performance of adaptive systems by tuning design parameters. Because of such advantages, research on decentralized adaptive control using backstepping technique has also received great attention. In Wen & Soh (1997), decentralized adaptive tracking for linear systems was considered. In Jiang (2000), decentralized adaptive tracking of nonlinear systems was addressed, where the interaction functions satisfy global Lipschitz condition and the proposed controllers are partially decentralized. In Wen & Zhou (2007); Zhou & Wen (2008a;b), systems with higher order nonlinear interactions were considered by using backstepping technique.
Stabilization and control problem for time-delay systems have received much attention, see for examples, Jankovic (2001); Luo et al. (1997); Wu (1999), etc. The Lyapunov-Krasovskii method and Lyapunov-Razumikhin method are always employed. The results are often obtained via linear matrix inequalities. Some fruitful results have been achieved in the past when dealing with stabilizing problem for time-delay systems using backstepping technique. In Ge et al. (2003), neural network control cooperating with iterative backstepping was constructed for a class of nonlinear system with unknown but constant time delays. Jiao & Shen (2005) and Wu (2002) considered the control problem of the class of time-invariant large-scale interconnected systems subject to constant delays. In Chou & Cheng (2003), a decentralized model reference adaptive variable structure controller was proposed for a large-scale time-delay system, where the time-delay function is known and linear. In Hua et al. (2005), the robust output feedback control problem was considered for a class of nonlinear time-varying delay systems, where the nonlinear time-delay functions are bounded by known functions. In Shyu et al. (2005), a decentralized state-feedback variable structure controller was proposed for large-scale systems with time delay and dead-zone nonlinearity. However, in Shyu et al. (2005), the time delay is constant and the parameters of the dead-zone are

known. Due to state feedback, no filter is required for state estimation. Furthermore, only the stabilization problem was considered. A decentralized feedback control approach for a class of large scale stochastic systems with time delay was proposed in Wu et al. (2006). In Hua et al. (2007) a result of backstepping adaptive tracking in the presence of time delay was established. In Zhou (2008), we develop a totally decentralized controller for large scale time-delays systems with dead-zone input. In Zhou et al. (2009), adaptive backstepping control is developed for uncertain systems with unknown input time-delay.

In fact, the existence of time-delay phenomenon usually deteriorates the system performance. The stabilization and control problem for time-delay systems is a topic of great importance and has received increasing attention. Due to the difficulties on considering the effects of interconnections and time delays, extension of single-loop results to multi-loop interconnected systems is still a challenging task, especially for decentralized tracking. In this chapter, the decentralized adaptive stabilization is addressed for a class of interconnected systems with subsystems having arbitrary relative degrees, with unknown time-varying delays, and with unknown parameter uncertainties. The nonlinear time-delay functions are unknown and are allowed to satisfy a nonlinear bound. Also, the interactions between subsystems satisfy a nonlinear bound by nonlinear models. As system output feedback is employed, a state observer is required. Practical control is carried out in the backstepping design to compensate the effects of unknown interactions and unknown time-delays. In our design, the term multiplying the control effort and the system parameters are not assumed to be within known intervals. Besides showing stability of the system, the transient performance, in terms of L_2 norm of the system output, is shown to be an explicit function of design parameters and thus our scheme allows designers to obtain closed-loop behavior by tuning design parameters in an explicit way.

The main contributions of the chapter include: (i) the development of adaptive compensation to accommodate the effects of time-delays and interactions; (ii) the use of new Lyapunov-Krasovskii function in eliminating the unknown time-varying delays.

2. Problem formulation

Considered a system consisting of N interconnected subsystems modelled as follows:

$$\dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{\Phi}_i(y_i)\boldsymbol{\theta}_i + \begin{bmatrix} \mathbf{0} \\ \mathbf{b}_i \end{bmatrix} u_i + \sum_{j=1}^{N} \mathbf{h}_{ij}(y_j(t-\tau_j(t))) + \sum_{j=1}^{N} \mathbf{f}_{ij}(t, y_j), \tag{1}$$

$$y_i = \mathbf{c}_i^T x_i, \ for\ i = 1, \ldots, N, \tag{2}$$

$$\mathbf{A}_i = \begin{bmatrix} 0 & & \\ \vdots & \mathbf{I}_{(n_i-1)\times(n_i-1)} & \\ 0 & & \\ 0 & \ldots & 0 \end{bmatrix}, \mathbf{b}_i = \begin{bmatrix} b_{i,m_i} \\ \vdots \\ b_{i,0} \end{bmatrix}, \ \mathbf{\Phi}_i(y_i) = \begin{bmatrix} \mathbf{\Phi}_{i,1}(y_i) \\ \vdots \\ \mathbf{\Phi}_{i,n_i}(y_i) \end{bmatrix},$$

$$\mathbf{c}_i = [1, 0, \ldots, 0]^T. \tag{3}$$

where $x_i \in \Re^{n_i}$, $u_i \in \Re^1$ and $y_i \in \Re^1$ are the states, input and output of the ith subsystem, respectively, $\boldsymbol{\theta}_i \in \Re^{r_i}$ and $\mathbf{b}_i \in \Re^{m_i+1}$ are unknown constant vectors, $\mathbf{\Phi}_i(y_i) \in \Re^{n_i \times r_i}$ is a known smooth function, $\mathbf{f}_{ij}(t, y_j) = [f_{ij}^1(t, y_j), ..., f_{ij}^{n_i}(t, y_j)]^T \in \Re^{n_i}$ denotes the nonlinear interactions from the jth subsystem to the ith subsystem for $j \neq i$, or a nonlinear un-modelled part of the ith subsystem for $j = i$, $\mathbf{h}_{ij} = [h_{ij}^1, ..., h_{ij}^{n_i}]^T \in R^{n_i}$ is an unknown function, the unknown scalar function $\tau_j(t)$ denotes any nonnegative, continuous and bounded time-varying delay satisfying

$$\dot{\tau}_j(t) \leq \bar{\tau}_j < 1, \tag{4}$$

where $\bar{\tau}_j$ are known constants. For each decoupled local system, we make the following assumptions.

Assumption 1: The triple $(A_i,\ b_i,\ c_i)$ are completely controllable and observable.

Assumption 2: For every $1 \leq i \leq N$, the polynomial $b_{i,m_i}s^{m_i} + \cdots + b_{i,1}s + b_{i,0}$ is Hurwitz. The sign of b_{i,m_i} and the relative degree $\rho_i(= n_i - m_i)$ are known.

Assumption 3: The nonlinear interaction terms satisfy

$$|f_{ij}(t, y_j)| \leq \bar{\gamma}_{ij}\bar{f}_j(t, y_j)y_j, \tag{5}$$

where $\bar{\gamma}ij$ are constants denoting the strength of interactions, and $\bar{f}_j(y_j), j = 1, 2, \ldots, N$ are known positive functions and differentiable at least ρ_i times.

Assumption 4: The unknown functions $h_{ij}(y_j(t))$ satisfy the following properties

$$|h_{ij}(y_j(t))| \leq \bar{\iota}_{ij}\bar{h}_j(y_j(t))y_j, \tag{6}$$

where $\bar{h}_j$ are known positive functions and differentiable at least ρ_i times, and $\bar{\iota}_{ij}^k$ are positive constants.

Remark 1. *The effects of the nonlinear interactions f_{ij} and time-delay functions h_{ij} from other subsystems to a local subsystem are bounded by functions of the output of this subsystem. With these conditions, it is possible for the designed local controller to stabilize the interconnected systems with arbitrary strong subsystem interactions and time-delays.*

The control objective is to design a decentralized adaptive stabilizer for a large scale system (1) with unknown time-varying delay satisfying Assumptions 1-4 such that the closed-loop system is stable.

3. Design of adaptive controllers

3.1 Local state estimation filters

In this section, decentralized filters using only local input and output will be designed to estimate the unmeasured states of each local system. For the ith subsystem, we design the filters as

$$\dot{\mathbf{v}}_{i,\iota} = \mathbf{A}_{i,0}\mathbf{v}_{i,\iota} + \mathbf{e}_{n_i,(n_i-\iota)}u_i, \quad \iota = 0, \ldots, m_i \tag{7}$$

$$\dot{\boldsymbol{\xi}}_{i,0} = \mathbf{A}_{i,0}\boldsymbol{\xi}_{i,0} + \mathbf{k}_i y_i, \tag{8}$$

$$\dot{\Xi}_i = \mathbf{A}_{i,0}\Xi_i + \mathbf{\Phi}_i(y_i), \tag{9}$$

where $\mathbf{v}_{i,\iota} \in \Re^{n_i}, \boldsymbol{\xi}_{i,0} \in \Re^{n_i}, \Xi_i \in \Re^{n_i \times r_i}$, the vector $\mathbf{k}_i = [k_{i,1}, \ldots, k_{i,n_i}]^T \in \Re^{n_i}$ is chosen such that the matrix $\mathbf{A}_{i,0} = \mathbf{A}_i - \mathbf{k}_i(\mathbf{e}_{n_i,1})^T$ is Hurwitz, and $\mathbf{e}_{i,k}$ denotes the kth coordinate vector in $\Re^i$. There exists a $\mathbf{P}_i$ such that $\mathbf{P}_i\mathbf{A}_{i,0} + (\mathbf{A}_{i,0})^T\mathbf{P}_i = -3\mathbf{I},\ \mathbf{P}_i = \mathbf{P}_i^T > 0$. With these designed filters, our state estimate is

$$\hat{\mathbf{x}}_i(t) = \boldsymbol{\xi}_{i,0} + \Xi_i\boldsymbol{\theta}_i + \sum_{k=0}^{m_i} b_{i,k}\mathbf{v}_{i,k}, \tag{10}$$

and the state estimation error $\boldsymbol{\epsilon}_i = \mathbf{x}_i - \hat{\mathbf{x}}_i$ satisfies

$$\dot{\boldsymbol{\epsilon}}_i = \mathbf{A}_{i,0}\boldsymbol{\epsilon}_i + \sum_{j=1}^{N} \mathbf{f}_{ij}(t, y_j) + \sum_{j=1}^{N} \mathbf{h}_{ij}(y_j(t - \tau_j(t))). \tag{11}$$

Let $V_{\epsilon_i} = \boldsymbol{\epsilon}_i^T \mathbf{P}_i \boldsymbol{\epsilon}_i$. It can be shown that

$$\dot{V}_{\epsilon_i} \leq -\boldsymbol{\epsilon}_i^T \boldsymbol{\epsilon}_i + 2N \parallel \mathbf{P}_i \parallel^2 \sum_{j=1}^{N} \parallel \mathbf{f}_{ij}(t, y_j) \parallel^2 + 2N \parallel \mathbf{P}_i \parallel^2 \sum_{j=1}^{N} \parallel \mathbf{h}_{ij}(y_j(t - \tau_j(t))) \parallel^2 . \tag{12}$$

Now system (1) is expressed as

$$\begin{aligned}
\dot{y}_i &= b_{i,m_i} v_{i,(m_i,2)} + \xi_{i,(0,2)} + \bar{\boldsymbol{\delta}}_i^T \boldsymbol{\Theta}_i + \epsilon_{i,2} + \sum_{j=1}^{N} f_{ij,1}(t, y_j) \\
&\quad + \sum_{j=1}^{N} h_{ij,1}\left(y_j(t - \tau_j(t))\right), &(13)\\
\dot{v}_{i,(m_i,q)} &= v_{i,(m_i,q+1)} - k_{i,q} v_{i,(m_i,1)}, \quad q = 2, \ldots, \rho_i - 1 &(14)\\
\dot{v}_{i,(m_i,\rho_i)} &= v_{i,(m_i,\rho_i+1)} - k_{i,\rho_i} v_{i,(m_i,1)} + u_i, &(15)
\end{aligned}$$

where

$$\bar{\delta}_i = [0, v_{i,(m_i-1,2)}, \ldots, v_{i,(0,2)}, \Xi_{i,2} + \boldsymbol{\Phi}_{i,1}]^T, \ \boldsymbol{\Theta}_i = [b_{i,m_i}, \ldots, b_{i,0}, \boldsymbol{\theta}_i^T]^T, \tag{16}$$

and $v_{i,(m_i,2)}, \epsilon_{i,2}, \xi_{i,(0,2)}, \Xi_{i,2}$ denote the second entries of $\mathbf{v}_{i,m_i}, \epsilon_i, \boldsymbol{\xi}_{i,0}, \Xi_i$ respectively, $f_{ij,1}(t, y_j)$ and $h_{ij,1}(y_j(t - \tau_j(t)))$ are respectively the first elements of vectors $\mathbf{f}_{ij}(t, y_j)$ and $\mathbf{h}_{ij}(y_j(t - \tau_j(t)))$.

Remark 2. *It is worthy to point out that the inputs to the designed filters (7)-(9) are only the local input u_i and output y_i and thus totally decentralized.*

Remark 3. *Even though the estimated state is given in (10), it is still unknown and thus not employed in our controller design. Instead, the outputs $\mathbf{v}_{i,\iota}, \boldsymbol{\xi}_{i,0}$ and Ξ_i from filters (7)-(9) are used to design controllers, while the state estimation error (11) will be considered in system analysis.*

3.2 Adaptive decentralized controller design

In this section, we develop an adaptive backstepping design scheme for decentralized output tracking. There is no a priori information required from system parameter $\boldsymbol{\Theta}_i$ and thus they can be allowed totally uncertain. As usual in backstepping approach in Krstic et al. (1995), the following change of coordinates is made.

$$\begin{aligned}
z_{i,1} &= y_i, &(17)\\
z_{i,q} &= v_{i,(m_i,q)} - \alpha_{i,q-1}, \ q = 2, 3, \ldots, \rho_i, &(18)
\end{aligned}$$

where $\alpha_{i,q-1}$ is the virtual control at the *q-th step* of the ith loop and will be determined in later discussion, $\hat{p}_i$ is the estimate of $p_i = 1/b_{i,m_i}$.

To illustrate the controller design procedures, we now give a brief description on the first step.

• *Step* 1: Starting with the equations for the tracking error $z_{i,1}$ obtained from (13), (17) and (18), we get

$$\begin{aligned}\dot{z}_{i,1} &= b_{i,m_i} v_{i,(m_i,2)} + \xi_{i,(0,2)} + \bar{\boldsymbol{\delta}}_i^T \boldsymbol{\Theta}_i + \epsilon_{i,2} + \sum_{j=1}^{N} f_{ij,1}(t, y_j) \\ &\quad + \sum_{j=1}^{N} h_{ij,1}(t, y_j(t-\tau_j(t))) \\ &= b_{i,m_i}\alpha_{i,1} + b_{i,m_i} z_{i,2} + \xi_{i,(0,2)} + \bar{\boldsymbol{\delta}}_i^T \boldsymbol{\Theta}_i + \epsilon_{i,2} + \sum_{j=1}^{N} f_{ij,1}(t, y_j) \\ &\quad + \sum_{j=1}^{N} h_{ij,1}(t, y_j(t-\tau_j(t))). \end{aligned} \tag{19}$$

The virtual control law $\alpha_{i,1}$ is designed as

$$\alpha_{i,1} = \hat{p}_i \bar{\alpha}_{i,1}\,, \tag{20}$$

$$\bar{\alpha}_{i,1} = -(c_{i,1} + l_{i,1}) z_{i,1} - l_i^* z_{i,1} \big(\bar{f}_i(y_i)\big)^2 - \lambda_i^* z_{i,1} \big(\bar{h}_i(y_i)\big)^2 - \xi_{i,(0,2)} - \bar{\boldsymbol{\delta}}_i^T \hat{\boldsymbol{\Theta}}_i, \tag{21}$$

where $c_{i,1}$, $l_{i,1}$, $l_i^*$and λ_i^* are positive design parameters, $\hat{\boldsymbol{\Theta}}_i$ and $\hat{p}_i$ are the estimates of $\boldsymbol{\Theta}_i$ and p_i, respectively. Using $\tilde{p}_i = p_i - \hat{p}_i$, we obtain

$$b_{i,m_i}\alpha_{i,1} = b_{i,m_i}\hat{p}_i\bar{\alpha}_{i,1} = \bar{\alpha}_{i,1} - b_{i,m_i}\tilde{p}_i\bar{\alpha}_{i,1}, \tag{22}$$

$$\begin{aligned}\bar{\boldsymbol{\delta}}_i^T \tilde{\boldsymbol{\Theta}}_i + b_{i,m_i} z_{i,2} &= \bar{\boldsymbol{\delta}}_i^T \tilde{\boldsymbol{\Theta}}_i + \tilde{b}_{i,m_i} z_{i,2} + \hat{b}_{i,m_i} z_{i,2} \\ &= \bar{\boldsymbol{\delta}}_i^T \tilde{\boldsymbol{\Theta}}_i + (v_{i,(m_i,2)} - \alpha_{i,1})(\mathbf{e}_{(r_i+m_i+1),1})^T \tilde{\boldsymbol{\Theta}}_i + \hat{b}_{i,m_i} z_{i,2} \\ &= (\boldsymbol{\delta}_i - \hat{p}_i\bar{\alpha}_{i,1}\mathbf{e}_{(r_i+m_i+1),1})^T \tilde{\boldsymbol{\Theta}}_i + \hat{b}_{i,m_i} z_{i,2}, \end{aligned} \tag{23}$$

where

$$\boldsymbol{\delta}_i = [v_{i,(m_i,2)}, v_{i,(m_i-1,2)}, \ldots, v_{i,(0,2)}, \boldsymbol{\xi}_{i,2} + \boldsymbol{\Phi}_{i,1}]^T. \tag{24}$$

From (20)-(23), (19) can be written as

$$\begin{aligned}\dot{z}_{i,1} &= -c_{i,1} z_{i,1} - l_{i,1} z_{i,1} - l_i^* z_{i,1}\big(\bar{f}_i(y_i)\big)^2 - \lambda_i^* z_{i,1}\big(\bar{h}_i(y_i)\big)^2 \\ &\quad + \epsilon_{i,2} + (\boldsymbol{\delta}_i - \hat{p}_i\bar{\alpha}_{i,1}\mathbf{e}_{r_i+m_i+1,1})^T \tilde{\boldsymbol{\Theta}}_i - b_{i,m_i}\bar{\alpha}_{i,1}\tilde{p}_i + \hat{b}_{i,m_i} z_{i,2} \\ &\quad + \sum_{j=1}^{N} f_{ij,1}(t, y_j) + \sum_{j=1}^{N} h_{ij,1}(t, y_j(t-\tau_j(t))), \end{aligned} \tag{25}$$

where $\tilde{\boldsymbol{\Theta}}_i = \boldsymbol{\Theta}_i - \hat{\boldsymbol{\Theta}}_i$, and $\mathbf{e}_{(r_i+m_i+1),1} \in \Re^{r_i+m_i+1}$. We now consider the Lyapunov function

$$V_i^1 = \frac{1}{2}(z_{i,1})^2 + \frac{1}{2}\tilde{\boldsymbol{\Theta}}_i^T \boldsymbol{\Gamma}_i^{-1} \tilde{\boldsymbol{\Theta}}_i + \frac{|b_{i,m_i}|}{2\gamma_i'}(\tilde{p}_i)^2 + \frac{1}{2\bar{l}_{i,1}} V_{\epsilon_i}, \tag{26}$$

where Γ_i is a positive definite design matrix and $\gamma_i^{'}$ is a positive design parameter. Examining the derivative of V_i^1 gives

$$\begin{aligned}
\dot{V}_i^1 &= z_{i,1}\dot{z}_{i,1} - \tilde{\boldsymbol{\Theta}}_i^T \boldsymbol{\Gamma}_i^{-1}\dot{\hat{\boldsymbol{\Theta}}}_i - \frac{|b_{i,m_i}|}{\gamma_i^{'}}\tilde{p}_i\dot{\hat{p}}_i + \frac{1}{2\bar{l}_{i,1}}\dot{V}_{\epsilon_i} \\
&\leq -c_{i,1}(z_{i,1})^2 - l_{i,1}(z_{i,1})^2 - l_i^*(z_{i,1})^2\big(\bar{f}_i(z_{i,1})\big)^2 - \lambda_i^*(z_{i,1})^2\big(\bar{h}_i(y_i)\big)^2 \\
&\quad -\frac{1}{2\bar{l}_{i,1}}\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i + \hat{b}_{i,m_i}z_{i,1}z_{i,2} - |b_{i,m_i}|\tilde{p}_i\frac{1}{\gamma_i^{'}}[\gamma_i^{'}sgn(b_{i,m_i})\bar{\alpha}_{i,1}z_{i,1} + \dot{\hat{p}}_i] \\
&\quad +\tilde{\boldsymbol{\Theta}}_i^T\boldsymbol{\Gamma}_i^{-1}[\boldsymbol{\Gamma}_i(\boldsymbol{\delta}_i - \hat{p}_i\bar{\alpha}_{i,1}\mathbf{e}_{(r_i+m_i+1),1})z_{i,1} - \dot{\hat{\boldsymbol{\Theta}}}_i] \\
&\quad +(\sum_{j=1}^{N} f_{ij,1}(t,y_j) + \sum_{j=1}^{N} h_{ij,1}(t,y_j(t-\tau_j(t))) + \epsilon_{i,2})z_{i,1} \\
&\quad +\frac{1}{\bar{l}_{i,1}}N \parallel \mathbf{P}_i \parallel^2 \big(\parallel \sum_{j=1}^{N} h_{ij}(t,y_j(t-\tau_j(t))) \parallel^2 + \sum_{j=1}^{N} \parallel \mathbf{f}_{ij}(t,y_j) \parallel^2 \big). \qquad (27)
\end{aligned}$$

Then we choose

$$\dot{\hat{p}}_i = -\gamma_i^{'}sgn(b_{i,m_i})\bar{\alpha}_{i,1}z_{i,1}, \qquad (28)$$

$$\tau_{i,1} = \left(\boldsymbol{\delta}_i - \hat{p}_i\bar{\alpha}_{i,1}\mathbf{e}_{(r_i+m_i+1),1}\right)z_{i,1}. \qquad (29)$$

Let $l_{i,1} = 3\bar{l}_{i,1}$ and using Young's inequality we have

$$-\bar{l}_{i,1}(z_{i,1})^2 + \sum_{j=1}^{N} f_{ij,1}(t,y_j)z_{i,1} \leq \frac{N}{4\bar{l}_{i,1}}\sum_{j=1}^{N} \parallel f_{ij,1}(t,y_j) \parallel^2, \qquad (30)$$

$$-\bar{l}_{i,1}(z_{i,1})^2 + \sum_{j=1}^{N} h_{ij,1}(t,y_j(t-\tau_j(t)))z_{i,1} \leq \frac{N}{4\bar{l}_{i,1}} \parallel \sum_{j=1}^{N} h_{ij,1}(t,y_j(t-\tau_j(t))) \parallel^2, \qquad (31)$$

$$\begin{aligned}
-\bar{l}_{i,1}(z_{i,1})^2 + \epsilon_{i,2}z_{i,1} - \frac{1}{4\bar{l}_{i,1}}\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i &\leq -\bar{l}_{i,1}(z_{i,1})^2 + \epsilon_{i,2}z_{i,1} - \frac{1}{4\bar{l}_{i,1}}(\epsilon_{i,2})^2 \\
&= -\bar{l}_{i,1}(z_{i,1} - \frac{1}{2\bar{l}_{i,1}}\epsilon_{i,2})^2 \leq 0. \qquad (32)
\end{aligned}$$

Substituting (28)-(32) into (27) gives

$$\begin{aligned}
\dot{V}_i^1 \leq& -c_{i,1}(z_{i,1})^2 - \frac{1}{4\bar{l}_{i,1}}\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i - l_i^*(z_{i,1})^2\big(\bar{f}_i(y_i)\big)^2 - \lambda_i^*(z_{i,1})^2\big(\bar{h}_i(y_i)\big)^2 + \hat{b}_{i,m_i}z_{i,1}z_{i,2} \\
&+\tilde{\boldsymbol{\Theta}}_i^T(\tau_{i,1} - \boldsymbol{\Gamma}_i^{-1}\dot{\hat{\boldsymbol{\Theta}}}_i) + \frac{N}{\bar{l}_{i,1}} \parallel \mathbf{P}_i \parallel^2 \sum_{j=1}^{N} \parallel \mathbf{f}_{ij}(t,y_j) \parallel^2 + \frac{N}{4\bar{l}_{i,1}}\sum_{j=1}^{N} \parallel f_{ij,1}(t,y_j) \parallel^2 \\
&+\frac{N}{\bar{l}_{i,1}} \parallel \mathbf{P}_i \parallel^2 \sum_{j=1}^{N} \parallel \mathbf{h}_{ij}(t,y_j(t-\tau_j(t))) \parallel^2 + \frac{N}{4\bar{l}_{i,1}} \parallel \sum_{j=1}^{N} h_{ij,1}(t,y_j(t-\tau_j(t))) \parallel^2 . \qquad (33)
\end{aligned}$$

• *Step q* ($q = 2, \dots, \rho_i$, $i = 1, \dots, N$): Choose virtual control laws

$$\alpha_{i,2} = -\hat{b}_{i,m_i} z_{i,1} - \left(c_{i,2} + l_{i,2} \left(\frac{\partial \alpha_{i,1}}{\partial y_i} \right)^2 \right) z_{i,2} + \bar{B}_{i,2} + \frac{\partial \alpha_{i,1}}{\partial \hat{\boldsymbol{\Theta}}_i} \boldsymbol{\Gamma}_i \tau_{i,2}, \tag{34}$$

$$\alpha_{i,q} = -z_{i,q-1} - \left(c_{i,q} + l_{i,q} \left(\frac{\partial \alpha_{i,q-1}}{\partial y_i} \right)^2 \right)] z_{i,q} + \bar{B}_{i,q} + \frac{\partial \alpha_{i,q-1}}{\partial \hat{\boldsymbol{\Theta}}_i} \boldsymbol{\Gamma}_i \tau_{i,q}$$
$$- \left(\sum_{k=2}^{q-1} z_{i,k} \frac{\partial \alpha_{i,k-1}}{\partial \hat{\boldsymbol{\Theta}}_i} \right) \boldsymbol{\Gamma}_i \frac{\partial \alpha_{i,q-1}}{\partial y_i} \boldsymbol{\delta}_i, \tag{35}$$

$$\tau_{i,q} = \tau_{i,q-1} - \frac{\partial \alpha_{i,q-1}}{\partial y_i} \boldsymbol{\delta}_i z_{i,q}, \tag{36}$$

where $c_i^q, l_{i,q}, q = 3, \dots, \rho_i$ are positive design parameters, and $\bar{B}_{i,q}, q = 2, \dots, \rho_i$ denotes some known terms and its detailed structure can be found in Krstic et al. (1995).
Then the local control and parameter update laws are finally given by

$$u_i = \alpha_{i,\rho_i} - v_{i,(m_i,\rho_i+1)}, \tag{37}$$

$$\dot{\hat{\boldsymbol{\Theta}}}_i = \boldsymbol{\Gamma}_i \tau_{i,\rho_i}. \tag{38}$$

Remark 4. *The crucial terms $l_i^* z_{i,1} \left(\bar{f}_i(y_i) \right)^2$ in (21) and $\lambda_i^* z_{i,1} \left(\bar{h}_i(y_i) \right)^2$ are proposed in the controller design to compensate for the effects of interactions from other subsystems or the un-modelled part of its own subsystem, and for the effects of time-delay functions, respectively. The detailed analysis will be given in Section 4.*

Remark 5. *When going through the details of the design procedures, we note that in the equations concerning $\dot{z}_{i,q}, q = 1, 2, \dots, \rho_i$, just functions $\sum_{j=1}^{N} f_{ij,1}(t, y_j)$ from the interactions and $\sum_{j=1}^{N} h_{ij,1}(t, y_j(t - \tau_j(t)))$ appear, and they are always together with $\epsilon_{i,2}$. This is because only $\dot{y}_i$ from the plant model (1) was used in the calculation of $\dot{\alpha}_{i,q}$ for steps $q = 2, \dots, \rho_i$.*

4. Stability analysis

In this section, the stability of the overall closed-loop system consisting of the interconnected plants and decentralized controllers will be established.
Now we define a Lyapunov function of decentralized adaptive control system as

$$V_i = \sum_{q=1}^{\rho_i} \left(\frac{1}{2} (z_{i,q})^2 + \frac{1}{2\bar{l}_{i,q}} \boldsymbol{\epsilon}_i^T \mathbf{P}_i \boldsymbol{\epsilon}_i \right) + \frac{1}{2} \tilde{\boldsymbol{\Theta}}_i^T \boldsymbol{\Gamma}_i^{-1} \tilde{\boldsymbol{\Theta}}_i + \frac{|b_{i,m_i}|}{2\gamma_i'} \tilde{p}_i^2. \tag{39}$$

From (12), (20), (33), (35)-(38), and (49), the derivative of V_i in (39) satisfies

$$\dot{V}_i \le - \sum_{q=1}^{\rho_i} c_{i,q} (z_{i,q})^2 - l_i^* (z_{i,1})^2 (\bar{f}_i(y_i))^2 - \lambda_i^* (z_{i,1})^2 \left(\bar{h}_i(y_i) \right)^2$$
$$+ \sum_{q=1}^{\rho_i} \frac{1}{\bar{l}_{i,q}} N \parallel \mathbf{P}_i \parallel^2 \left(\sum_{j=1}^{N} \parallel \mathbf{h}_{ij}(t, y_j(t - \tau_j)) \parallel^2 + \sum_{j=1}^{N} \parallel \mathbf{f}_{ij}(t, y_j) \parallel^2 \right)$$

$$+\frac{1}{4\bar{l}_{i,1}}\left(N\sum_{j=1}^{N}\parallel f_{ij,1}(t,y_j)\parallel^2+N\sum_{j=1}^{N}\parallel h_{ij,1}(t,y_j(t-\tau_j))\parallel^2\right)$$
$$-\frac{1}{4\bar{l}_{i,1}}\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i+\sum_{q=2}^{\rho_i}\left[-l_{i,q}\left(\frac{\partial\alpha_{i,q-1}}{\partial y_i}\right)^2(z_{i,q})^2-\frac{1}{2\bar{l}_{i,q}}\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i\right.$$
$$\left.+\frac{\partial\alpha_{i,q-1}}{\partial y_i}\left(\sum_{j=1}^{N}f_{ij,1}(t,y_j)+\sum_{j=1}^{N}h_{ij,1}\left(t,y_j(t-\tau_j)\right)+\epsilon_{i,2}\right)z_{i,q}\right]. \tag{40}$$

Using Young's inequality and let $l_{i,q}=3\bar{l}_{i,q}$, we have

$$-\bar{l}_{i,q}\left(\frac{\partial\alpha_{i,q-1}}{\partial y_i}\right)^2(z_{i,q})^2+\frac{\partial\alpha_{i,q-1}}{\partial y_i}\sum_{j=1}^{N}f_{ij,1}(t,y_j)z_{i,q}\leq\frac{N}{4\bar{l}_{i,q}}\sum_{j=1}^{N}\parallel f_{ij,1}(t,y_j)\parallel^2, \tag{41}$$

$$-\bar{l}_{i,q}\left(\frac{\partial\alpha_{i,q-1}}{\partial y_i}\right)^2(z_{i,q})^2+\frac{\partial\alpha_{i,q-1}}{\partial y_i}\epsilon_{i,2}z_{i,q}-\frac{1}{4\bar{l}_{i,q}}\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i\leq 0, \tag{42}$$

$$-\bar{l}_{i,q}\left(\frac{\partial\alpha_{i,q-1}}{\partial y_i}\right)^2(z_{i,q})^2+\frac{\partial\alpha_{i,q-1}}{\partial y_i}\sum_{j=1}^{N}h_{ij,1}(t,y_j(t-\tau_j))z_{i,q}$$
$$\leq\frac{N}{4\bar{l}_{i,q}}\sum_{j=1}^{N}\parallel h_{ij,1}(t,y_j(t-\tau_j))\parallel^2. \tag{43}$$

Then from (40),

$$\dot{V}_i\leq-\sum_{q=1}^{\rho_i}c_{i,q}(z_{i,q})^2-\sum_{q=1}^{\rho_i}\frac{1}{4\bar{l}_{i,q}}\boldsymbol{\epsilon}_i^T\boldsymbol{\epsilon}_i-l_i^*(z_{i,1})^2\big(\bar{f}_i(y_i)\big)^2-\lambda_i^*(z_{i,1})^2\big(\bar{h}_i(y_i(t)\big)^2$$
$$+\sum_{q=1}^{\rho_i}\frac{N}{4\bar{l}_{i,q}}\left(4\parallel\mathbf{P}_i\parallel^2\sum_{j=1}^{N}\parallel\mathbf{f}_{ij}(t,y_j)\parallel^2+\sum_{j=1}^{N}\parallel f_{ij,1}(t,y_j)\parallel^2\right)$$
$$+\sum_{q=1}^{\rho_i}\frac{N}{4\bar{l}_{i,q}}\left(4\parallel\mathbf{P}_i\parallel^2\sum_{j=1}^{N}\parallel\mathbf{h}_{ij}(t,y_j(t-\tau_j))\parallel^2+\sum_{j=1}^{N}\parallel h_{ij,1}(t,y_j(t-\tau_j))\parallel^2\right). \tag{44}$$

From Assumptions 3 and 4, we can show that

$$\sum_{q=1}^{\rho_i}\frac{N}{4\bar{l}_{i,q}}\left(4\parallel\mathbf{P}_i\parallel^2\sum_{j=1}^{N}\parallel\mathbf{f}_{ij}(t,y_j)\parallel^2+\sum_{j=1}^{N}\parallel f_{ij,1}(t,y_j)\parallel^2\right)\leq\sum_{j=1}^{N}\gamma_{ij}\big(\bar{f}_j(y_j)\big)^2(y_j)^2, \tag{45}$$

$$\sum_{q=1}^{\rho_i}\frac{N}{4\bar{l}_{i,q}}\left(4\parallel\mathbf{P}_i\parallel^2\sum_{j=1}^{N}\parallel\mathbf{h}_{ij}(t,y_j(t-\tau_j))\parallel^2+\sum_{j=1}^{N}\parallel h_{ij,1}(t,y_j(t-\tau_j))\parallel^2\right)$$
$$\leq\sum_{j=1}^{N}\iota_{ij}\big(\bar{h}_j(y_j)(t-\tau_j)\big)^2(y_j(t-\tau_j))^2, \tag{46}$$

where $\gamma_{ij} = O(\bar{\gamma}_{ij}^2)$ indicates the coupling strength from the jth subsystem to the ith subsystem depending on $\bar{l}_{i,q}, \| \mathbf{P}_i \|$ and $O(\bar{\gamma}_{ij}^2)$ denotes that γ_{ij} and $O(\bar{\gamma}_{ij}^2)$ are in the same order mathematically, and $\iota_{ij} = O(\bar{\iota}_{ij}^2)$.
Then the derivative of V_i is given as

$$\dot{V}_i \leq -\sum_{q=1}^{\rho_i} c_{i,q}(z_{i,q})^2 - \sum_{q=1}^{\rho_i} \frac{1}{4\bar{l}_{i,q}} \boldsymbol{\epsilon}_i^T \boldsymbol{\epsilon}_i - l_i^*(z_{i,1})^2\left(\bar{f}_i(y_i)\right)^2 - \lambda_i^*(z_{i,1})^2\left(\bar{h}_i(y_i(t)\right)^2 + \sum_{j=1}^{N} \gamma_{ij}\left(\bar{f}_j(y_j)y_j\right)^2 + \sum_{j=1}^{N} \iota_{ij}\left(\bar{h}_j(y_j)(t-\tau_j)y_j(t-\tau_j)\right)^2. \tag{47}$$

To tackle the unknown time-delay problem, we introduce the following Lyapunov-Krasovskii function

$$W_i = \sum_{j=1}^{N} \frac{\iota_{ij}}{1-\bar{\tau}_j} \int_{t-\tau_j(t)}^{t} \left(\bar{h}_j^1\left(y_j(s)\right)y_j(s)\right)^2 ds. \tag{48}$$

The time derivative of W_i is given by

$$\dot{W}_i \leq \sum_{j=1}^{N}\left(\frac{\iota_{ij}}{1-\bar{\tau}_j}\left[\bar{h}_j\left(y_j(t)\right)y_j(t)\right]^2 - \iota_{ij}\left[\bar{h}_j\left(y_j(t-\tau_j(t))\right)y_j(t-\tau_j(t))\right]^2\right). \tag{49}$$

Now define a new control Lyapunov function for each local subsystem

$$\begin{aligned} V_i^{\rho} &= V_i + W_i \\ &= \sum_{q=1}^{\rho_i}\left(\frac{1}{2}(z_{i,q})^2 + \frac{1}{2\bar{l}_{i,q}}\boldsymbol{\epsilon}_i^T \mathbf{P}_i \boldsymbol{\epsilon}_i\right) + \frac{1}{2}\tilde{\boldsymbol{\Theta}}_i^T \boldsymbol{\Gamma}_i^{-1} \tilde{\boldsymbol{\Theta}}_i + \frac{|b_{i,m_i}|}{2\gamma_i'}\tilde{p}_i^2 \\ &\quad + \sum_{j=1}^{N} \frac{\iota_{ij}}{1-\bar{\tau}_j} \int_{t-\tau_j(t)}^{t} \left(\bar{h}_j^1\left(y_j(s)\right)y_j(s)\right)^2. \end{aligned} \tag{50}$$

Therefore, the derivative of V_i^{ρ}

$$\dot{V}_i^{\rho} \leq -\sum_{q=1}^{\rho_i} c_{i,q}(z_{i,q})^2 - \sum_{q=1}^{\rho_i} \frac{1}{4\bar{l}_{i,q}} \boldsymbol{\epsilon}_i^T \boldsymbol{\epsilon}_i - l_i^*\left(\bar{f}_i(y_i)z_{i,1}\right)^2 - \lambda_i^*\left(\bar{h}_i(y_i(t)z_{i,1}\right)^2 + \sum_{j=1}^{N} \gamma_{ij}\left(\bar{f}_j(y_j)y_j\right)^2 + \sum_{j=1}^{N} \frac{\iota_{ij}}{1-\bar{\tau}_j}\left(\bar{h}_j(y_j)y_j\right)^2. \tag{51}$$

Clearly there exists a constant γ_{ij}^* such that for each γ_{ij} satisfying $\gamma_{ij} \leq \gamma_{ij}^*$, and

$$l_i^* \geq \sum_{j=1}^{N} \gamma_{ji} \;\; if \;\; l_i^* \geq \sum_{j=1}^{N} \gamma_{ji}^*. \tag{52}$$

Constant γ_{ij}^* stands for a upper bound of γ_{ij}.
Simialy, there exists a constant ι_{ij}^* such that for each ι_{ij} satisfying $\iota_{ij} \leq \iota_{ij}^*$, and

$$\lambda_i^* \geq \sum_{j=1}^{N} \iota_{ji}\frac{1}{1-\bar{\tau}_i} \;\; if \;\; \lambda_i^* \geq \sum_{j=1}^{N} \iota_{ji}^*\frac{1}{1-\bar{\tau}_i}. \tag{53}$$

Now we define a Lyapunov function of overall system

$$V = \sum_{i=1}^{N} V_i^{\rho}. \tag{54}$$

Now taking the summation of the last four terms in (51) and using (52) and (53), we get

$$\sum_{i=1}^{N} \left[-l_i^* \left(\bar{f}_i(y_i) z_{i,1}\right)^2 - \lambda_i^* \left(\bar{h}_i(y_i(t) z_{i,1}\right)^2 + \sum_{j=1}^{N} \gamma_{ij} \left(\bar{f}_j(y_j) y_j\right)^2 + \sum_{j=1}^{N} \frac{\iota_{ij}}{1-\bar{\tau}_j} \left(\bar{h}_j(y_j) y_j\right)^2 \right]$$
$$= \sum_{i=1}^{N} \left[-\left((l_i^* - \sum_{j=1}^{N} \gamma_{ji} \right) \left(\bar{f}_i(y_i) y_i\right)^2 - \left(\lambda_i^* - \sum_{j=1}^{N} \frac{\iota_{ji}}{1-\bar{\tau}_i} \right) \left(\bar{h}_i(y_i) y_i\right)^2 \right] \leq 0. \tag{55}$$

Therefore,

$$\dot{V} \leq -\sum_{i=1}^{N} \sum_{q=1}^{\rho_i} c_{i,q} (z_{i,q})^2 - \sum_{i=1}^{N} \sum_{q=1}^{\rho_i} \frac{1}{4\bar{l}_{i,q}} \boldsymbol{\epsilon}_i^T \boldsymbol{\epsilon}_i \leq 0. \tag{56}$$

This shows that V is uniformly bounded. Thus $z_{i,1}, \ldots, z_{i,\rho_i}, \hat{p}_i, \hat{\boldsymbol{\Theta}}_i, \boldsymbol{\epsilon}_i$ are bounded. Since $z_{i,1}$ is bounded, y_i is also bounded. Because of the boundedness of y_i, variables $\mathbf{v}_{i,j}$, $\boldsymbol{\xi}_{i,0}$ and Ξ_i are bounded as $\mathbf{A}_{i,0}$ is Hurwitz. Following similar analysis to Wen & Zhou (2007), we can show that all the states associated with the zero dynamics of the ith subsystem are bounded under Assumption 2. In conclusion, boundedness of all signals is ensured as formally stated in the following theorem.

Theorem 1. *Consider the closed-loop adaptive system consisting of the plant (1) under Assumptions 1-4, the controller (37), the estimator (28) and (38), and the filters (7)-(9). There exist a constant γ_{ij}^* such that for each constant γ_{ij} satisfying $\gamma_{ij} \leq \gamma_{ij}^*$ and ι_{ij} satisfying $\iota_{ij} \leq \iota_{ij}^*$ $i, j = 1, \ldots, N$, all the signals in the system are globally uniformly bounded.*

We now derive a bound for the vector $z_i(t)$ where $z_i(t) = [z_{i,1}, z_{i,2}, \ldots, z_{i,\rho_i}]^T$. Firstly, the following definitions are made.

$$c_i^0 = min_{1 \leq q \leq \rho_i} c_{i,q} \tag{57}$$

$$\| z_i \|_2 = \sqrt{\int_0^{\infty} \| z_i(t) \|^2 dt}. \tag{58}$$

From (56), the derivative of V can be given as

$$\dot{V} \leq -c_i^0 \| z_i \|^2 . \tag{59}$$

Since V is nonincreasing, we obtain

$$\| z_i \|_2^2 = \int_0^{\infty} \| z_i(t) \|^2 dt \leq \frac{1}{c_i^0} \left(V(0) - V_(\infty) \right) \leq \frac{1}{c_i^0} V(0). \tag{60}$$

Similarly, the output y_i is bounded by

$$\| y_i \|_2^2 = \int_0^{\infty} (y_i(t))^2 dt \leq \frac{1}{c_{i,1}} V(0). \tag{61}$$

Theorem 2. *The L_2 norm of the state z_i is bounded by*

$$\| z_i(t) \|_2 \leq \frac{1}{\sqrt{c_i^0}} \sqrt{V(0)}, \tag{62}$$

$$\| y_i \|_2^2 \leq \frac{1}{\sqrt{c_{i,1}}} \sqrt{V(0)}. \tag{63}$$

Remark 6. *Regarding the output bound in (63), the following conclusions can be drawn by noting that $\tilde{\mathbf{\Theta}}_i(0)$, $\tilde{p}_i(0)$, $\boldsymbol{\epsilon}_i(0)$ and $y_i(0)$ are independent of $c_{i,1}, \mathbf{\Gamma}_i, \gamma_i^{'}$.*

- *The transient output performance in the sense of truncated norm given in (62) depends on the initial estimation errors $\tilde{\mathbf{\Theta}}_i(0), \tilde{p}_i(0)$ and $\boldsymbol{\epsilon}_i(0)$. The closer the initial estimates to the true values, the better the transient output performance.*
- *This bound can also be systematically reduced down to a lower bound by increasing $\mathbf{\Gamma}_i, \gamma_i^{'}, c_{i,1}$.*

5. Simulation example

We consider the following interconnected system with two subsystems.

$$\dot{\mathbf{x}}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 2y_1 & y_1^2 \\ 0 & y_1 \end{bmatrix} \boldsymbol{\theta}_1 + \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u_1 + \mathbf{f}_1 + \mathbf{h}_1, \;\; y_1 = x_{1,1} \tag{64}$$

$$\dot{\mathbf{x}}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_2 + \begin{bmatrix} 0 & 0 \\ y_2 & 1 + y_2 \end{bmatrix} \boldsymbol{\theta}_2 + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u_2 + \mathbf{f}_2 + \mathbf{h}_2, \;\; y_2 = x_{2,1}, \tag{65}$$

where $\boldsymbol{\theta}_1 = [1,\ 1]^T, \boldsymbol{\theta}_2 = [0.5,\ 1]^T$, $b_1 = b_2 = 1$, the nonlinear interaction terms $\mathbf{f}_1 = [0,\ y_2^2 + \sin(y_1)]^T$,$\mathbf{f}_2 = [0.2y_1^2 + y_2,\ 0]^T$, the external disturbance $\mathbf{h}_1 = \mathbf{0}, \mathbf{h}_2 = [y_1(t - \tau_1),\ y_2(t - \tau_2(t)]^T$. The parameters and the interactions are not needed to be known. The objective is to make the outputs y_1 and y_2 converge to zero.
The design parameters are chosen as $c_{1,1} = c_{1,2} = 2, c_{2,1} = c_{2,2} = 3, l_{1,1} = l_{1,2} = 1, l_{2,1} = l_{2,2} = 2, l_1^* = l_2^* = 5, \lambda_1^* = \lambda_2^* = 5,\ \gamma_1^{'} = 2, \gamma_2^{'} = 2, \mathbf{\Gamma}_1 = 0.5\mathbf{I}_3, \mathbf{\Gamma}_2 = \mathbf{I}_3,\ l_{i,p} = l_{i,\Theta} = 1$, $p_{1,0} = p_{2,0} = 1$, $\mathbf{\Theta}_{1,0} = [1, 1, 1]^T, \mathbf{\Theta}_{2,0} = [0.6, 1, 1]^T$. The initials are set as $y_1(0) = 0.5, y_2(0) = 1$, $\hat{\mathbf{\Theta}}_1(0) = [0.5, 0.8, 0.8]^T, \hat{\mathbf{\Theta}}_2(0) = [0.6, 0.8, 0.8]^T$. The block diagram in Figure 1 shows the proposed control structure for each subsystem. The input signals to the designed ith local adaptive controller are $y_i, \boldsymbol{\xi}_{i,0}, \Xi_i, \mathbf{v}_{i,0}$. Figures 2-3 show the system outputs y_1 and y_2. Figures 4-5 show the system inputs u_1 and $u_2(t)$. All the simulation results verify that our proposed scheme is effective to cope with nonlinear interactions and time-delay.

6. Conclusion

In this chapter, a new scheme is proposed to design totally decentralized adaptive output stabilizer for a class of unknown nonlinear interconnected system in the presence of time-delays. Unknown time-varying delays are compensated by using appropriate Lyapunov-Krasovskii functionals. It is shown that the designed decentralized adaptive controllers can ensure the stability of the overall interconnected systems. An explicit bound in terms of L_2 norms of the output is also derived as a function of design parameters. This implies that the transient the output performance can be adjusted by choosing suitable design parameters.

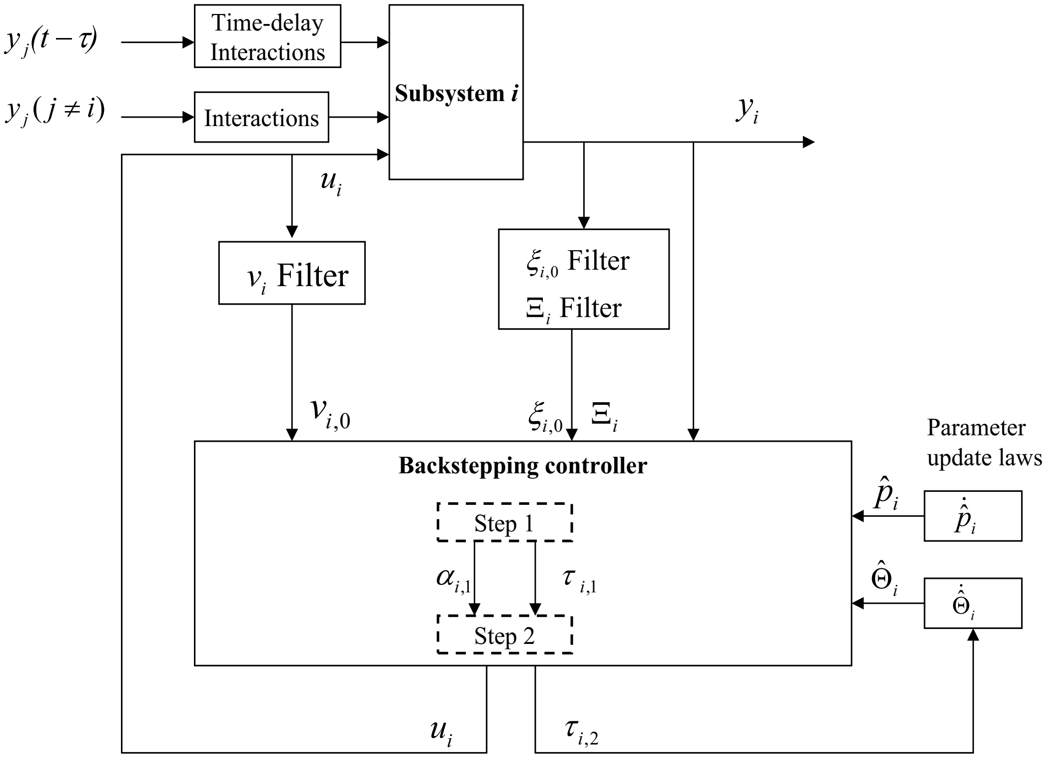

Fig. 1. Control block diagram.

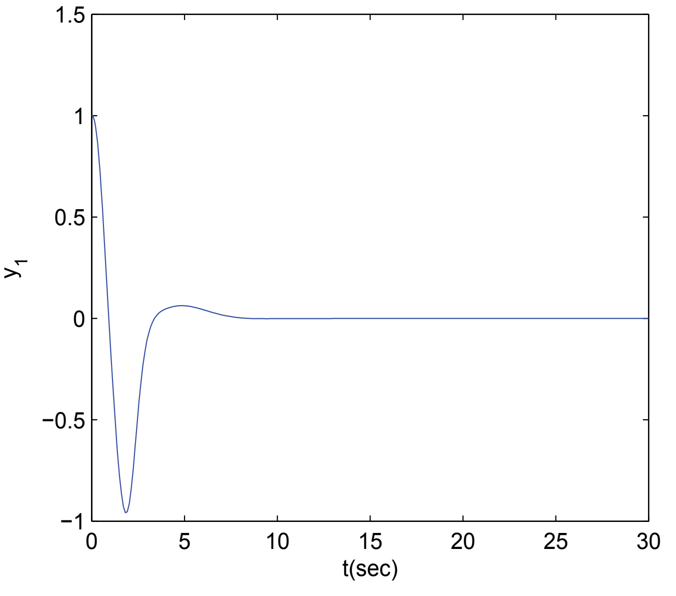

Fig. 2. Output y_1.

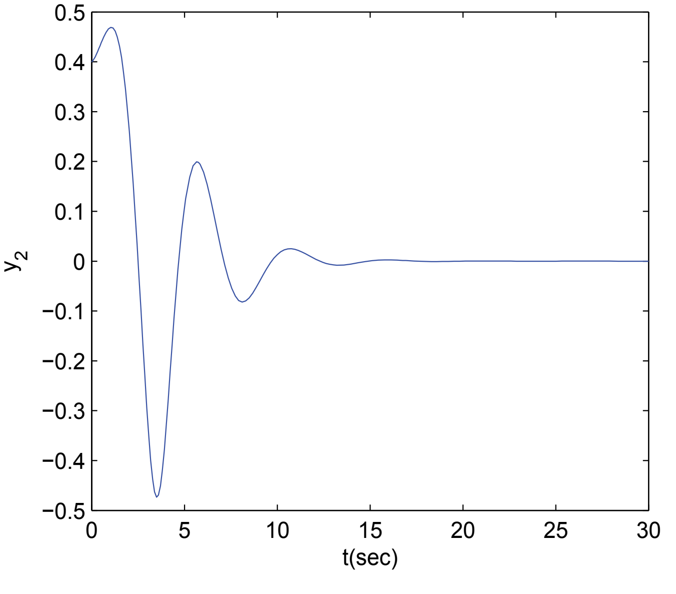

Fig. 3. Output y_2

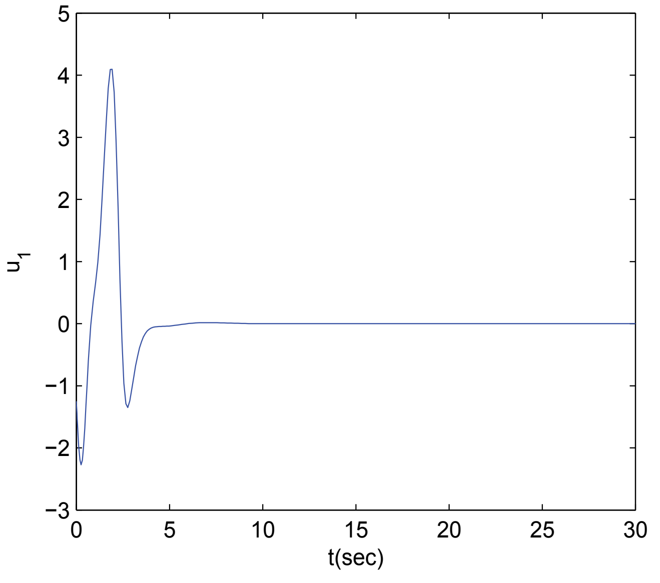

Fig. 4. Input u_1.

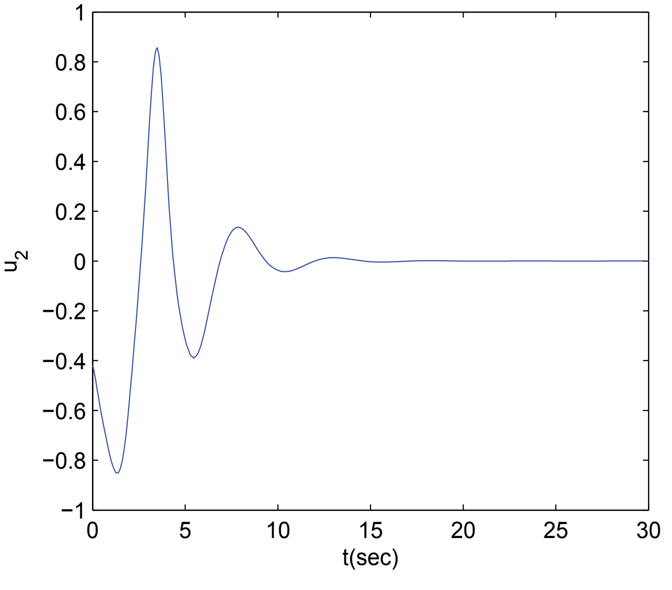

Fig. 5. Input u_2.

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8

Resilient Adaptive Control of Uncertain Time-Delay Systems

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1. Introduction

Time delay systems are widely encountered in many real applications, such as chemical processes and communication networks. Hence, the problem of controlling time-delay systems has been investigated by many researchers in the past few decades. It has been found that controlling time-delay systems can be a challenging task, especially in the presence of uncertainties and parameter variations. Several techniques have been studied in the analysis and design of time delay systems with parameter uncertainties. Such techniques include robust control Mahmoud (2000; 2001), H_∞ control Fridman & Shaked (2002); Mahmoud & Zribi (1999); Yang & Wang (2001); Yang et al. (2000), and sliding mode control Choi (2001; 2003); Edwards et al. (2001); Gouaisbaut et al. (2002); Xia & Jia (2003). For time-delay systems with parametric uncertainties Nounou & Mahmoud (2006); Nounou et al. (2007), adaptive control schemes have been developed. The main contribution in Nounou & Mahmoud (2006) is the development of two delay-independent adaptive controllers. The first one is an adaptive state feedback controller when no uncertainties appear in the controller's state feedback gain. This adaptive controller stabilizes the closed-loop system in the sense of uniform ultimate boundedness. The second controller is an adaptive state feedback controller when uncertainties also appear in the controller's state feedback gain. This adaptive controller guarantees asymptotic stabilization of the closed-loop system. In Nounou et al. (2007), the authors focused on the stabilization of the class of time-delay systems with parametric uncertainties and time varying state delay when the states are not assumed to be measurable. For this class of systems, the authors developed two controllers. The first one is a robust output feedback controller when a sliding-mode observer is used to estimate the states of the system, and the second one is an adaptive output feedback controller when a sliding-mode observer is used to estimate the states of the system, such that the uncertainties also appear in the gain of the sliding-mode observer. In the case where uncertain time-delay systems include a nonlinear perturbation, several adaptive control approaches have been introduced Cheres et al. (1989); Wu (1995; 1996; 1997; 1999; 2000). In Cheres et al. (1989); Wu (1996), the authors developed state feedback controllers when the state vector is available for measurement and the upper bound on the delayed state perturbation vector is known. For the case where the upper bound of the nonlinear perturbation is known, more stabilizing controllers with stability conditions have been derived in Wu (1995; 1997). However, in many real control problems, the bounds of the uncertainties are unknown. For such a class of systems, the author in Wu (1999) has developed a continuous time state feedback adaptive controller to guarantee uniform ultimate boundedness for systems with partially known uncertainties. For a class of systems with multiple uncertain state delays that are assumed to satisfy the matching condition, an adaptive law that guarantees uniform ultimate boundedness has been introduced in Wu (2000). In all of the papers discussed above, the authors investigated delay-independent stabilization and control of time-delay systems. Delay-dependent stabilization and H_∞ control of time-delay systems have been studied in De Souza & Li (1999); Fridman (1998); Fridman & Shaked (2003); He et al. (1998); Lee et al. (2004); Mahmoud (2000); Wang (2004). In Mahmoud (2000), the author discussed stabilization conditions and analyzed passivity of continuous and discrete time-delay systems with time-varying delay and norm-bounded parameter uncertainties. The results in Mahmoud (2000) have been extended in Nounou (2006) to consider designing delay-dependent adaptive controllers for a class of uncertain time-delay systems with time-varying delays in the presence of nonlinear perturbation. In Nounou (2006), the nonlinear perturbation is assumed to be bounded by a weighted norm of the state vector, and for this problem adaptive controllers have been developed for the two cases where the upper bound of the weight is assumed to be known and unknown.

An inherent assumption in the design of all of the above control algorithms is that the controller will be implemented perfectly. Here, the results in Nounou (2006) are extended to investigate the resilient control problem Haddad & Corrado (1997; 1998); Keel & Bhattacharyya (1997), where perturbation in controller state feedback gain is considered.

Here, It is assumed that the nonlinear perturbation is bounded by a weighted norm of the state such that the weight is a positive constant, and the norm of the uncertainty of the state feedback gain is assumed to be bounded by a positive constant. Under these assumptions, adaptive controllers are developed for all combinations when the upper bound of the nonlinear perturbation weight is known and unknown, and when the value of the upper bound of the state feedback gain perturbation is known and unknown. For all these cases, asymptotically stabilizing adaptive controllers are derived.

This chapter is organized as follows. In Section 2, the problem statement is defined. Then, in Section 3, the main stability results are presented. In Section 4, the design schemes are illustrated via a numerical example, and finally in Section 5, some concluding remarks are outlined.

Notations and Facts: In the sequel, the Euclidean norm is used for vectors. We use $W^\top$, W^{-1}, and $||W||$ to denote, respectively, the transpose of, the inverse of, and the induced norm of any square matrix W. We use $W > 0$ $(\geq, <, \leq 0)$ to denote a symmetric positive definite (positive semidefinite, negative, negative semidefinite) matrix W, and I to denote the $n \times n$ identity matrix. The symbol $\bullet$ will be used in some matrix expressions to induce a symmetric structure, that is if the matrices $L = L^\top$ and $R = R^\top$ of appropriate dimensions are given, then

$$\begin{bmatrix} L & N \\ \bullet & R \end{bmatrix} = \begin{bmatrix} L & N \\ N^\top & R \end{bmatrix}.$$

Now, we introduce the following facts that will be used later on to establish the stability results.

Fact 1: Mahmoud (2000) Given matrices Σ_1 and Σ_2 with appropriate dimensions, it follows that

$$\Sigma_1\Sigma_2 + \Sigma_2^\top\Sigma_1^\top \leq \alpha^{-1}\,\Sigma_1\Sigma_1^\top + \alpha\,\Sigma_2^\top\Sigma_2, \;\forall\, \alpha > 0.$$

Fact 2 (Schur Complement): Boukas & Liu (2002); Mahmoud (2000) Given constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^\top$ and $0 < \Omega_2 = \Omega_2^\top$ then $\Omega_1 + \Omega_3^\top \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^\top \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^\top & \Omega_1 \end{bmatrix} < 0.$$

2. Problem statement

Consider the class of dynamical systems with state delay

$$\dot{x}(t) = A_o x(t) + A_d x(t-\tau) + B_o u(t) + E\left(x(t), t\right) \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $E\left(x(t), t\right) : \Re^n \times \Re \to \Re^n$ is an unknown continuous vector function that represents a nonlinear perturbation, and τ is some unknown time-varying state delay factor satisfying $0 \le \tau \le \tau^+$, where the bound τ^+ is a known constant. The matrices A_o, A_d, and B_o are known real constant matrices of appropriate dimensions. The nonlinear perturbation function is defined to satisfy the following assumption.

Assumption 2.1. *The nonlinear perturbation function $E\left(x(t), t\right)$ satis□es the following inequality*

$$\|E\left(x(t), t\right)\| \le \theta^* \, \|x(t)\|, \tag{2}$$

where θ^* is some positive constant.

In this chapter, resilient delay-dependent adaptive stabilization results are established for the system (1) when uncertainties appear in the state feedback gain of the following control law:

$$u(t) \quad = \quad (K + \Delta K)\, x(t) + \mu(t) \mathcal{I} x(t), \tag{3}$$

where $\mathcal{I} \in \Re^{m \times n}$ is a matrix whose elements are all ones, $\mu(t) \in \Re$ is adapted such that closed-loop asymptotic stabilization is guaranteed, $K \in \Re^{m \times n}$ is a state feedback gain, and $\Delta K(t) \in \Re^{m \times n}$ is the time varying uncertainty of the state feedback gain that satisfies the following assumption.

Assumption 2.2. *The uncertainty of the state feedback gain satis□es the following inequality*

$$\|\Delta K(t)\| \le \rho^*, \tag{4}$$

where ρ^* is some positive constant.

Before we proceed, we start be expressing the delayed state as Mahmoud (2000)

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{-\tau}^{0} \dot{x}(t+s) ds \qquad (5) \\ &= x(t) - \int_{-\tau}^{0} \left[A_o\, x(t+s) + A_d\, x(t-\tau+s) + B_o\, u(t+s) - E\left(x(t+s), t+s\right)\right] ds \end{aligned}$$

Hence, if we define $A_{od} = A_o + A_d$, then the system (1) can be expressed as

$$\begin{aligned} \dot{x}(t) &= A_{od}\, x(t) + A_d\, \eta(t) + B_o u(t) + E\left(x(t), t\right), \qquad (6) \\ \eta(t) &= -\int_{-\tau}^{0} \left[A_o\, x(t+s) + A_d\, x(t-\tau+s) + B_o\, u(t+s) + E\left(x(t+s), t+s\right)\right]\, ds. \end{aligned}$$

Here, resilient delay-dependent stabilization results are established for the system (6) considering the following cases:

1. The nonlinear perturbation function satisfies Assumption 2.1 such that θ^* is assumed to be *a known* positive constant, and the uncertainty of the state feedback gain satisfies Assumption 2.2 such that ρ^* is assumed to be *a known* positive constant.
2. The nonlinear perturbation function satisfies Assumption 2.1 such that θ^* is assumed to be *a known* positive constant, and the uncertainty of the state feedback gain satisfies Assumption 2.2 such that ρ^* is assumed to be *an unknown* positive constant.
3. The nonlinear perturbation function satisfies Assumption 2.1 such that θ^* is assumed to be *an unknown* positive constant, and the uncertainty of the state feedback gain satisfies Assumption 2.2 such that ρ^* is assumed to be *a known* positive constant.
4. The nonlinear perturbation function satisfies Assumption 2.1 such that θ^* is assumed to be *an unknown* positive constant, and the uncertainty of the state feedback gain satisfies Assumption 2.2 such that ρ^* is assumed to be *an unknown* positive constant.

3. Main results

In the sequel, the main design results will be presented.

3.1 Adaptive control when both θ^* and ρ^* are known

Here, we wish to stabilize the system (6) considering the control law (3) when both θ^* and ρ^* are known. Let us define $z(t) = \mu(t)x(t)$, and let the Lyapunov-Krasovskii functional for the transformed system (6) be selected as:

$$V_a(x) \triangleq V_1(x) + V_2(x) + V_3(x) + V_4(x) + V_5(x) + V_6(x) + V_7(x) + V_8(x), \quad (7)$$

where

$$\begin{aligned}
V_1(x) &= x^\top(t)Px(t), && (8)\\
V_2(x) &= r_1 \int_{-\tau}^{0}\int_{t+s}^{t} x^\top(\alpha)A_o^\top A_o x(\alpha)d\alpha ds, && (9)\\
V_3(x) &= r_2 \int_{-\tau}^{0}\int_{t+s-\tau}^{t} x^\top(\alpha)\ A_d^\top A_d\ x(\alpha)\ d\alpha\ ds, && (10)\\
V_4(x) &= r_3 \int_{-\tau}^{0}\int_{t+s}^{t} x^\top(\alpha)\ K^\top B_o^\top B_o K\ x(\alpha)\ d\alpha\ ds, && (11)\\
V_5(x) &= r_4 \int_{-\tau}^{0}\int_{t+s}^{t} x^\top(\alpha)\ \Delta K^\top(t) B_o^\top B_o \Delta K(t)\ x(\alpha)\ d\alpha\ ds, && (12)\\
V_6(x) &= r_5 \int_{-\tau}^{0}\int_{t+s}^{t} z^\top(\alpha)\ \mathcal{I}^\top B_o^\top B_o \mathcal{I}\ z(\alpha)\ d\alpha\ ds, && (13)\\
V_7(x) &= r_6 \int_{-\tau}^{0}\int_{t+s}^{t} E^\top(x,\alpha)\ E(x,\alpha)\ d\alpha\ ds, && (14)\\
V_8(x) &= \mu^2(t), && (15)
\end{aligned}$$

where $r_1 > 0$, $r_2 > 0$, $r_3 > 0$, $r_4 > 0$, $r_5 > 0$ and $r_6 > 0$ are positive scalars, and $P = P^\top \in \Re^{n\times n} > 0$. It can be shown that the time derivative of the Lyapunov-Krasovskii functional is

$$\dot{V}_a(x) = \dot{V}_1(x) + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x) + \dot{V}_6(x) + \dot{V}_7(x) + \dot{V}_8(x), \quad (16)$$

where

$$\dot{V}_1(x) = x^\top(t)P\dot{x}(t) + \dot{x}^\top(t)Px(t), \tag{17}$$

$$\dot{V}_2(x) = \tau r_1 x^\top(t)A_o^\top A_o x(t) - r_1 \int_{-\tau}^{0} x^\top(t+s)A_o^\top A_o x(t+s)ds, \tag{18}$$

$$\dot{V}_3(x) = \tau r_2 x^\top(t)A_d^\top A_d x(t) - r_2 \int_{-\tau}^{0} x^\top(t+s-\tau)A_d^\top A_d x(t+s-\tau)ds, \tag{19}$$

$$\dot{V}_4(x) = \tau r_3 x^\top(t)K^\top B_o^\top B_o K x(t) - r_3 \int_{-\tau}^{0} x^\top(t+s)K^\top B_o^\top B_o K x(t+s)ds, \tag{20}$$

$$\begin{aligned}\dot{V}_5(x) &= \tau r_4 x^\top(t)\Delta K(t)^\top B_o^\top B_o \Delta K(t)x(t) \\ &\quad - r_4 \int_{-\tau}^{0} x^\top(t+s)\Delta K^\top(t+s)B_o^\top B_o \Delta K(t+s)x(t+s)ds,\end{aligned} \tag{21}$$

$$\dot{V}_6(x) = \tau r_5 z^\top(t)\mathcal{I}^\top B_o^\top B_o \mathcal{I} z(t) - r_5 \int_{-\tau}^{0} z^\top(t+s)\mathcal{I}^\top B_o^\top B_o \mathcal{I} z(t+s)ds, \tag{22}$$

$$\dot{V}_7(x) = \tau r_6\, E^\top(x,t)\, E(x,t) - r_6 \int_{-\tau}^{0} E^\top(x,t+s)\, E(x,t+s)\, ds, \tag{23}$$

$$\dot{V}_8(x) = 2\,\mu(t)\,\dot{\mu}(t). \tag{24}$$

The next Theorem provides the main results for this case.

Theorem 1: *Consider system (6). If there exist matrices* $0 < \mathcal{X} = \mathcal{X}^\top \in \Re^{n\times n}$, $\mathcal{Y} \in \Re^{m\times n}$, $\mathcal{Z} \in \Re^{n\times n}$, *and scalars* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > \varepsilon$, $\varepsilon_5 > \varepsilon$ *and* $\varepsilon_6 > \varepsilon$ *(where* ε *is an arbitrary small positive constant) such that the following LMI*

$$\begin{bmatrix} \begin{array}{c} A_{od}\mathcal{X} + \mathcal{X}A_{od} + B_o\mathcal{Y} + \mathcal{Y}^\top B_o^\top \\ +\tau^+(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6)A_d A_d^\top \end{array} & \tau^+\mathcal{X}A_o^\top & \tau^+\mathcal{X}A_d^\top & \tau^+\mathcal{Z} \\ \bullet & -\tau^+\varepsilon_1 I & 0 & 0 \\ \bullet & \bullet & -\tau^+\varepsilon_2 I & 0 \\ \bullet & \bullet & \bullet & -\tau^+\varepsilon_3 I \end{bmatrix} < 0, \tag{25}$$

has a feasible solution, and $K = \mathcal{Y}\mathcal{X}^{-1}$, *and* $\mu(t)$ *is adapted subject to the adaptive law*

$$\dot{\mu}(t) = Proj\left\{\alpha_1\, sgn\,(\mu(t))\ ||x(t)||^2 + \alpha_2\, \mu(t)\ ||x(t)||^2, \mu(t)\right\}, \tag{26}$$

where $Proj\{\cdot\}$ *Krstic et al. (1995) is applied to ensure that* $|\mu(t)| \geq 1$ *as follows*

$$\mu(t) = \begin{cases} \mu(t) & \text{if } |\mu(t)| \geq 1 \\ 1 & \text{if } 0 \leq \mu(t) < 1 \\ -1 & \text{if } -1 < \mu(t) < 0, \end{cases}$$

and the adaptive law parameters are selected such that

$$\alpha_1 < -\frac{1}{2}\left[\tau^+ r_4 (\rho^*)^2 ||B_o^\top B_o|| + \tau^+ r_6 (\theta^*)^2 + 2\rho^*||PB_o|| + 2||PB_o|| + 2\theta^*||P||\right], \tag{27}$$

and

$$\alpha_2 < -\frac{1}{2}\tau^+ r_5 ||\mathcal{I}^\top B_o^\top B_o \mathcal{I}||, \tag{28}$$

then the control law (3) will guarantee asymptotic stabilization of the closed-loop system.

Proof As shown in (16), the time derivative of $V_a(x)$ is

$$\begin{aligned}\dot{V}_a(x) &= \dot{V}_1(x) + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x) + \dot{V}_6(x) + \dot{V}_7(x) + \dot{V}_8(x), \\ &= x^\top(t)P\dot{x}(t) + \dot{x}^\top(t)Px(t) + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x) + \dot{V}_6(x) \\ &\quad + \dot{V}_7(x) + \dot{V}_8(x). \end{aligned} \tag{29}$$

Using the system equation defined in (6) and the control law (3), we have

$$\begin{aligned}\dot{V}_a(x) &= x^\top(t)\left[PA_{od} + A_{od}^\top P + PB_oK + K^\top B_o^\top P\right]x(t) \\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} A_o x(t+s)ds - 2x^\top(t)PA_d\int_{-\tau}^{0} A_d x(t-\tau+s)ds \\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} B_oKx(t+s)ds - 2x^\top(t)PA_d\int_{-\tau}^{0} B_o\Delta K(t+s)x(t+s)ds \\ &\quad -2x^\top(t)PA_d\int_{-\tau}^{0} \mu(t+s)B_o\mathcal{I}x(t+s)ds - 2x^\top(t)PA_d\int_{-\tau}^{0} E(x,t+s)ds \\ &\quad +2x^\top(t)PB_o\Delta K(t)x(t) + 2\mu(t)x^\top(t)PB_o\mathcal{I}x(t) + 2x^\top(t)PE(x,t) \\ &\quad + \dot{V}_2(x) + \dot{V}_3(x) + \dot{V}_4(x) + \dot{V}_5(x) + \dot{V}_6(x) + \dot{V}_7(x) + \dot{V}_8(x). \end{aligned} \tag{30}$$

By applying Fact 1, we have

$$\begin{aligned} -2x^\top(t)PA_d\int_{-\tau}^{0} A_o x(t+s)ds &\le r_1^{-1}\int_{-\tau}^{0} x^\top(s)PA_dA_d^\top Px(s)ds \\ &\quad + r_1\int_{-\tau}^{0} x^\top(t+s)A_o^\top A_o x(t+s)ds \\ &\le \tau^+ r_1^{-1}x^\top(t)PA_dA_d^\top Px(t) \\ &\quad + r_1\int_{-\tau}^{0} x^\top(t+s)A_o^\top A_o x(t+s)ds, \end{aligned} \tag{31}$$

where r_1 is a positive scalar. Similarly, if r_2, r_3 and r_4 are positive scalars, we have

$$\begin{aligned} -2x^\top(t)PA_d\int_{-\tau}^{0} A_d x(t-\tau+s)ds &\le \tau^+ r_2^{-1}x^\top(t)PA_dA_d^\top Px(t) \\ &\quad + r_2\int_{-\tau}^{0} x^\top(t-\tau+s)A_d^\top A_d x(t-\tau+s)ds, \end{aligned} \tag{32}$$

$$\begin{aligned} -2x^\top(t)PA_d\int_{-\tau}^{0} B_oKx(t+s)ds &\le \tau^+ r_3^{-1}x^\top(t)PA_dA_d^\top Px(t) \\ &\quad + r_3\int_{-\tau}^{0} x^\top(t+s)K^\top B_o^\top B_oKx(t+s)ds, \end{aligned} \tag{33}$$

and

$$\begin{aligned} -2x^\top(t)PA_d\textstyle\int_{-\tau}^{0} B_o\Delta K(t+s)x(t+s)ds \le \tau^+ r_4^{-1}x^\top(t)PA_dA_d^\top Px(t) \\ + r_4\textstyle\int_{-\tau}^{0} x^\top(t+s)\Delta K^\top(t+s)B_o^\top B_o\Delta K(t+s)x(t+s)ds. \end{aligned} \tag{34}$$

Now, let r_5 be a positive scalar, then using Fact 1 we have

$$-2x^\top(t)PA_d\int_{-\tau}^{0}\mu(t+s)B_o\mathcal{I}x(t+s)ds = -2x^\top(t)PA_d\int_{-\tau}^{0}B_o\mathcal{I}z(t+s)ds$$
$$\leq \tau^+ r_5^{-1}x^\top(t)PA_dA_d^\top Px(t) + r_5\int_{-\tau}^{0}z^\top(t+s)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t+s)ds. \quad (35)$$

Also, if r_6 is a positive scalar, then using Fact 1 we have

$$-2x^\top(t)PA_d\int_{-\tau}^{0}E(x,t+s)ds \leq \tau^+ r_6^{-1}x^\top(t)PA_dA_d^\top Px(t)$$
$$+r_6\int_{-\tau}^{0}E^\top(x,t+s)E(x,t+s)ds. \quad (36)$$

It is known that

$$2\mu(t)x^\top(t)PB_o\mathcal{I}x(t) \leq 2||PB_o\mathcal{I}||\ |\mu(t)|\ ||x(t)||^2. \quad (37)$$

Also, using Assumption 2.1, it can be shown that

$$2x^\top(t)PE(x,t) \leq 2||P||\ \theta^*\ ||x(t)||^2. \quad (38)$$

Using equations (31)- (38) and equations (17)- (24) (with the fact that $0 \leq \tau \leq \tau^+$) in (30), we have

$$\begin{aligned}\dot{V}_a(x) \quad \leq \quad & x^\top(t)\Xi x(t) + \tau^+ r_4 x^\top(t)\Delta K^\top(t)B_o^\top B_o\Delta K(t)x(t)\\ & +\tau^+ r_5 z^\top(t)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t) + \tau^+ r_6 E^\top(x,t)E(x,t) + 2\rho^*||PB_o||\ ||x(t)||^2\\ & +2||PB_o\mathcal{I}||\ |\mu(t)|\ ||x(t)||^2 + 2\theta^*||P||\ ||x(t)||^2 + 2\ \mu(t)\ \dot{\mu}(t). \end{aligned} \quad (39)$$

where

$$\begin{aligned}\Xi \quad = \quad & PA_{od} + A_{od}^\top P + PB_oK + K^\top B_o^\top P + \tau^+ r_1 A_o^\top A_o + \tau^+ r_2 A_d^\top A_d + \tau^+ r_3 B_oKK^\top B_o^\top\\ & +\tau^+\left(r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1} + r_5^{-1} + r_6^{-1}\right)PA_dA_d^\top P. \end{aligned} \quad (40)$$

To guarantee that $x^\top(t)\Xi x(t) < 0$, it sufficient to show that $\Xi < 0$. Let us introduce the linearizing terms, $\mathcal{X} = P^{-1}$, $\mathcal{Y} = K\mathcal{X}$, and $\mathcal{Z} = \mathcal{X}B_oK$. Also, let $\varepsilon_1 = r_1^{-1}$, $\varepsilon_2 = r_2^{-1}$, $\varepsilon_3 = r_3^{-1}$, $\varepsilon_4 = r_4^{-1}$, $\varepsilon_5 = r_5^{-1}$ and $\varepsilon_6 = r_6^{-1}$. Now, by pre-multiplying and post-multiplying Ξ by $\mathcal{X}$ and invoking the Schur complement, we arrive at the LMI (25) which guarantees that $\Xi < 0$, and consequently $x^\top(t)\Xi x(t) < 0$. Now, we need to show that the remaining terms of (39) are negative definite. Using the definition of $z(t) = \mu(t)x(t)$, we know that

$$\tau^+ r_5 z^\top(t)\mathcal{I}^\top B_o^\top B_o\mathcal{I}z(t) \leq \tau^+ r_5\ ||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||\ \mu^2(t)\ ||x(t)||^2. \quad (41)$$

Also, using Assumptions 2.1 and 2.2 , we have

$$\tau^+ r_6 E^\top(x,t)E(x,t) \leq \tau^+ r_6\,(\theta^*)^2\ ||x(t)||^2, \quad (42)$$

and

$$\tau^+ r_4 x^\top(t)\Delta K^\top(t)B_o^\top B_o\Delta K(t)x(t) \leq \tau^+ r_4\,(\rho^*)^2\ ||B_o^\top B_o||\ ||x(t)||^2. \quad (43)$$

Now, using (41)- (43), the adaptive law (26), and the fact that $|\mu(t)| \geq 1$, equation (39) becomes

$$\begin{aligned} \dot{V}_a(x) \quad \leq \quad & x^\top(t)\Xi x(t) + \tau^+ r_4\,(\rho^*)^2\ \ ||B_o^\top B_o||\ ||x(t)||^2 + \tau^+ r_5\ ||\mathcal{I}^\top B_o^\top B_o \mathcal{I}||\ \mu^2(t)\ ||x(t)||^2 \\ & + \tau^+ r_6\,(\theta^*)^2\ \ ||x(t)||^2 + 2\rho^*||PB_o||\ ||x(t)||^2 + 2||PB_o\mathcal{I}||\ |\mu(t)|\ ||x(t)||^2 \\ & + 2\theta^*||P||\ ||x(t)||^2 + 2\alpha_1\ |\mu(t)|\ ||x(t)||^2 + 2\alpha_2\ \mu^2(t)\ ||x(t)||^2. \end{aligned} \tag{44}$$

It can be easily shown that by selecting α_1 and α_2 as in (27) and (28), we guarantee that

$$\dot{V}_a(x) \quad \leq \quad x^\top(t)\Xi x(t), \tag{45}$$

where $\Xi < 0$. Hence, $\dot{V}_a(x) < 0$ which guarantees asymptotic stabilization of the closed-loop system. ∎

3.2 Adaptive control when θ^* is known and ρ^* is unknown

Here, we wish to stabilize the system (6) considering the control law (3) when θ^* is known and ρ^* is unknown. Before we present the stability results for this case, let us define $\tilde{\rho}(t) = \hat{\rho}(t) - \rho^*$, where $\hat{\rho}(t)$ is the estimate of ρ^*, and $\tilde{\rho}(t)$ is error between the estimate and the true value of ρ^*. Let the Lyapunov-Krasovskii functional for the transformed system (6) be selected as:

$$V_b(x) \quad \stackrel{\Delta}{=} \quad V_a(x)\ +\ V_9(x), \tag{46}$$

where $V_a(x)$ is defined in equations (7), and $V_9(x)$ is defined as

$$V_9(x) = (1+\rho^*)\,[\tilde{\rho}(t)]^2\,, \tag{47}$$

where its time derivative is

$$\dot{V}_9(x) = 2\ \ (1+\rho^*)\,\tilde{\rho}(t)\ \dot{\tilde{\rho}}(t). \tag{48}$$

Since $\tilde{\rho}(t) = \hat{\rho}(t) - \rho^*$, then $\dot{\tilde{\rho}}(t) = \dot{\hat{\rho}}(t)$. Hence, equation (48) becomes

$$\dot{V}_9(x) = 2\ \ (1+\rho^*)\,[\hat{\rho}(t) - \rho^*]\ \ \dot{\hat{\rho}}(t). \tag{49}$$

The next Theorem provides the main results for this case.

Theorem 2: *Consider system (6). If there exist matrices $0 < \mathcal{X} = \mathcal{X}^\top \in \Re^{n\times n}$, $\mathcal{Y} \in \Re^{m\times n}$, $\mathcal{Z} \in \Re^{n\times n}$, and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > \varepsilon$, $\varepsilon_5 > \varepsilon$ and $\varepsilon_6 > \varepsilon$ (where ε is an arbitrary small positive constant) such that the LMI (25) has a feasible solution, and $K = \mathcal{Y}\mathcal{X}^{-1}$, and $\mu(t)$ and $\hat{\rho}(t)$ are adapted subject to the adaptive laws*

$$\begin{aligned} \dot{\mu}(t) \quad &= \quad Proj\left\{[\beta_1\ \ sgn\,(\mu(t))\ \ + \beta_2\ \ \mu(t) +\ \ \beta_3\,sgn\,(\mu(t))\ \ \hat{\rho}(t)\ ]\ ||x(t)||^2, \mu(t)\right\} \qquad (50) \\ \dot{\hat{\rho}}(t) \quad &= \quad \gamma\ ||x(t)||^2, \qquad (51) \end{aligned}$$

where $Proj\{\cdot\}$ Krstic et al. (1995) is applied to ensure that $|\mu(t)| \geq 1$ as follows:

$$\mu(t) = \begin{cases} \mu(t) & if\ |\mu(t)| \geq 1 \\ 1 & if\ 0 \leq \mu(t) < 1 \\ -1 & if\ -1 < \mu(t) < 0, \end{cases}$$

and the adaptive law parameters are selected such that $\beta_1 < -\frac{1}{2}\left[\ \tau^+ r_6\,(\theta^*)^2 + 2\ ||PB_o\mathcal{I}||\ \ + 2\theta^*\ ||P||\ \right]$, $\beta_2 < -\frac{1}{2}\tau^+ r_5||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||$, $\gamma > \frac{1}{2}\tau^+ r_4||B_o^\top B_o||$,

$\beta_3 < -\gamma$, and $\hat{\rho}(0) > 1$, then the control law (3) will guarantee asymptotic stabilization of the closed-loop system.

Proof The time derivative of $V_b(x)$ is

$$\dot{V}_b(x) \;=\; \dot{V}_a(x) + \dot{V}_9(x). \tag{52}$$

Following the steps used in the proof of Theorem 1 and using equation (49), it can be shown that

$$\begin{aligned}\dot{V}_b(x) \;\le\; & x^\top(t)\Xi x(t) + \tau^+ r_4 (\rho^*)^2\; ||B_o^\top B_o||\; ||x(t)||^2 + \tau^+ r_5\; ||\mathcal{I}^\top B_o^\top B_o \mathcal{I}||\; \mu^2(t)\; ||x(t)||^2 \\ & + \tau^+ r_6 (\theta^*)^2\; ||x(t)||^2 + 2\rho^* ||PB_o||\; ||x(t)||^2 + 2||PB_o\mathcal{I}||\; |\mu(t)|\; ||x(t)||^2 \\ & + 2\theta^* ||P||\; ||x(t)||^2 + 2\,\mu(t)\,\dot{\mu}(t) + 2\,(1+\rho^*)\,[\hat{\rho}(t) - \rho^*]\;\dot{\hat{\rho}}(t),\end{aligned} \tag{53}$$

where Ξ is defined in equation (40). Using the linearization procedure and invoking the Schur complement (as in the proof of Theorem 1), it can be shown that Ξ is guaranteed to be negative definite whenever the LMI (25) has a feasible solution. Using the adaptive laws (50)- (51) in (53) and the fact that $|\mu(t)| \ge 1$, we get

$$\begin{aligned}\dot{V}_b(x) \;\le\; & x^\top(t)\Xi x(t) + \tau^+ r_4 (\rho^*)^2\; ||B_o^\top B_o||\; ||x(t)||^2 + \tau^+ r_5\; ||\mathcal{I}^\top B_o^\top B_o \mathcal{I}||\; \mu^2(t)\; ||x(t)||^2 \\ & + \tau^+ r_6 (\theta^*)^2\; ||x(t)||^2 + 2\rho^* ||PB_o||\; ||x(t)||^2 \\ & + 2||PB_o\mathcal{I}||\; |\mu(t)|\; ||x(t)||^2 + 2\theta^* ||P||\; ||x(t)||^2 \\ & + 2\beta_1\, |\mu(t)|\; ||x(t)||^2 + 2\beta_2\, \mu^2(t)\; ||x(t)||^2 + 2\beta_3\, \hat{\rho}(t)\, |\mu(t)|\; ||x(t)||^2 + 2\gamma\hat{\rho}(t)\; ||x(t)||^2 \\ & - 2\gamma\rho^*\; ||x(t)||^2 - 2\gamma\rho^*\hat{\rho}(t)\; ||x(t)||^2 - 2\gamma\,(\rho^*)^2\; ||x(t)||^2.\end{aligned} \tag{54}$$

Using the fact that $|\mu(t)| > 1$ and arranging terms of equation (54), it can be shown that $\dot{V}_b(x) < 0$ if we select $\beta_1 < -\frac{1}{2}\left[\tau^+ r_6 (\theta^*)^2 + 2\,||PB_o\mathcal{I}|| + 2\theta^*\,||P||\right]$, $\beta_2 < -\frac{1}{2}\tau^+ r_5 ||\mathcal{I}^\top B_o^\top B_o \mathcal{I}||$, and $\beta_3 < -\gamma$, where γ needs to be selected to satisfy the following two conditions:

$$\gamma > \frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||, \tag{55}$$

and

$$2||PB_o|| - 2\gamma + 2\gamma\hat{\rho}(t) < 0. \tag{56}$$

Hence, we need to select γ such that

$$\gamma \;>\; max\left\{\frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||\,,\; \frac{||PB_o||}{1-\hat{\rho}(t)}\right\}. \tag{57}$$

It is clear that when $\hat{\rho}(t) > 1$, we only need to ensure that $\gamma > \frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||$. Note that from equation (51), $\hat{\rho}(t) > 1$ can be easily ensured by selecting $\hat{\rho}(0) > 1$ and $\gamma > \frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||$ to guarantee that $\hat{\rho}(t)$ in equation (51) is monotonically increasing. Hence, we guarantee that

$$\dot{V}_b(x) \;\le\; x^\top(t)\;\Xi\; x(t), \tag{58}$$

where $\Xi < 0$. Hence, $\dot{V}_b(x) < 0$ which guarantees asymptotic stabilization of the closed-loop system. ∎

3.3 Adaptive control when θ^* is unknown and ρ^* is known

Here, we wish to stabilize the system (6) considering the control law (3) when θ^* is unknown and ρ^* is known. Since θ^* is unknown, let us define $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$, where $\hat{\theta}(t)$ is the estimate of θ^*, and $\tilde{\theta}(t)$ is error between the estimate and the true value of θ^*. Also, let the Lyapunov-Krasovskii functional for the transformed system (6) be selected as:

$$V_c(x) \quad \stackrel{\Delta}{=} \quad V_a(x) \; + \; V_{10}(x), \tag{59}$$

where

$$V_{10}(x) = (1+\theta^*) \left[\tilde{\theta}(t)\right]^2, \tag{60}$$

where its time derivative is

$$\begin{aligned} \dot{V}_{10}(x) &= 2\ (1+\theta^*)\,\tilde{\theta}(t)\,\dot{\tilde{\theta}}(t), \\ &= 2\ (1+\theta^*) \left[\hat{\theta}(t) - \theta^*\right]\ \dot{\hat{\theta}}(t). \end{aligned} \tag{61}$$

The next Theorem provides the main results for this case.

Theorem 3: *Consider system (6). If there exist matrices $0 < \mathcal{X} = \mathcal{X}^\top \in \Re^{n\times n}$, $\mathcal{Y} \in \Re^{m\times n}$, $\mathcal{Z} \in \Re^{n\times n}$, and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > \varepsilon$, $\varepsilon_5 > \varepsilon$ and $\varepsilon_6 > \varepsilon$ (where ε is an arbitrary small positive constant) such that the LMI (25) has a feasible solution, and $K = \mathcal{Y}\mathcal{X}^{-1}$, and $\mu(t)$ is adapted subject to the adaptive laws*

$$\dot{\mu}(t) \;=\; Proj\left\{\delta_1\ sgn\left(\mu(t)\right)\ ||x(t)||^2 + \delta_2\ \mu(t)\ ||x(t)||^2 + \delta_3\ \ sgn\left(\mu(t)\right)\ \hat{\theta}(t)\ ||x(t)||^2, \mu(t)\right\} \tag{62}$$

$$\dot{\hat{\theta}}(t) \;=\; \kappa\ ||x(t)||^2, \tag{63}$$

where $Proj\{\cdot\}$ Krstic et al. (1995) is applied to ensure that $|\mu(t)| \geq 1$ as follows

$$\mu(t) = \begin{cases} \mu(t) & if\,|\mu(t)| \geq 1 \\ 1 & if\, 0 \leq \mu(t) < 1 \\ -1 & if\, -1 < \mu(t) < 0, \end{cases}$$

and the adaptive law parameters are selected such that $\delta_1 < -\left[||PB_o\mathcal{I}|| + \tau^+ r_4\,(\rho^*)^2\,||B_o^\top B_o|| + \rho^*||PB_o||\right]$, $\delta_2 < -\frac{1}{2}\tau^+ r_5||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||$, $\delta_3 < -\kappa$, $\kappa > \frac{1}{2}\tau^+ r_6$ and $\hat{\theta}(0) > 1$, then the control law (3) will guarantee asymptotic stabilization of the closed-loop system.

Proof The time derivative of $V_c(x)$ is

$$\dot{V}_c(x) \quad = \quad \dot{V}_a(x) \; + \; \dot{V}_{10}(x). \tag{64}$$

Following the steps used in the proof of Theorem 1 and using equation (61), it can be shown that

$$\begin{aligned} \dot{V}_c(x) \;\leq\; & x^\top(t)\Xi x(t) + \tau^+ r_4 x^\top(t)\Delta K^\top(t) B_o^\top B_o \Delta K(t) x(t) + \tau^+ r_5 z^\top(t)\mathcal{I}^\top B_o^\top B_o \mathcal{I} z(t) \\ & + \tau^+ r_6 E^\top(x,t)E(x,t) + 2\rho^*||PB_o||\ ||x(t)||^2 + 2||PB_o\mathcal{I}||\ |\mu(t)|\ ||x(t)||^2 \\ & + 2\theta^*||P||\ ||x(t)||^2 + 2\ \mu(t)\ \dot{\mu}(t) + 2\ \ (1+\theta^*)\left[\hat{\theta}(t) - \theta^*\right]\ \dot{\hat{\theta}}(t), \end{aligned} \tag{65}$$

where Ξ is defined in equation (40). Using the linearization procedure and invoking the Schur complement (as in the proof of Theorem 1), it can be shown that Ξ is guaranteed to be negative definite whenever the LMI (25) has a feasible solution. Now, we need to show that the remaining terms of (65) are negative definite. Using the definition of $z(t) = \mu(t)x(t)$, we know that

$$\tau^{+} r_5 z^{\top}(t)\mathcal{I}^{\top} B_o^{\top} B_o \mathcal{I} z(t) \leq \tau^{+} r_5 \, ||\mathcal{I}^{\top} B_o^{\top} B_o \mathcal{I}|| \, \mu^2(t) \, ||x(t)||^2. \tag{66}$$

Also, using Assumptions 2.1 and 2.2 , we have

$$\tau^{+} r_6 E^{\top}(x,t)E(x,t) \leq \tau^{+} r_6 \, (\theta^*)^2 \, ||x(t)||^2, \tag{67}$$

and

$$\tau^{+} r_4 x^{\top}(t)\Delta K^{\top}(t) B_o^{\top} B_o \Delta K(t) x(t) \leq \tau^{+} r_4 \, (\rho^*)^2 \, ||B_o^{\top} B_o|| \, ||x(t)||^2. \tag{68}$$

Now, using (66)- (68), the adaptive laws (62)- (63), and the fact that $|\mu(t)| \geq 1$, equation (65) becomes

$$\begin{aligned} \dot{V}_c(x) \quad \leq \quad & x^{\top}(t)\Xi x(t) + \tau^{+} r_4 \, (\rho^*)^2 \, ||B_o^{\top} B_o|| \, ||x(t)||^2 + \tau^{+} r_5 \, ||\mathcal{I}^{\top} B_o^{\top} B_o \mathcal{I}|| \, \mu^2(t) \, ||x(t)||^2 \\ & + \tau^{+} r_6 \, (\theta^*)^2 \, ||x(t)||^2 + 2\rho^* ||P B_o|| \, ||x(t)||^2 + 2||P B_o \mathcal{I}|| \, |\mu(t)| \, ||x(t)||^2 \\ & 6 + 2\theta^* ||P|| \, ||x(t)||^2 + 2\delta_1 \, |\mu(t)| \, ||x(t)||^2 + 2\delta_2 \, \mu^2(t) \, ||x(t)||^2 \\ & + 2\delta_3 \, |\mu(t)| \, \hat{\theta}(t) \, ||x(t)||^2 + 2\kappa \, |\mu(t)| \, \hat{\theta}(t) \, ||x(t)||^2 - 2\kappa \, \theta^* \, ||x(t)||^2 \\ & + 2\kappa \, \theta^* \, \hat{\theta}(t) \, ||x(t)||^2 - 2\kappa \, (\theta^*)^2 \, ||x(t)||^2. \end{aligned} \tag{69}$$

It can be shown that $\dot{V}_c(x) < 0$ if the adaptive law parameters δ_1, δ_2, and δ_3 are selected as stated in Theorem 3, and κ is selected to satisfy the following two conditions: $\kappa > \frac{1}{2}\tau^{+} r_6$ and $||P|| - \kappa + \kappa\hat{\theta}(t) < 0$. Hence, we need to select κ such that

$$\kappa \; > \; max \left\{ \frac{1}{2}\tau^{+} r_6 \, , \; \frac{||P||}{1 - \hat{\theta}(t)} \right\}. \tag{70}$$

It is clear that when $\hat{\theta}(t) > 1$, we only need to ensure that $\kappa > \frac{1}{2}\tau^{+} r_6$. Note that from equation (63), $\hat{\theta}(t) > 1$ can be easily ensured by selecting $\hat{\theta}(0) > 1$ and $\kappa > \frac{1}{2}\tau^{+} r_6$ to guarantee that $\hat{\theta}(t)$ in equation (63) is monotonically increasing. Hence, we guarantee that

$$\dot{V}_c(x) \quad \leq \quad x^{\top}(t)\Xi x(t), \tag{71}$$

where $\Xi < 0$. Hence, $\dot{V}_c(x) < 0$ which guarantees asymptotic stabilization of the closed-loop system. ∎

3.4 Adaptive control when both θ^* and ρ^* are unknown

Here, we wish to stabilize the system (6) considering the control law (3) when both θ^* and ρ^* are unknown. Here, the following Lyapunov-Krasovskii functional is used

$$V_d(x) \quad = \quad V_c(x) \; + \; V_{11}(x), \tag{72}$$

where $V_c(x)$ is defined in equations (59), and $V_{11}(x)$ is defined as

$$V_{11}(x) = (1 + \rho^*) \, [\tilde{\rho}(t)]^2 \, , \tag{73}$$

where its time derivative is

$$\dot{V}_{11}(x) = 2\ (1+\rho^*)\,\tilde{\rho}(t)\,\dot{\tilde{\rho}}(t). \tag{74}$$

Since $\tilde{\rho}(t) = \hat{\rho}(t) - \rho^*$, then $\dot{\tilde{\rho}}(t) = \dot{\hat{\rho}}(t)$. Hence, equation (74) becomes

$$\dot{V}_{11}(x) = 2\ (1+\rho^*)\,[\hat{\rho}(t) - \rho^*]\ \dot{\hat{\rho}}(t). \tag{75}$$

The next Theorem provides the main results for this case.

Theorem 4: *Consider system (6). If there exist matrices $0 < \mathcal{X} = \mathcal{X}^\top \in \Re^{n\times n}$, $\mathcal{Y} \in \Re^{m\times n}$, $\mathcal{Z} \in \Re^{n\times n}$, and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > \varepsilon$, $\varepsilon_5 > \varepsilon$ and $\varepsilon_6 > \varepsilon$ (where ε is an arbitrary small positive constant) such that the LMI (25) has a feasible solution, and $K = \mathcal{Y}\mathcal{X}^{-1}$, and $\mu(t)$ is adapted subject to the adaptive laws*

$$\begin{aligned} \dot{\mu}(t) &= Proj\Big\{\lambda_1\ sgn\,(\mu(t))\ \ ||x(t)||^2 + \lambda_2\ \mu(t)\ ||x(t)||^2 \\ &\quad +\lambda_3\ \ sgn\,(\mu(t))\ \ \hat{\theta}(t)\ ||x(t)||^2 + \lambda_4\ \ sgn\,(\mu(t))\ \ \hat{\rho}(t)\ ||x(t)||^2, \mu(t)\Big\}, \end{aligned} \tag{76}$$

$$\dot{\hat{\theta}}(t) = \sigma\,||x(t)||^2, \tag{77}$$

$$\dot{\hat{\rho}}(t) = \varsigma\,||x(t)||^2, \tag{78}$$

where $Proj\{\cdot\}$ Krstic et al. (1995) is applied to ensure that $|\mu(t)| \geq 1$ as follows

$$\mu(t) = \begin{cases} \mu(t) & if\,|\mu(t)| \geq 1 \\ 1 & if\,0 \leq \mu(t) < 1 \\ -1 & if\,-1 < \mu(t) < 0, \end{cases}$$

and the adaptive law parameters are selected such that $\lambda_1 < -[||PB_o\mathcal{I}||]$, $\lambda_2 < -\frac{1}{2}\tau^+ r_5||\mathcal{I}^\top B_o^\top B_o\mathcal{I}||$, $\lambda_3 < -\sigma$, $\lambda_4 < -\varsigma$, $\sigma > \frac{1}{2}\tau^+ r_6$, $\varsigma > \frac{1}{2}\tau^+ r_4||B_o^\top B_o||$, $\hat{\theta}(0) > 1$ and $\hat{\rho}(0) > 1$, then the control law (3) will guarantee asymptotic stabilization of the closed-loop system.

Proof The time derivative of $V_d(x)$ is

$$\dot{V}_d(x) \ \ = \ \ \dot{V}_c(x)\ +\ \dot{V}_{11}(x). \tag{79}$$

Following the steps used in the proof of Theorem 3 and using equation (75), it can be shown that

$$\begin{aligned} \dot{V}_d(x) \leq\ & x^\top(t)\Xi x(t) + \tau^+ r_4 x^\top(t)\Delta K^\top(t) B_o^\top B_o \Delta K(t) x(t) \\ & +\tau^+ r_5 z^\top(t)\mathcal{I}^\top B_o^\top B_o\mathcal{I} z(t) + \tau^+ r_6 E^\top(x,t)E(x,t) + 2\rho^*||PB_o||\ ||x(t)||^2 \\ & +2||PB_o\mathcal{I}||\ |\mu(t)|\ ||x(t)||^2 + 2\theta^*||P||\ ||x(t)||^2 + 2\ \mu(t)\ \dot{\mu}(t) \\ & +2\ (1+\theta^*)\left[\hat{\theta}(t) - \theta^*\right]\ \dot{\hat{\theta}}(t) + 2\ (1+\rho^*)\,[\hat{\rho}(t) - \rho^*]\ \dot{\hat{\rho}}(t), \end{aligned} \tag{80}$$

where Ξ is defined in equation (40). Using the linearization procedure and invoking the Schur complement (as in the proof of Theorem 1), it can be shown that Ξ is guaranteed to be negative definite whenever the LMI (25) has a feasible solution. Using the adaptive laws (76)- (78)

in (80) and the fact that $|\mu(t)| \geq 1$, we get

$$\begin{aligned}
\dot{V}_b(x) \leq\ & x^\top(t)\Xi x(t) + \tau^+ r_4 (\rho^*)^2\ ||B_o^\top B_o||\ ||x(t)||^2 + \tau^+ r_5\ ||\mathcal{I}^\top B_o^\top B_o \mathcal{I}||\ \mu^2(t)\ ||x(t)||^2 \\
& + \tau^+ r_6 (\theta^*)^2\ ||x(t)||^2 + 2\rho^* ||PB_o||\ ||x(t)||^2 + 2||PB_o\mathcal{I}||\ |\mu(t)|\ ||x(t)||^2 \\
& + 2\theta^* ||P||\ ||x(t)||^2 + 2\lambda_1\ |\mu(t)|\ ||x(t)||^2 + 2\lambda_2\ \mu^2(t)\ ||x(t)||^2 \\
& + 2\lambda_3\ |\mu(t)|\ \hat{\theta}(t)\ ||x(t)||^2 + 2\lambda_4\ |\mu(t)|\ \hat{\rho}(t)\ ||x(t)||^2 + 2\sigma\ |\mu(t)|\ \hat{\theta}(t)\ ||x(t)||^2 \\
& - 2\sigma\, \theta^*\ ||x(t)||^2 + 2\sigma\, \theta^*\ \hat{\theta}(t)\ ||x(t)||^2 - 2\sigma\ (\theta^*)^2\ ||x(t)||^2 + 2\varsigma\ |\mu(t)|\ \hat{\rho}(t)\ ||x(t)||^2 \\
& - 2\varsigma\, \rho^*\ ||x(t)||^2 + 2\varsigma\, \rho^*\ \hat{\rho}(t)\ ||x(t)||^2 - 2\varsigma\ (\rho^*)^2\ ||x(t)||^2.
\end{aligned} \tag{81}$$

Arranging terms of equation (81), it can be shown that $\dot{V}_d(x) < 0$ if the adaptive law parameters λ_1, λ_2, λ_3, and λ_4 are selected as stated in Theorem 4, and σ and ς are selected to satisfy the following conditions: $\sigma > \frac{1}{2}\tau^+ r_6$, $2||P|| - \sigma + \sigma\hat{\theta}(t) < 0$, $\varsigma > \frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||$, and $||PB_o|| - \varsigma + \varsigma\hat{\rho}(t) < 0$. Hence, we need to select σ and ς such that

$$\sigma > max\left\{\frac{1}{2}\tau^+ r_6\ ,\ \frac{||P||}{1-\hat{\theta}(t)}\right\}, \tag{82}$$

$$\varsigma > max\left\{\frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||\ ,\ \frac{||PB_o||}{1-\hat{\rho}(t)}\right\}. \tag{83}$$

It is clear that when $\hat{\theta}(t) > 1$ and $\hat{\rho}(t) > 1$, we only need to ensure that $\sigma > \frac{1}{2}\tau^+ r_6$ and $\varsigma > \frac{1}{2}\tau^+ r_4 ||B_o^\top B_o||$. Note that from equations (77)- (78), $\hat{\theta}(t) > 1$ and $\hat{\rho}(t) > 1$ can be easily ensured by selecting $\hat{\theta}(0) > 1$ and $\hat{\rho}(0) > 1$ and σ and ς as stated in Theorem 4 to guarantee that $\hat{\theta}(t)$ and $\hat{\rho}(t)$ are monotonically increasing. Hence, we guarantee that

$$\dot{V}_d(x) \leq x^\top(t)\ \Xi\ x(t), \tag{84}$$

where $\Xi < 0$. Hence, $\dot{V}_d(x) < 0$ which guarantees asymptotic stabilization of the closed-loop system. ∎

Remarks:

1. The results obtained in all theorems stated above are sufficient stabilization results, that is asymptotic stabilization results are guaranteed only if all of the conditions in the theorems are satisfied.
2. The projection for μ may introduce chattering for μ and control input u Utkin (1992). The chattering phenomenon can be undesirable for some applications since it involves high control activity. It can, however, be reduced for easier implementation of the controller. This can be achieved by smoothing out the control discontinuity using, for example, a low pass filter. This, however, affects the robustness of the proposed controller.

4. Simulation example

Consider the second order system in the form of (1) such that

$$A_o = \begin{bmatrix} 2 & 1.1 \\ 2.2 & -3.3 \end{bmatrix},\quad B_o = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix},\ A_d = \begin{bmatrix} -0.5 & 0 \\ 0 & -1.2 \end{bmatrix}, \tag{85}$$

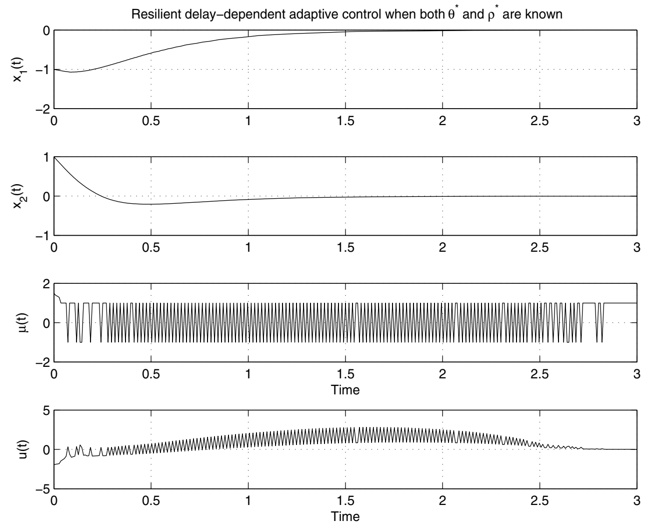

Fig. 1. Closed-loop response when both θ^* and ρ^* are known

and $\tau^* = 0.1$. Using the LMI control toolbox of MATLAB, when the following scalars are selected as $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 1$, the LMI (25) is solved to find the following matrices:

$$\mathcal{X} = \begin{bmatrix} 0.7214 & 0.1639 \\ 0.1639 & 0.2520 \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} -1.7681 & -1.1899 \end{bmatrix}. \qquad (86)$$

Using the fact that $K = \mathcal{Y}\mathcal{X}^{-1}$, K is found to be $K = \begin{bmatrix} -1.6173 & -3.6695 \end{bmatrix}$. Here, for simulation purposes, the nonlinear perturbation function is assumed to be $E(x(t)) = \begin{bmatrix} 1.2\,|x_1(t)|\,, & 1.2\,|x_2(t)| \end{bmatrix}^\top$, where $x(t) = \begin{bmatrix} x_1(t)\,, \, x_2(t) \end{bmatrix}^\top$. Based on Assumption 2.1, it can be shown that $\theta^* = 1.2$. Also, the uncertainty of the state feedback gain is assumed to be $\Delta K(t) = \begin{bmatrix} 0.1sin(t) & 0.1cos(t) \end{bmatrix}$. Hence, based on Assumption 2.2, it can be shown that $\rho^* = 0.1$.

4.1 Simulation results when both θ^* and ρ^* are Known

For this case, the control law (3) is employed subject to the initial conditions $x(0) = [-1\,,\,1]^\top$ and $\mu(0) = 1.5$. To satisfy the conditions of Theorem 1, the adaptive law parameters are selected as $\alpha_1 = -10$ and $\alpha_2 = -0.5$. The closed-loop response of this case is shown in Fig. 1, where the upper two plots show the response of the two states $x_1(t)$ and $x_2(t)$, and third and fourth plots show the projected signal $\mu(t)$ and the control $u(t)$.

4.2 Simulation results when θ^* is known and ρ^* is unknown

For this case, the control law (3) is employed subject to the initial conditions $x(0) = [-1\,,\,1]^\top$ and $\mu(0) = 1.5$ and $\hat{\rho}(0) = 1.1$. To satisfy the conditions of Theorem 2, the adaptive law parameters are selected as $\beta_1 = -10$, $\beta_2 = -0.5$, $\beta_3 = -0.2$, and $\gamma = 0.1$. For this case, the closed-loop response is shown in Fig. 2, where the upper two plots show the response of the two states $x_1(t)$ and $x_2(t)$, third plot shows the projected signal $\mu(t)$, the fourth plot shows $\hat{\rho}(t)$ and the fifth plot shows the control $u(t)$.

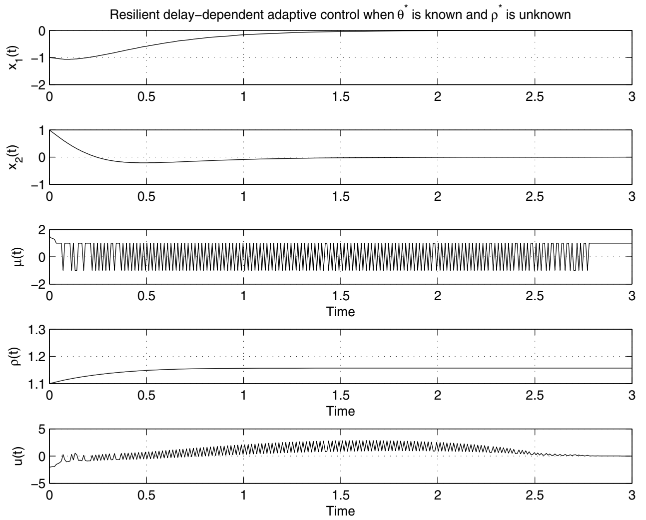

Fig. 2. Closed-loop response when θ^* is known and ρ^* is unknown

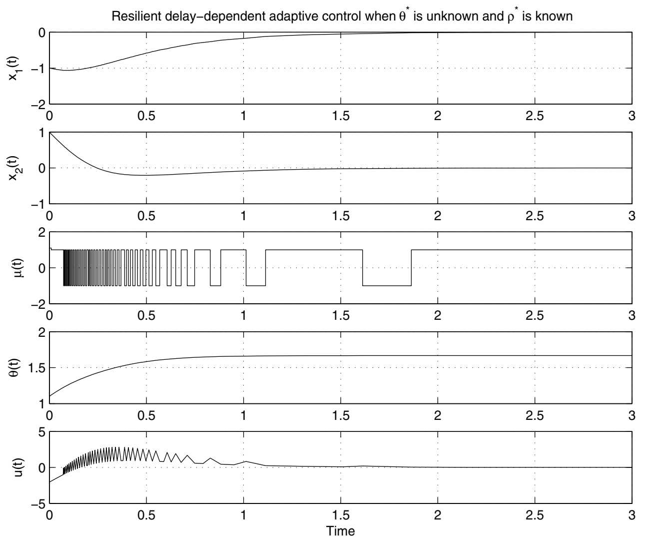

Fig. 3. Closed-loop response when θ^* is unknown and ρ^* is known

4.3 Simulation results when θ^* is unknown and ρ^* is known

For this case, the control law (3) is employed subject to the initial conditions $x(0) = [-1\ ,\ 1]^\top$ and $\mu(0) = 1.1$ and $\hat{\theta}(0) = 1.1$. To satisfy the conditions of Theorem 3, the adaptive law parameters are selected as $\delta_1 = -5$, $\delta_2 = -2$, $\delta_3 = -1.5$ and $\kappa = 1$. For this case, the closed-loop response is shown in Fig. 3, where the upper two plots show the response of the two states $x_1(t)$ and $x_2(t)$, third plot shows the projected signal $\mu(t)$, the fourth plot shows $\hat{\theta}(t)$ and the fifth plot shows the control $u(t)$.

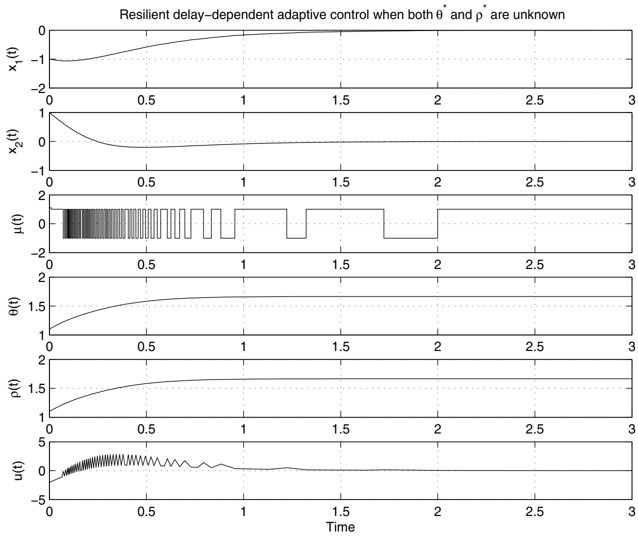

Fig. 4. Closed-loop response when both θ^* and ρ^* are unknown

4.4 Simulation results when both θ^* and ρ^* are unknown

For this case, the control law (3) is employed subject to the initial conditions $x(0) = [-1\ ,\ 1]^\top$ and $\mu(0) = 1.1$, $\hat{\theta}(0) = 1.1$ and $\hat{\rho}(0) = 1.1$. To satisfy the conditions of Theorem 4, the adaptive law parameters are selected as $\lambda_1 = -5$, $\lambda_2 = -1$, $\lambda_3 = -1.5$, $\lambda_4 = -1.5$, $\sigma = 1$, and $\varsigma = 1$. For this case, the closed-loop response is shown in Fig. 4, where the upper two plots show the response of the two states $x_1(t)$ and $x_2(t)$, third plot shows the projected signal $\mu(t)$, the fourth plot shows $\hat{\theta}(t)$, the fifth plot shows $\hat{\rho}(t)$, and the sixth plot shows the control $u(t)$.

5. Conclusion

In this chapter, we investigated the problem of designing resilient delay-dependent adaptive controllers for a class of uncertain time-delay systems with time-varying delays and a nonlinear perturbation when perturbations also appear in the state feedback gain of the controller. It is assumed that the nonlinear perturbation is bounded by a weighted norm of the state vector such that the weight is a positive constant, and the norm of the uncertainty of the state feedback gain is assumed to be bounded by a positive constant. Under these assumptions, adaptive controllers have been developed for all combinations when the upper bound of the nonlinear perturbation weight is known and unknown, and when the value of the upper bound of the state feedback gain perturbation is known and unknown. For all these cases, asymptotically stabilizing adaptive controllers have been derived. Also, a numerical simulation example, that illustrates the design approaches, is presented.

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9

Sliding Mode Control for a Class of Multiple Time-Delay Systems

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1. Introduction

Time-delay frequently occurs in many practical systems, such as chemical processes, manufacturing systems, long transmission lines, telecommunication and economic systems, etc. Since time-delay is a main source of instability and poor performance, the control problem of time-delay systems has received considerable attentions in literature, such as [1]-[9]. The design approaches adopt in these literatures can be divided into the delay-dependent method [1]-[5] and the delay-independent method [6]-[9]. The delay-dependent method needs an exactly known delay, but the delay-independent method does not. In other words, the delay-independent method is more suitable for practical applications. Nevertheless, most literatures focus on linear time-delay systems due to the fact that the stability analysis developed in the two methods is usually based on linear matrix inequality techniques [10]. To deal with nonlinear time-delay systems, the Takagi-Sugeno (TS) fuzzy model-based approaches [11]-[12] extend the results of controlling linear time-delay systems to more general cases. In addition, some sliding-mode control (SMC) schemes have been applied to uncertain nonlinear time-delay systems in [13]-[15]. However, these SMC schemes still exist some limits as follows: i) specific form of the dynamical model and uncertainties [13]-[14]; ii) an exactly known delay time [15]; and iii) a complex gain design [13]-[15]. From the above, we are motivated to further improve SMC for nonlinear time-delay systems in the presence of matched and unmatched uncertainties.
The fuzzy control and the neural network control have attractive features to keep the systems insensitive to the uncertainties, such that these two methods are usually used as a tool in control engineering. In the fuzzy control, the TS fuzzy model [16]-[18] provides an efficient and effective way to represent uncertain nonlinear systems and renders to some straightforward research based on linear control theory [11]-[12], [16]. On the other hand, the neural network has good capabilities in function approximation which is an indirect compensation of uncertainties. Recently, many fuzzy neural network (FNN) articles are proposed by combining the fuzzy concept and the configuration of neural network, e.g., [19]-[23]. There, the fuzzy logic system is constructed from a collection of fuzzy If-Then rules while the training algorithm adjusts adaptable parameters. Nevertheless, few results using FNN are proposed for time-delay nonlinear systems due to a large computational load and a vast amount of feedback data, for example, see [22]-[23]. Moreover, the training algorithm is difficultly found for time-delay systems.

In this paper, an adaptive TS-FNN sliding mode control is proposed for a class of nonlinear time-delay systems with uncertainties. In the presence of mismatched uncertainties, we introduce a novel sliding surface design to keep the sliding motion insensitive to uncertainties and time-delay. Although the form of the sliding surface is as similar as conventional schemes [13]-[15], a delay-independent sufficient condition for the existence of the asymptotic sliding surface is obtained by appropriately using the Lyapunov-Krasoviskii stability method and LMI techniques. Furthermore, the gain condition is transformed in terms of a simple and legible LMI. Here less limitation on the uncertainty is required. When the asymptotic sliding surface is constructed, the ideal and TS-FNN-based reaching laws are derived. The TS-FNN combining TS fuzzy rules and neural network provides a near ideal reaching law. Meanwhile, the error between the ideal and TS-FNN reaching laws is compensated by adaptively gained switching control law. The advantages of the proposed TS-FNN are: i) allowing fewer fuzzy rules for complex systems (since the Then-part of fuzzy rules can be properly chosen); and ii) a small switching gain is used (since the uncertainty is indirectly cancelled by the TS-FNN). As a result, the adaptive TS-FNN sliding mode controller achieves asymptotic stabilization for a class of uncertain nonlinear time-delay systems.

This paper is organized as follows. The problem formulation is given in Section 2. The sliding surface design and ideal sliding mode controller are given in Section 3. In Section 4, the adaptive TS-FNN control scheme is developed to solve the robust control problem of time-delay systems. Section 5 shows simulation results to verify the validity of the proposed method. Some concluding remarks are finally made in Section 6.

2. Problem description

Consider a class of nonlinear time-delay systems described by the following differential equation:

$$\begin{aligned}\dot{x}(t) &= (A+\Delta A)x(t)+\textstyle\sum_{k=1}^{h}(A_{dk}+\Delta A_{dk})x(t-d_k)\\ &\quad +Bg^{-1}(x)(u(t)+h(\bar{x}))\end{aligned}$$

$$x(t)=\psi(t), t\in[-d_{\max}\ \ 0] \tag{1}$$

where $x(t)\in R^n$ and $u(t)\in R$ are the state vector and control input, respectively; $d_k\in R$ ($k=1,\ 2,\ \ldots,\ h$) is an unknown constant delay time with upper bounded $d_{\max}$; A and A_{dk} are nominal system matrices with appropriate dimensions; ΔA and ΔA_{dk} are time-varying uncertainties; $x(t)$ is defined as $\bar{x}(t)=[x(t)\quad x(t-d_1)\quad \cdots \quad x(t-d_h)]^T$; $h(\cdot)$ is an unknown nonlinear function containing uncertainties; B is a known input matrix; $g(\cdot)$ is an unknown function presenting the input uncertainties; and $\psi(t)$ is the initial of state. In the system (1), for simplicity, we assume the input matrix $B=[0\quad \cdots \quad 0\quad 1]^T$ and partition the state vector $x(t)$ into $[x_1(t)\quad x_2(t)]^T$ with $x_1(t)\in R^{n-1}$ and $x_2(t)\in R$. Accompanying the state partition, the system (1) can be decomposed into the following:

$$\begin{aligned}\dot{x}_1(t) &= (A_{11}+\Delta A_{11})x(t)+\textstyle\sum_{k=1}^{h}(A_{dk11}+\Delta A_{dk11})x_1(t-d_k)\\ &\quad +(A_{12}+\Delta A_{12})x(t)+\textstyle\sum_{k=1}^{h}(A_{dk12}+\Delta A_{dk12})x_2(t-d_k)\end{aligned} \tag{2}$$

$$\dot{x}_2(t) = (A_{21} + \Delta A_{21})x_1(t) + \sum_{k=1}^{h}(A_{dk11} + \Delta A_{dk11})x_1(t - d_k)$$
$$+(A_{22} + \Delta A_{22})x_2(t) + \sum_{k=1}^{h}(A_{dk22} + \Delta A_{dk22})x_2(t - d_k) \tag{3}$$
$$+g^{-1}(x)(u(t) + h(\overline{x}(t)))$$

where A_{ij}, A_{dkij}, ΔA_{ij}, and ΔA_{dkij} (for $i, j = 1,\ 2$ and $k = 1,\ ...,\ h$) with appropriate dimension are decomposed components of A, A_{dk}, ΔA, and ΔA_{dk}, respectively.
Throughout this study we need the following assumptions:
Assumption 1: For controllability, $g(x) > 0$ for $x(t) \in U_c$, where $U_c \subset \mathrm{R}^n$. Moreover, $g(x) \in L_\infty$ if $x(t) \in L_\infty$.
Assumption 2: The uncertainty $h(\overline{x})$ is bounded for all $\overline{x}(t)$.
Assumption 3: The uncertain matrices satisfy

$$[\Delta A_{11} \quad \Delta A_{12}] = D_1C_1\left[E_{11} \quad E_{12}\right] \tag{4}$$

$$[\Delta A_{dk11} \quad \Delta A_{dk12}] = D_2C_2\left[E_{dk11} \quad E_{dk12}\right] \tag{5}$$

for some known matrices D_i, C_i, E_{1i}, and E_{dk1i} (for $i = 1,\ 2$) with proper dimensions and unknown matrices C_i satisfying $\|C_i\| \le 1$ (for $i = 1,\ 2$).
Note that most nonlinear systems satisfy the above assumptions, for example, chemical processes or stirred tank reactor systems, etc. If $g(x)$ is negative, the matrix B can be modified such that Assumption 1 is obtained. Assumption 3 often exists in robust control of uncertainties. Since uncertainties ΔA and ΔA_d are presented, the dynamical model is closer to practical situations which are more complex than the cases considered in [13]-[15].
Indeed, the control objective is to determine a robust adaptive fuzzy controller such that the state $x(t)$ converges to zero. Since high uncertainty is considered here, we want to derive a sliding-mode control (SMC) based design for the control goal. Note that the system (1) is not the Isidori-Bynes canonical form [21], [24] such that a new design approaches of sliding surface and reaching control law is proposed in the following.

3. Sliding surface design

Due to the high uncertainty and nonlinearity in the system (1), an asymptotically stable sliding surface is difficultly obtained in current sliding mode control. This section presents an alternative approach to design an asymptotic stable sliding surface below.
Without loss of generality, let the sliding surface denote

$$S(t) = [-\Lambda \quad 1\ ]x(t) = \tilde{\Lambda}x(t) = 0 \tag{6}$$

where $\Lambda \in R^{(n-1)}$ and $\tilde{\Lambda} = \left[-\Lambda \quad 1\right]$ determined later. In the surface, we have $x_2(t) = \Lambda x_1(t)$. Thus, the result of sliding surface design is stated in the following theorem.
Theorem 1: Consider the system (1) lie in the sliding surface (6). The sliding motion is asymptotically stable independent of delay, i.e., $\lim_{t\to\infty} x_1(t),\ x_2(t) = 0$, if there exist positive symmetric matrices X, $\overline{Q}_k$ and a parameter Λ satisfying the following LMI:

Given $\varepsilon > 0$,
Subject to $X > 0$, $\bar{Q}_k > 0$

$$\begin{bmatrix} N_{11} & (*) \\ N_{21} & -I_{\varepsilon} \end{bmatrix} < 0 \tag{8}$$

where

$$N_{11} = \begin{bmatrix} N_0 & (*) & (*) & (*) \\ XA_{d111}^T + K^T A_{d112}^T & -\bar{Q}_1 & (*) & (*) \\ \vdots & \vdots & \ddots & (*) \\ XA_{dh11}^T + K^T A_{dh12}^T & 0 & \cdots & -\bar{Q}_h \end{bmatrix}$$

$$N_{21} = \begin{bmatrix} E_{11}X + A_{12}K & 0 & 0 & 0 \\ 0 & E_{111}X + E_{112}K & \cdots & E_{h11}X + E_{h12}K \\ D_1^T & 0 & \ddots & 0 \\ D_2^T & 0 & \cdots & 0_h \end{bmatrix}$$

$$N_0 = A_{11}X + XA_{11}^T + A_{12}K + K^T A_{12}^T + \sum_{k=1}^{h} \bar{Q}_k \ ;$$

$K = \Lambda X$; $I_{\varepsilon} = diag\{\varepsilon I_a, \varepsilon I_a, \varepsilon^{-1} I_b, \varepsilon^{-1} I_b\}$ in which I_a, I_b are identity matrices with proper dimensions; and (*) denotes the transposed elements in the symmetric positions. ■

Proof: When the system (1) lie in the sliding surface (6), the sliding motion is described by the dynamics (7). To analysis the stability of the sliding motion, let us define the following Lyapunov-Krasoviskii function

$$V(t) = x_1^T(t) P x_1(t) + \sum_{k=1}^{h} \int_{t-d_k}^{t} x_1^T(v) Q_k x_1(v) dv$$

where $P > 0$ and $Q_k > 0$ are symmetric matrices. The time derivative of $V(t)$ along the dynamics (7) is

$$\dot{V}(t) = \bar{x}^T(t)(\Omega_1 + \Omega_2)\bar{x}(t)$$

where

$$\Omega_1 = \begin{bmatrix} \Omega_{10} & (*) & (*) & (*) \\ (A_{d111} + A_{d112}\Lambda)^T P & -Q_1 & (*) & (*) \\ \vdots & \vdots & \ddots & (*) \\ (A_{dh11} + A_{dh12}\Lambda)^T P & 0 & \cdots & -Q_h \end{bmatrix}$$

$$\Omega_{10} = (A_{11} + \Lambda A_{12})^T P + P(A_{11} + \Lambda A_{12}) + \sum_{k=1}^{h} Q_k$$

$$\Omega_2 = \begin{bmatrix} \Omega_{20} & (*) & (*) & (*) \\ \left[D_2C_2(E_{111}+E_{112}\Lambda)\right]^T P & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \left[D_2C_2(E_{h11}+E_{h12}\Lambda)\right]^T P & 0 & \cdots & 0 \end{bmatrix}$$

$$\Omega_{20} = PD_1C_1(E_{11}+E_{12}\Lambda) + \left[D_1C_1(E_{11}+E_{12}\Lambda)\right]^T P$$

Note that the second term Ω_2 can be further rewritten in the form:

$$\Omega_2 = \bar{D}\bar{C}\bar{E} + \bar{E}^T\bar{C}^T\bar{D}^T$$

where

$$\bar{D} = \begin{bmatrix} PD_1 & PD_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \ \bar{C}^T = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

$$\bar{E} = \begin{bmatrix} E_{11}+E_{12}\Lambda & 0 & 0 & 0 \\ 0 & E_{111}+E_{112}\Lambda & \cdots & E_{h11}+E_{h12}\Lambda \end{bmatrix}$$

with $\bar{C}$ satisfies $\bar{C}^T\bar{C} \le I_d$ for identity matrix I_d from Assumption 3. According to the matrix inequality lemma [25] (see Appendix I) and the decomposition (9), the stability condition $\Omega < 0$ is equivalent to

$$\Omega_1 + \begin{bmatrix} \bar{E}^T & \bar{D} \end{bmatrix} I_\varepsilon^{-1} \begin{bmatrix} \bar{E} \\ \bar{D}^T \end{bmatrix} < 0$$

After applying the Schur complement to the above inequality, we further have

$$\begin{bmatrix} \Omega_1 & (*) \\ M_{21} & -I_\varepsilon \end{bmatrix} < 0$$

where

$$M_{21} = \begin{bmatrix} E_{11}+E_{12}\Lambda & 0 & 0 & 0 \\ 0 & E_{121}+E_{122}\Lambda & \cdots & E_{h21}+E_{h22}\Lambda \\ D_1^T P & 0 & \ddots & 0 \\ D_2^T P & 0 & \cdots & 0_h \end{bmatrix}$$

By premultiplying and postmultiplying above inequality by a symmetric positive-definite matrix $diag\{X\bar{I}_a, \bar{I}_b\}$ with $\bar{I}_a$, $\bar{I}_b$ are identity matrices with proper dimensions, the LMI addressed in (8) is obtained with $X = P^{-1}$ and $\bar{Q}_k = XQ_kX$. Therefore, if the LMI problem has a feasible solution, then the sliding dynamical system (7) is asymptotically stable, i.e., $\lim_{t\to\infty} x_1(t) = 0$. In turn, from the fact $x_2(t) = \Lambda x_1(t)$ in the sliding surface, the state $x_2(t)$ will asymptotically converge to zero as $t \to \infty$. Moreover, since the gain condition (8) does not contain the information of the delay time, the stability is independent of the delay.

After solving the LMI problem (8), the sliding surface is constructed by $\Lambda = KP$. Therefore, the LMI-based sliding surface design is completed for uncertain time-delay systems.

Note that the main contribution of Theorem 1 is solving the following problems: i) the sliding surface gain Λ appears in the delayed term $x_1(t-d_k)$ such that the gain design is highly coupled; and ii) the mismatched uncertainties (e.g., ΔA_{11}, ΔA_{dk11}, ΔA_{12}, ΔA_{dk12}) is considered in the design. Compared to current literature, this study proposes a valid and straightforward LMI-based sliding mode control for highly uncertain time-delay systems.

The design of exponentially stable sliding surface, a coordinate transformation is used $\sigma(t) = e^{\gamma t} x_1(t)$ with an attenuation rate $\gamma > 0$. When $\sigma(t)$ is asymptotically stable, the state $x_1(t)$ exponentially stable is guaranteed (see **Appendix II or [26],[28] in detail)**.

Based on Theorem 1, the control goal becomes to drive the system (1) to the sliding surface defined in (6). To this end, let us choose a Lyapunov function candidate $V_s = g(x)S^2/2$. Taking the derivative the Lyapunov V_s along with (1), it renders to

$$\begin{aligned}\dot{V}_s(t) &= g(x)S(t)\dot{S}(t) + \dot{g}(x)S^2(t)/2 \\ &= S(t)\Big[g(x)\Lambda\Big\{(A+\Delta A)x(t) + \sum_{k=1}^{h}(A_{dk}+\Delta A_{dk})x(t-d_k)\Big\}\Big] \\ &\quad + \dot{g}(x)S^2(t)/2 + u(t) + h(\bar{x})\end{aligned}$$

If the plant dynamics and delay-time are exactly known, then the control problem can be solved by the so-called feedback linearization method [24]. In this case, the ideal control law u^* is set to

$$\begin{aligned}u^*(t) = -\{g(x)\Lambda\Big[&\sum_{k=1}^{h}(A_{dk}+\Delta A_{dk})x(t-d_k) \\ &+(A+\Delta A)x\Big](t) + g(x)S^2(t)/2 + k_f S(t) + h(\bar{x})\}\end{aligned} \tag{10}$$

where k_f is a positive control gain. Then the ideal control law (10) yields $\dot{V}_s(t)$ satisfying $\dot{V}_s(t) < 0$.

Since $V_s(t) > 0$ and $\dot{V}_s(t) < 0$, the error signal $S(t)$ converges to zero in an asymptotic manner, i.e., $\lim_{t\to\infty} S(t) = 0$. This implies that the system (1) reaches the sliding surface $S(t) = 0$ for any start initial conditions. Therefore, the ideal control law provides the following result.

Unfortunately, the ideal control law (10) is unrealizable in practice applications due to the poor modeled dynamics. To overcome this difficulty, we will present a robust reaching control law by using an adaptive TS-FNN control in next section.

4. TS-FNN-based sliding mode control

In control engineering, neural network is usually used as a tool for modeling nonlinear system functions because of their good capabilities in function approximation. In this section, the TS-FNN [26] is proposed to approximate the ideal sliding mode control law $u^*(t)$. Indeed, the FNN is composed of a collection of T-S fuzzy IF-THEN rules as follows:

Rule i :

$$IF\ \bar{z}_1 \text{ is } \bar{G}_{i1} \text{ and } \cdots and\ \bar{z}_{ni} \text{ is } \bar{G}_{ni} \text{ THEN}$$

$$u_n(t) = z_0 v_{i0} + z_1 v_{i1} + \cdots + z_{nv} v_{inv} \ = \mathrm{z}^{\mathrm{T}} v_i$$

for $i = 1, 2, \cdots, n_R$, where n_R is the number of fuzzy rules; $\bar{z}_1 \sim \bar{z}_{ni}$ are the premise variables composed of available signals; u_n is the fuzzy output with tunable $v_i = \begin{bmatrix} v_{i0} & v_{i1} & \cdots & v_{inv} \end{bmatrix}^T$ and properly chosen signal $z = \begin{bmatrix} z_0 & z_1 & \cdots & z_{nv} \end{bmatrix}^T$; $\bar{G}_{ij}(\bar{z}_j)$ ($j = 1, 2, \cdots, n_i$) are the fuzzy sets with Gaussian membership functions which have the form $\bar{G}_{ij}(\bar{z}_j) = \exp(-(\bar{z}_j - m_{ij})^2 / (\sigma_{ij}^2))$ where m_{ij} is the center of the Gaussian function; and σ_{ij} is the variance of the Gaussian function.

Using the singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the inferred output of the fuzzy neural network is $u_n = \sum_{i=1}^{nr} \mu_i(\bar{z}) z^T v_i$ where $\mu_i(\bar{z}) = \bar{\omega}_i(\bar{z}) / \sum_{i=1}^{nr}(\bar{z})$, $\bar{z} = \begin{bmatrix} \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_{ni} \end{bmatrix}^T$ and $\bar{\omega}_i(\bar{z}) = \prod_{j=1}^{ni} G_{ij}(\bar{z}_j)$. For simplification, define two auxiliary signals

$$\xi = \begin{bmatrix} z^T \mu_1 & z^T \mu_2 & \cdots & z^T \mu_{nR} \end{bmatrix}^T$$

$$\theta = \begin{bmatrix} v_1^T & v_2^T & \cdots & v_{nR}^T \end{bmatrix}^T .$$

In turn, the output of the TS-FNN is rewritten in the form:

$$u_n(t) = \xi^T \theta \tag{13}$$

Thus, the above TS-FNN has a simple structure, which is easily implemented in comparison of traditional FNN. Moreover, the signal z can be appropriately selected for more complex function approximation. In other words, we can use less fuzzy rules to achieve a better approximation.

According to the uniform approximation theorem [19], there exists an optimal parametric vector θ^* of the TS-FNN which arbitrarily accurately approximates the ideal control law $u^*(t)$. This implies that the ideal control law can be expressed in terms of an optimal TS-FNN as $u^*(t) = \xi^T \theta + \bar{\varepsilon}(x)$ where $\bar{\varepsilon}(x)$ is a minimum approximation error which is assumed to be upper bounded in a compact discussion region. Meanwhile, the output of the TS-FNN is further rewritten in the following form:

$$u_n = u^* \xi^T \tilde{\theta} - \bar{\varepsilon}(x) \tag{14}$$

where $\tilde{\theta} = \theta - \theta^*$ is the estimation error of the optimal parameter. Then, the tuning law of the FNN is derived below.

Based on the proposed TS-FNN, the overall control law is set to

$$u(t) = u_n(t) + u_c(t) \tag{15}$$

where $u_n(t)$ is the TS-FNN controller part defined in (13); and $u_c(t)$ is an auxiliary compensation controller part determined later. The TS-FNN control $u_n(t)$ is the main tracking controller part that is used to imitate the idea control law $u^*(t)$ due to high uncertainties, while the auxiliary controller part $u_c(t)$ is designed to cope with the difference between the idea control law and the TS-FNN control. Then, applying the control law (15) and the expression form of $u_n(t)$ in (10), the error dynamics of S is obtained as follows:

$$\begin{aligned}
&g(x)\dot{S}(t) \\
&= g(x)\Lambda\left[(A+\Delta A)x(t)+\sum_{k=1}^{h}(A_{dk}+\Delta A_{dk})x_1(t-d_k)\right] \\
&+h(\bar{x})+u^*(t)+\xi^T\tilde{\theta}-\bar{\varepsilon}(x)+u_c(t) \\
&= -k_f S(t)-\frac{1}{2}\dot{g}(x)S(t)+\xi^T\tilde{\theta}-\bar{\varepsilon}(x)+u_c(t)
\end{aligned}$$

where the definition of $u^*(t)$ in (10) has been used. Now, the auxiliary controller part and tuning law of FNN are stated in the following.

Theorem 2: Consider the uncertain time-delay system (1) using the sliding surface designed by Theorem 1 and the control law (15) with the TS-FNN controller part (14) and the auxiliary controller part

$$u_n(t) = -\hat{\delta}\,\mathrm{sgn}(S(t))$$

The controller is adaptively tuned by

$$\dot{\theta}(t) = -\eta_\theta S(t)\xi \tag{17}$$

$$\dot{\hat{\delta}}(t) = -\eta_\delta |S(t)| \tag{18}$$

where η_θ and η_δ are positive constants. The closed-loop error system is guaranteed with asymptotic convergence of $S(t)$, $x_1(t)$, and $x_2(t)$, while all adaptation parameters are bounded. ■

Proof: Consider a Lyapunov function candidate as

$$V_n(t) = \frac{1}{2}\left(g(x)S^2(t)+\frac{1}{\eta_\theta}\tilde{\theta}(t)^T\tilde{\theta}(t)+\frac{1}{\eta_\delta}\tilde{\delta}^2(t)\right)$$

where $\tilde{\delta}(t)=\hat{\delta}(t)-\delta$ is the estimation error of the bound of $\bar{\varepsilon}(x)$ (i.e., $\sup_t|\bar{\varepsilon}(x)|\le\delta$). By taking the derivative the Lyapunov $V_n(t)$ along with (16), we have

$$\begin{aligned}
\dot{V}_n(t) &= \frac{1}{2}g(x)S(t)\dot{S}(t)+\frac{1}{2}\dot{g}(x)S^2(t)+\frac{1}{\eta_\theta}\tilde{\theta}(t)^T\dot{\tilde{\theta}}(t)+\frac{1}{\eta_\delta}\tilde{\delta}(t)\dot{\tilde{\delta}}(t) \\
&= -k_f S^2(t)-S(t)\bar{\varepsilon}(x)-\delta|S(t)|+S(t)\xi^T\tilde{\theta}(t)+\frac{1}{\eta_\theta}\tilde{\theta}(t)^T\dot{\tilde{\theta}}(t) \\
&-(\hat{\delta}(t)-\delta)|S(t)|+\frac{1}{\eta_\delta}\tilde{\delta}(t)\dot{\tilde{\delta}}(t)
\end{aligned}$$

When substituting the update laws (17), (18) into the above, $V_n(t)$ further satisfies

$$\begin{aligned}\dot{V}_n(t) &= -k_f S^2(t) - S(t)\bar{\varepsilon}(x) - \delta|S(t)| \\ &\le -k_f S^2(t) - (\delta - |\bar{\varepsilon}(x)|)|S(t)| \\ &\le -k_f S(t)\end{aligned}$$

Since $V_n(t) > 0$ and $\dot{V}_n(t) < 0$, we obtain the fact that $V_n(t) \le V_n(0)$, which implies all $S(t)$, $\tilde{\theta}(t)$ and $\tilde{\delta}(t)$ are bounded. In turn, $\dot{S}(t) \in L_\infty$ due to all bounded terms in the right-hand side of (16). Moreover, integrating both sides of the above inequality, the error signal $S(t)$ is L_2-gain stable as

$$k_f \int_0^t S^2(\tau)d\tau \le V_n(0) - V_n(t) \le V_n(0)$$

where $V_n(0)$ is bounded and $V_n(t)$ is non-increasing and bounded. As a result, combining the facts that $S(t)$, $\dot{S}(t) \in L_\infty$ and, $S(t) \in L_2$ the error signal $S(t)$ asymptotically converges to zero as $t \to \infty$ by Barbalat's lemma. Therefore, according to Theorem 1, the state $x(t)$ will is asymptotically sliding to the origin. The results will be similar when we replace another FNN[26] or NN[27] with the TS-FNN, but the slight different to transient.

5. Simulation results

In this section, the proposed TS-FNN sliding mode controller is applied to two uncertain time-delay system.

Example 1: Consider an uncertain time-delay system described by the dynamical equation (1) with $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}$,

$$A + \Delta A(t) = \begin{bmatrix} -10+\sin(t) & 1 & 1+\sin(t) \\ 1 & -8-\cos(t) & 1-\cos(t) \\ 5+\cos(t) & 4+\sin(t) & 2+\cos(t) \end{bmatrix}$$

$$A_d + \Delta A_d(t) = \begin{bmatrix} 1+\sin(t) & 0 & 1+\sin(t) \\ 0 & 1+\cos(t) & 1+\cos(t) \\ 3+\sin(t) & 4+\cos(t) & 2+\sin(t) \end{bmatrix}$$

$B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, $g(\bar{x}) = 1$ and $h(\bar{x}) = 0.5\|x\| + \|x(t-d)\| + \sin(t)$.

It is easily checked that Assumptions 1~3 are satisfied for the above system. Moreover, for Assumption 3, the uncertain matrices $\Delta A_{11}(t)$, $\Delta A_{12}(t)$, $\Delta A_{d11}(t)$, and $\Delta A_{d12}(t)$ are decomposed with

$$D_1 = D_2 = E_{11} = E_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ E_{12} = E_{22} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T,$$

$$C_1(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & -\cos(t) \end{bmatrix},\ C_2(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{bmatrix}$$

First, let us design the asymptotic sliding surface according to Theorem 1. By choosing $\varepsilon = 0.2$ and solving the LMI problem (8), we obtain a feasible solution as follows:

$$\Lambda = \begin{bmatrix} 0.4059 & 0.4270 \end{bmatrix}$$

$$P = \begin{bmatrix} 9.8315 & 0.2684 \\ 0.2684 & 6.1525 \end{bmatrix},$$

$$Q = \begin{bmatrix} 85.2449 & 2.9772 \\ 2.9772 & 51.2442 \end{bmatrix}$$

The error signal S is thus created from (6).

Next, the TS-FNN (11) is constructed with $n_i = 1$, $n_R = 8$, and $n_v = 4$. Since the T-S fuzzy rules are used in the FNN, the number of the input of the TS-FNN can be reduced by an appropriate choice of THEN part of the fuzzy rules. Here the error signal S is taken as the input of the TS-FNN, while the discussion region is characterized by 8 fuzzy sets with Gaussian membership functions as (12). Each membership function is set to the center $m_{ij} = -2 + 4(i-1)/(n_R - 1)$ and variance $\sigma_{ij} = 10$ for $i = 1,\ldots, n_R$ and $j = 1$. On the other hand, the basis vector of THEN part of fuzzy rules is chosen as $z = \begin{bmatrix} 1 & x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$. Then, the fuzzy parameters v_j are tuned by the update law (17) with all zero initial condition (i.e., $v_j(0) = 0$ for all j).

In this simulation, the update gains are chosen as $\eta_\theta = 0.01$ and $\eta_\delta = 0.01$. When assuming the initial state $x(0) = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}^T$ and delay time $d(t) = 0.2 + 0.15\cos(0.9t)$, the TS-FNN sliding controller (17) designed from Theorem 3 leads to the control results shown in Figs. 1 and 2. The trajectory of the system states and error signal $S(t)$ asymptotically converge to zero. Figure 3 shows the corresponding control effort.

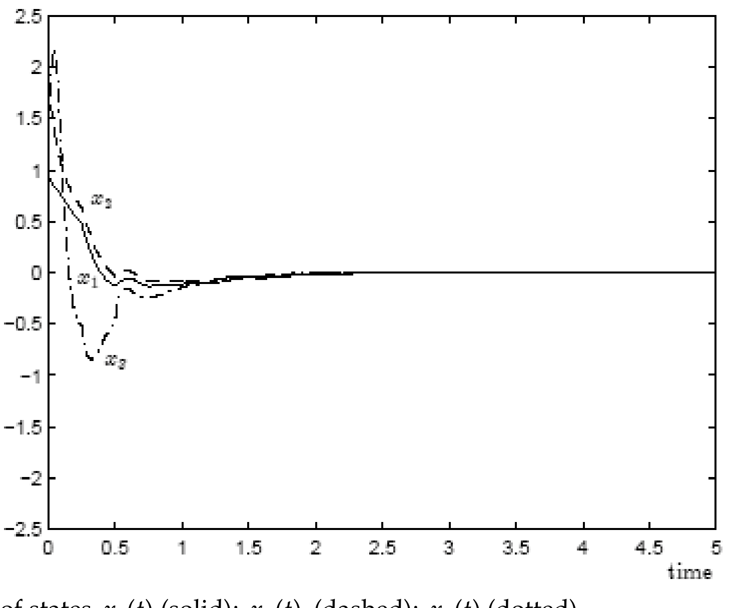

Fig. 1. Trajectory of states $x_1(t)$ (solid); $x_2(t)$ (dashed); $x_3(t)$ (dotted).

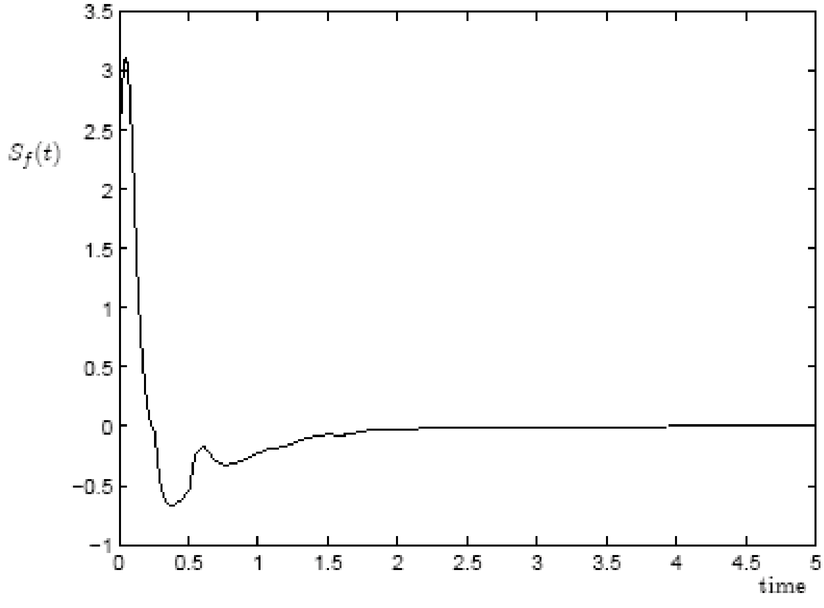

Fig. 2. Dynamic sliding surface $S(t)$.

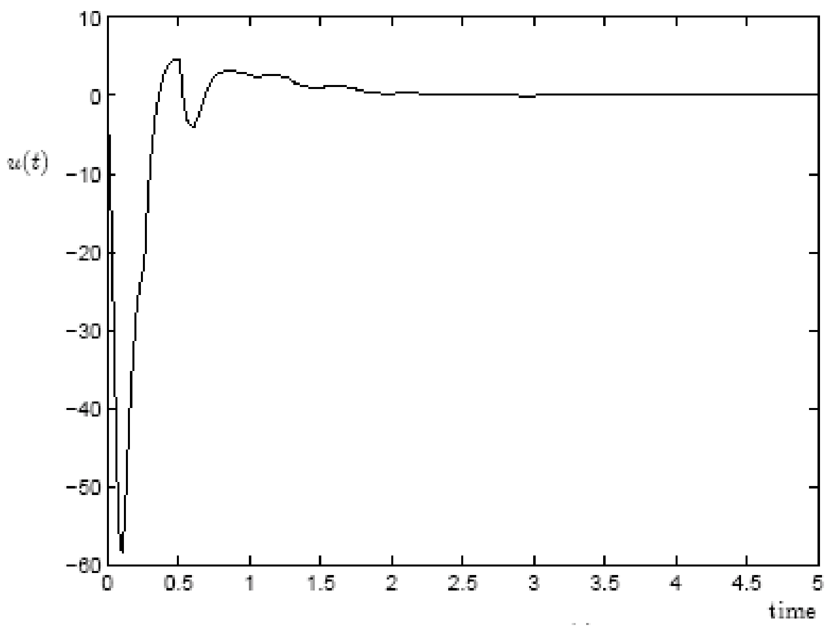

Fig. 3. Control effort $u(t)$.

Example 2: Consider a chaotic system with multiple time-dely system. The nonlinear system is described by the dynamical equation (1) with $x(t) = \left[x_1(t) \quad x_2(t)\right]$,

$$\begin{aligned}\dot{x}(t) &= (A + \Delta A)x(t) + Bg^{-1}(x)(u(t) + h(\bar{x})) \\ &+(A_{d1} + \Delta A_{d1})x(t - 0.02) \\ &+(A_{d2} + \Delta A_{d2})x(t - 0.015)\end{aligned}$$

where $g^{-1}(x) = 4.5$,

$$A = \begin{bmatrix} 0 & 2.5 \\ -\dfrac{1}{2.5} & -0.1 \end{bmatrix},\ \Delta A = \begin{bmatrix} \sin(t) & \sin(t) \\ 0 & 0 \end{bmatrix}$$

$$A_{d1} = \begin{bmatrix} 0 & 0 \\ 0.01 & 0.01 \end{bmatrix},\ \Delta A_{d1} = \begin{bmatrix} \cos(t) & \cos(t) \\ \cos(t) & \sin(t) \end{bmatrix}$$

$$A_{d2} = \begin{bmatrix} 0 & 0 \\ 0.01 & 0.01 \end{bmatrix},\ \Delta A_{d2} = \begin{bmatrix} \cos(t) & \cos(t) \\ \sin(t) & \cos(t) \end{bmatrix},$$

$$\begin{aligned}h(\bar{x}) &= \frac{1}{4.5}[-(\frac{1}{2.5}x_1(t))^3 + 0.01x_2^2(t - 0.02) \\ &+0.01x_2^2(t - 0.015) + 25\cos(t)]\end{aligned}$$

If both the uncertainties and control force are zero the nonlinear system is chaotic system (c.f. [23]). It is easily checked that Assumptions 1~3 are satisfied for the above system. Moreover, for Assumption 3, the uncertain matrices ΔA_{11}, ΔA_{12}, ΔA_{d111}, ΔA_{d112} ΔA_{d211}, and ΔA_{d212} are decomposed with

$$D_1 = D_2 = E_{11} = E_{21} = E_{111} = E_{112} = E_{211} = E_{212} = 1$$

$$C_1 = \sin(t)\ , C_2 = \cos(t)$$

First, let us design the asymptotic sliding surface according to Theorem 1. By choosing $\varepsilon = 0.2$ and solving the LMI problem (8), we obtain a feasible solution as follows: $\Lambda = 1.0014$, $P = 0.4860$, and $Q_1 = Q_2 = 0.8129$. The error signal $S(t)$ is thus created.

Next, the TS-FNN (11) is constructed with $n_i = 1$, $n_R = 8$, and. Since the T-S fuzzy rules are used in the FNN, the number of the input of the TS-FNN can be reduced by an appropriate choice of THEN part of the fuzzy rules. Here the error signal S is taken as the input of the TS-FNN, while the discussion region is characterized by 8 fuzzy sets with Gaussian membership functions as (12). Each membership function is set to the center $m_{ij} = -2 + 4(i-1)/(n_R - 1)$ and variance $\sigma_{ij} = 5$ for $i = 1, \ldots, n_R$ and $j = 1$. On the other hand, the basis vector of THEN part of fuzzy rules is chosen as $z = \left[75 \quad x_1(t) \quad x_2(t)\right]^T$. Then, the fuzzy parameters v_j are tuned by the update law (17) with all zero initial condition (i.e., $v_j(0) = 0$ for all j).

In this simulation, the update gains are chosen as $\eta_\theta = 0.01$ and $\eta_\delta = 0.01$. When assuming the initial state $x(0) = \left[2 \quad -2\right]$, the TS-FNN sliding controller (15) designed from Theorem 3 leads to the control results shown in Figs. 4 and 5. The trajectory of the system states and error signal S asymptotically converge to zero. Figure 6 shows the corresponding control effort. In addition, to show the robustness to time-varying delay, the proposed controller set above is also applied to the uncertain system with delay time $d_2(t) = 0.02 + 0.015\cos(0.9t)$. The trajectory of the states and error signal S are shown in Figs. 7 and 8, respectively. The control input is shown in Fig. 9

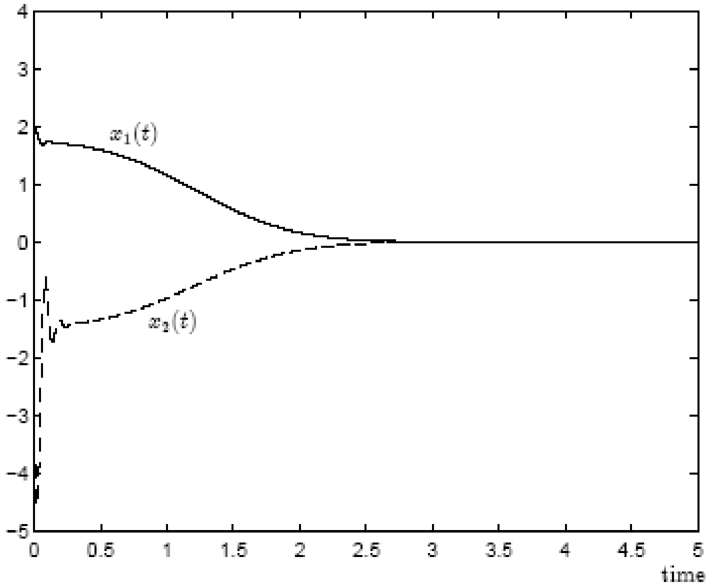

Fig. 4. Trajectory of states $x_1(t)$ (solid); $x_2(t)$ (dashed).

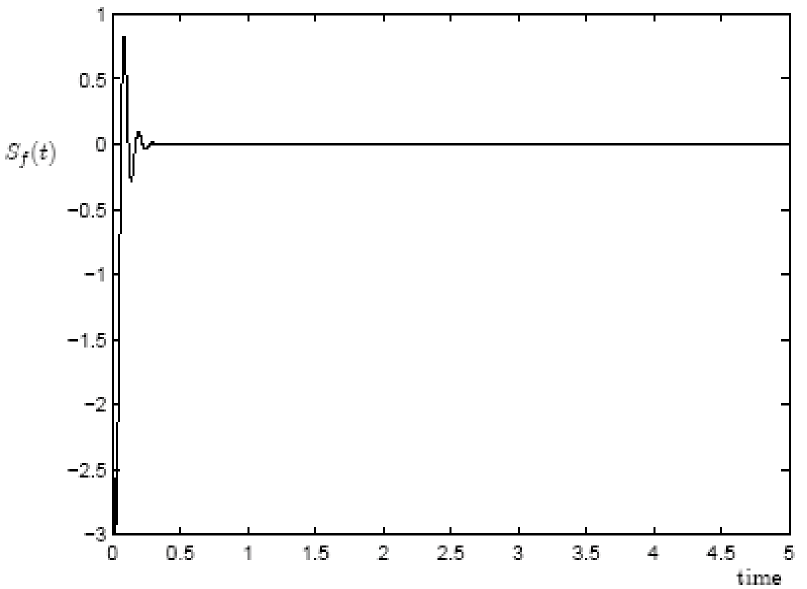

Fig. 5. Dynamic sliding surface $S(t)$.

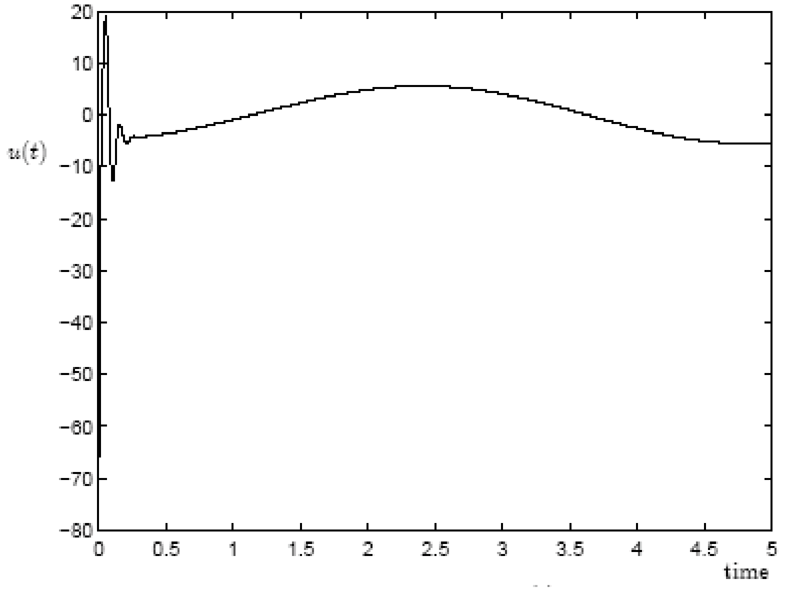

Fig. 6. Control effort $u(t)$.

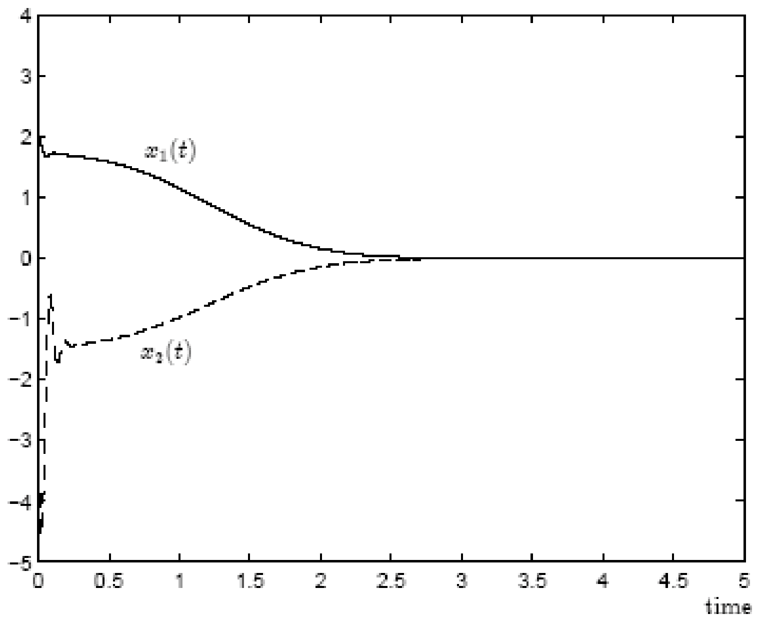

Fig. 7. Trajectory of states $x_1(t)$ (solid); $x_2(t)$ (dashed).

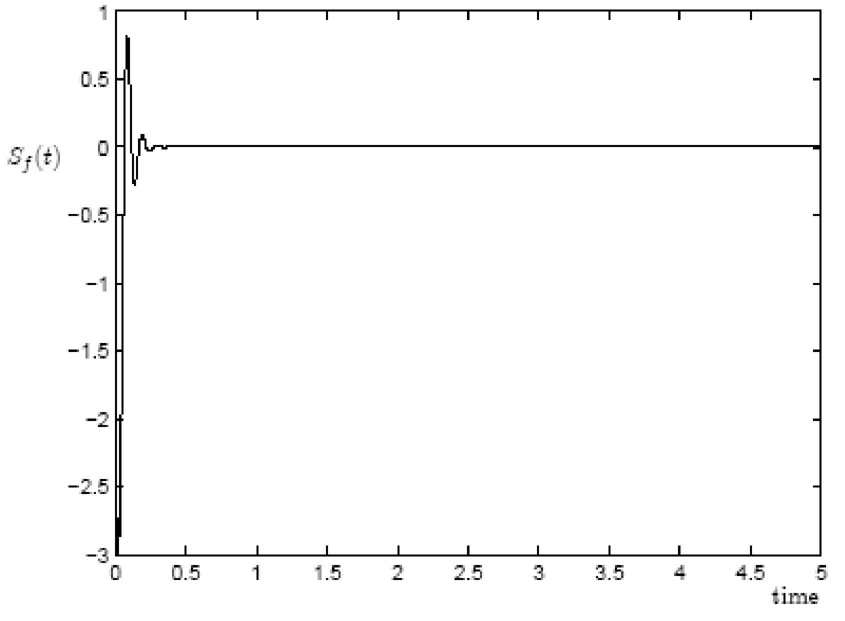

Fig. 8. Dynamic sliding surface $S(t)$.

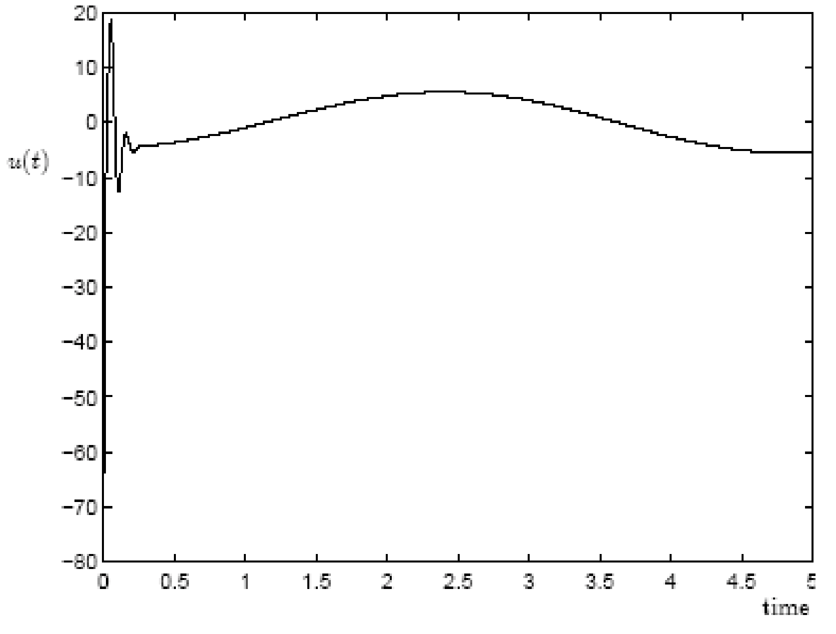

Fig. 9. Control effort $u(t)$.

5. Conclusion

In this paper, the robust control problem of a class of uncertain nonlinear time-delay systems has been solved by the proposed TS-FNN sliding mode control scheme. Although the system dynamics with mismatched uncertainties is not an Isidori-Bynes canonical form, the sliding surface design using LMI techniques achieves an asymptotic sliding motion. Moreover, the stability condition of the sliding motion is derived to be independent on the delay time. Based on the sliding surface design, and TS-FNN-based sliding mode control laws assure the robust control goal. Although the system has high uncertainties (here both state and input uncertainties are considered), the adaptive TS-FNN realizes the ideal reaching law and guarantees the asymptotic convergence of the states. Simulation results have demonstrated some favorable control performance by using the proposed controller for a three-dimensional uncertain time-delay system.

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Appendix I

Refer to the matrix inequality lemma in the literature [25]. Consider constant matrices D, E and a symmetric constant matrix G with appropriate dimension. The following matrix inequality

$$G + DC(t)E + E^T C^T(t) D^T < 0$$

for $C(t)$ satisfying $C^T(t)C(t) \le R$, if and only if, is equivalent to

$$G + \begin{bmatrix} E^T & D \end{bmatrix} \begin{bmatrix} \varepsilon^{-1}R & 0 \\ 0 & \varepsilon I \end{bmatrix} \begin{bmatrix} E \\ D^T \end{bmatrix} < 0$$

for some ε >0. ■

Appendix II

An exponential convergence is more desirable for practice. To design an exponential sliding mode, the following coordinate transformation is used

$$\sigma(t) = e^{\gamma t} x_1(t)$$

with an attenuation rate $\gamma > 0$. The equivalent dynamics to (2):

$$\dot{\sigma}(t) = \gamma e^{\gamma t} x_1(t) + e^{\gamma t} \dot{x}_1(t)$$

$$= A_\sigma \sigma(t) + \sum_{k=1}^{h} e^{\gamma d_k} \sigma(t - d_k) \qquad \text{(A.1)}$$

where

$$A_\sigma = \gamma I_{n-1} + A_{11} - A_{12}\Lambda - \Delta A_{11} - \Delta A_{12}\Lambda \,,$$

$$A_{\sigma d_k} = A_{d_k 11} - A_{d_k 12}\Lambda - \Delta A_{d_k 11} - \Delta A_{d_k 12}\Lambda$$

; the equation (A.1) and the fact $e^{\gamma d_k}\sigma(t - d_k) = e^{\gamma t} x_1(t - d_k)$ have been applied. If the system (A.1) is asymptotically stable, the original system (1) is exponentially stable with the decay

taking form of $x_1(t) = e^{-\gamma t}\sigma(t)$ and an attenuation rate γ. Therefore, the sliding surface design problem is transformed into finding an appropriate gain Λ such that the subsystem (A.1) is asymptotically stable.
Consider the following Lyapunov-Krasoviskii function

$$V(t) = \sigma^T(t)P\sigma(t) + \sum_{k=1}^{h} e^{2\gamma d_k} \int_{t-d_k}^{t} \sigma^T(v)xQ_k\sigma(v)dv$$

where $P > 0$ and $Q_k > 0$ are symmetric matrices.
Let the sliding surface $S(t) = 0$ with the definition (6). The sliding motion of the system (1) is delay-independent exponentially stable, if there exist positive symmetric matrices X, $\bar{Q}_k$ and a parameter Λ satisfying the following LMI:

Given $\varepsilon > 0$

Subject to $X > 0$, $\bar{Q}_k > 0$

$$\begin{bmatrix} N_{11} & (*) \\ N_{21} & -I_\varepsilon \end{bmatrix} < 0 \tag{A.2}$$

where

$$N_{11} = \begin{bmatrix} N_0 & (*) & (*) & (*) \\ XA_{d111}^T + K^T A_{d112}^T & -\bar{Q}_1 & (*) & (*) \\ \vdots & \vdots & \ddots & (*) \\ XA_{dh11}^T + K^T A_{dh12}^T & 0 & \cdots & -\bar{Q}_h \end{bmatrix}$$

$$N_{21} = \begin{bmatrix} E_{11}X + A_{12}K & 0 & 0 & 0 \\ 0 & E_{111}X + E_{112}K & \cdots & E_{h11}X + E_{h12}K \\ D_1^T & 0 & \ddots & 0 \\ D_2^T & 0 & \cdots & 0_h \end{bmatrix}$$

$$N_0 = A_{11}X + XA_{11}^T + A_{12}K + K^T A_{12}^T + \sum_{k=1}^{h} e^{2\gamma d_{\max}} \bar{Q}_k \ ;$$

$K = \Lambda X$; $I_\varepsilon = diag\{\varepsilon I_a, \varepsilon I_a, \varepsilon^{-1} I_b, \varepsilon^{-1} I_b\}$ in which I_a, I_b are identity matrices with proper dimensions; and (*) denotes the transposed elements in the symmetric positions.
If the LMI problem has a feasible solution, then we obtain $V(t) > 0$ and $\dot{V}(t) < 0$. This implies that the equivalent subsystem (A.1) is asymptotically stable, i.e., $\lim_{t\to\infty} \sigma(t) = 0$. In turn, the states $x_1(t)$ and $x_2(t)$ (here $x_2(t) = \Lambda x_1(t)$) will exponentially converge to zero as $t \to \infty$. As a result, the sliding motion on the manifold $S(t) = 0$ is exponentially stable.

Moreover, since the gain condition (A.2) does not contain the delay time d_k , the sliding surface is delay-independent exponentially stable. ■

10

Recent Progress in Synchronization of Multiple Time Delay Systems

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1. Introduction

The phenomenon of synchronization of dynamical systems was reported by the famous Dutch scientist Christiaan Huygens in 1665 on his observation of synchronization of two pendulum clocks. And, chaos theory has been aroused and developed very early (since 1960s) with efforts in many different research fields, such as mathematics (Li & Yorke, 1975; Ruelle, 1980; Sharkovskii, 1964; 1995), physics (Feigenbaum, 1978; Hénon, 1976; Rossler, 1976), chemistry (Zaikin & Zhabotinsky, 1970; Zhabotinsky, 1964), biology (May, 1976) and engineering (Lorenz, 1963a;b; Nakagawa, 1999), etc (Gleick, 1987; Stewart, 1990). However, until 1983, the idea of synchronization of chaotic systems was raised by Fujisaka and Yamada (Fujisaka & Yamada, 1983). There, the general stability theory of the synchronized motions of the coupled-oscillator systems with the use of the extended Lyapunov matrix approach, and the coupled Lorenz model was investigated as an typical example. A typical synchronous system can be seen in Fig.1. In 1990, Pecora and Carroll (Pecora & Carroll, 1990) realized chaos synchronization in the form of drive-response under the identical synchronous scheme. Since then, chaos synchronization has been aroused and it has become the subject of active research, mainly due to its potential applications in several engineering fields such as communications (Kocarev et al., 1992; Parlitz et al., 1992; Parlitz, Kocarev, Stojanovski & Preckel, 1996), lasers (Fabiny et al., 1993; Roy & Thornburg, 1994), ecology (Blasius et al., 1999), biological systems (Han et al., 1995), system identification (Parlitz, Junge & Kocarev, 1996), etc. The research evolution on chaos synchronization has led to several schemes of chaos synchronization proposed successively and pursued, i.e., generalized (Rulkov et al., 1995), phase (Rosenblum et al., 1996), lag (Rosenblum et al., 1997), projective (Mainieri & Rehacek, 1999), and anticipating (Voss, 2000) synchronizations. Roughly speaking, synchronization of coupled dynamical systems can be interpreted to mean that the master sends the driving signal to drive the slave, and there exists some functional relations in their trajectories during interaction. In fact, the difference between synchronous schemes is lied in the difference of functional relations in trajectories. In other words, a certain functional relation expresses the particular characteristic of corresponding synchronous scheme. When a synchronous regime is established, the expected functional relation is achieved and *synchronization manifold* is usually used to refer to such specific relation in a certain coupled systems.
Time delay systems have been studied in both theory (Krasovskii, 1963) and

application (Loiseau et al., 2009). The prominent feature of chaotic time-delay systems is that they have very complicated dynamics (Farmer, 1982). Analytical investigation on time-delay systems by Farmer has showed that it is very easy to generate chaotic behavior even in systems with a single equation with a single delay such as Mackey-Glass's, Ikeda's. Recently, researchers have been attracted by synchronization issues in coupled time-delay systems. Accordingly, several synchronous schemes have been proposed and pursued. However, up-to-date research works have been restricted to the synchronization models of single-delay (Pyragas, 1998a; Senthilkumar & Lakshmanan, 2005) and multiple time delay systems (MTDSs) (Shahverdiev, 2004; Shahverdiev et al., 2005; Shahverdiev & Shore, 2005). There, coupling (or driving) signals are in the form of either linear or single nonlinear transform of state variable. Those models of synchronization in coupled time-delay systems can be used in secure communications (Pyragas, 1998b), however, the security is not assured (Ponomarenko & Prokhorov, 2002; Zhou & Lai, 1999) due that there are several advanced reconstruction techniques which can infer the system's dynamics. From such the fact, synchronization of MTDSs has been intensively investigated (Hoang et al., 2005). In this chapter, recent development for synchronization in coupled MTDSs has been reported. The examples will illustrate the existence and transition in various synchronous schemes in coupled MTDSs.

The remainder of the chapter is organized as follows. Section 2 introduces the MTDSs and its complexity. The proposed synchronization models of coupled MTDSs with various synchronous schemes are described in Section 3. Numerical simulation for proposed synchronization models is illustrated in Section 4. The discussions and conclusions for the proposed models are given in the last two sections.

Fig. 1. A typical synchronous system.

2. Multiple time-delay systems

2.1 Overview of time-delay feedback systems

Let us consider the equation representing for a single time-delay system (STDS) as below

$$\frac{dx}{dt} = -\alpha x + f(x(t-\tau)) \tag{1}$$

where α and τ are positive real numbers, τ is a time length of delay applied to the state variable. $f(x) = \frac{x}{1+x^{10}}$ and $f(x) = sin(x)$ are well-known time-delay feedback systems; Mackey-Glass (Mackey & Glass, 1977) and Ikeda (Ikeda & Matsumoto, 1987) systems, respectively. α and/or τ can be used for controlling the complexity of chaotic dynamics (Farmer, 1982). An analog circuit model (Namajūnas et al., 1995) of STDSs is depicted in Fig. 2. The dynamical model of the circuit can be written as

$$\frac{dU}{dt} = \frac{U_{ND}(t) - U(t)}{C_0 R_0} \tag{2}$$

where $U_{ND}(t) = f(U(t-\tau))$. Apparently, the equations given in Eqs. (1) and (2) has the same form.

Chaos synchronization of coupled STDSs has been studied and experimented in several

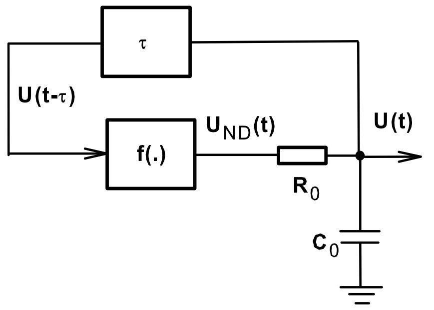


Fig. 2. Circuit model of single delay feedback systems

fields such as circuits (Kim et al., 2006; Kittel et al., 1998; Namajūnas et al., 1995; Sano et al., 2007; Voss, 2002), lasers (Celka, 1995; Goedgebuer et al., 1998; Lee et al., 2006; Masoller, 2001; S. Sivaprakasam et al., 2002; Zhu & WU, 2004), etc. So far, most of research works in this context have focused on synchronization models of STDSs (Pyragas, 1998a; Senthilkumar & Lakshmanan, 2005), in which Mackey-Glass (Mackey & Glass, 1977) and Ikeda (Ikeda & Matsumoto, 1987) systems have been employed as dynamical equations for specific examples. Up to date, there have been several coupling methods for synchronization models of STDSs, i.e., linear (Mensour & Longtin, 1998; Pyragas, 1998a) and single nonlinear coupling (Shahverdiev & Shore, 2005). In other words, the form of driving signals is either $x(t)$ or $f(x(t-\tau))$.

Recently, MTDSs have been interested and aroused (Shahverdiev, 2004). That is because of their potential applications in various fields. A general equation representing for MTDSs is as

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f(x(t-\tau_i)) \tag{3}$$

where $m_i, \tau_i \in (\tau_i \geq 0)\Re$. It is clear that MTDSs can been seen as an extension of STDSs. STDSs, MTDSs exhibit chaos.

Chaos synchronization models of MTDSs has been aroused by Shahverdiev *et al.* (Shahverdiev, 2004; Shahverdiev et al., 2005). So far, the studies are constrained to the cases that the coupling (or driving) signal is in the form of linear ($x(t)$) or single nonlinear transform of delayed state variable ($f(x(t-\tau))$). A synchronization model using STDSs with linear form of driving signal can be expressed by

Master:

$$\frac{dx}{dt} = -\alpha x + f(x(t-\tau)) \tag{4}$$

Driving signal:

$$DS(t) = kx \tag{5}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + f(y(t-\tau)) + DS(t) \tag{6}$$

where k is coupling strength. A synchronization model using MTDSs with linear form of driving signal can be expressed by

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f(x(t-\tau_i)) \tag{7}$$

Driving signal:

$$DS(t) = kx \tag{8}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P} n_i f(y(t-\tau_i)) + DS(t) \tag{9}$$

In the synchronous system given in Eqs. (7)-(9), if the driving signal is in the form of $DS(t) = kf(x(t-\tau))$, the synchronous system becomes synchronization of MTDSs with single nonlinear driving signal.

In theoretical, such above synchronization models do not offer advantages for the secure communication application due to the fact that their dynamics can be inferred easily by using conventional reconstruction methods (Prokhorov et al., 2005; Voss & Kurths, 1997). By such the reason, seeking for a non-reconstructed time-delay system is important for the chaotic secure communication application. One of the disadvantages of state-of-the-art reconstruction methods is that MTDSs can not be reconstructed if the measured time series is sum of multiple nonlinear transforms of delayed state variable, i.e. $\sum_j f(x(t-\tau_j))$. This is the key hint for proposing a new synchronization model of MTDSs. In the next section, the synchronization models of coupled MTDSs are investigated, in which the driving signal is sum of nonlinearly transformed components of delayed state variable, $\sum_j f(x(t-\tau_j))$. The conditions for synchronization in particular synchronous schemes are considered and proved under the Krasovskii-Lyapunov theory. The numerical simulation will demonstrate and verify the prediction in these contexts.

2.2 The complexity analysis for MTDSs

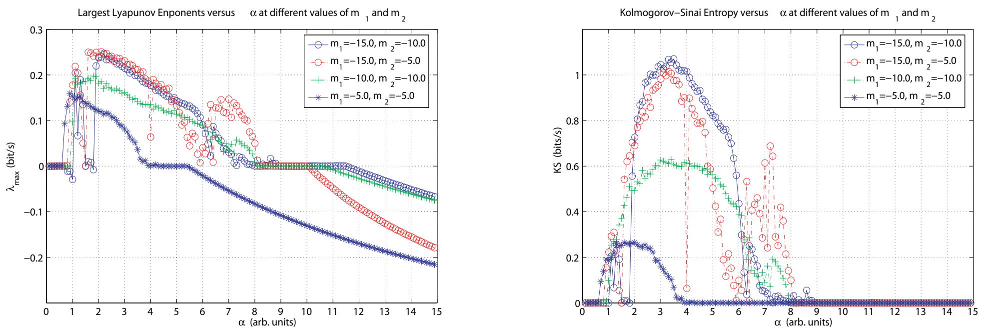
The complexity degree of MTDSs is confirmed that MTDSs not only exhibit hyperchaos, but also bring much more complicated dynamics in comparison with that in single delay systems. This will emphasis significances of MTDSs to the secure communication application. In order to illustrate the complicated dynamics of MTDSs, the Lyapunov spectrum and metric entropy are calculated. Lyapunov spectrum shows the complexity measure while metric entropy presents the predictability to chaotic systems. Here, Kolmogorov-Sinai entropy (Cornfeld et al., 1982) is estimated with

$$KS = \sum_i \lambda_i \quad for\ \lambda_i > 0 \tag{10}$$

The two-delays Mackey-Glass system given in Eq. (11) is studied for this purpose. The complexity degree with respect to values of parameters and of delays is shown by varying α, m_i and τ_i. There are several algorithms to calculate the Lyapunov exponents of dynamical systems as presented in (Christiansen & Rugh, 1997; Grassberger & Procaccia, 1983; Sano & Sawada, 1985; Zeng et al., 1991) and others. However, so far , all of the existing algorithms are inappropriate to deal with the case of MTDSs. Here, estimation of Lyapunov spectrum is based on the algorithm proposed by Masahiro Nakagawa (Nakagawa, 2007)

$$\frac{dx}{dt} = -\alpha x + m_1 \frac{x_{\tau_1}}{1 + x_{\tau_1}^{10}} + m_2 \frac{x_{\tau_2}}{1 + x_{\tau_2}^{10}} \tag{11}$$

Shown in Fig. 3(a) is largest Lyapunov exponents (LLE) and Kolmogorov-Sinai entropy with some couples of value of m_1 and m_2. In this case, the value of τ_1 and τ_2 is set at 2.5 and 5.0, respectively. The chaotic behavior exhibits in the specific value range of α. Moreover, the range seems to be wider with the increase in the value of $|m_1| + |m_2|$. It is clear to be seen from Fig.3 that the possible largest LEs and metric entropy in this system (approximately 0.3 for largest LEs and 1.4 for metric entropy) are larger in comparison with those of the single delay Mackey-Glass system studied by J.D. Farmer (Farmer, 1982) (approximately 0.07 for largest LEs and 0.1 for metric entropy). It means that the chaotic dynamics of MTDSs is much more complicated than that of single delay systems. As a result, it is hard to reconstruct and predict the motion of MTDSs.



(a) LLEs versus α

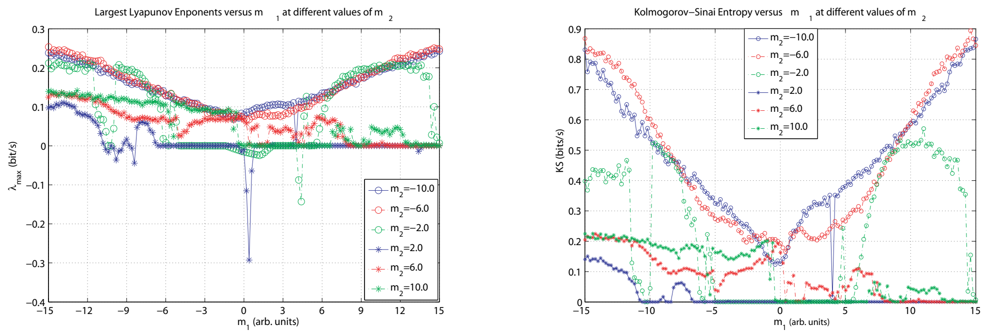
(b) Kolmogorov-Sinai entropy versus α

Fig. 3. Largest LEs and Kolmogorov-Sinai entropy versus α of the two delays Mackey-Glass system.

Illustrated in Figs. 4 and 5 is the largest LEs as well as Kolmogorov-Sinai entropy with respect to the value of m_1 and m_2. There, the value of α, τ_1 and τ_2 is kept constant at 2.1, 2.5 and 5.0, respectively. Noticeably, the two-delays system is with weak feedback in the range of small value of m_1, or the two-feedbacks system tends to single feedback one. In such the range, the curves of largest LEs and Kolmogorov-Sinai entropy are in 'V' shape for negative value of m_2 as depicted in Figs. 4(a) and 4(b). This is also observed in the curves of metric entropy in Fig. 4(b). In other words, the dynamics of MTDSs are intuitively more complicated than that of STDSs. By observing the curves in Figs. 5(a) and 5(b), these characteristics in the case of changing the value of m_2 are a bit different. The range of m_2 offers the 'V' shape is around 3.0 for large negative values of m_1, i.e., -14.5 and -9.5. It can be interpreted that this characteristic depends on the value of delays associated with m_i. As a particular case, the result shows that the trend of largest LEs and metric entropy depends on the value of m_1 and m_2.

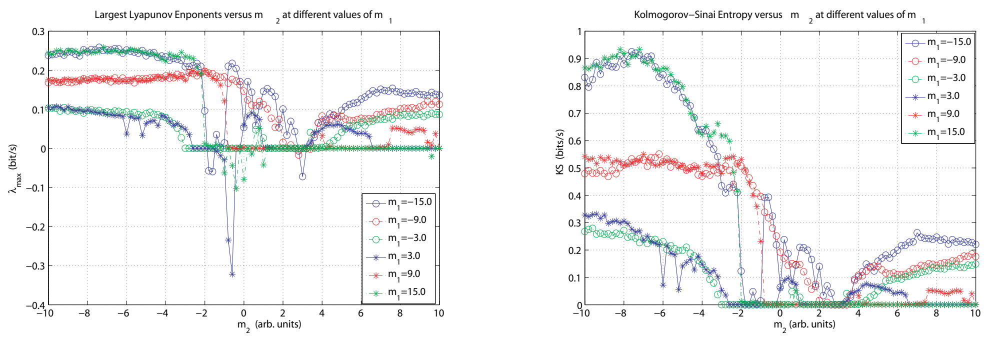
In Fig. 6, the largest LEs and Kolmogorov-Sinai entropy related to the value of τ_1 and τ_2 are presented, and it is clear that they strongly depend on the value of τ_1 and τ_2. There, the value of other parameters is set at $m_1 = -15.0$, $m_2 = -10.0$ and $\alpha = 2.1$. The system still exhibits chaotic dynamics even though the dependence of largest LEs and metric entropy on the value of τ_1 and τ_2 is observed.

In summary, the dynamics of MTDSs is firmly more complicated than that of STDSs. In other words, MTDSs present significances to the secure communication application.



(a) LLEs versus m_1 (b) Kolmogorov-Sinai entropy versus m_1

Fig. 4. Largest LEs and Kolmogorov-Sinai entropy versus m_1 of the two-delays Mackey-Glass system.



(a) LLEs versus m_2 (b) Kolmogorov-Sinai entropy versus m_2

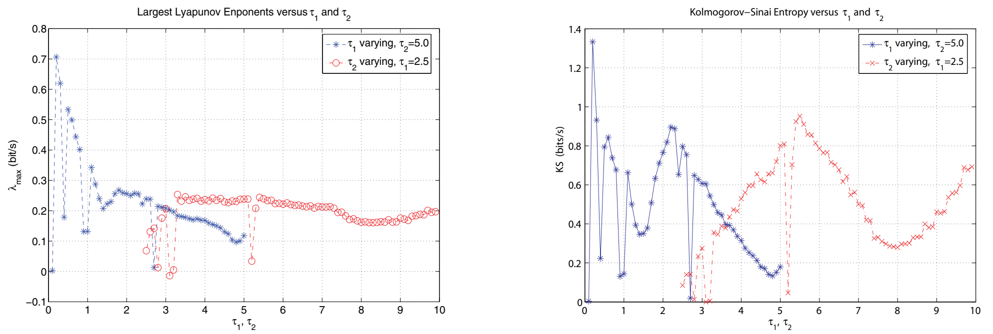
Fig. 5. Largest LEs and Kolmogorov-Sinai entropy versus m_2 of the two-delays Mackey-Glass system.

3. The proposed synchronization models of coupled MTDSs

We consider synchronization models of coupled MTDSs with restriction to the only one state variable. In addition, various schemes of synchronization are investigated on such the synchronization models. The main differences between these proposed models and conventional ones are that dynamical equations for the master and slave are in the form of multiple time delays and the driving signal is constituted by sum of nonlinear transforms of delayed state variable. The condition for synchronization is still based on the Krasovskii-Lyapunov theory. Proofs of the sufficient condition for considered synchronous schemes will also be shown.

3.1 Synchronization of coupled identical MTDSs

We start considering the synchronization of MTDSs with the dynamical equations in the form of one dimension defined by



(a) LLEs versus τ_1 and τ_2 (arb. units)

(b) Kolmogorov-Sinai entropy versus τ_1 and τ_2 (arb. units)

Fig. 6. Largest LEs and Kolmogorov-Sinai entropy versus τ_1 and τ_2 of the two-delays Mackey-Glass system.

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f(x_{\tau_i}) \tag{12}$$

Driving signal:

$$DS(t) = \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}}) \tag{13}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + DS(t) \tag{14}$$

where $\alpha, m_i, n_i, k_j, \tau_i(\tau_i \geq 0) \in \Re$; integers P, Q $(Q \leq P)$, $f(.)$ is the differentiable generic nonlinear function. x_{τ_i} and y_{τ_i} stand for delayed state variables $x(t - \tau_i)$ and $y(t - \tau_i)$, respectively. Note that, the form of $f(.)$ and the value of P are shared in both the master's and slave's equations. As shown in Eq. (13), the driving signal is constituted by sum of multiple nonlinear transforms of delayed state variable, and it is generated by driving signal generator (DSG) as illustrated in Fig. 7. The master's and slave's equations in Eqs. (12) and (14) with $\{P = 1,\ f(x) = \frac{x}{1+x^b}\}$ and $\{P = 1,\ f(x) = sin(x)\}$ turn out being the well-known Mackey-Glass (Mackey & Glass, 1977) and Ikeda systems (Ikeda & Matsumoto, 1987), respectively.

It is clear to observe from the proposed synchronization model given in Eqs. (12)-(14) that

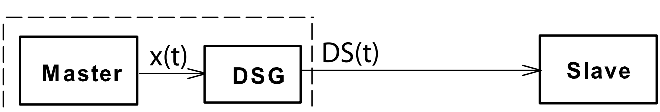


Fig. 7. The proposed synchronization model of MTDSs.

the structure of slave is identical to that of master, except for the presence of driving signal in the dynamical equation of slave.

In consideration to the synchronization condition, so far, there are two strategies (Pyragas, 1998a) which are used for dealing with synchronization of time-delay systems. The first one is based on the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) while the other one is based on the perturbation theory (Pyragas, 1998a). Note that, perturbation theory is used suitably for the case that the delay length is large ($\tau_i \to \infty$). And, the Krasovskii-Lyapunov theory can be used for the case of multiple time-delays as in this context. In the following subsections, this model is investigated with different synchronous schemes, i.e., lag, anticipating, projective-lag, and projective-anticipating.

3.1.1 Lag synchronization

As a general case, lag synchronization refers to the means that the slave's state variable is retarded with a time length in compared to the master's. Here, lag synchronization has been studied in coupled MTDSs described in Eqs. (12)-(14) with the desired synchronization manifold defined by

$$y(t) = x(t - \tau_d) \tag{15}$$

where $\tau_d \in \Re^+$ is a time-delay, called a manifold's delay. We define the synchronization error upon expected synchronization manifold in Eq. (15) as

$$\Delta(t) = y(t) - x(t - \tau_d) \tag{16}$$

And, the dynamics of synchronization error is

$$\frac{d\Delta}{dt} = \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt} \tag{17}$$

By applying the delay of τ_d to Eq. (12), we get $\frac{dx(t-\tau_d)}{dt} = -\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})$. Then, substituting $\frac{dx(t-\tau_d)}{dt}$, $y_{\tau_i} = x_{\tau_i+\tau_d} + \Delta_{\tau_i}$, and Eq. (14) into Eq. (17), the dynamics of synchronization error becomes

$$\begin{aligned}
\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt} \\
&= \left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - \left[-\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})\right] \\
&= -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i+\tau_d} + \Delta_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}}) - \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})
\end{aligned} \tag{18}$$

$$\tau_{P+j} = \tau_i + \tau_d \tag{19}$$

It is assumed that delays in Eq. (18) are chosen so that Eq. (19) is satisfied. Hence, Eq. (18) is rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i+\tau_d} + \Delta_{\tau_i}) - \sum_{i=1,j=1}^{P,Q} \left[m_i - k_j\right] f(x_{\tau_i+\tau_d}) \tag{20}$$

$$m_i - k_j = n_i \tag{21}$$

It is easy to realize that the derivative of $f(x+\delta) - f(x) = f'(x)\delta$ exists if $f(.)$ is differentiable, bounded, and δ is small enough. Also suppose that the value of coefficients in Eq. (20) is adopted so as the relation in Eq. (21) is fulfilled in pair. Note from Eqs. (19) and (21) that only some components in the master's and slave's equations are selected for such the relations. Therefore, Eq. (20) reduces to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f'(x_{\tau_i+\tau_d} + \Delta_{\tau_i})\Delta_{\tau_i} \tag{22}$$

By applying the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) to the case of multiple time-delays, the sufficient condition to achieve $\lim_{t\to\infty} \Delta(t) = 0$ from Eq. (22) is expressed as

$$\alpha > \sum_{i=1}^{P} |n_i| \sup \left| f'(x_{\tau_i+\tau_d}) \right| \tag{23}$$

where $\sup |f'(.)|$ stands for the supreme limit of $|f'(.)|$. It is easy to see that the sufficicent condition for synchronization is obtained under a series of assumptions. Noticably, the linear delayed system of Δ given in Eq. (22) is with time-dependent coefficients. The specific example shown in Section 4 with coupled modified Mackey-Glass systems will demonstrate and verify for the case.

Next, combination synchronous scheme will be presented, there, the mentioned synchronous scheme of coupled MTDSs is associated with projective one.

3.1.2 Projective-lag synchronization

In this section, the lag synchronization of coupled MTDSs is investigated in a way that the master's and slave's state variables correlate each other upon a scale factor. The dynamical equations for synchronous system are defined in Eqs. (12)- (14). The desired projective-lag manifold is described by

$$ay(t) = bx(t - \tau_d) \tag{24}$$

where a and b are nonzero real numbers, and τ_d is the time lag by which the state variable of the master is retarded in comparison with that of the slave. The synchronization error can be written as

$$\Delta(t) = ay(t) - bx(t - \tau_d), \tag{25}$$

And, dynamics of synchronization error is

$$\frac{d\Delta}{dt} = a\frac{dy}{dt} - b\frac{dx(t-\tau_d)}{dt}. \tag{26}$$

By substituting appropriate components to Eq. (26), the dynamics of synchronization error can be rewritten as

$$\frac{d\Delta}{dt} = a\left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - b\left[-\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})\right] \tag{27}$$

Moreover, y_{τ_i} can be deduced from Eq. (25) as

$$y_{\tau_i} = \frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a} \tag{28}$$

And, Eq. (27) can be represented as

$$\frac{d\Delta}{dt} = a\left[-\alpha y + \sum_{i=1}^{P} n_i f\left(\frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a}\right) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - b\left[-\alpha x(t-\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i+\tau_d})\right] \tag{29}$$

Let us assume that the relation of delays is as given in Eq. (19), $\tau_{P+j} = \tau_i + \tau_d$. The error dynamics in Eq. (29) becomes

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q}\left[an_i f\left(\frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a}\right) - (bm_i - ak_j) f(x_{\tau_i+\tau_d})\right] \tag{30}$$

The right-hand side of Eq. (28) can be represented as

$$\frac{bx_{\tau_i+\tau_d} + \Delta_{\tau_i}}{a} = x_{\tau_i+\tau_d} + \Delta_{\tau_i^{(app)}} \tag{31}$$

where $\tau_i^{(app)}$ is a time-delay at which the synchronization error satisfies Eq. (31). By replacing right-hand side of Eq. (31) to Eq. (30), The error dynamics can be rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q}\left[an_i f(x_{\tau_i+\tau_d} + \Delta_{\tau_i^{(app)}}) - (bm_i - ak_j) f(x_{\tau_i+\tau_d})\right] \tag{32}$$

Suppose that the relation of parameters in Eq. (32) as follows

$$bm_i - ak_j = an_i \tag{33}$$

If $\Delta_{\tau_i^{(app)}}$ is small enough and $f(.)$ is differentiable, bounded, then Eq. (32) can be reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} an_i f'(x_{\tau_i+\tau_d})\Delta_{\tau_i^{(app)}} \tag{34}$$

By applying the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) to this case, the sufficient condition for synchronization is expressed as

$$\alpha > \sum_{i=1}^{P} |an_i| \sup \left|f'(x_{\tau_i+\tau_d})\right| \tag{35}$$

It is clear that the main difference of this scheme in comparison with lag synchronization is the existence of scale factor. This leads to the change in the synchronization condition. In fact, projective-lag synchronization becomes lag synchronization when scale factor is equivalent to unity, but the relative value of α is changed in the sufficient condition regarding to the bound. This allows us to arrange multiple slaves with the same structure which are synchronized with a certain master under various scale factors. Anyways, the value of n_i and k_j must be adjusted correspondingly. This can not be the case by using lag synchronization as presented in the previous section, that is, only one slave with a certain structure is satisfied.

3.1.3 Anticipating synchronization

In this section, anticipating synchronization of coupled MTDSs is presented, in which the master's motion can be anticipated by the slave's. The proposed model given in Eqs. (12)-(14) is investigated with the desired synchronization manifold of

$$y(t) = x(t + \tau_d) \tag{36}$$

where $\tau_d \in \Re^+$ is the time length of anticipation. It is also called a manifold's delay because the master's state variable is retarded in compared with the slave's. Synchronization error in this case is

$$\Delta(t) = y(t) - x(t + \tau_d) \tag{37}$$

Similar to the scheme of lag synchronization, the dynamics of synchronization error is written as

$$\frac{d\Delta}{dt} = \frac{dy}{dt} - \frac{dx(t + \tau_d)}{dt} \tag{38}$$

By substituting $\frac{dx(t+\tau_d)}{dt} = -\alpha x(t + \tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i - \tau_d})$, $y_{\tau_i} = x_{\tau_i - \tau_d} + \Delta_{\tau_i}$, and $\frac{dy}{dt}$ into Eq. (38), the dynamics of synchronization error is described by

$$\begin{aligned}
\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t + \tau_d)}{dt} \\
&= \left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}}) \right] - \left[-\alpha x(t + \tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i - \tau_d}) \right] \\
&= -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i - \tau_d} + \Delta_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}}) - \sum_{i=1}^{P} m_i f(x_{\tau_i - \tau_d})
\end{aligned} \tag{39}$$

Assume that τ_{P+j} in Eq. (39) are fulfilled the relation of

$$\tau_{P+j} = \tau_i - \tau_d \tag{40}$$

delays must be non-negative, thus, τ_i must be equal to or greater than τ_d in Eq. (19). Equation (39) is represented as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f(x_{\tau_i - \tau_d} + \Delta_{\tau_i}) - \sum_{i=1, j=1}^{P, Q} \left[m_i - k_j \right] f(x_{\tau_i - \tau_d}) \tag{41}$$

Applying the same reasoning in lag synchronization to this case, parameters satisfies the relation given in Eq. (21). Equation (41) reduces to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} n_i f'(x_{\tau_i - \tau_d}) \Delta_{\tau_i} \tag{42}$$

And, the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) is applied to Eq. (42), hence, the sufficient condition for synchronization for anticipating synchronization is

$$\alpha > \sum_{i=1}^{P} |n_i| \sup \left| f'(x_{\tau_i - \tau_d}) \right| \tag{43}$$

It is clear from (35) and (43) that there is small difference made to the relation of delays in comparison to lag synchronization, and a completely new scheme is resulted. Therefore, the switching between schemes of lag and anticipating synchronization can be obtained in such a simple way. This may be exploited for various purposes including secure communications.

3.1.4 Projective-anticipating synchronization

Obviously, projective-anticipating synchronization is examined in a very similar way to that dealing with the scheme of projective-lag synchronization. The dynamical equations for synchronous system are as given in Eq. (12)- (14). The considered projective-anticipating manifold is as

$$ay(t) = bx(t + \tau_d) \tag{44}$$

where a and b are nonzero real numbers, and τ_d is the time lag by which the state variable of the slave is retarded in comparison with that of the master. The synchronization error is defined as

$$\Delta = ay - bx(t + \tau_d) \tag{45}$$

Dynamics of synchronization error is as

$$\frac{d\Delta}{dt} = a\frac{dy}{dt} - b\frac{dx(t+\tau_d)}{dt}. \tag{46}$$

By substituting $\frac{dy}{dt}$ and $\frac{dx(t+\tau_d)}{dt}$ to Eq. (46), the dynamics of synchronization error becomes

$$\frac{d\Delta}{dt} = a\left[-\alpha y + \sum_{i=1}^{P} n_i f(y_{\tau_i}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] - b\left[-\alpha x(t+\tau_d) + \sum_{i=1}^{P} m_i f(x_{\tau_i - \tau_d})\right] \tag{47}$$

It is clear that y_{τ_i} can be deduced from Eq. (45) as

$$y_{\tau_i} = \frac{bx_{\tau_i - \tau_d} + \Delta_{\tau_i}}{a} \tag{48}$$

Hence, Eq. (47) can be represented as

$$\begin{aligned} \frac{d\Delta}{dt} &= a\left[-\alpha y + \sum_{i=1}^{P} n_i f(\frac{bx_{\tau_i - \tau_d} + \Delta_{\tau_i}}{a}) + \sum_{j=1}^{Q} k_j f(x_{\tau_{P+j}})\right] \\ &- b\left[-\alpha x_{\tau_d} + \sum_{i=1}^{P} m_i f(x_{\tau_i - \tau_d})\right] \end{aligned} \tag{49}$$

Similar to anticipating synchronization, the relation of delays is chosen as given in Eq. (40), $\tau_{P+j} = \tau_i - \tau_d$. The error dynamics in Eq. (49) is rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q}\left[an_i f(\frac{bx_{\tau_i-\tau_d} + \Delta_{\tau_i}}{a}) - (bm_i - ak_j) f(x_{\tau_i - \tau_d})\right] \tag{50}$$

The right-hand side of Eq. (48) can be equivalent to

$$\frac{bx_{\tau_i-\tau_d} + \Delta_{\tau_i}}{a} = x_{\tau_i - \tau_d} + \Delta_{\tau_i^{(app)}} \tag{51}$$

where $\tau_i^{(app)}$ is a time-delay satisfying Eq. (51). Therefore, the error dynamics can be rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1,j=1}^{P,Q} \left[an_i f(x_{\tau_i-\tau_d} + \Delta_{\tau_i^{(app)}}) - (bm_i - ak_j) f(x_{\tau_i-\tau_d}) \right] \tag{52}$$

Suppose that the relation of parameters in Eq. (52) is as given in Eq. (33), $bm_i - ak_j = an_i$. $\Delta_{\tau_i^{(app)}}$ is small enough, $f(.)$ is differentiable and bounded, hence, Eq. (52) is reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{P} an_i f'(x_{\tau_i-\tau_d})\Delta_{\tau_i^{(app)}} \tag{53}$$

The sufficient condition for synchronization can be expressed as

$$\alpha > \sum_{i=1}^{P} |an_i| \sup \left| f'(x_{\tau_i-\tau_d}) \right| \tag{54}$$

It is easy to see that the change from anticipating into projective-anticipating synchronization is similar to that from lag to projective-lag one. It is realized that transition from the lag to anticipating is simply done by changing the relation of delays. This is easy to be observed on their sufficient conditions.

3.2 Synchronization of coupled nonidentical MTDSs

It is easy to observe from the synchronization model presented in Eqs. (12)-(14) that the value of P and the function form of $f(.)$ are shared in the master's and slave's equations. It means that the structure of the master is identical to that of slave. In other words, the proposed synchronization model above is not a truly general one. In this section, the proposed synchronization model of coupled nonidentical MTDSs is presented, there, the similarity in the master's and slave's equations is removed. The dynamical equations representing for the synchronization are defined as
Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}}) \tag{55}$$

Driving signal:

$$DS(t) = \sum_{j=1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) \tag{56}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + DS(t) \tag{57}$$

where $\alpha, m_i, n_i, k_j, \tau_i^{(M)}, \tau_j^{(DS)}, \tau_i^{(S)} \in \Re$; P, Q and R are integers. The delayed state variables $x_{\tau_i^{(M)}}$, $x_{\tau_j^{(DS)}}$ and $y_{\tau_i^{(S)}}$ stand for $x(t-\tau_i^{(M)})$, $x(t-\tau_j^{(DS)})$ and $y(t-\tau_i^{(S)})$, respectively. $f_i^{(M)}(.)$, $f_j^{(DS)}(.)$ and $f_i^{(S)}(.)$ are differentiable, generic, and nonlinear functions. The superscripts (M), (S) and (DS) associated with main symbols (delay, function, set of function forms) indicate that they are belonged to the master, slave and driving signal, respectively.

The non-identicalness between the master's and slave's configuration can be clarified by defining the set of function forms, $S = \{F_i; i = 1..N\}$, in which F_i ($i = 1..N$) represents for the function form of $f_i^{(M)}(.)$, $f_j^{(DS)}(.)$ and $f_i^{(S)}(.)$ in Eqs. (55)-(57). The subsets of S_M, S_S and S_{DSG} are collections of function forms of the master, slave and DSG, respectively. It is assumed that the relation among subsets is $S_{DSG} \subseteq S_M \cup S_S$. It is easy to realize that the structure of master is completely nonidentical to that of slave if $S_I = S_M \cap S_S \equiv \Phi$. Otherwise, if there are I components of nonlinear transforms whose function forms and value of delays are shared between the master's and slave's equations, i.e., $S_I = S_M \cap S_S \neq \Phi$ and $\tau_i^{(M)} = \tau_i^{(S)}$ for $i = 1..I$. These components are called *identicalness* ones which make pairs of $\{f^{(M)}(x_{\tau_i^{(M)}})$ *vs.* $f^{(S)}(y_{\tau_i^{(S)}})\}$ for $i = 1..I$.

Therefore, there are two cases needed to consider specifically: *(i)* the structure of master is partially identical to that of slave by means of identicalness components, and *(ii)* the structure of master is completely nonidentical to that of slave. In any cases, it is easy to realize from the relation among S_M, S_S and S_{DSG} that the difference between the master's and slave's equations can be complemented by the DSG's equation. In other words, function forms and value of parameters will be chosen appropriately for the driving signal's equation so that the Krasovskii-Lyapunov theory can be used for considering the synchronization condition in a certain case. For simplicity, only scheme of lag synchronization with the synchronization manifold of $y(t) = x(t - \tau_d)$ is studied, and other schemes can be extended as in a way of synchronization of coupled identical MTDSs.

3.2.1 Structure of master partially identical to that of slave

Suppose that there are I identicalness components shared between the master's and slave's equations, hence, Eqs. (55) and (57) can be decomposed as

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{I} m_i f_i^{(M)}(x_{\tau_i^{(M)}}) + \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}}) \tag{58}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{I} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + DS(t) \tag{59}$$

where $f_i^{(M)}$ is with the form identical to $f_i^{(S)}$ and $\tau_i^{(M)} = \tau_i^{(S)}$ for $i = 1..I$. They are pairs of identicalness components. The driving signal's equation in Eq. (56) is chosen in the following form

$$DS(t) = \sum_{j=1}^{I} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) + \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) \tag{60}$$

where forms of $f_j^{(DS)}(.)$ for $j = 1..I$ are, in pair, identical to that of $f_i^{(M)}$ as well as of $f_i^{(S)}$ for $i = 1..I$. Let's consider the lag synchronization manifold of

$$y(t) = x(t - \tau_d) \tag{61}$$

And, the synchronization error is

$$\Delta(t) = y(t) - x(t - \tau_d) \tag{62}$$

Hence, the dynamics of synchronization error is expressed by

$$\begin{aligned}\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt} \\ &= -\alpha y + \sum_{i=1}^{I} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{j=1}^{I} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) + \\ &+ \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) + \alpha x(t-\tau_d) - \sum_{i=1}^{I} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) - \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})\end{aligned} \tag{63}$$

By applying delay of $\tau_i^{(S)}$ to Eq. (62), $y_{\tau_i^{(S)}}$ can be deduced as

$$y_{\tau_i^{(S)}} = x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}} \tag{64}$$

By substituting $y_{\tau_i}^{(S)}$ to Eq. (63), the dynamics of synchronization error can be rewritten as

$$\begin{aligned}\frac{d\Delta}{dt} &= -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{j=1}^{I} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) \\ &+ \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) - \sum_{i=1}^{I} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) - \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})\end{aligned} \tag{65}$$

Suppose that the relation of delays in the fourth and sixth terms at the right-hand side of Eq. (65) is

$$\tau_j^{(DS)} = \tau_i^{(M)} + \tau_d \quad (\equiv \tau_i^{(S)} + \tau_d) \quad for\ j,i = 1..I \tag{66}$$

Hence, Eq. (65) can be reduced to

$$\begin{aligned}\frac{d\Delta}{dt} &= -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) - \sum_{i=1}^{I} (m_i - k_i) f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) + \sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) - \\ &- \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}})\end{aligned} \tag{67}$$

Also suppose that function forms and value of parameters of the fourth term of Eq. (67) (the second right-hand term of Eq. (60)) are chosen so that the last three terms of Eq. (67) satisfy the following equation

$$\sum_{j=I+1}^{Q} k_j f_j^{(DS)}(x_{\tau_j^{(DS)}}) - \sum_{i=I+1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) + \sum_{i=I+1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) = 0 \tag{68}$$

Let us assume that $Q = P + R - I$. The first left-hand term is decomposed, and Eq. (68) becomes

$$\begin{aligned} &\sum_{j1=1}^{P-I} k_{I+j1} f_{I+j1}^{(DS)}(x_{\tau_{I+j1}^{(DS)}}) + \sum_{j2=1}^{R-I} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) \\ &- \sum_{i=1}^{P-I} m_{I+i} f_{I+i}^{(M)}(x_{\tau_{I+i}^{(M)}+\tau_d}) + \sum_{i=1}^{R-I} n_{I+i} f_{I+i}^{(S)}(x_{\tau_{I+i}^{(S)}+\tau_d} + \Delta_{\tau_{I+i}^{(S)}}) = 0 \end{aligned} \tag{69}$$

Undoubtedly, Eq. (69) can be fulfilled if following assumptions are made: $k_{I+j1} = m_{I+i}$, $\tau_{I+j1}^{(DS)} = \tau_{I+i}^{(M)} + \tau_d$ and forms of $f_{I+j1}^{(DS)}(.)$ are identical to that of $f_{I+i}^{(M)}(.)$ for $i, j1 = 1..(P-I)$, and $k_{P+j2} = -n_{I+i}$, $\tau_{P+j2}^{(DS)} = \tau_{I+i}^{(S)} + \tau_d$, $\Delta_{\tau_{I+i}^{(S)}}$ is equal to zero as well as the form of $f_{P+j2}^{(DS)}(.)$ is identical to that of $f_{I+i}^{(S)}(.)$ for $i, j2 = 1..(R-I)$. Thus, Eq. (67) can be represented as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) - \sum_{i=1}^{I}(m_i - k_i) f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) \tag{70}$$

According to above assumptions, $\tau_i^{(S)} = \tau_i^{(M)}$ and forms of $f_i^{(M)}(.)$ being identical to those of $f_i^{(M)}(.)$ for $i = 1..I$ have been made. Here, further suppose that functions $f_i^{(M)}(.)$ and $f_i^{(S)}(.)$ are bounded. If synchronization errors $\Delta_{\tau_i^{(S)}}$ are small enough and $m_i - k_j = n_i$ for $i = 1..I$, Eq. (70) can be reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{I} n_i f_i^{(S)'}(x_{\tau_i^{(S)}+\tau_d})\Delta_{\tau_i^{(S)}} \tag{71}$$

where $f_i^{(S)'}(.)$ is the derivative of $f_i^{(S)}(.)$. By applying the Krasovskii-Lyapunov theory (Hale & Lunel, 1993; Krasovskii, 1963) to the case of multiple time-delays in Eq. (71), the sufficient condition for synchronization can be expressed as

$$\alpha > \sum_{i=1}^{I} |n_i| \sup \left| f_i^{(S)'}(x_{\tau_i^{(S)}+\tau_d}) \right| \tag{72}$$

It turns out that the difference in the structures of the master and slave can be complemented in the equation of driving signal. In order to test the proposed scheme, Example 5 is demonstrated in Section 4, in which the master's equation is in the heterogeneous form and the slave's is in the multiple time-delay Ikeda equation.

3.2.2 Structure of master completely nonidentical to that of slave

In this section, the synchronous system given in Eqs. (58)-(59) is examined, in which there is no identicalness component shared between the master's and slave's equations. In other words, the function set is of $S_I = S_M \cap S_S = \Phi$. Therefore, the driving signal's equation must contain all function forms of the master's and slave's equations or $S_{DSG} = S_M \cup S_S$ and $Q = P + R$. The driving signal's equation Eq. (56) can be decomposed to

$$DS(t) = \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) \tag{73}$$

And, the synchronization error Eq. (62) can be represented as below

$$\begin{aligned}
\frac{d\Delta}{dt} &= \frac{dy}{dt} - \frac{dx(t-\tau_d)}{dt} \\
&= -\alpha y + \sum_{i=1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) \\
&+ \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) + \alpha x(t-\tau_d) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) \\
&= -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)}(y_{\tau_i^{(S)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}} \\
&+ \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})
\end{aligned} \tag{74}$$

By substituting $y_{\tau_s^{(S)}}$ from Eq. (64) into Eq. (74), the dynamics of synchronization error is rewritten as

$$\begin{aligned}
\frac{d\Delta}{dt} &= -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) \\
&+ \sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d})
\end{aligned} \tag{75}$$

Assume that value of parameters and function forms of the first right-hand term of Eq. (73) are chosen so that the relation between the last two right-hand terms of Eq. (75) is as

$$\sum_{j1=1}^{P} k_{j1} f_{j1}^{(DS)}(x_{\tau_{j1}^{(DS)}}) - \sum_{i=1}^{P} m_i f_i^{(M)}(x_{\tau_i^{(M)}+\tau_d}) = 0 \tag{76}$$

Equation Eq. (76) is fulfilled if the relation is as $k_{j1} = m_i$, $\tau_{j1}^{(DS)} = \tau_i^{(M)} + \tau_d$ and the form of $f_{j1}^{(DS)}(.)$ is identical to that of $f_i^{(M)}(.)$ for $i, j1 = 1..P$. At this point, the dynamics of synchronization error in (75) can be reduced to

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)}(x_{\tau_i^{(S)}+\tau_d} + \Delta_{\tau_i^{(S)}}) + \sum_{j2=1}^{R} k_{P+j2} f_{P+j2}^{(DS)}(x_{\tau_{P+j2}^{(DS)}}) \tag{77}$$

As mentioned, the form of $f_i^{(S)}(.)$ is identical to that of $f_{P+j2}^{(DS)}(.)$ in pair. Here, we suppose that coefficients and delays in Eq. (77) are adopted as $k_{P+j2} = -n_i$ and $\tau_{P+j2}^{(DS)} = \tau_i^{(S)} + \tau_d$ for $i, j2 = 1..P$. If $\Delta_{\tau_i^{(S)}}$ is small enough and functions $f_i^{(S)}$ are bounded, Eq. (77) can be rewritten as

$$\frac{d\Delta}{dt} = -\alpha\Delta + \sum_{i=1}^{R} n_i f_i^{(S)\prime}(x_{\tau_i^{(S)}+\tau_d})\Delta_{\tau_i^{(S)}} \tag{78}$$

where $f_i^{(S)'}(.)$ is the derivative of $f_i^{(S)}(.)$. Similarly, the synchronization condition is obtained by applying the Krasovskii-Lyapunov (Hale & Lunel, 1993; Krasovskii, 1963) theory to Eq. (78); that is

$$\alpha > \sum_{i=1}^{R} |n_i| \sup \left| f_i^{(S)'}(x_{\tau_i^{(S)}+\tau_d}) \right| \tag{79}$$

It is undoubtedly that for a certain master and slave in the form of MTDS, we always obtained synchronous regime. Example 6 in Section 4 is given to verify for synchronization of completely nonidentical MTDSs; the multidelay Mackey-Glass and multidelay Ikeda systems.

4. Numerical simulation for synchronous schemes on the proposed models

In this subsection, a number of specific examples demonstrate and verify for the general description. Each example will correspond to a proposal in above section.

Example 1:

This example illustrates the lag synchronous scheme in coupled identical MTDSs given in Section 3.1.1. Let's consider the synchronization of coupled six-delays Mackey-Glass systems with the coupling signal constituted by the four-delays components. The dynamical equations are as

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P=6} m_i \frac{x_{\tau_i}}{1 + x_{\tau_i}^b} \tag{80}$$

Driving signal:

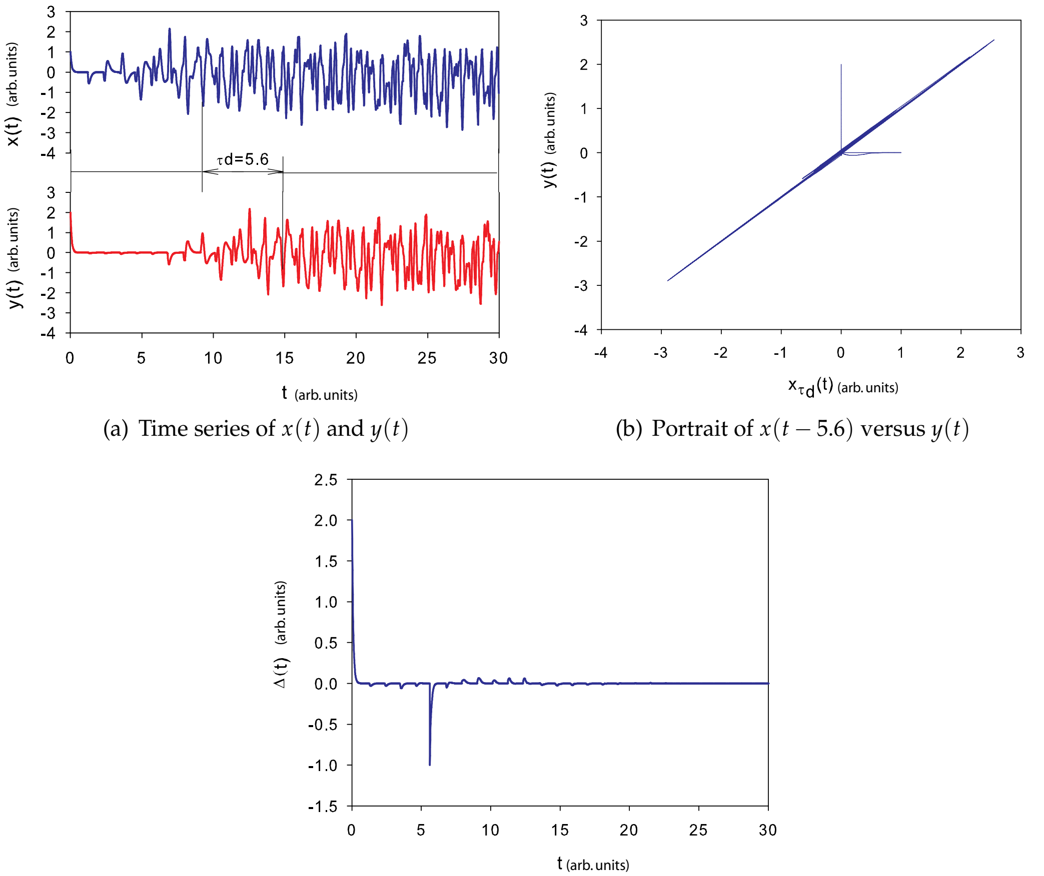
$$DS(t) = \sum_{j=1}^{Q=4} k_j \frac{x_{P+j}}{1 + x_{\tau_{P+j}}^b} \tag{81}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P=6} n_i \frac{x_{\tau_i}}{1 + x_{\tau_i}^b} + DS(t) \tag{82}$$

Moreover, the supreme limit of the function $f'(x)$ is equal to $\frac{(b-1)^2}{4b}$ at $x = \left(\frac{b+1}{b-1}\right)^{\frac{1}{b}}$ (Pyragas, 1998a). The relation of delays and of parameters is chosen as: $\tau_7 = \tau_1 + \tau_d$, $\tau_8 = \tau_2 + \tau_d$, $\tau_9 = \tau_4 + \tau_d$, $\tau_{10} = \tau_5 + \tau_d$, $m_1 - k_1 = n_1$, $m_2 - k_2 = n_2$, $m_3 = n_3$, $m_4 - k_3 = n_4$, $m_5 - k_4 = n_5$, $m_6 = n_6$.

The value of delays and parameters are adopted as: $b = 10$, $\alpha = 12.3$, $m_1 = -20.0$, $m_2 = -15.0$, $m_3 = -1.0$, $m_4 = -16.0$, $m_5 = -25.0$, $m_6 = -1.0$, $n_1 = -1.0$, $n_2 = -1.0$, $n_3 = -1.0$, $n_4 = -1.0$, $n_5 = -1.0$, $n_6 = -1.0$, $k_1 = -19.0$, $k_2 = -14.0$, $k_3 = -15.0$, $k_4 = -24.0$, $\tau_d = 5.6$, $\tau_1 = 1.2$, $\tau_2 = 2.3$, $\tau_3 = 3.4$, $\tau_4 = 4.5$, $\tau_5 = 5.6$, $\tau_6 = 6.7$, $\tau_7 = 6.8$, $\tau_8 = 7.9$, $\tau_9 = 10.1$, $\tau_{10} = 11.2$. Illustrated in Fig. 8 is the simulation result for the synchronization manifold of $y(t) = x(t - 5.6)$. Obviously, the lag existing in the state variables is observed in Fig. 8(a). Establishment of the synchronization manifold can be seen through the portrait of $x(t - 5.6)$ versus $y(t)$ in Fig. 8(b). Moreover, the synchronization error vanishes in time evolution as shown in Fig. 8(c). As a result, the desired synchronization manifold is firmly achieved.



(a) Time series of $x(t)$ and $y(t)$

(b) Portrait of $x(t-5.6)$ versus $y(t)$

(c) Synchronization error $\Delta(t) = y(t) - x(t-5.6)$

Fig. 8. Simulation result of lag synchronization of coupled six-delays Mackey-Glass systems.

Example 2:
This example demonstrates the description of anticipating synchronization of coupled identical MTDSs given in Section 3.1.3. The anticipating synchronous scheme is examined in coupled four-delays Ikeda systems with the dynamical equations given as follows
Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P=4} m_i \sin x_{\tau_i} \tag{83}$$

Driving signal:

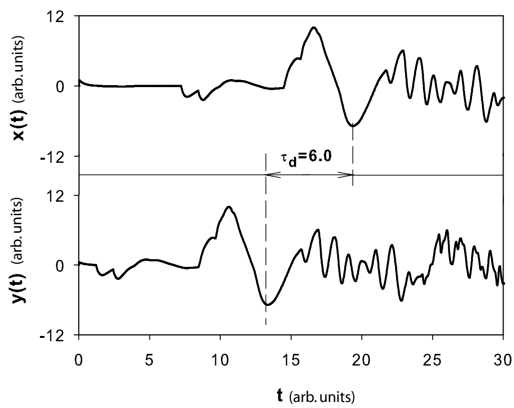
$$DS(t) = \sum_{j=1}^{Q=2} k_j \sin x_{\tau_{P+j}} \tag{84}$$

Slave:

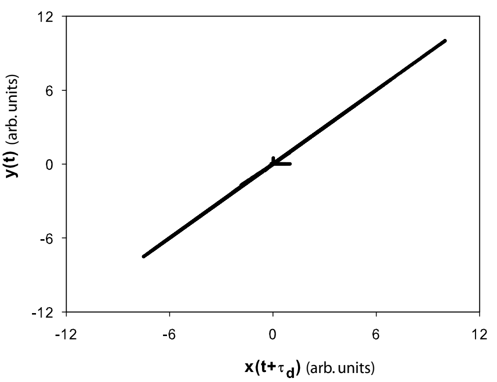
$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P=4} n_i \sin y_{\tau_i} + DS(t) \tag{85}$$

Following to above description, the relation of parameters and delays is chosen as: $m_1 = n_1$, $m_2 - k_1 = n_2$, $m_3 = n_3$, $m_4 - k_2 = n_4$, $\tau_5 = \tau_2 - \tau_d$, $\tau_6 = \tau_4 - \tau_d$. Anticipating synchronization manifold considered in this example is $y(t) = x(t + \tau_d)$, and chosen $\tau_d = 6.0$. The adopted value of parameters and delays for simulation are as: $\alpha = 2.5$, $m_1 = -0.5$, $m_2 = -13.5$, $m_3 = -0.6$, $m_4 = -14.6$, $n_1 = -0.5$, $n_2 = -0.9$, $n_3 = -0.6$, $n_4 = -0.2$, $k_1 = -12.6$, $k_2 = -14.4$, $\tau_1 = 1.5$, $\tau_2 = 7.2$, $\tau_3 = 2.6$, $\tau_4 = 8.4$, $\tau_5 = 1.2$, $\tau_6 = 2.4$.

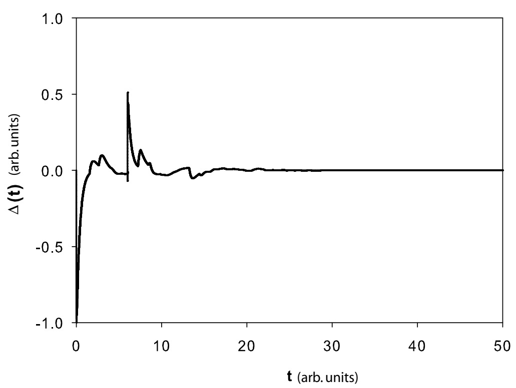
The simulation result is displayed in Fig. 9. It is realized from Fig. 9(a) that the slave anticipates the master's motion, and the synchronization manifold of $y(t) = x(t + 6.0)$ is established as illustrated in Fig. 9(b), with vanishing synchronization error as depicted in Fig. 9(c).



(a) Time series of $x(t)$ and $y(t)$



(b) Portrait of $x(t + 6.0)$ versus $y(t)$



(c) Synchronization error $\Delta(t) = y(t) - x(t + 6.0)$

Fig. 9. Simulation result of anticipating synchronization of coupled four-delays Ikeda systems

Example 3:

To support for projective-lag synchronization as given in Section 3.1.2, this example deals with synchronization of coupled six-delays Mackey-Glass systems with the driving signal constituted by the four-delays components. The dynamical equations are expressed in Eqs. (80)- (82). For the synchronization manifold of $ay(t) = bx(t - \tau_d)$, the relations between the value of delays and parameters are chosen as $\tau_7 = \tau_d + \tau_1$, $\tau_8 = \tau_d + \tau_2$, $\tau_9 = \tau_d + \tau_4$, $\tau_{10} = \tau_d + \tau_6$, $bm_1 - ak_1 = an_1$, $bm_2 - ak_2 = an_2$, $m_3 = n_3$, $bm_4 - ak_3 = an_4$, $m_5 = n_5$, $bm_6 - ak_4 = an_6$. According to Eq. (35), the sufficient condition for synchronization is

$$\alpha > \sum_{i=1}^{P=6} |an_i| \sup \left| f'(x_{\tau_i+\tau_d}) \right|. \tag{86}$$

The value of delays and parameters adopted for simulation are $a = 1.0$, $b = 3.0$, $c = 10$, $\alpha = 6.3$, $\tau_d = 5.6$, $\tau_1 = 6.7$, $\tau_2 = 3.4$, $\tau_3 = 4.5$, $\tau_4 = 5.6$, $\tau_5 = 2.3$, $\tau_6 = 1.2$, $\tau_7 = 12.3$, $\tau_8 = 9.0$, $\tau_9 = 11.2$, $\tau_{10} = 6.8$, $m_1 = -8.0$, $m_2 = -7.0$, $m_3 = -0.3$, $m_4 = -6.7$, $m_5 = -0.2$, $m_6 = -5.4$, $n_1 = -0.6$, $n_2 = -0.5$, $n_3 = -0.3$, $n_4 = -0.8$, $n_5 = -0.2$, $n_6 = -0.7$, $k_1 = -23.4$, $k_2 = -20.5$, $k_3 = -19.3$, and $k_4 = -15.5$.

The simulation result is illustrated in Fig. 10 with synchronization manifold of $1.0y(t) = 3.0x(t-5.6)$. The scale factor can be seen by means of the scale of vertical axes in Fig. 10(a). The scale factor can also be observed via the slope of the synchronization line in the portrait of $x(t-5.6)$ versus $y(t)$ shown in Fig. 10(b). Moreover, the synchronization error is reduced with respect to time as displayed in Figs. 10(c). However, the level of $\Delta_{\tau_i}^{(app)}$ in the linear approximation given in Eq. (31) is dependent on the difference between the value of a and b, $\delta = a - b$. Therefore, examination on the impact of $\delta = a - b$ on the synchronization error is necessary. As presented in Fig. 10(d) is the relation between the means square error (MSE) of the synchronization error in whole synchronizing time and $\delta = a - b$. It is clear that synchronization error is lowest when $\delta = 0$ or $a = b$.

Example 4:

The description given in Section 3.1.4 is illustrated in this example. Projective-anticipating synchronization of coupled five-delays Mackey-Glass systems is examined with three-delays driving signal. The dynamical equations are as

Master:

$$\frac{dx}{dt} = -\alpha x + \sum_{i=1}^{P=5} m_i \frac{x_{\tau_i}}{1 + x_{\tau_i}^c} \tag{87}$$

Driving signal:

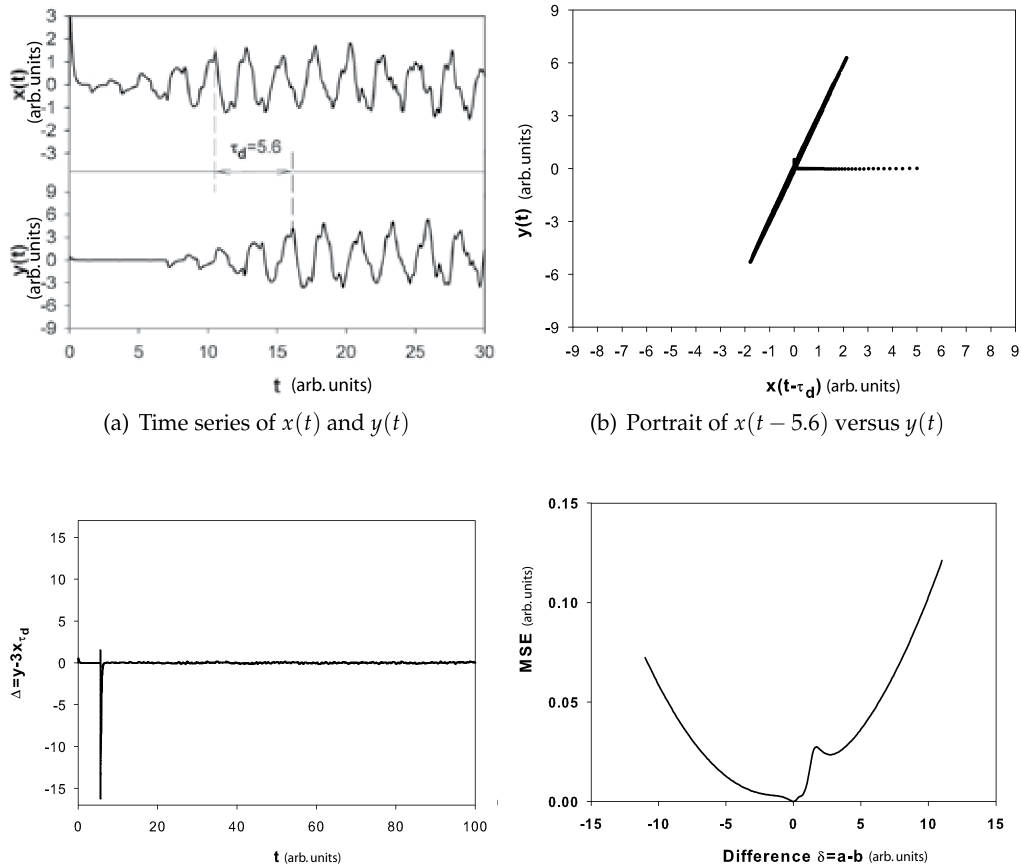
$$DS(t) = \sum_{j=1}^{Q=3} k_j \frac{x_{\tau_{P+j}}}{1 + x_{\tau_{P+j}}^c} \tag{88}$$

Slave:

$$\frac{dy}{dt} = -\alpha y + \sum_{i=1}^{P=5} n_i \frac{y_{\tau_i}}{1 + y_{\tau_i}^c} + DS(t) \tag{89}$$

The synchronization manifold of $ay(t) = bx(t + \tau_d)$ is studied with the relation of delays and parameters chosen as: $\tau_6 = \tau_1 - \tau_d$, $\tau_7 = \tau_3 - \tau_d$, $\tau_8 = \tau_5 - \tau_d$, $bm_1 - ak_1 = an_1$, $m_2 = n_2$, $bm_3 - ak_2 = an_3$, $m_4 = n_4$, $bm_5 - ak_3 = an_5$. The value of parameters and delays for simulation is set at: $a = -2.5$, $b = 1.5$, $\alpha = 16.3$, $c = 10$, $m_1 = -16.2$, $m_2 = -0.3$, $m_3 = -14.5$, $m_4 = -1.0$, $m_5 = -18.6$, $n_1 = -0.4$, $n_2 = -0.3$, $n_3 = -0.8$, $n_4 = -1.0$, $n_5 = -0.7$, $k_1 = 10.12$, $k_2 = 9.5$, $k_3 = 11.86$, $\tau_d = 4.6$, $\tau_1 = 4.8$, $\tau_2 = 3.8$, $\tau_3 = 6.2$, $\tau_4 = 5.5$, $\tau_5 = 4.6$, $\tau_6 = 0.6$, $\tau_7 = 2.0$, $\tau_8 = 0.4$.

The simulation result is depicted in Fig. 11 with the synchronization manifold of $-2.5y(t) = 1.5x(t + 4.6)$. It is easy to observed the scale factor by means of the scale of vertical axes in



(a) Time series of $x(t)$ and $y(t)$

(b) Portrait of $x(t-5.6)$ versus $y(t)$

(c) Synchronization error $\Delta(t) = y - 3x(t-5.6)$

(d) The relation between $\delta = a - b$ and means square error of $\triangle = y - 3x_{\tau_d}$

Fig. 10. Simulation result of projective-lag synchronization of coupled six-delays Mackey-Glass systems

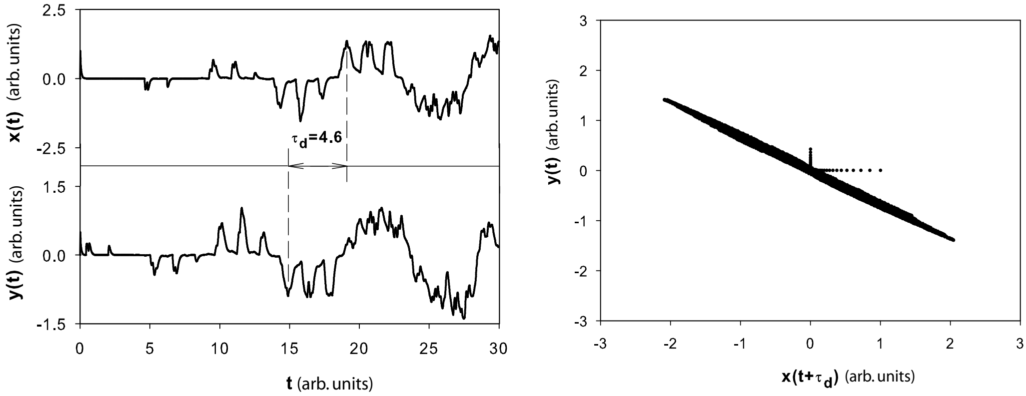
Fig. 11(a). The scale factor can also be observed via the slope of the line illustrated in the portrait of $x(t + 4.6)$ versus $y(t)$ in Fig. 11(b).

Example 5:

Synchronization model in this example demonstrate the lag synchronization of partially identical MTDSs with the general description has been presented in Section 3.2.1. The master's and slave's equations are chosen as

Master:

$$\frac{dx}{dt} = -\alpha x + m_1 sinx_{\tau_1^{(M)}} + m_2 sinx_{\tau_2^{(M)}} + m_3 sinx_{\tau_3^{(M)}} + \\ + m_4 \frac{x_{\tau_4^{(M)}}}{1 + x^8_{\tau_4^{(M)}}} + m_5 \frac{x_{\tau_5^{(M)}}}{1 + x^{10}_{\tau_5^{(M)}}} \tag{90}$$



(a) Time series of $x(t)$ and $y(t)$ (b) Portrait of $x(t+4.6)$ versus $y(t)$

Fig. 11. Simulation result of projective-anticipating synchronization of coupled five-delays Mackey-Glass systems

Slave:

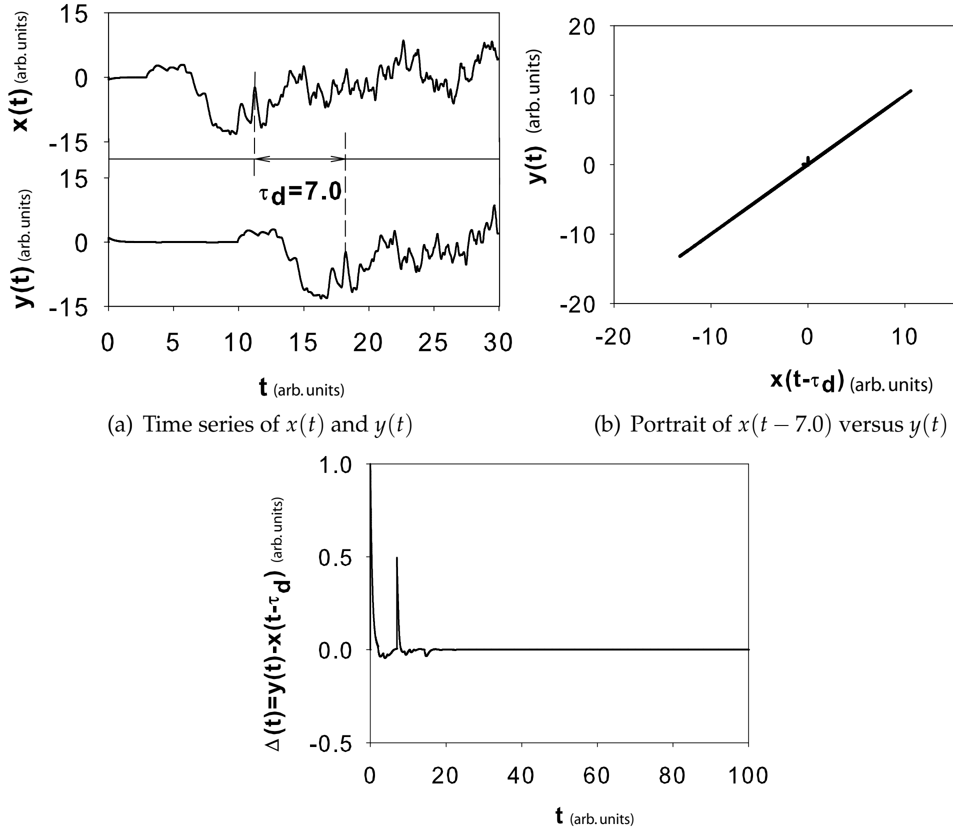
$$\begin{aligned}\frac{dy}{dt} &= -\alpha y + n_1 siny_{\tau_1^{(S)}} + n_2 siny_{\tau_2^{(S)}} + \\ &+ n_3 siny_{\tau_3^{(S)}} + n_4 siny_{\tau_4^{(S)}} + DS(t)\end{aligned} \tag{91}$$

It is easy to observe that the sets of function forms are $S_M = \{sinz, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$, $S_S = \{sinz\}$. Thus, $S_I = S_M \cap S_S = \{sinz\}$ and $S_{DSG} \subseteq S_M \cup S_S = \{sinz, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$. It is assumed that $\tau_1^{(M)} = \tau_1^{(S)}$ and $\tau_2^{(M)} = \tau_2^{(S)}$, thus, the pairs of identicalness components are $\{sinx_{\tau_1^{(M)}}\ vs.\ siny_{\tau_1^{(S)}}\}$ and $\{sinx_{\tau_2^{(M)}}\ vs.\ siny_{\tau_2^{(S)}}\}$. Therefore, the equation for driving signal must be chosen as

$$\begin{aligned}DS(t) &= k_1 sinx_{\tau_1^{(DS)}} + k_2 sinx_{\tau_2^{(DS)}} + k_3 sinx_{\tau_3^{(DS)}} + \\ &+ k_4 \frac{x_{\tau_4^{(DS)}}}{1+x^8_{\tau_4^{(DS)}}} + k_5 \frac{x_{\tau_5^{(DS)}}}{1+x^{10}_{\tau_5^{(DS)}}} + k_6 sinx_{\tau_6^{(DS)}} + k_7 sinx_{\tau_7^{(DS)}}\end{aligned} \tag{92}$$

Following to the assumption described in the above description for the manifold of $y(t) = x(t-\tau_d)$, the relation of delays and coefficients is chosen as: $m_1 - k_1 = n_1, m_2 - k_2 = n_2, k_3 = m_3, k_4 = m_4, k_5 = m_5, k_6 = -n_3, k_7 = -n_4, \tau_1^{(DS)} = \tau_1^{(M)} + \tau_d\ (= \tau_1^{(S)} + \tau_d), \tau_2^{(DS)} = \tau_2^{(M)} + \tau_d\ (= \tau_2^{(S)} + \tau_d), \tau_3^{(DS)} = \tau_3^{(M)} + \tau_d, \tau_4^{(DS)} = \tau_4^{(M)} + \tau_d, \tau_5^{(DS)} = \tau_5^{(M)} + \tau_d, \tau_6^{(DS)} = \tau_3^{(S)} + \tau_d$, and $\tau_7^{(DS)} = \tau_4^{(S)} + \tau_d$. In simulation, the value of parameters are adopted as: $\alpha = 2.0$, $m_1 = -15.4$, $m_2 = -16.0$, $m_3 = -0.35$, $m_4 = -20.0$, $m_5 = -18.5$, $n_1 = -0.2$, $n_2 = -0.1$, $n_3 = -0.25$, $n_4 = -0.4$, $k_1 = -15.2$, $k_2 = -15.9$, $k_3 = -0.35$, $k_4 = -20.0$, $k_5 = -18.5$, $k_6 = 0.25$, $k_7 = 0.4$, $\tau_1^{(M)} = 3.4$, $\tau_2^{(M)} = 4.5$, $\tau_3^{(M)} = 6.5$, $\tau_4^{(M)} = 5.3$, $\tau_5^{(M)} = 2.9$, $\tau_1^{(S)} = 3.4$, $\tau_2^{(S)} = 4.5$, $\tau_3^{(S)} = 2.0$, $\tau_4^{(S)} = 7.3$, $\tau_1^{(DS)} = 10.4$, $\tau_2^{(DS)} = 11.5$, $\tau_3^{(DS)} = 13.5$, $\tau_4^{(DS)} = 12.3$, $\tau_5^{(DS)} = 9.9$, $\tau_6^{(DS)} = 9.0$, and $\tau_7^{(DS)} = 14.3$.

The simulation result illustrated in Fig. 12 shows that the manifold of $y(t) = x(t-7.0)$ is established and maintained. The manifold's delay can be seen in Fig. 12(a) and Fig. 12(b). The synchronization error vanishes eventually as given in Fig. 12(c), it confirms the synchronous regime of nonidentical MTDSs.



(a) Time series of $x(t)$ and $y(t)$

(b) Portrait of $x(t-7.0)$ versus $y(t)$

(c) Synchronization error $\Delta(t) = y(t) - x(t-7.0)$

Fig. 12. Simulation result of lag synchronization of partially identical MTDSs.

Example 6:

In this example, the demonstration for lag synchronization of completely nonidentical MTDSs given in Section 3.2.2 is presented. the equations representing for the master and slave are as Master:

$$\frac{dx}{dt} = -\alpha x + m_1 \frac{x_{\tau_1^{(M)}}}{1 + x^6_{\tau_1^{(M)}}} + m_2 \frac{x_{\tau_2^{(M)}}}{1 + x^8_{\tau_2^{(M)}}} + m_3 \frac{x_{\tau_3^{(M)}}}{1 + x^{10}_{\tau_3^{(M)}}} \tag{93}$$

Slave:

$$\begin{aligned} \frac{dy}{dt} &= -\alpha y + n_1 siny_{\tau_1^{(S)}} + n_2 siny_{\tau_2^{(S)}} + n_3 siny_{\tau_3^{(S)}} + \\ &+ n_4 siny_{\tau_4^{(S)}} + DS(t) \end{aligned} \tag{94}$$

It is clear that the sets of function forms are $S_M = \{\frac{z}{1+z^6}, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$, $S_S = \{sinz\}$, $S_I = S_M \cap S_S \equiv \Phi$. Thus, the subset of function form for DSG is $S_{DSG} \subseteq S_M \cup S_S =$

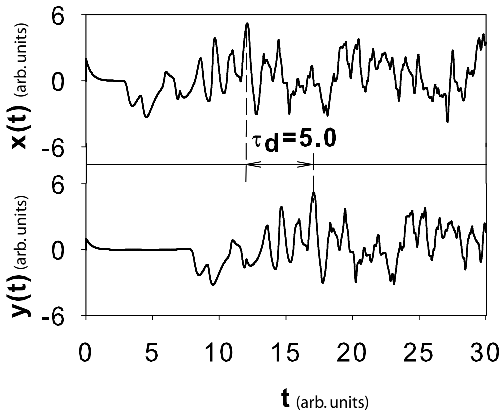
$\{sinz, \frac{z}{1+z^6}, \frac{z}{1+z^8}, \frac{z}{1+z^{10}}\}$, and the driving signal's equation must be chosen as

$$DS(t) = k_1 \frac{x_{\tau_1^{(DS)}}}{1 + x^6_{\tau_1^{(DS)}}} + k_2 \frac{x_{\tau_2^{(DS)}}}{1 + x^8_{\tau_2^{(DS)}}} + k_3 \frac{x_{\tau_3^{(DS)}}}{1 + x^{10}_{\tau_3^{(DS)}}} + \\ + k_4 sinx_{\tau_4^{(DS)}} + k_5 sinx_{\tau_5^{(DS)}} + k_6 sinx_{\tau_6^{(DS)}} + k_7 sinx_{\tau_7^{(DS)}} \tag{95}$$

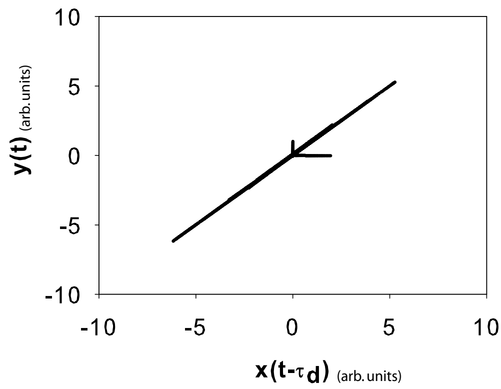
Following to the general description above, the chosen relation of delays and coefficients for the manifold of $y(t) = x(t - \tau_d)$ are as: $k_1 = m_1, k_2 = m_2, k_3 = m_3, k_4 = -n_1, k_5 = -n_2, k_6 = -n_3, k_7 = -n_4$, $\tau_1^{(DS)} = \tau_1^{(M)} + \tau_d$, $\tau_2^{(DS)} = \tau_2^{(M)} + \tau_d$, $\tau_3^{(DS)} = \tau_3^{(M)} + \tau_d$, $\tau_4^{(DS)} = \tau_1^{(S)} + \tau_d$, $\tau_5^{(DS)} = \tau_2^{(S)} + \tau_d$, $\tau_6^{(DS)} = \tau_3^{(S)} + \tau_d$, and $\tau_7^{(DS)} = \tau_4^{(S)} + \tau_d$. And, the value of parameters and delays are adopted for simulation as: $\alpha = 2.5$, $m_1 = -15.5$, $m_2 = -20.2$, $m_3 = -18.4$, $n_1 = -0.3$, $n_2 = -0.2$, $n_3 = -0.4$, $n_4 = -0.6$, $k_1 = -15.5$, $k_2 = -20.2$, $k_3 = -18.4$, $k_4 = 0.3$, $k_5 = 0.2$, $k_6 = 0.4$, $k_7 = 0.6$, $\tau_d = 5.0$, $\tau_1^{(M)} = 2.8$, $\tau_2^{(M)} = 6.4$, $\tau_3^{(M)} = 3.9$, $\tau_1^{(S)} = 1.7$, $\tau_2^{(S)} = 6.5$, $\tau_3^{(S)} = 4.1$, $\tau_4^{(S)} = 8.0$, $\tau_1^{(DS)} = 7.8$, $\tau_2^{(DS)} = 11.4$, $\tau_3^{(DS)} = 8.9$, $\tau_4^{(DS)} = 6.7$, $\tau_5^{(DS)} = 11.5$, $\tau_6^{(DS)} = 9.1$, and $\tau_7^{(DS)} = 13.0$.
Shown in Fig. 13 is the time series of state variables, the portrait of $x(t - 5.0)$ versus $y(t)$ and synchronization error $\Delta(t) = y(t) - x(t - 5.0)$, and it is easy to realize that the desired manifold is created and maintained.

5. Discussion

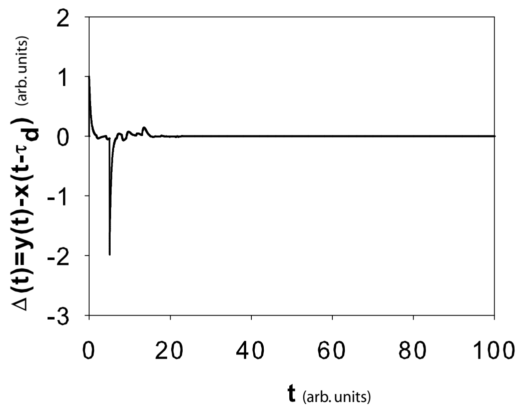
In this section, the discussion is given on four aspects, i.e., the sufficient condition for synchronization, the connection between the synchronous schemes in the proposed models, the form of driving signal and the complicated dynamics of MTDSs in compared to STDSs. These will confirm the application of the proposed synchronization model in secure communications.
Firstly, the sufficient conditions for synchronization given in Eqs. (23), (35), (43), (54), (72) and (79) are loose for adopting value of parameters and delays. It is dependent on value of parameters and not on delays since $f'(x)$ is not a piecewise function with respect to x. This allows to arrange multiple slaves being synchronized with one master at the same time.
Secondly, it is easy to realize from the connection between the synchronous schemes that transition from lag synchronization to anticipating one can be done by changing the relation between delays in DSG from $\tau_{P+j} = \tau_i + \tau_d$ to $\tau_{P+j} = \tau_i - \tau_d$ (see Eqs. (19) and (40)). Moreover, the sufficient condition for lag synchronization is identical to that for anticipating synchronization as presented in Eqs. (23) and (43). Besides, transition from lag synchronization with the synchronization manifold of $y(t) = x(t - \tau_d)$ in Eq. (15) to projective-lag synchronization with the manifold of $ay(t) = bx(t - \tau_d)$ given in Eq. (24) has been done by changing the relation between parameters from $m_i - k_j = n_i$ to $bm_i - ak_j = an_i$ (see Eqs. (21) and (33)); a, b are nonzero real numbers. Similar to the case of transition from lag synchronization to anticipating one, projective-anticipating synchronization has been achieved by changing the relation between delays in projective-lag synchronization from $\tau_{P+j} = \tau_i + \tau_d$ to $\tau_{P+j} = \tau_i - \tau_d$ (see Eqs. (19) and (40)) whereas the relation between parameters and the sufficient condition for synchronization have been kept intact (see Eqs. (33), (35) and (54)). As a special case, if the value of τ_d is set to zero, then lag and anticipating synchronization will become the scheme of complete synchronization of MTDSs and the schemes of projective-lag and projective-anticipating synchronizations turn into the



(a) Time series of $x(t)$ and $y(t)$



(b) Portrait of $x(t-5.0)$ versus $y(t)$



(c) Synchronization error $\Delta(t) = y(t) - x(t-5.0)$

Fig. 13. Simulation for lag synchronization of completely nonidentical MTDSs.

projective synchronization of MTDSs.

Thirdly, in the proposed model of identical MTDSs, it is observed that the driving signals given in Eqs. (13) and (56) are in the form of sum of nonlinear transforms, and they are commonly used for considering all the synchronous schemes. The reason for choosing such the form is to obtain synchronization error dynamics being in the linear form. Then, the Krasovskii-Lyapunov theory is applied to get sufficient condition for synchronization. Assumptions made to $f(.)$ being differentiable and bounded as well as obliged relations made to parameters and delays are also for this reason. This must be appropriate to given forms of the master and slave.

Lastly, earlier part of the paper has been mentioned the prediction that MTDSs may hold more complicated dynamics than STDSs do. This has been confirmed from the result of numerical simulation given in Section 2.2. It is well-known that Lyapunov exponents and metric entropy are measure of complexity degree for chaotic dynamics. That is, in the specific example of two-delays Mackey-Glass system, it is possible to obtain dynamics with LLE of approximate 0.7 and metric entropy of around 1.4 as shown in Fig. 6 by adopting suitable value of parameters and delays. Recall that, in the specific example of single time-delay Mackey-Glass system examined by J.D. Farmer (Farmer, 1982), LLE and metric entropy were reported at around 0.07 and 0.1, respectively. The 'V' shape of LLE and metric entropy with respect to m_1 and m_2 in Figs. 4 and 5 illustrates more intuitively. At small value of m_i, the two-delays system tends to be single time-delay system due to weak feedback. The shift of 'V' shape in the case of $m_3 = 3.0$ can be interpreted that there is some correlation to value of delays. Here, τ_2 associated with m_2 holds largest value. Undoubtedly, MTDSs holds dynamics which is more complicated than that of STDSs.

6. Conclusion

In this chapter, the synchronization model of coupled identical MTDSs has been presented, in which the coupling signal is sum of nonlinear transforms of delayed state variable. The synchronous schemes of lag, anticipating, projective-lag and projective-anticipating have been examined in the proposed models. In addition, the synchronization model of coupled nonidentical MTDS has been studied in two cases, i.e., partially identical and completely nonidentical. The scheme of lag synchronization has been used for demonstrating and verifying the cases. The simulation result has consolidated the general description to the proposed synchronous schemes. Noticeably, combination between synchronous schemes of projective and lag/anticipating is first time mentioned and investigated.

The transition between the lag and anticipating synchronization as well as between the projective-lag to projective-anticipating synchronization can be yielded simply by adjusting the relation between delays while the change from the lag to projective-lag synchronization and from the anticipating to projective-anticipating synchronization has been realized by modifying the relation between coefficients. Similarly, other synchronous schemes of coupled nonidentical MTDSs can be investigated as ways dealing in the synchronization models of identical MTDSs, and synchronous regimes will also be established as expected. This allows the synchronization models becoming flexible in selection of working scheme and switch among various schemes.

In summary, the proposed synchronization models present advantages to the application of secure communications in comparison with conventional ones. Advantages lie in both the complexity of driving signal and infinite-dimensional dynamics.

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11

T-S Fuzzy H_∞ Tracking Control of Input Delayed Robotic Manipulators

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1. Introduction

Time delays are often encountered by practical control systems while they are acquiring, processing, communicating, and sending signals. Time delays may affect the system stability and degrade the control system performance if they are not properly dealt with. Taking the classical robot control problem as an example, the significant effect of time delay on the closed-loop system stability has been highlighted in the bilateral teleoperation, where the communication delay transmitted through a network medium has been received widespread attention and different approaches have been proposed to address this problem (Hokayem and Spong, 2006). In addition, examples like processing delays in visual systems and communication delay between different computers on a single humanoid robot are also main sources that may cause time delays in a robotic control system (Chopra, 2009), and the issue of time delay for robotic systems has been studied through the passivity property.

For systems with time delays, both delay dependent and delay independent control strategies have been extensively studied in recent years, see for example (Xu and Lam, 2008) and references therein. For the control of nonlinear time delay systems, model based Takagi-Sugeno (T-S) fuzzy control (Tanaka and Wang, 2001; Feng, 2006; Lin et al., 2007) is regarded as one of the most effective approach because some of linear control theory can be applied directly. Conditions for designing such kinds of controllers are generally expressed as linear matrix inequalities (LMIs) which can be efficiently solved by using most available software like Matlab LMI Toolbox, or bilinear matrix inequalities (BMIs) which could be transferred to LMIs by using algorithms like iteration algorithm or cone complementary linearisation algorithm. From the theoretical point of view, one of the current focus on the control of time delay systems is to develop less conservative approaches so that the controller can stabilise the systems or can achieve the defined control performance under bigger time delays (Chen et al., 2009; Liu et al., 2010).

Tracking control of robotic manipulators is another important topic which receives considerable attention due to its significant applications. Over the decades, various approaches in tracking control of nonlinear systems have been investigated, such as adaptive control approach, variable structure approach, and feedback linearisation approach, etc. Fuzzy control technique through T-S fuzzy model approach is also one effective approach in tracking control of nonlinear systems (Ma and Sun, 2000; Tong et al., 2002; Lin et al., 2006), and in particular, for robotic systems (Tseng et al., 2001; Begovich et al., 2002; Ho et al., 2007).

In spite of the significance on tracking control of robotic systems with input time delays, few studies have been found in the literature up to the date. This chapter attempts to propose an H_∞ controller design approach for tracking control of robotic manipulators with input delays. As a robotic manipulator is a highly nonlinear system, to design a controller such that the tracking performance in the sense of H_∞ norm can be achieved with existing input time delays, the T-S fuzzy control strategy is applied. Firstly, the nonlinear robotic manipulator model is represented by a T-S fuzzy model. And then, sufficient conditions for designing such a controller are derived with taking advantage of the recently proposed method (Li and Liu, 2009) in constructing a Lyapunov-Krasovskii functional and using a tighter bounding technology for cross terms and the free weighting matrix approach to reduce the issue of conservatism. The control objective is to stabilise the control system and to minimise the H_∞ tracking performance, which is related to the output tracking error for all bounded reference inputs, subject to input time delays. With appropriate derivation, all the required conditions are expressed as LMIs. Finally, simulation results on a two-link manipulator are used to validate the effectiveness of the proposed approach. The main contributions of this chapter are: 1) to propose an effective controller design method for tracking control of robotic manipulator with input time delays; 2) to apply advanced techniques in deriving less conservative conditions for designing the required controller; 3) to derive the conditions properly so that they can be expressed as LMIs and can be solved efficiently.

This chapter is organised as follows. In section 2, the problem formulation and some preliminaries on manipulator model, T-S fuzzy model, and tracking control problem are introduced. The conditions for designing a fuzzy H_∞ tracking controller are derived in section 3. In section 4, the simulation results on stability control and tracking control of a nonlinear two-link robotic manipulator are discussed. Finally, conclusions are summarised in section 5.

The notation used throughout the paper is fairly standard. For a real symmetric matrix W, the notation of W >0 (W <0) is used to denote its positive- (negative-) definiteness. $\|.\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm. I is used to denote the identity matrix of appropriate dimensions. To simplify notation, * is used to represent a block matrix which is readily inferred by symmetry.

2. Preliminaries and problem statement

2.1 Manipulator dynamics model

To simplify the problem formulation, a two-link robot manipulator as shown in Fig. 1 is considered.

The dynamic equation of the two-link robot manipulator is expressed as (Tseng, Chen and Uang, 2001)

$$M(q)\ddot{q}+V(q,\dot{q})\dot{q}+G(q)=u \tag{1}$$

where

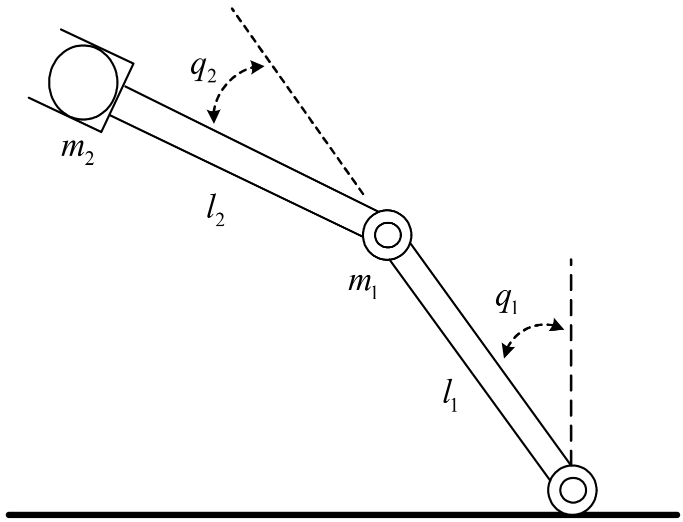

Fig. 1. Two-link robotic manipulator.

$$M(q)=\begin{bmatrix} (m_1+m_2)l_1^2 & m_1l_1l_2(s_1s_2+c_1c_2) \\ m_2l_1l_2(s_1s_2+c_1c_2) & m_2l_2^2 \end{bmatrix}$$

$$V(q,\dot{q})=m_2l_1l_2(c_1c_2-s_1c_2)\begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}$$

$$G(q)=\begin{bmatrix} -(m_1+m_2)l_1gs_1 \\ -m_2l_2gs_2 \end{bmatrix}$$

and $q=[q_1,q_2]^T$ and $u=[u_1,u_2]^T$ denote the generalised coordinates (radians) and the control torques (N-m), respectively. M(q) is the moment of inertia, $V(q,\dot{q})$ is the centripetal-Coriolis matrix, and G(q) is the gravitational vector. m_1 and m_2 (in kilograms) are link masses, l_1 and l_2 (in meters) are link lengths, g=9.8 (m/s^2) is the acceleration due to gravity, and $s_1=\sin(q_1)$, $s_2=\sin(q_2)$, $c_1=\cos(q_1)$, and $c_2=\cos(q_2)$. After defining $x_1=q_1$, $x_2=\dot{q}_1$, $x_3=q_2$, and $x_4=\dot{q}_2$, equation (1) can be rearranged as

$$\begin{aligned} \dot{x}_1&=x_2+w_1 \\ \dot{x}_2&=f_1(x)+g_{11}(x)u_1+g_{12}(x)u_2+w_2 \\ \dot{x}_3&=x_4+w_3 \\ \dot{x}_4&=f_2(x)+g_{21}(x)u_1+g_{22}(x)u_2+w_4 \end{aligned} \tag{2}$$

where w_1, w_2, w_3, w_4 denote external disturbances, and

$$f_1(x)=\frac{(s_1c_2-c_1s_2)}{l_1l_2[(m_1+m_2)-m_2(s_1s_2+c_1c_2)^2][m_2l_1l_2[(s_1s_2+c_1c_2)x_2^2-m_2l_2^2x_4^2]}$$
$$+\frac{1}{l_1l_2[(m_1+m_2)-m_2(s_1s_2+c_1c_2)^2][(m_1+m_2)l_2gs_1-m_2l_2gs_2(s_1s_2+c_1c_2)]}$$

$$f_2(x)=\frac{(s_1c_2\text{-}c_1s_2)}{l_1l_2[(m_1+m_2)\text{-}m_2(s_1s_2+c_1c_2)^2][\text{-}(m_1+m_2)l_1^2x_2^2+m_2l_1l_2(s_1s_2+c_1c_2)x_4^2]}$$
$$+\frac{1}{l_1l_2[(m_1+m_2)\text{-}m_2(s_1s_2+c_1c_2)^2][\text{-}(m_1+m_2)l_1gs_1(s_1s_2+c_1c_2)+(m_1+m_2)l_1gs_2]}$$
$$g_{11}(x)=\frac{m_2l_2^2}{m_2l_1^2l_2^2[(m_1+m_2)\text{-}m_2(s_1s_2+c_1c_2)^2]}$$
$$g_{12}(x)=\frac{\text{-}m_2l_1l_2(s_1s_2+c_1c_2)}{m_2l_1^2l_2^2[(m_1+m_2)\text{-}m_2(s_1s_2+c_1c_2)^2]}$$
$$g_{21}(x)=\frac{\text{-}m_2l_1l_2(s_1s_2+c_1c_2)}{m_2l_1^2l_2^2[(m_1+m_2)\text{-}m_2(s_1s_2+c_1c_2)^2]}$$
$$g_{22}(x)=\frac{(m_1+m_2)l_1^2}{m_2l_1^2l_2^2[(m_1+m_2)\text{-}m_2(s_1s_2+c_1c_2)^2]}$$

Note that the time variable t is omitted in the above equations for brevity.

2.2 T-S fuzzy model

The above described robotic manipulator is a nonlinear system. To deal with the controller design problem for the nonlinear system, the T-S fuzzy model is employed to represent the nonlinear system with input delays as follows:
Plant rule i

IF $\theta_1(t)$ is N_{i1} , …, $\theta_p(t)$ is N_{ip} THEN

$$\begin{aligned}&\dot{x}(t)=A_ix(t)+B_iu(t\text{-}\tau)+Ew(t)\\&y(t)=Cx(t)\\&x(0)=x_0,u(t)=\varphi(t),t\in[\text{-}\bar{\tau},0],i=1,2,...,k\end{aligned} \tag{3}$$

where N_{ij} is a fuzzy set, $\theta(t)=[\theta_1(t),...,\theta_p(t)]^T$ are the premise variables, x(t) is the state vector, and w(t) is external disturbance vector, A_i and B_i are constant matrices. Scalar k is the number of IF-THEN rules. It is assumed that the premise control variables do not depend on the input u(t). The input delay τ is an unknown constant time-delay, and the constant $\bar{\tau}>0$ is an upper bound of τ .
Given a pair of (x(t),u(t)), the final output of the fuzzy system is inferred as follows

$$\begin{aligned}&\dot{x}(t)=\sum_{i=1}^{k}h_i(\theta(t))(A_ix(t)+B_iu(t\text{-}\tau)+Ew(t))\\&y(t)=Cx(t)\\&x(0)=x_0,u(t)=\varphi(t),t\in[\text{-}\bar{\tau},0]\end{aligned} \tag{4}$$

where $h_i(\theta(t))=\frac{\mu_i(\theta(t))}{\sum_{i=1}^{k}\mu_i(t))}$, $\mu_i(\theta_j(t))=\prod_{j=1}^{p}N_{ij}(\theta_j(t))$ and $N_{ij}(\theta_j(t))$ is the degree of the membership of $\theta_j(t)$ in N_{ij}. In this paper, we assume that $\mu_i(\theta_j(t))\geq 0$ for i=1,2,…,k and $\sum_{i=1}^{k}\mu_i(\theta(t))>0$ for all t. Therefore, $h_i(\theta(t))\geq 0$ for i=1,2,…,k, and $\sum_{i=1}^{k}h_i(\theta(t))=1$.

2.1 Tracking control problem

Consider a reference model as follows

$$\begin{aligned}\dot{x}_r(t)&=A_r x_r(t)+r(t)\\ y_r(t)&=C_r x_r(t)\end{aligned} \tag{5}$$

where $x_r(t)$ and $r(t)$ are reference state and energy-bounded reference input vectors, respectively, A_r and C_r are appropriately dimensioned constant matrices. It is assumed that both $x(t)$ and $x_r(t)$ are online measurable.

For system model (3) and reference model (5), based on the parallel distributed compensation (PDC) strategy, the following fuzzy control law is employed to deal with the output tracking control problem via state feedback.

Control rule

IF $\theta_1(t)$ is N_{i1} , …, $\theta_p(t)$ is N_{ip} THEN

$$u(t)=K_{1i}x(t)+K_{2i}x_r(t),\ i=1,2,...,k \tag{6}$$

Hence, the overall fuzzy control law is represented by

$$u(t)=\sum_{i=1}^{k} h_i(\theta(t))[K_{1i}x(t)+K_{2i}x_r(t)]=\sum_{i=1}^{k} h_i(\theta(t))K_i\bar{x}(t) \tag{7}$$

where K_{1i}, and K_{2i}, i=1,2,…,k, are the local control gains, and $K_i=[K_{1i}\ \ K_{2i}]$ and $\bar{x}(t)=[x^T(t),x_r^T(t)]^T$. When there exists an input delay τ, we have that $u(t-\tau)=\sum_{i=1}^{k} h_i(\theta(t-\tau))[K_{1i}x(t-\tau)+K_{2i}x_r(t-\tau)]$, so, it is natural and necessary to make an assumption that the functions $h_i(\theta(t))$, i=1,2,… ,k, are well defined for all $t\in[-\tau,0]$, and satisfy the following properties $h_i(\theta(t-\tau))\geq 0$ for i=1,2,…,k and $\sum_{i=1}^{k} h_i(\theta(t-\tau))=1$. For convenience, let $h_i=h_i(\theta(t))$, $h_i(\tau)=h_i(\theta(t-\tau))$, $x(\tau)=x(t-\tau)$, and $u(\tau)=u(t-\tau)$. From here, unless confusion arises, time variable t will be omitted again for notational convenience.

With the control law (7), the augmented closed-loop system can be expressed as follows

$$\begin{aligned}\dot{\bar{x}}&=\sum_{i,j=1}^{k} h_i h_j(\tau)[\bar{A}_i\bar{x}+\bar{B}_{ij}\bar{x}(\tau)+\bar{E}\bar{v}]\\ e&=\bar{C}\bar{x}\end{aligned} \tag{8}$$

where

$$\bar{A}_i=\begin{bmatrix}A_i & 0\\ 0 & A_r\end{bmatrix},\bar{B}_{ij}=\begin{bmatrix}B_iK_{1j} & B_iK_{2j}\\ 0 & 0\end{bmatrix}=\begin{bmatrix}B_i\\ 0\end{bmatrix}[K_{1j}\quad K_{2j}]=\hat{B}_iK_j,\bar{E}=\begin{bmatrix}E & 0\\ 0 & I\end{bmatrix},\bar{C}=[C\quad -C_r],\bar{v}=\begin{bmatrix}w\\ r\end{bmatrix},e=y-y_r .$$

The tracking requirements are expressed as follows

1. The augmented closed-loop system in (8) with $\bar{v}=0$ is asymptotically stable;
2. The H_∞ tracking performance related to tracking error e is attenuated below a desired level, i.e., it is required that

$$\|e\|_2 < \gamma \|\bar{v}\|_2 \tag{9}$$

for all nonzero $\bar{v} \in L_2[0,\infty)$ under zero initial condition, where $\gamma>0$.
Our purpose is to find the feedback gains K_i (i=1,2,…,k) such that the above mentioned two requirements are met.

3. Tracking controller design

To derive the conditions for designing the required controller, the following lemma will be used.

Lemma 1: (Li and Liu, 2009) For any constant matrices $S_{11} \geq 0$, S_{12}, $S_{22} \geq 0$, $\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} \geq 0$, scalar $\tau \leq \bar{\tau}$ and vector function $\dot{x}:[-\bar{\tau},0] \to R^n$ such that the following integration is well defined, then

$$-\bar{\tau}\int_{t-\bar{\tau}}^{t}[x^T(s),\dot{x}^T(s)]\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}\begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} \leq \begin{bmatrix} x \\ x(\tau) \\ \int_{t-\tau}^{t} x(s)ds \end{bmatrix}^T \begin{bmatrix} -S_{22} & S_{22} & -S_{12}^T \\ S_{22} & -S_{22} & S_{12}^T \\ -S_{12} & S_{12} & -S_{11} \end{bmatrix} \begin{bmatrix} x \\ x(\tau) \\ \int_{t-\tau}^{t} x(s)ds \end{bmatrix} \tag{10}$$

We now choose a delay-dependent Lyapunov-Krasovksii functional candidate as

$$V=\bar{x}^T P\bar{x}+\bar{\tau}\int_{t-\bar{\tau}}^{t}(s-(t-\bar{\tau}))\eta^T(s)S\eta(s)ds \tag{11}$$

where $\eta(s)=\left[\bar{x}^T(s),\dot{\bar{x}}^T(s)\right]^T$, P>0, $S=\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$, $S_{11}>0$, $S_{22}>0$, $\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}>0$.
The derivative of V along the trajectory of (8) satisfies

$$\dot{V}=2\bar{x}^T P\dot{\bar{x}}+\bar{\tau}^2\eta^T S\eta-\bar{\tau}\int_{t-\bar{\tau}}^{t}\eta^T(s)S\eta(s)ds \tag{12}$$

If follows from (8) that

$$0=2[\bar{x}^T T_1+\bar{x}^T(\tau)T_2+\dot{\bar{x}}^T T_3+d_4\bar{v}^T]\left(\sum_{i,j=1}^{k}h_i h_j(\tau)[\bar{A}_i\bar{x}+\bar{B}_{ij}\bar{x}(\tau)+\bar{E}\bar{v}]-\dot{\bar{x}}\right) \tag{13}$$

i.e.,

$$0=2\sum_{i,j=1}^{k}h_i h_j(\tau)[\bar{x}^T \quad \bar{x}^T(\tau) \quad \dot{\bar{x}}^T \quad \bar{v}^T]\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ d_4 I \end{bmatrix}\begin{bmatrix} \bar{A}_i & \bar{B}_{ij} & -I & \bar{E} \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{x}(\tau) \\ \dot{\bar{x}} \\ \bar{v} \end{bmatrix}$$

$$=\sum_{i,j=1}^{k}h_ih_j(\tau)[\bar{x}^T \quad \bar{x}^T(\tau) \quad \dot{\bar{x}}^T \quad \bar{v}^T]\left(\begin{bmatrix} T_1\bar{A}_i & T_1\bar{B}_{ij} & -T_1 & T_1\bar{E} \\ T_2\bar{A}_i & T_2\bar{B}_{ij} & -T_2 & T_2\bar{E} \\ T_3\bar{A}_i & T_3\bar{B}_{ij} & -T_3 & T_3\bar{E} \\ d_4\bar{A}_i & d_4\bar{B}_{ij} & -d_4I & d_4\bar{E}\end{bmatrix} + \begin{bmatrix} T_1\bar{A}_i & T_1\bar{B}_{ij} & -T_1 & T_1\bar{E} \\ T_2\bar{A}_i & T_2\bar{B}_{ij} & -T_2 & T_2\bar{E} \\ T_3\bar{A}_i & T_3\bar{B}_{ij} & -T_3 & T_3\bar{E} \\ d_4\bar{A}_i & d_4\bar{B}_{ij} & -d_4I & d_4\bar{E}\end{bmatrix}^T\right)\begin{bmatrix}\bar{x} \\ \bar{x}(\tau) \\ \dot{\bar{x}} \\ \bar{v}\end{bmatrix} \tag{14}$$

where T_1, T_2, and T_3 are constant matrices, and d_4 is a constant scalar. Note that d_4 is introduced as a scalar not a matrix because it is convenient to get the LMI conditions later. Using the above given equality (14) and Lemma 1, and adding two sides of (12) by $e^Te-\gamma^2\bar{v}^T\bar{v}$, it is obtained that

$$\begin{aligned}\dot{V}+e^Te-\gamma^2\bar{v}^T\bar{v} \le\ & 2\bar{x}^TP\dot{\bar{x}}+\bar{\tau}[\bar{x}^T,\dot{\bar{x}}^T]\begin{bmatrix}S_{11} & S_{12} \\ * & S_{22}\end{bmatrix}\begin{bmatrix}\bar{x} \\ \dot{\bar{x}}\end{bmatrix}+e^Te-\gamma^2\bar{v}^T\bar{v} \\ & +\begin{bmatrix}x \\ x(\tau) \\ \int_{t-\tau}^{t}x(s)ds\end{bmatrix}^T\begin{bmatrix}-S_{22} & S_{22} & -S_{12}^T \\ S_{22} & -S_{22} & S_{12}^T \\ -S_{12} & S_{12} & -S_{11}\end{bmatrix}\begin{bmatrix}x \\ x(\tau) \\ \int_{t-\tau}^{t}x(s)ds\end{bmatrix} \\ & +2\sum_{i,j=1}^{k}h_ih_j(\tau)[\bar{x}^T \quad \bar{x}^T(\tau) \quad \dot{\bar{x}}^T \quad \bar{v}^T]\begin{bmatrix}T_1 \\ T_2 \\ T_3 \\ d_4I\end{bmatrix}\begin{bmatrix}\bar{A}_i & \bar{B}_{ij} & -I & \bar{E}\end{bmatrix}\begin{bmatrix}\bar{x} \\ \bar{x}(\tau) \\ \dot{\bar{x}} \\ \bar{v}\end{bmatrix} \\ =\ & \sum_{i,j=1}^{k}h_ih_j(\tau)\xi^T\Sigma_{ij}\xi\end{aligned} \tag{15}$$

where $\xi^T=\begin{bmatrix}\bar{x}^T & \bar{x}^T(\tau) & \left(\int_{t-\tau}^{t}\bar{x}(s)ds\right)^T & \dot{\bar{x}}^T & \bar{v}^T\end{bmatrix}$ and

$$\Sigma_{ij}=\begin{bmatrix} \bar{\tau}^2S_{11}-S_{22}+T_1\bar{A}_i+\bar{A}_i^TT_1^T+\bar{C}^T\bar{C} & S_{22}+T_1\bar{B}_{ij}+\bar{A}_i^TT_2^T & -S_{12}^T & P+\bar{\tau}^2S_{12}-T_1+\bar{A}_i^TT_3^T & T_1\bar{E}+d_4\bar{A}_i^T \\ * & -S_{22}+T_2\bar{B}_{ij}+\bar{B}_{ij}^TT_2^T & S_{12}^T & -T_2+\bar{B}_{ij}^TT_3^T & T_2\bar{E}+d_4\bar{B}_{ij}^T \\ * & * & -S_{11} & 0 & 0 \\ * & * & * & \bar{\tau}^2S_{22}-T_3-T_3^T & T_3\bar{E}-d_4I \\ * & * & * & * & d_4\bar{E}+d_4\bar{E}^T-\gamma^2I\end{bmatrix} \tag{16}$$

It can be seen from (15) that if $\Sigma_{ij} < 0$, then $\dot{V}+e^Te-\gamma^2\bar{v}^T\bar{v}<0$ can be deduced and therefore $\|e\|_2<\gamma\|\bar{v}\|_2$ can be established with the zero initial condition. When the disturbance is zero, i.e., $\bar{v}=0$, it can be inferred from (15) that if $\Xi_{ij} < 0$, then $\dot{V}<0$, and the closed-loop system (8) is asymptotically stable.

By denoting $T_2=d_2T_1, T_3=d_3T_1$, where d_2 and d_3 are given constants, pre and post-multiplying both side of (16) with diag[Q, Q, Q, I, Q] and their transpose, defining new variables $Q=T_1^{-1}$, $\bar{S}_{11}=QS_{11}Q^T$, $\bar{S}_{12}=QS_{12}Q^T$, $\bar{S}_{22}=QS_{22}Q^T$, $\bar{P}=QPQ^T$, and $\bar{K}_j=K_jQ^T$, $\Sigma_{ij} < 0$ is equivalent to

$$\begin{bmatrix} \bar{\tau}^2\bar{S}_{11}-\bar{S}_{22}+\bar{A}_iQ^T+Q\bar{A}_i^T+Q\bar{C}^T\bar{C}Q^T & \bar{S}_{22}+\hat{B}_i\bar{K}_j+d_2Q\bar{A}_i^T & -\bar{S}_{12}^T & \bar{P}+\bar{\tau}^2\bar{S}_{12}-Q^T+d_3Q\bar{A}_i^T & \bar{E}+d_4Q\bar{A}_i^T \\ * & -\bar{S}_{22}+d_2\hat{B}_i\bar{K}_j+d_2\bar{K}_j^T\hat{B}_i^T & \bar{S}_{12}^T & d_3\bar{K}_j^T\hat{B}_i^T-d_2Q^T & d_2\bar{E}+d_4\bar{K}_j^T\hat{B}_i^T \\ * & * & -\bar{S}_{11} & 0 & 0 \\ * & * & * & \bar{\tau}^2\bar{S}_{22}-d_3Q-d_3Q^T & d_3\bar{E}-d_4Q \\ * & * & * & * & d_4\bar{E}+d_4\bar{E}^T-\gamma^2I \end{bmatrix} < 0 \quad (17)$$

which is further equivalent to $\Xi_{ij} < 0$ by the Schur complement, where

$$\Xi_{ij}=\begin{bmatrix} \bar{\tau}^2\bar{S}_{11}-\bar{S}_{22}+\bar{A}_iQ^T+Q\bar{A}_i^T & \bar{S}_{22}+\hat{B}_i\bar{K}_j+d_2Q\bar{A}_i^T & -\bar{S}_{12}^T & \bar{P}+\bar{\tau}^2\bar{S}_{12}-Q^T+d_3Q\bar{A}_i^T & \bar{E}+d_4Q\bar{A}_i^T & Q\bar{C}^T \\ * & -\bar{S}_{22}+d_2\hat{B}_i\bar{K}_j+d_2\bar{K}_j^T\hat{B}_i^T & \bar{S}_{12}^T & d_3\bar{K}_j^T\hat{B}_i^T-d_2Q^T & d_2\bar{E}+d_4\bar{K}_j^T\hat{B}_i^T & 0 \\ * & * & -\bar{S}_{11} & 0 & 0 & 0 \\ * & * & * & \bar{\tau}^2\bar{S}_{22}-d_3Q-d_3Q^T & d_3\bar{E}-d_4Q & 0 \\ * & * & * & * & d_4\bar{E}+d_4\bar{E}^T-\gamma^2I & 0 \\ * & * & * & * & * & -I \end{bmatrix} \quad (18)$$

In terms of the above given analysis, we now summarise the proposed tracking controller design procedure as:

i. define value for $\bar{\tau}$ and choose appropriate values for d_2, d_3, and d_4.
ii. solve the following LMIs

$$\Xi_{ii} < 0 \quad (19)$$

$$\Xi_{ij} + \Xi_{ji} < 0 \quad (20)$$

$$\begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} \\ * & \bar{S}_{22} \end{bmatrix} \geq 0 \tag{21}$$

If there exist $\bar{P}>0$, $\bar{S}_{11}>0$, $\bar{S}_{22}>0$ and real matrices Q, $\bar{S}_{12}$, $\bar{K}_j$ (j=1,...k) satisfying LMIs (19-21), then the closed-loop system (8) is asymptotically stable for any $0 \leq \tau \leq \bar{\tau}$ and the tracking performance defined in (9) can be achieved.

iii. obtain the control gain matrices as

$$K_j = \bar{K}_j (Q^T)^{-1} \tag{22}$$

4. Numerical example

This section takes two-link robotic manipulator as an example and evaluates the proposed controller design approach through numerical simulations. In the reference (Tseng, Chen and Uang, 2001), the T-S fuzzy model with nine rules is used to represent the original nonlinear manipulator system with acceptable accuracy when link masses $m_1=m_2=1$ (kg), link lengths $l_1=l_2=1$ (m), and angular positions are constrained within $[-\pi/2, \pi/2]$, where triangle type membership functions are used for all the rules.

To show the effectiveness of the proposed controller design method, the stability control of the robotic manipulator with and without input delays is firstly evaluated. For comparison purpose, we introduce a so-called robust controller from (Sun, et al., 2007), which was designed using a region based rule reduction approach and obtained with one rule to reduce the complexity caused by the number of fuzzy rules. The design result for this controller with a decay rate 0.5 was given as

$$K = \begin{bmatrix} -115.6439 & -49.9782 & -13.4219 & -3.7453 \\ 14.6547 & -3.4203 & -62.7788 & -22.1846 \end{bmatrix} \tag{23}$$

The simulation results for the nonlinear model (1) with initial condition $x(0)=[1.2,0,-1.2,0]^T$ and controller (23) without input delays are shown in Fig. 2.

It is seen from Fig. 2 that all the state variables converge to the equilibrium states from initial conditions quickly. We now introduce input delays to the two control inputs. As an example, input delays for both control inputs are given as 24 ms, and the simulation results for all state variables are shown in Fig. 3.

It is observed that the state variables do not converge to equilibrium states in this case and hence controller (23) is not able to stabilise the system when input time delays are given as 24 ms.

Following the similar idea given in (Sun, et al., 2007), a robust controller which uses only one rule and considers the fuzzy model as a polytopic uncertain model can also be designed using the presented conditions (19-21). We now use the reference model as

$$A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix},$$

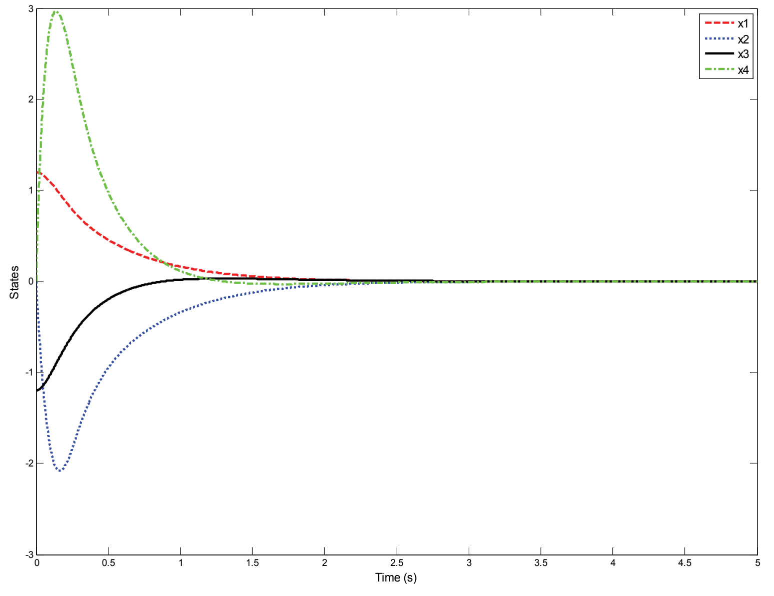

Fig. 2. State responses for controller (23) without input delays.

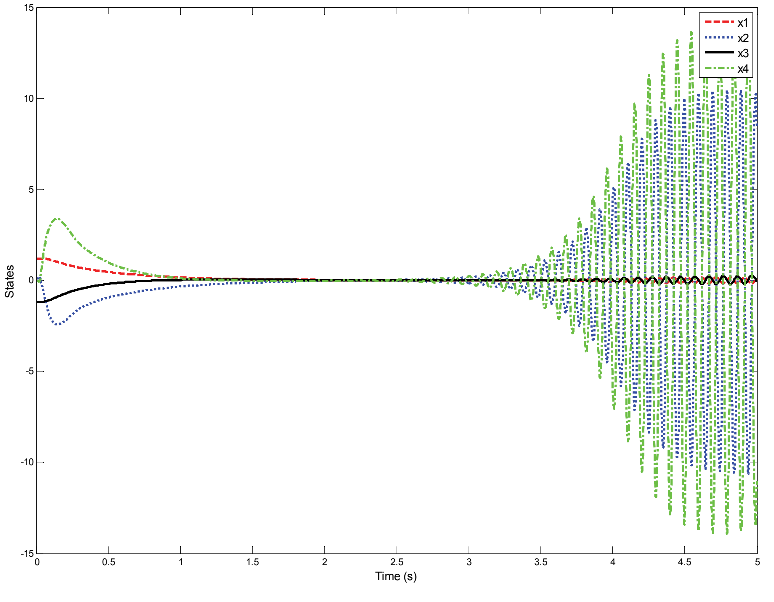

Fig. 3. State responses for controller (23) with input delay as 24 ms.

choose $\bar{\tau}$ =30 ms, d_2=0.1, d_3=0.1, d_4=0.1, and define $\bar{C}=\begin{bmatrix} 10 & 0 & 0 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & -10 & 0 \end{bmatrix}$, which aims on reducing tracking errors on state variables x_1 and x_3, the LMIs (19-21) are feasible to find a solution, and the controller gain matrix is obtained as

$$K=\begin{bmatrix} -52.5581 & -14.8674 & 0.7159 & -0.0785 & 33.3479 & 5.8168 & -5.0603 & -0.6409 \\ -0.6312 & -0.5382 & -31.8608 & -8.5689 & -1.9704 & -0.2084 & 22.7118 & 3.7215 \end{bmatrix} \tag{24}$$

To check the stability control performance of the designed controller (24), the reference input and external disturbances are all set as zero, and the initial conditions are same to the above used values. The simulation results with controller (24) are now shown in Fig. 4 when input delays are given as 30 ms.

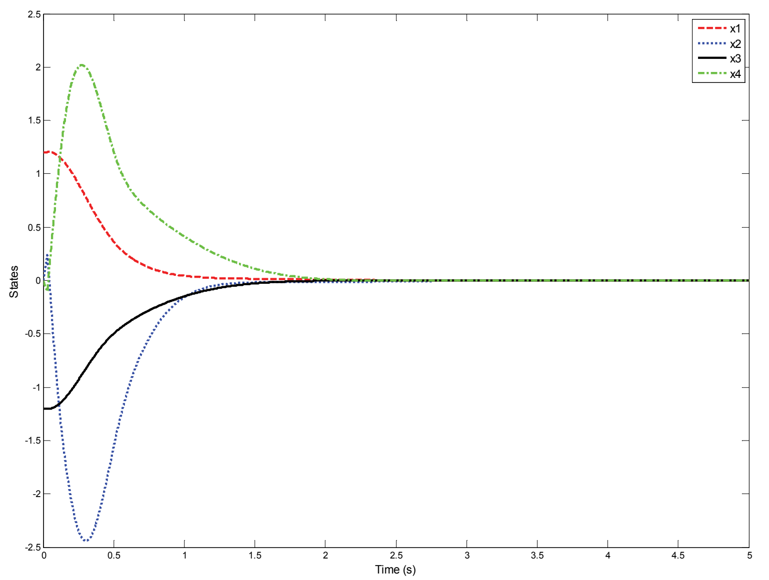

Fig. 4. State responses for controller (24) with input delays as 30 ms.

It is seen from Fig. 4 that all the state variables converge to equilibrium states no matter the existence of the input time delays, which shows the effectiveness of the designed controller (24) when the input time delays are considered in the controller design procedure.

As controller (24) is designed using the tracking controller design conditions (19-21), its tracking control performance can be checked as well when the reference inputs are provided. As those done in (Tseng, Chen and Uang, 2001), we define reference input as $r(t)=\left[0, 8\sin(t), 0, 8\cos(t)\right]^T$ and to validate its robustness, the external disturbances are given as w_1=0.1sin(2t), w_2=0.1cos(2t), w_3=0.1cos(2t), and w_4=0.1sin(2t). The initial condition is assumed to be $[x_1(0),x_2(0),x_3(0),x_4(0)]^T=[0.5,\ 0,\ -0.5,\ 0]^T$, and the input time delays are assumed to be 30 ms. Under these conditions, the simulation responses for both the reference state variables and actual state variables are shown in Fig. 5 for x_1 (left) and x_3 (right), respectively. From Fig. 5, it is observed that the actual state variables are able to track the reference state variables although there is a big difference at the beginning due to different initial values. It proves that the designed controller (24), in spite of its simplicity in structure, can stabilise the nonlinear manipulator system and can basically track the reference state variables.

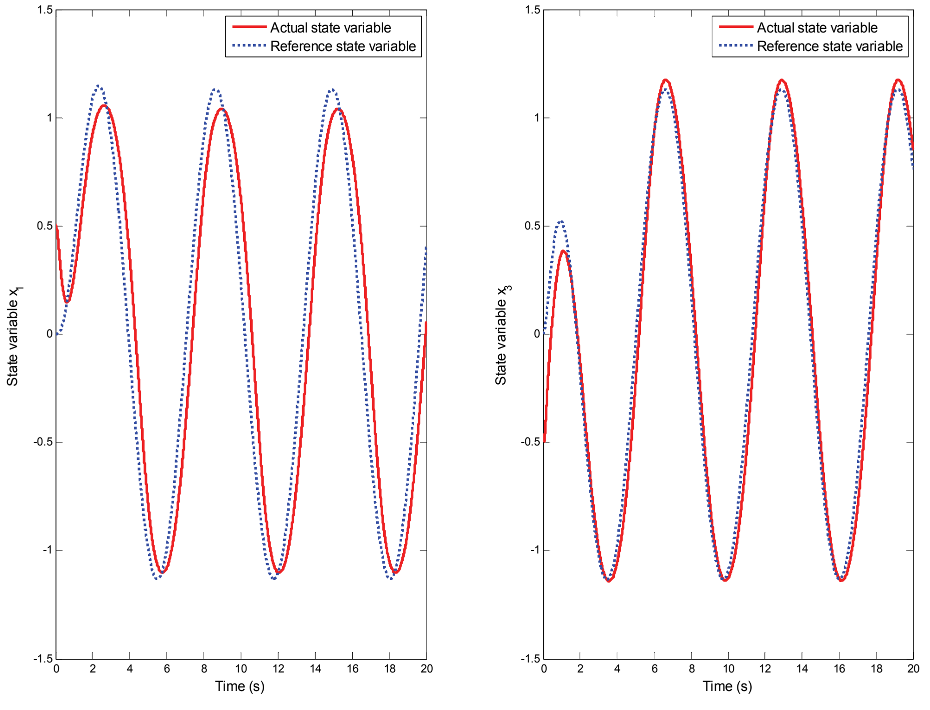

Fig. 5. State responses for the designed controller (24) with input delays as 30 ms.

Nevertheless, from Fig. 5, it is also seen that the tracking performance is not really desirable as the differences between the reference state variables and the actual state variables can be easily identified, in particular, for state variable x_1 (left). The poor tracking performance realised by controller (24) comes from the reasons that it is one rule based controller and therefore it is weak in achieving good performance for the original model which is approximated with nine rules.

We now design a fuzzy tracking controller through PDC strategy by using the proposed approach. Using the same parameter values for $\bar{\tau}$, d_2, d_3, d_4, and $\bar{C}$, the LMIs (19-21) are feasible to find a solution, and the controller gain matrices for nine rules are given as

$$K_1=\begin{bmatrix} -115.9265 & -19.4020 & -51.6975 & -9.0525 & 101.1323 & 12.6747 & 45.3281 & 5.8894 \\ -53.0984 & -9.4817 & -58.7058 & -9.9765 & 48.3992 & 6.1958 & 51.9449 & 6.5429 \end{bmatrix}$$

$$K_2=\begin{bmatrix} -141.9683 & -23.5791 & 0.2777 & -0.3129 & 124.8768 & 15.4512 & -2.2731 & 0.0976 \\ -3.4846 & -0.5815 & -88.7399 & -14.9675 & 0.2146 & 0.2869 & 80.2204 & 9.8727 \end{bmatrix}$$

$$K_3=\begin{bmatrix} -115.5704 & -19.0475 & 55.4697 & 9.1381 & 102.6000 & 12.5192 & -52.2332 & -6.1268 \\ 54.2377 & 9.4285 & -55.4358 & -9.5060 & -51.7672 & -6.2518 & 52.6118 & 6.3387 \end{bmatrix}$$

$$K_4=\begin{bmatrix} -146.4229 & -23.9600 & 0.6201 & -0.1205 & 126.9068 & 15.6729 & -3.3843 & -0.0513 \\ 1.2587 & -0.3380 & -90.8831 & -15.0851 & -0.9041 & 0.1750 & 80.8721 & 9.9366 \end{bmatrix}$$

$$K_5=\begin{bmatrix} -121.6095 & -19.5529 & -50.2643 & -8.9250 & 101.9231 & 12.7272 & 44.6201 & 5.8019 \\ -51.8299 & -9.3220 & -62.0814 & -10.0800 & 47.5336 & 6.0782 & 52.4858 & 6.5645 \end{bmatrix}$$

$$K_6=\begin{bmatrix} -145.5178 & -23.8434 & -0.1544 & -0.2041 & 126.2843 & 15.5980 & -2.9193 & 0.0002 \\ 0.3571 & -0.4417 & -90.2942 & -15.0410 & -0.5286 & 0.2197 & 80.6263 & 9.9110 \end{bmatrix}$$

$$K_7=\begin{bmatrix} -115.6616 & -19.0441 & 55.1637 & 9.1441 & 102.6047 & 12.5166 & -52.2722 & -6.1372 \\ 54.5435 & 9.4487 & -55.3558 & -9.5095 & -51.7828 & -6.2513 & 52.6340 & 6.3439 \end{bmatrix}$$

$$K_8=\begin{bmatrix} -142.5607 & -23.6137 & 0.4674 & -0.2926 & 125.0556 & 15.4695 & -2.3887 & 0.0840 \\ -2.8051 & -0.5295 & -89.1068 & -14.9940 & 0.1127 & 0.2772 & 80.3652 & 9.8887 \end{bmatrix}$$

$$K_9=\begin{bmatrix} -116.9415 & -19.5309 & -50.5115 & -8.8252 & 101.8508 & 12.7564 & 44.0836 & 5.7350 \\ -52.3753 & -9.3980 & -59.5804 & -10.1199 & 47.9301 & 6.1423 & 52.7279 & 6.6383 \end{bmatrix}$$

The tracking performance implemented by this fuzzy controller is shown in Fig. 6. It can be seen that the differences between the reference state variables and the actual state variables are largely reduced for both state variables. The tracking performance is therefore improved even with the existence of input time delays.

It is noted that in the proposed controller design approach, several parameters like d_2, d_3, and d_4, need to be defined before starting to solve the LMIs. These parameters could be optimised in terms of the tolerable maximum input delays $\bar{\tau}$, tracking performance γ, and feasible solutions to LMIs (19-21), etc. The weights on matrix $\bar{C}$ will also play an important role in obtaining a good tracking performance. Higher weight value on one state variable will generally result in a controller which can reduce the tracking error on this state variable in comparison to other variables. However, these parameters need to be considered altogether and some possible optimisation algorithms, such as genetic algorithms (GAs), could be used to find the sub-optimal parameters, which, however, is beyond the scope of this chapter, and will not be further discussed.

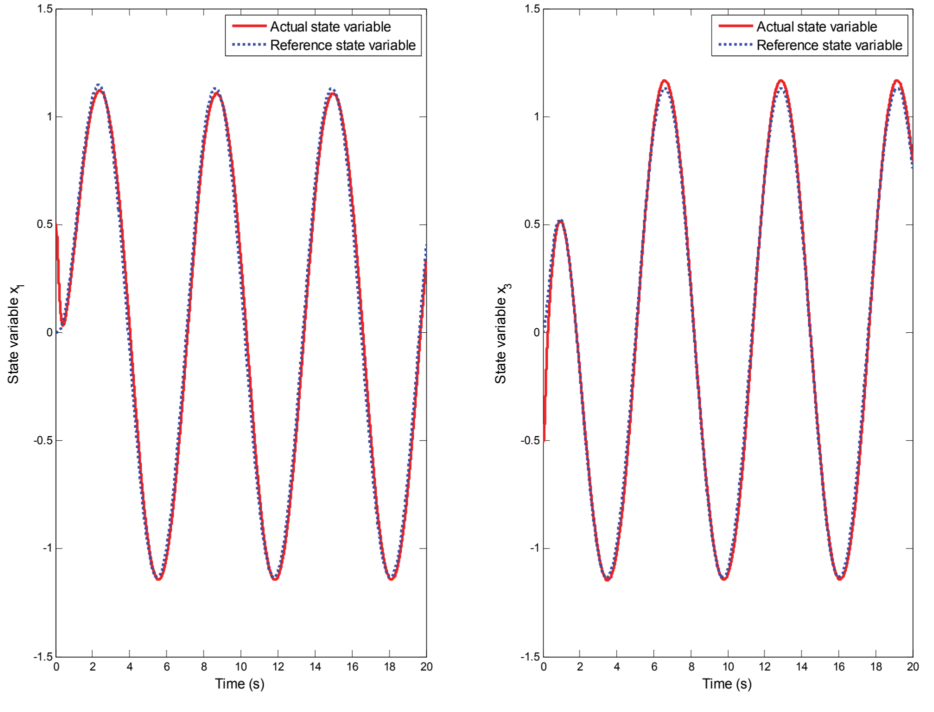

Fig. 6. State responses for the proposed fuzzy tracking controller with input delays as 30 ms.

5. Conclusions

In this chapter, the tracking control problem for a robotic manipulator with input time delays is studied. To deal with the nonlinear dynamics of robotic manipulator, the T-S fuzzy control strategy is applied. To reduce the conservativeness in deriving conditions for designing such a tracking controller, the most advanced techniques in defining Lyapunov-Krasovskii functional and in solving cross terms are used. To achieve good tracking performance, the tracking error in the sense of H_∞ norm is minimised. The sufficient conditions are derived as delay-dependent LMIs, which can be solved efficiently using currently available software like Matlab LMI Toolbox. The solution is also dependent to the values of d2, d3, d4, and the weights on matrix $\bar{C}$, which may further provide the space to improve the performance of the designed controller. Numerical simulations are applied to validate the performance of the proposed approach. The results show that the designed controller can achieve good tracking performance regardless of the existence of input time delays. This topic is going to be further studied with considering modelling errors, parameter uncertainties, and actuator saturations.

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Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, robotics, etc.

The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability.

Consequently, the problem of controllability, observability, robustness, optimization, adaptive control, pole placement and particularly stability and robustness stabilization for this class of systems, has been one of the main interests for many scientists and researchers during the last five decades.

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