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Recent Developments in the Solution of Nonlinear Differential Equations

Edited by Bruno Carpentieri

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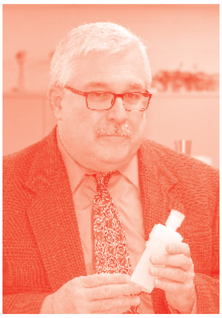



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Meet the editor

Bruno Carpentieri obtained a laurea degree in Applied Mathematics in 1997 from Bari University, Italy. He obtained a Ph.D. in Computer Science from the Institut National Polytechnique de Toulouse (INPT), France. After some post-doctoral experiences, Dr. Carpentieri served as an assistant professor at the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, the Netherlands, and as a reader in Applied Mathematics at Nottingham Trent University, United Kingdom. Since May 2017, he has been an associate professor of Applied Mathematics at the Faculty of Computer Science, Free University of Bozen-Bolzano, Italy. His research interests include applied mathematics, numerical linear algebra, and high-performance computing. Dr. Carpentieri has served on several scientific advisory boards in computational mathematics. He is an editorial board member of the *Journal of Applied Mathematics*, an editorial committee member of *Mathematical Reviews* (American Mathematical Society) and a reviewer for about thirty scientific journals in numerical analysis. He has co-authored fifty publications in peer-reviewed scientific journals.

Contents

Preface

Nonlinear differential equations are ubiquitous in computational science and in engineering modeling, for example, in fluid dynamics, finance, quantum mechanics, material science, medical applications, and biology, among other areas. Nowadays, solving challenging problems in an industrial setting requires a continuous interplay between the theoretical analysis of such systems (investigation of the existence and stability of analytical solutions, study of bifurcation and of chaotic dynamics, etc.) and the development and use of sophisticated computational methods that can guide and support the theoretical findings by practical computer simulations.

This book is not a standard textbook on the solution of nonlinear differential equations. There is already an extensive treatment on the subject on the market. The purpose of this volume is to discuss some significant developments of the last years on the definition of new theories, models, computer algorithms, and applications relating to the solution of nonlinear differential equations in various scientific areas. It collects research papers written by leading world experts in the field, highlighting ongoing trends, progress, and open problems in this critically important area of mathematics and modern science.

The book includes contributions that contain both theory-oriented chapters and more applied ones. As such, it will lead to a deeper understanding and appreciation of the research produced in this fascinating field. Researchers, engineers, and graduate students in both pure and applied mathematics will benefit from reading the papers collected in this volume.

We express appreciation to IntechOpen for professional support and Author Service Manager Dr. Kristina Kardum Cvitan for her tireless help in the preparation of this book.

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Chapter 1

Using the Boundary Element Method to Simulate Visco-Elastic Deformations of Rough Fractures

Hao Kang

Abstract

In many engineering applications, such as tribology and rock mechanics, it is very important to understand the deformation of rough fractures to evaluate the safety and profitability of the project. Since a lot of materials can be characterized as visco-elastic materials, it is very significant to simulate the visco-elastic deformation of rough fractures. This chapter focuses on using the boundary element method to simulate visco-elastic deformations of rough fractures. First, the principles and procedures of the above-mentioned method will be introduced. Then, one example will be given in detail. This example investigates the effect of surface geometry on visco-elastic deformations of rough rock fractures under normal compressive stresses. The rock fracture surfaces are assumed to be self-affine, and synthetic rough surfaces are generated by systematically changing three surface roughness parameters: the Hurst exponent, root mean square roughness, and mismatch length. The results indicate that by decreasing the Hurst exponent or increasing the root mean square roughness or increasing the mismatch length, the fracture mean aperture increases, and the contact ratio (the number of contacting cells/total number of cells) increases slower with time. Finally, the limitations and possible future research directions will be briefly discussed.

Keywords: visco-elastic deformation, fast Fourier transform, Boussinesq's solution, linear viscoelasticity, rough fracture, self-affine

1. Introduction

A lot of natural and engineering materials can be categorized as visco-elastic materials, such as rock, elastomers, and rubbers. In engineering applications, it is very important to understand and simulate the visco-elastic deformation of rough fractures. For example, in hydrocarbon extraction, we need to accurately simulate the visco-elastic deformation of rock fractures to predict production rates. In biomedical devices, we need to simulate the visco-elastic deformation of artificial joints to evaluate safety and effectiveness. Due to the geometrical complexity of rough fractures and the time-dependent properties of engineering materials, it is extremely difficult to obtain closed-form mathematical solutions. Thus, numerical models are required to simulate the time-dependent behavior of rough fractures.

The boundary element method (BEM) has been extensively used in solving rough surface contacting problems for distinct advantages compared with the traditional finite element method (FEM). First, it only requires discretization and calculation on the boundaries of the calculation domain, which is two-dimensional. On the contrary, FEM requires discretization and calculation for the whole calculation domain. As a result, to achieve the same stress calculation resolution, BEM requires much fewer numbers of elements and therefore, much less calculation time. In addition, since all the approximations are limited to the boundary, BEM has better stress calculation accuracy compared with FEM.

In recent years, researchers have been combining the BEM and fast numerical algorithms to achieve more efficient numerical simulations for contact problems. Stanley and Kato [1] published the first paper using the fast Fourier Transform (FFT) method to calculate the elastic deformation of rough surfaces under normal stresses. The FFT method makes the BEM simulation more efficient because FFT turns complicated convolution into simple matrix multiplication. Later, Polonsky and Keer [2] proposed the conjugate gradient (CG) method and combined it with the FFT method to further improve the efficiency. Liu et al. [3] improved the drawbacks of the FFT method proposed by Stanley and Kato [1]. Then, the CG and FFT methods have been applied to simulate plastic and visco-elastic deformations of rough fractures. Jacq et al. [4] and Sahlin et al. [5] considered perfect plasticity to simulate deformations of rough metal surfaces; and Wang et al. [6] considered strain-hardening plasticity.

For visco-elasticity, Chen et al. [7] first used the CG and FFT method to simulate visco-elastic deformations of rough fracture surfaces. They simulated three load-driven scenarios: rigid sphere indenting into PMMA surface, contact area evolution under constant load, and contact area evolution under harmonic cyclic load. Spinu and Cerlinca [8] applied different cut-off values for contact pressure to account for the plastic deformation of contacting asperities.

However, it appears that there is not much work that systematically simulates the visco-elastic deformation of rock fracture surfaces. Kang et al. [9] reported that for Musandam limestone fractures, the effect of mechanical compression on rock fracture time-dependent deformation is non-negligible, and should be systematically investigated. In addition, previous articles suggest that the fracture surface geometry has a significant effect on fracture time-dependent deformation. Therefore, we should systematically study the effect of surface geometry on rock fracture visco-elastic deformations.

Brown [10] proposed a simple probabilistic model to describe rock fracture surface geometry. In his model, the rock fracture surface geometry can be completely described by three key parameters: the Hurst exponent, the root mean square (RMS) roughness, and the mismatch length scale. In this research, his model will be used to generate synthetic fracture surface pairs, and the three key parameters will be changed systematically. The numerical method proposed by Chen et al. [7] will be used to simulate the visco-elastic deformation of synthetic fracture surfaces.

This chapter is organized as follows. Section 2 introduces and explains the principles and procedures of the numerical method. Section 3 provides a detailed example. The method for generating synthetic rough surfaces is introduced, and the effect of surface geometry parameters on the creep deformation is shown and discussed. Section 4 mentions the limitations of this method. Section 5 summarizes the findings.

2. BEM solution for visco-elastic deformations of rough fractures

2.1 Method for calculating fracture elastic deformation

Before explaining the method for visco-elastic deformation calculation, it is essential to introduce the method for elastic deformation calculation. The author has developed an in-house numerical code, which is similar to the algorithm proposed by Polonsky and Keer [2]. In this section, only the key mathematical concepts will be shown; the details can be found in their work [2]. It is worth noting that only the compressive stress (stress normal to the fracture surface) is considered; the shear stress (stress parallel to the fracture surface) is not considered.

First, the aperture (surface gap between two rough surfaces) distribution h (x,y) needs to be defined:

$$\mathrm{h}(\mathrm{x},\mathrm{y}) = h_0(x,y) - u_e(x,y) - \delta \tag{1}$$

where $h_0(x,y)$ is the initial aperture distribution, $u_e(x,y)$ is the elastic deformation of fracture surfaces, and δ is the rigid body displacement between two surfaces under applied normal stress. Here, compressive stress and fracture closure are defined as positive.

The boundary conditions are expressed as:

$$\mathrm{p}(\mathrm{x},\mathrm{y}) = 0 \textit{ and } h(x,y) > 0 \tag{2}$$

$$\mathrm{p}(\mathrm{x},\mathrm{y}) > 0 \textit{ and } h(x,y) = 0 \tag{3}$$

where p(x,y) is the contacting stress (normal to the surface) acting on location (x,y). Eqs. (2) and (3) indicate that the contacting stress is larger than zero at contacting regions, and is zero at non-contacting regions.

Boussinesq and Cerrutti [11] stated that the vertical displacement u_e (x,y) can be calculated as:

$$u_e(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(x,y,x',y')p(\mathrm{x}',y')dx'dy' \tag{4}$$

where p(x′, y′) is the normal pressure acting on location (x′, y′), K is the influence matrix, which represents the normal displacement at location (x, y) caused by unit normal pressure acting on location (x′, y′), and u_e (x,y) is the elastic displacement at location (x, y). The influence matrix K can be expressed as:

$$K(x,y,x',y') = \frac{1-\nu}{2\pi G}\frac{1}{\sqrt{(x-x')^2+(y-y')^2}} \tag{5}$$

where G is the shear modulus, and v is the Poisson's ratio.

As mentioned in the introduction section, it is difficult to obtain the analytical solution for rough surface deformation under normal stress. However, the numerical solution can be obtained. To obtain the numerical solution, the fracture surface area needs to be discretized into rectangular grids:

$$x_i = i\Delta x, i = 1, 2, \ldots, N \tag{6}$$

$$y_j = j\Delta y, j = 1, 2, \ldots, M \tag{7}$$

where x_i, y_j are x and y coordinates, respectively; N and M are total number of grids in x- and y-direction, respectively; and Δx and Δy are the grid dimensions in x- and y-direction, respectively. After discretization, the aperture distribution function and boundary conditions can be expressed as:

$$h_{i,j} = (h_0)_{i,j} + (u_e)_{i,j} - \delta \tag{8}$$

$$p_{i,j} = 0 \; if \; h_{i,j} > 0 \tag{9}$$

$$p_{i,j} > 0 \; if \; h_{i,j} = 0 \tag{10}$$

Love [12] first discretized Eqs. (4) and (5) as:

$$(u_e)_{i,j} = \sum_{l=1}^{M} \sum_{k=1}^{N} K_{i,k,j,l} \times p_{k,l} \tag{11}$$

$$K_{i,k,j,l} = \frac{1-\nu}{2\pi G} \left(a \ln \frac{c + \sqrt{a^2 + c^2}}{d + \sqrt{a^2 + d^2}} + b \ln \frac{d + \sqrt{b^2 + d^2}}{c + \sqrt{b^2 + c^2}} + c \ln \frac{a + \sqrt{a^2 + c^2}}{b + \sqrt{b^2 + c^2}} + d \ln \frac{b + \sqrt{b^2 + d^2}}{a + \sqrt{a^2 + d^2}} \right) \tag{12}$$

where

$$a = x_i - x_k + {}^{\Delta x}\!/_2, b = x_i - x_k - {}^{\Delta x}\!/_2, c = y_j - y_l + {}^{\Delta y}\!/_2, d = y_j - y_l - {}^{\Delta y}\!/_2 \tag{13}$$

As mentioned before, Stanley and Kato [1] first the FFT method to solve Eq. (11) to make the calculation more efficient. The FFT method turns complicated convolution into simple matrix multiplication. By using the FFT method, Eq. (11) becomes:

$$(u_e)_{i,j} = IFFT\left[FFT\left(K_{i,k,j,l}\right) \times FFT\left(p_{k,l}\right)\right] \tag{14}$$

where IFFT represents the inverse of Fourier transform. The FFT method reduces the number of operations from $N^2 * M^2$ to N*M*log(N*M) [1]. Therefore, when N and M are large, the FFT method can greatly reduce the calculation time.

The force balance over the entire fracture surface needs to be satisfied:

$$F_{total} = \sum_{k=1}^{N} \sum_{l=1}^{M} p_{k,l} \tag{15}$$

Eqs. (8)–(10), (14), and (15) are solved iteratively using the CG method proposed by Polonsky and Keer [2].

2.2 Method for calculating fracture visco-elastic deformation

As described before, Chen et al. [7] first combined the FFT and CG method to simulate visco-elastic deformations of rough fractures. The author has developed an in-house numerical code, which is similar to the algorithm described by Chen et al. [7]. In this section, only the key mathematical aspects will be introduced; the rest can be found in their work [7].

In this simulation, the rock materials are assumed to be linear viscoelastic. Therefore, is it essential to introduce the concept of linear viscoelasticity first. For linear viscoelastic materials, the stress/strain response scales linearly with the strain/stress input, and the behavior follows the rule of linear superposition. Mathematically, the stress/strain at time t can be expressed as:

$$\sigma(t) = \int_0^t E(t-\tau) \frac{d\varepsilon(\tau)}{dt} d\tau \tag{16}$$

$$\varepsilon(t) = \int_0^t J(t-\tau) \frac{d\sigma(\tau)}{dt} d\tau \tag{17}$$

where J(t) and E(t) are the creep compliance function and the relaxation modulus function, respectively. J(t) represents the time-dependent strain change with a step change in stress, and E(t) represents the time-dependent stress change with a step change in strain. Based on Eq. (17), the Boussinesq and Cerrutti equation can be modified to represent linear viscoelasticity by adding the creep compliance function:

$$u_e(x,y,t) = \int_0^t J(t-\tau)\frac{\partial}{\partial \tau}\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{p(x',y',\tau)(1-\nu)}{\pi\sqrt{(x-x')^2+(y-y')^2}}dx'dy'\right]d\tau \tag{18}$$

In Eq. (18), the creep compliance function J(t-τ) replaces the term 1/2G. Rearranging Eq. (18) gives:

$$u_e(x,y,t) = \int_0^t\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{J(t-\tau)(1-\nu)}{\pi\sqrt{(x-x')^2+(y-y')^2}}\frac{\partial p(x',y',\tau)}{\partial \tau}dx'dy'd\tau \tag{19}$$

Eq. (19) cannot be solved analytically for rough fracture surfaces. However, if the time integration term can be de-coupled with the pressure integration term, Eq. (19) will become a linear equation system, and can therefore be solved numerically. To de-couple the time integration term, the time duration t is discretized into N_t time steps. The time interval is uniform, and is termed as Δt. The time interval is assumed to be sufficiently small that the pressure distribution field within each time interval does not change. In addition, based on the fundamental theorem of calculus, the term ∂p(x′, y′, τ) dτ/ ∂τ can be substituted by a finite difference p(x′, y′, τ + dτ) – p(x′, y′, τ). Therefore, Eq. (19) becomes:

$$u_e(x,y,\alpha\Delta t) = \sum_{\alpha'=1}^{\alpha}\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{J(\alpha\Delta t-\alpha'\Delta t)(1-\nu)}{\pi\sqrt{(x-x')^2+(y-y')^2}}[p(x',y',\alpha')-p(x',y',\alpha'-1)]dx'dy'\right\} \tag{20}$$

where α = 1, 2, … , N_t.

In addition, within each time interval, the pressure distribution field does not change. Therefore, the pressure distribution field can be removed from the integration term:

$$u_e(x,y,\alpha\Delta t) = \sum_{\alpha'=1}^{\alpha}[p(x',y',\alpha')-p(x',y',\alpha'-1)]\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{J(\alpha\Delta t-\alpha'\Delta \mathrm{t})(1-\nu)}{\pi\sqrt{(x-x')^2+(y-y')^2}}dx'dy'\right\} \tag{21}$$

Eq. (21) indicates that the time integration term is de-coupled with the pressure integration term. The pressure integration term can then be discretized, similar to Eq. (11). From Eqs. (4), (5), and (11), the Boussinesq equation can be discretized as:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{(1-\nu)}{2\pi G\sqrt{(x-x')^2+(y-y')^2}}dx'dy' \xrightarrow{Discretize} \sum_{l=1}^{M}\sum_{k=1}^{N}K_{i,k,j,l} \tag{22}$$

Based on Eq. (22), the integration part of Eq. (21) can then be discretized:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{J(\alpha\Delta t-\alpha'\Delta t)(1-\nu)}{\pi\sqrt{(x-x')^2+(y-y')^2}}dx'dy' \overset{Discretize}{\rightarrow} \sum_{l=1}^{M}\sum_{k=1}^{N}2GJ((\alpha-\alpha')\Delta t)K_{i,k,j,l} \quad (23)$$

Therefore, Eq. (21) can be discretized as:

$$u_e(i,j,\alpha\Delta t)=\sum_{\alpha'=1}^{\alpha}\sum_{l=1}^{M}\sum_{k=1}^{N}2GJ((\alpha-\alpha')\Delta t)K_{i,k,j,l}\left(p_{k,l,\alpha'}-p_{k,l,\alpha'-1}\right) \quad (24)$$

To implement FFT, Eq. (24) can be decoupled into two equations:

$$u_e(i,j,\alpha\Delta t)=\sum_{\alpha'=1}^{\alpha}(u_e)_{\alpha'} \quad (25)$$

and

$$(u_e)_{\alpha'}=\sum_{l=1}^{M}\sum_{k=1}^{N}2GJ((\alpha-\alpha')\Delta t)K_{i,k,j,l}\left(p_{k,l,\alpha'}-p_{k,l,\alpha'-1}\right) \quad (26)$$

Eq. (26) can be solved by the FFT method, similar to Eqs. (13) and (14):

$$(u_e)_{\alpha'}=IFFT\left[FFT\left(2GJ((\alpha-\alpha')\Delta t)K_{i,k,j,l}\right)\times FFT\left(p_{k,l,\alpha'}-p_{k,l,\alpha'-1}\right)\right] \quad (27)$$

Within each time step, Eqs. (8)–(10), (15), (25), and (27) are solved using the CG method. The pressure distribution field is obtained and stored. Then, a new time step will be added (α will be increased by one), and the new deformation and pressure fields will be solved based on the historical pressure fields. **Figure 1** summarizes the main calculation algorithm based on the above mathematical concepts.

2.3 Model validation

Before simulating visco-elastic deformations of rough rock fractures, it is essential to validate the numerical code against analytical solutions. In this research, the analytical solutions provided by Radok and Lee [14] will be used for validation. In their solutions, a rigid spherical indenter is indented into a flat visco-elastic surface; and the visco-elastic models for the flat surface are the Maxwell and Standard Linear Solid (SLS) model. **Figure 2** illustrates the geometry setup for the analytical solution, and **Figure 3** shows the concepts of the Maxwell and SLS model.

The Maxwell model consists of a dashpot and a spring. The dashpot represents viscosity, with a viscosity of η; the spring represents elasticity, with a shear modulus of G. Under constant stress σ_0, the strain can be obtained:

$$\varepsilon(t)=\sigma_0\left(\frac{1}{G}+\frac{t}{\eta}\right) \quad (28)$$

Eq. (28) implies that under constant stress, the strain rate does not change with time. The creep compliance can be expressed as:

$$J(t)=\frac{1}{G}+\frac{t}{\eta} \quad (29)$$

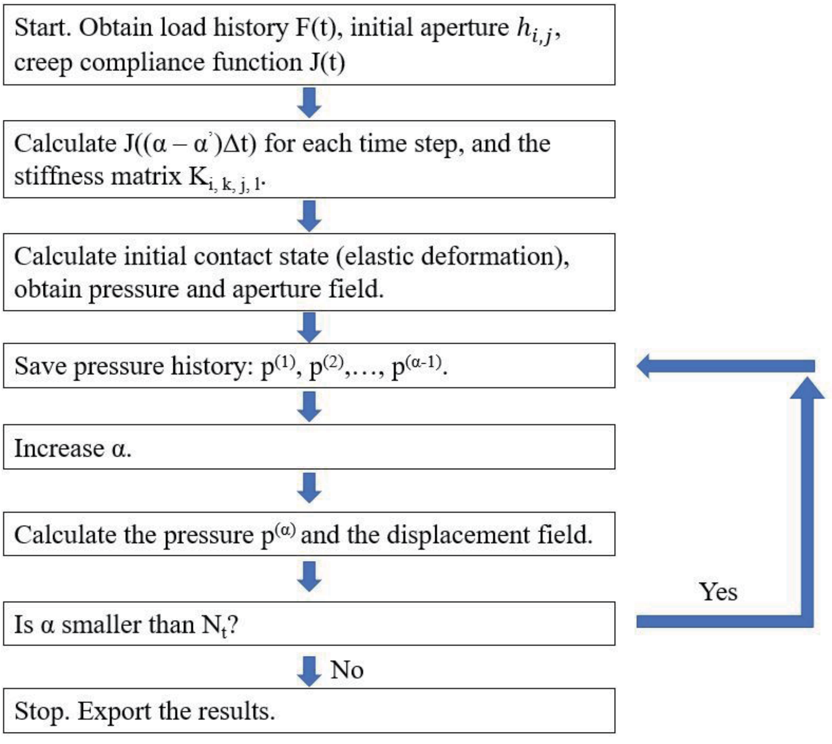

Figure 1.
Summary of the calculation algorithm (Kang et al. [13]).

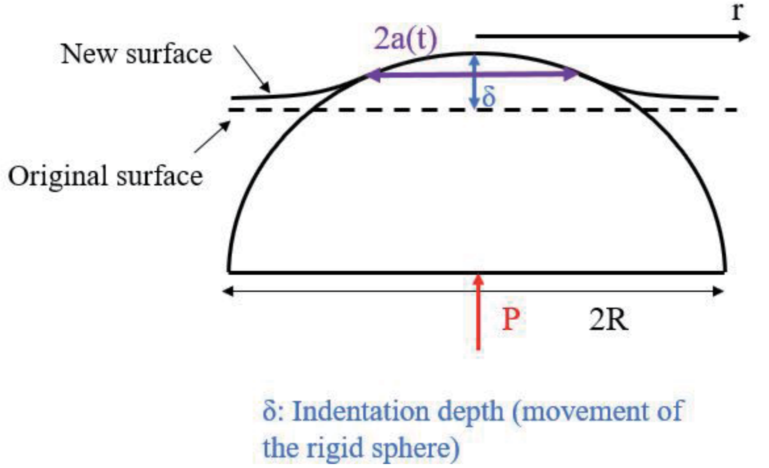

Figure 2.
Geometry setup for the analytical solution (Kang et al. [13]). R is the radius of the spherical rigid indenter, P is the total load, δ is the indentation depth, t is the time duration, and a(t) is the radius of the contacting region.

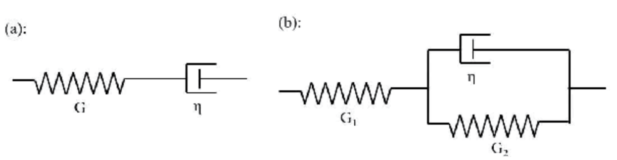

Figure 3.
Concepts of the Maxwell and SLS model (Kang et al. [13]). (a): Schematic of the Maxwell model; (b): Schematic of the SLS model.

Another parameter, the relaxation time T, is defined as:

$$T = \eta / G \tag{30}$$

In the numerical simulation, Eq. (29) will be implemented into Eq. (27), and the displacement and pressure field will be solved as described in Sections 2.1 and 2.2. For the geometry setup shown in **Figure 2**, the analytical solution for the contacting region radius and pressure field can be obtained:

$$p(t,r) = \frac{2}{\pi R(1-\upsilon)} \int_0^t G e^{-(t-t')G/\eta} \frac{d}{dt'} \left[a^2(t') - r^2\right]^{1/2} dt' \tag{31}$$

and

$$a(t) = \sqrt[3]{\frac{3(1-\nu)RP}{4}\left(\frac{1}{G} + \frac{t}{\eta}\right)} \tag{32}$$

where p is the pressure field, t is the total time duration, υ is the Poisson's ratio, and r is the distance from the center of the contacting region.

The SLS model consists of one dashpot and two springs. The dashpot represents viscosity, with a viscosity η; the two springs represent elasticity, with a shear modulus of G_1 and G_2, respectively. Under constant stress σ_0, the strain can be obtained:

$$\sigma(\mathrm{t}) = \frac{G_1 G_2}{G_1 + G_2} \varepsilon(t) + \frac{G_1 \eta}{G_1 + G_2} \frac{d\varepsilon(t)}{dt} \tag{33}$$

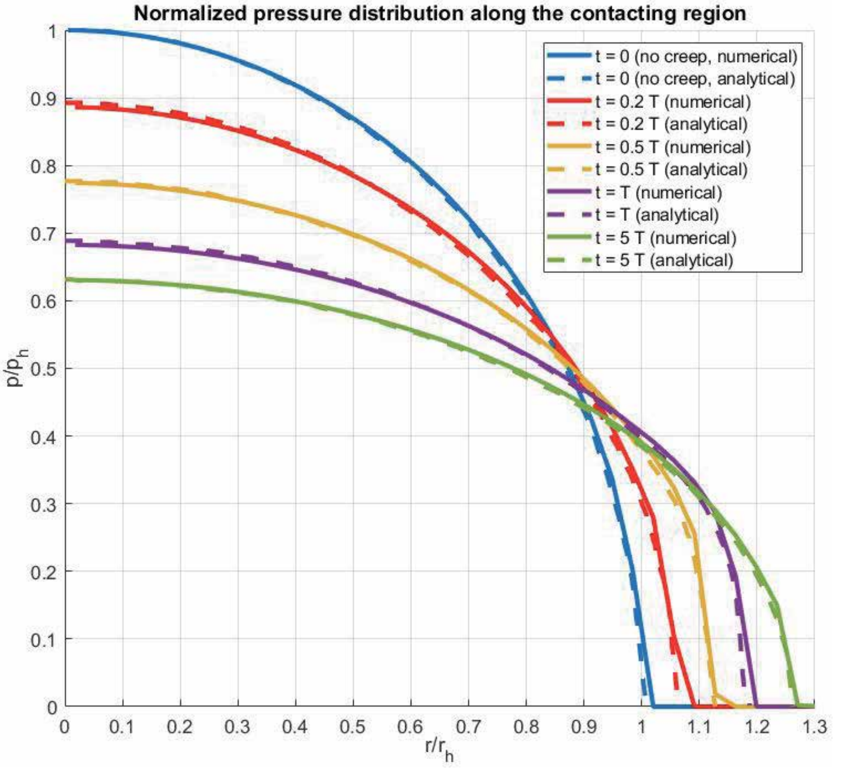

Figure 4.
Numerical and analytical solutions for the SLS model (Kang et al. [13]).

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The creep compliance J(t) is expressed as:

$$J(t) = \frac{1}{G_1} + \frac{1 - e^{-tG_2/\eta}}{G_2} \tag{34}$$

The relaxation time T is defined as:

$$T = \eta/G_2 \tag{35}$$

In the numerical simulation, Eq. (34) will be implemented into Eq. (27), and the displacement and pressure field will be solved as described in Sections 2.1 and 2.2. For the geometry setup shown in **Figure 2**, the analytical solution for the contacting region radius and pressure field can be obtained:

$$p(t,r) = \frac{2}{\pi R(1-\upsilon)} \int_0^t \frac{G_1}{G_1 + G_2} \left[G_2 + G_1 e^{-(t-t')(G_1+G_2)/\eta} \right] \frac{d}{dt'} \left[a^2(t') - r^2 \right]^{1/2} dt' \tag{36}$$

and

$$a(t) = \sqrt[3]{\frac{3(1-\nu)RP}{4} \left(\frac{1}{G_1} + \frac{1}{G_2} \left(1 - e^{-tG_2/\eta} \right) \right)} \tag{37}$$

where p is the pressure field, t is the total time duration, υ is the Poisson's ratio, and r is the distance from the center of the contacting region.

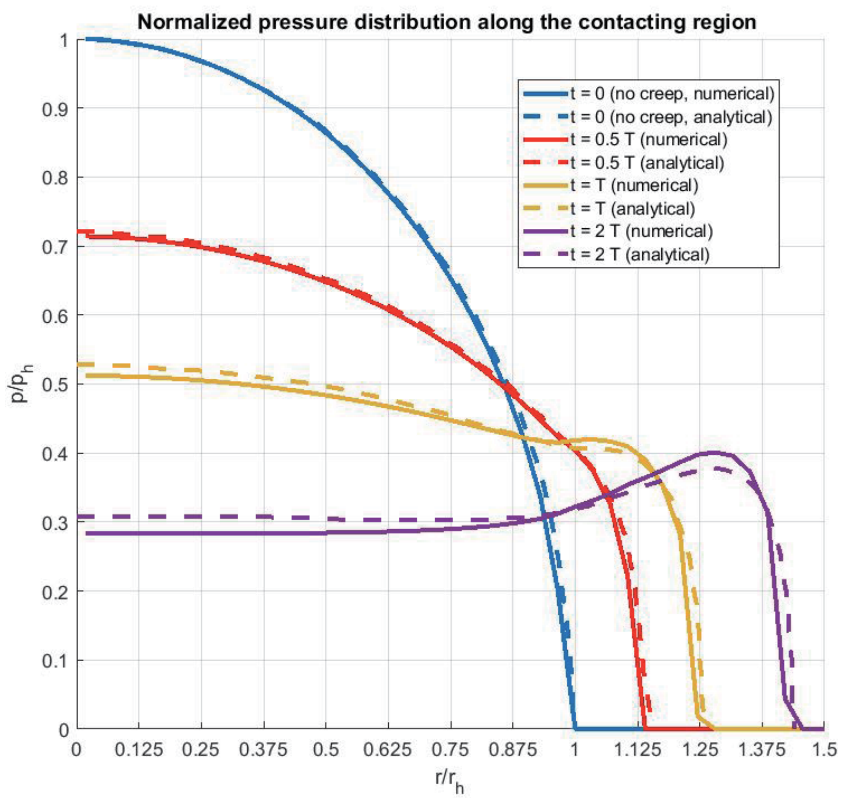

Figure 5.
Numerical and analytical solutions for the Maxwell model (Kang et al. [13]).

Johnson [15] solved Eqs. (31), (32), (36), and (37), **Figures 4** and **5** compare the numerical and analytical solutions for the SLS and Maxwell models, respectively. The solid lines are the numerical solutions obtained by the author, and the dashed lines are the analytical solutions solved by Johnson [15]. In **Figures 4** and **5**, r_h is the contacting region at time zero, p_h is the maximum contacting pressure at time zero, and T is the relaxation time.

Figures 4 and **5** indicate the deviation between the numerical and analytical results is less than 10%. Therefore, the numerical code can be used to simulate the visco-elastic deformations of rough fractures. For the two validation cases, the numerical simulation accuracy is not strongly dependent on the total number of elements, but on the time interval Δt. The deviation between numerical and analytical solutions will be smaller if the time interval Δt is reduced.

3. One example: visco-elastic deformations of rough rock fractures

3.1 Brief introduction of Brown's (1995) model

In this chapter, synthetic fracture surface pairs are generated based on Brown's model [10]. Brown's probabilistic model assumes that the surface is self-affine, and the surface height distribution follows Gaussian distribution [10]. The surface geometry can be completely described by three parameters: the Hurst exponent H, the mismatch length λ_c, and the root mean square roughness RMS.

Mathematically, a self-affine surface is defined as:

$$z(x) \sim \varepsilon^{-H} z(\varepsilon x) \tag{38}$$

where H is the Hurst exponent, z is the surface height, and ε is a constant for scaling at the x-direction. The H value is between 0 and 1, and it describes local roughness. A smaller H value corresponds to a rougher local surface profile.

The H value can be obtained from the power spectrum of surface height. The power spectrum of a surface can be obtained by decomposing the surface profile into a series of sinusoidal waves via Fourier transform, and each sinusoidal wave has its own amplitude A, wavelength λ, and phase. **Figure 6** shows the schematic of the decomposition process. The power (A^2) is defined as the square of the amplitude A; and the plot of power versus the wavelength number (the inverse of wavelength, which is 2π/λ) is defined as the power spectrum. **Figure 7** shows the schematic of power spectrum.

For a self-affine fracture surface, the power C (=A^2) can be related to the wavelength number q (=2π/λ) as:

$$\mathrm{C(q)} \sim q^{-2(1+H)} \tag{39}$$

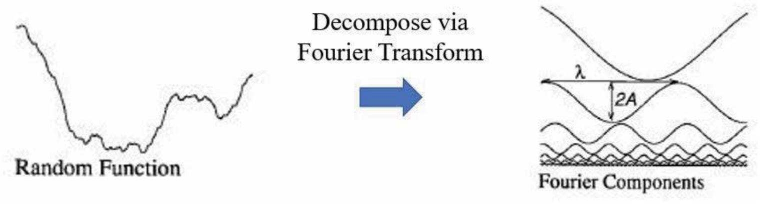

Figure 6.
Schematic of wave decomposition via Fourier transform (Kang et al. [13]).

In **Figure 7**, the q has an upper bound and a lower bound. For the lower bound, $q_{min} = 2\pi / \lambda_L$, where λ_L is the surface dimension; for the upper bound, $q_{max} = 2\pi/\lambda_1$, where λ_1 is the surface measurement resolution.

The second parameter is the mismatch length, λ_c. As illustrated in **Figure 6**, each wave component has its own wavelength λ. Glover et al. [16] and Brown [10, 17, 18] stated that for most natural rock joints, the two surfaces have relative shear displacements. At long wavelengths, the wave components match well; at short wavelengths, the wave components are not identical. Based on the above observation, Brown [10] proposed a parameter: critical wavelength λ_c, which is also called the mismatch length scale. Brown [10] assumed that above the mismatch wavelength, the decomposed wave components of two surfaces match perfectly; they have the same amplitudes, wavelengths, and phases. On the contrary, below the mismatch wavelength, the decomposed wave components of two surfaces do not match; they have the same amplitudes and wavelengths, but the phases are independent. **Figure 8** illustrates the concept of the mismatch wavelength.

The third parameter is the root mean square roughness, RMS. It represents the absolute scale of surface asperity elevation. Mathematically, the RMS is defined as:

$$\sigma^2 = \int_{q_{min}}^{q_{max}} C(q)dq \tag{40}$$

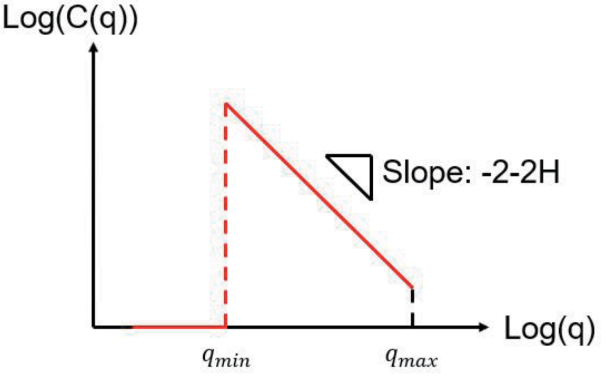

Figure 7.
Schematic of a power spectrum (Kang et al. [13]).

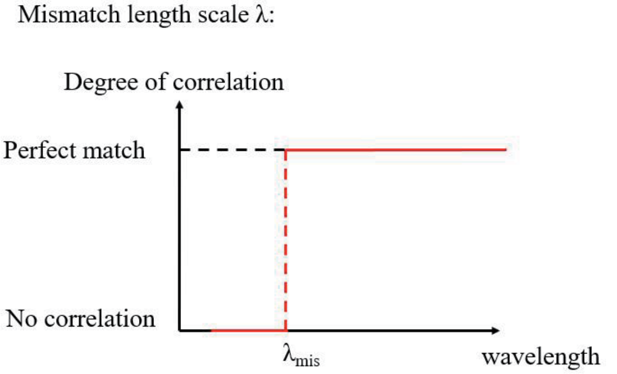

Figure 8.
Illustration of the mismatch wavelength (Kang et al. [13]).

where C is the power, q is the wavelength number, and σ is the RMS value. When generating the synthetic surface, the surface heights are normalized by its own RMS value, σ_{ini}, and then multiplied by the designated RMS value, σ_{des}:

$$z_{des} = z_{ini}\frac{\sigma_{des}}{\sigma_{ini}} \tag{41}$$

where z_{ini} is the initial surface height and z_{des} is the surface height after linear scaling. In this chapter, only the key mathematical concepts of Brown's [10] model is introduced; other details can be found in [10].

3.2 Generated synthetic surface pairs

Brown [10] measured the Hurst exponent H, mismatch length λ_c, and RMS for 23 natural rock joints. His measurement results imply that the H value is normally between 0.5 and 1.0; the normalized λ_c value (λ_c/fracture profile length) is normally between 0.02 and 0.2, and the normalized RMS value (RMS/fracture profile length) is normally between 0.005 and 0.015. Based on the above conclusion, seven synthetic fracture surface pairs are generated, with different H, λ_c, and RMS values. **Table 1** summarizes the parameters of the seven synthetic surface pairs. It is worth noting that surface pair No. 2 is the reference surface pair.

Table 1 shows that between surface pairs 1, 2, and 3, the H value is varied; between surface pairs 2, 4, and 5, the λ_c value is varied; between surface pairs 2, 6, and 7, the RMS value is varied. For each surface pair, the aperture distribution field can be plotted. **Figure 9** plots the aperture fields for surface pairs 1, 2, and 3; **Figure 10** plots the aperture fields for surface pairs 2, 4, and 5, and **Figure 11** plots the aperture fields for surface pairs 2, 6, and 7.

Based on **Figures 9–11**, we have the following observations:

1. According to **Figure 9**, when H increases, the average and standard deviation of the aperture decreases;

2. According to **Figure 10**, when λ_c deceases, the average and standard deviation of aperture decreases;

3. According to **Figure 11**, the average and standard deviation of aperture scales linearly with the RMS value.

Surface Pair No.	Profile length L (mm)	H	λ_c		RMS	
			λ_c/L	Absolute value (μm)	RMS/L	Absolute value (μm)
1	10	0.6	0.1	1000	0.005	50
2	10	0.8	0.1	1000	0.005	50
3	10	1.0	0.1	1000	0.005	50
4	10	0.8	0.2	2000	0.005	50
5	10	0.8	0.3	3000	0.005	50
6	10	0.8	0.1	1000	0.010	100
7	10	0.8	0.1	1000	0.015	150

Table 1.
The parameters of the seven synthetic surface pairs.

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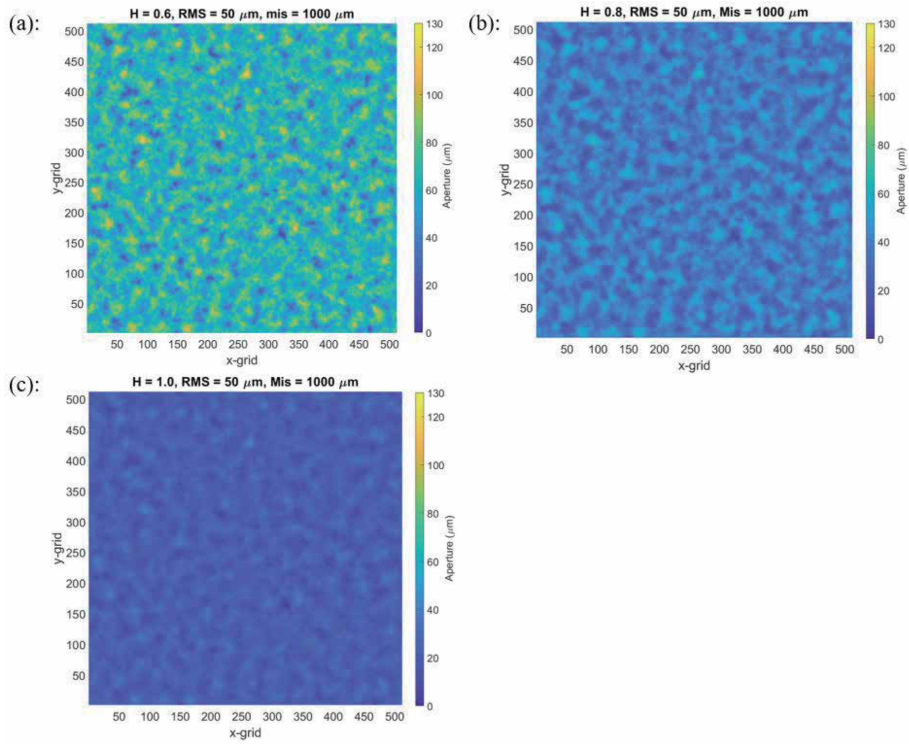

Figure 9.
Aperture fields for different H values (Kang et al. [13]). (a): aperture field for surface pair 1; (b): aperture field for surface pair 2; (c): aperture field for surface pair 3. The color bar scales are identical.

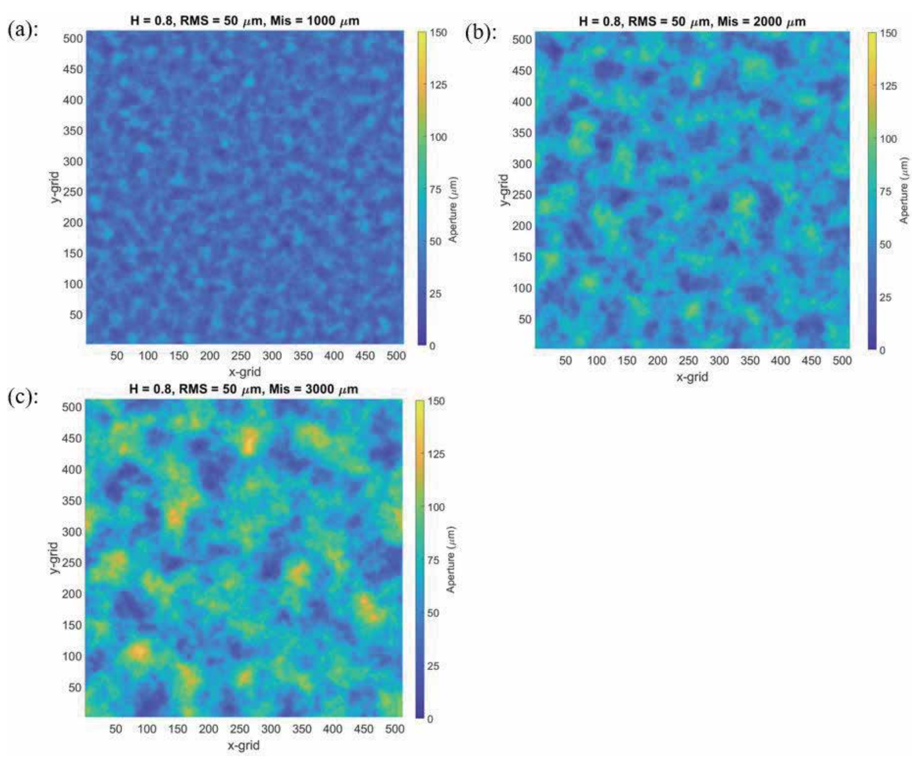

Figure 10.
Aperture fields for different λ_c values (Kang et al. [13]). (a): aperture field for surface pair 2; (b): aperture field for surface pair 4; (c): aperture field for surface pair 5. The color bar scales are identical.

Table 2 summarizes the mean and standard deviation of aperture for each surface pair. In the numerical code, each calculated aperture field (shown in **Figures 9–11**) is considered as the initial aperture field.

3.3 Creep simulation results for the Maxwell model

The author uses the Maxwell model to calculate the visco-elastic deformation of seven synthetic surface pairs. The mechanical properties of Vaca Muerta Shale measured by Mighani et al. [19] are used as the input parameters, and those properties are summarized in **Table 3**.

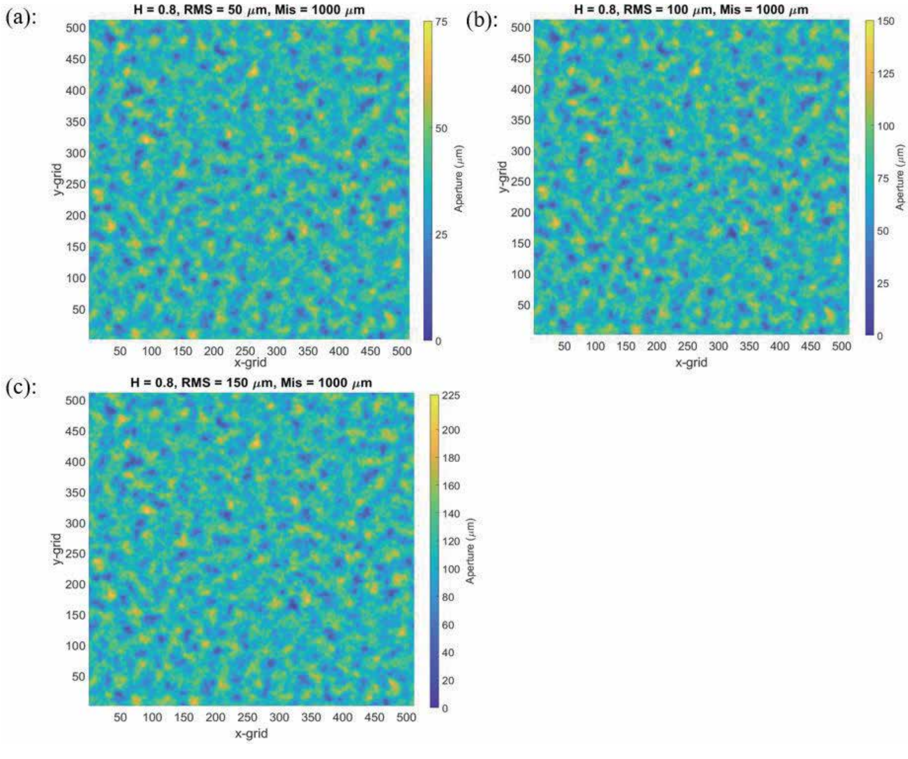

Figure 11.
Aperture fields for different RMS values (Kang et al. [13]). (a): aperture field for surface pair 2; (b): aperture field for surface pair 6; (c): aperture field for surface pair 7. The color bar scales scale linearly with the RMS value.

Surface pair No.	H	λ_c (μm)	RMS (μm)	Average aperture (μm)	Standard deviation of aperture (μm)
1	0.6	1000	50	63.41	14.29
2	0.8	1000	50	37.30	8.57
3	1.0	1000	50	21.89	5.14
4	0.8	2000	50	55.94	15.01
5	0.8	3000	50	66.10	20.12
6	0.8	1000	100	74.59	17.15
7	0.8	1000	150	111.89	25.72

Table 2.
The average and standard deviation of seven synthetic surface pairs.

Before showing the results, two parameters are introduced: macroscopic stress σ and contact ratio:

1. The macroscopic stress σ = total force applied to the fracture/fracture surface area;

2. Contact ratio = 100 * (the number of grids in contact/total number of grids).

Figures 12 and **13** show the mean aperture and contact ratio evolving with time for seven synthetic surface pairs, respectively. The total time duration is 2τ, and the macroscopic stress σ = 10 MPa. The initial changes of the mean aperture and contact ratio correspond to fracture elastic deformation.

Based on **Figures 12** and **13**, several conclusions can be drawn:

1. As H decreases, the mean aperture increases, and the contact ratio increases slower with time;

Parameters	Value
Shear modulus, G (GPa)	7.0
Poisson's ratio, υ	0.25
Viscosity, η (GPa*s)	2.0×10^7
Relaxation time, τ = η / G (s)	2.857×10^6

Table 3.
Input parameters for the Maxwell model.

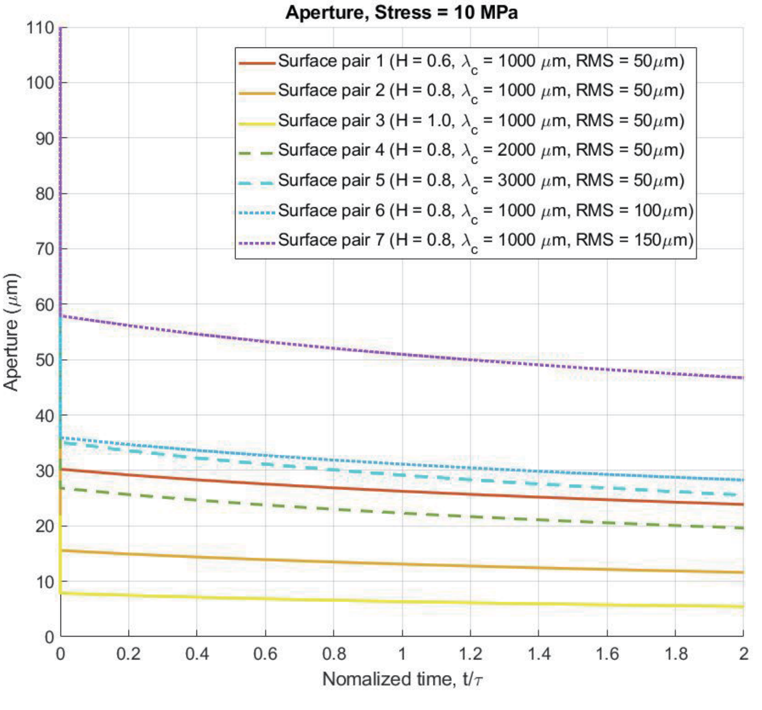

Figure 12.
Mean aperture changing with time (Kang et al. [13]). The time duration is normalized by τ.

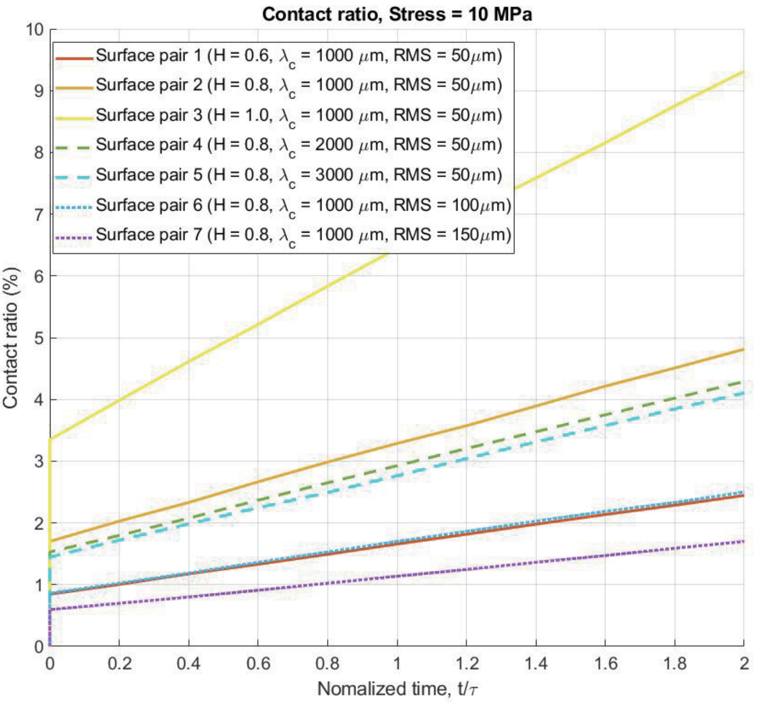

Figure 13.
Contact ratio changing with time (Kang et al. [13]). The time duration is normalized by τ.

Parameters	Average aperture		Contact ratio	
	Initial value	**Decrease rate**	**Initial value**	**Increase rate**
H↓	↑	↑	↓	↓
λ_c↑	↑	↑	↓	↓
RMS↑	↑	↑	↓	↓

Table 4.
Effect of surface parameters on the mean aperture and contact ratio.

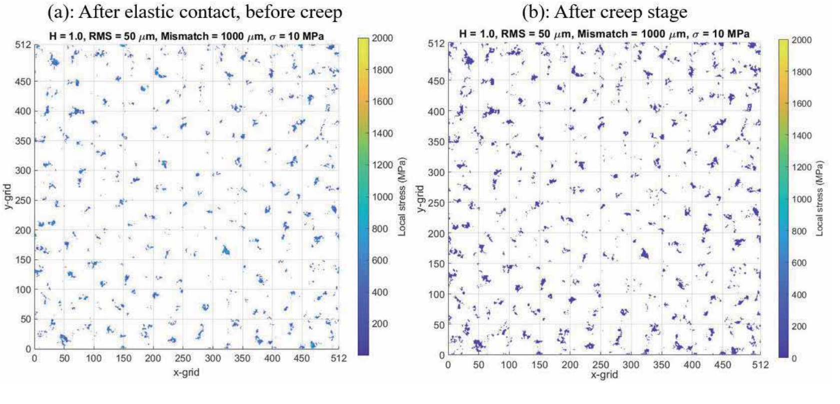

Figure 14.
Contact region and local contacting stress evolution before and after the creep stage (Kang et al. [13]). (a): before the creep stage; (b): after the creep stage. In both x- and y-directions, the number of grids is 512. The contact area increase is qualitatively shown.

2. As RMS increases, the mean aperture increases, and the contact ratio increases slower with time;

3. As λ_c increases, the mean aperture increases, and the contact ratio increases slower with time.

4. Under current macroscopic stress, time duration, and surface parameters, the contact ratio is generally less than 9.5%.

Table 4 summarizes the effect of surface parameters on the mean aperture and contact ratio.

Figure 14 shows the contact region and local contacting stress evolution of surface pair **3** before and after the creep stage. The macroscopic stress is 10 MPa and

Parameters	Value
Shear modulus, G_1 (GPa)	7.0
Shear modulus, G_2 (GPa)	7.0
Poisson's ratio, υ	0.25
Viscosity, η (GPa*s)	2.0×10^7
Relaxation time, $\tau = \eta / G_2$ (s)	2.857×10^6

Table 5.
Input parameters for the SLS model.

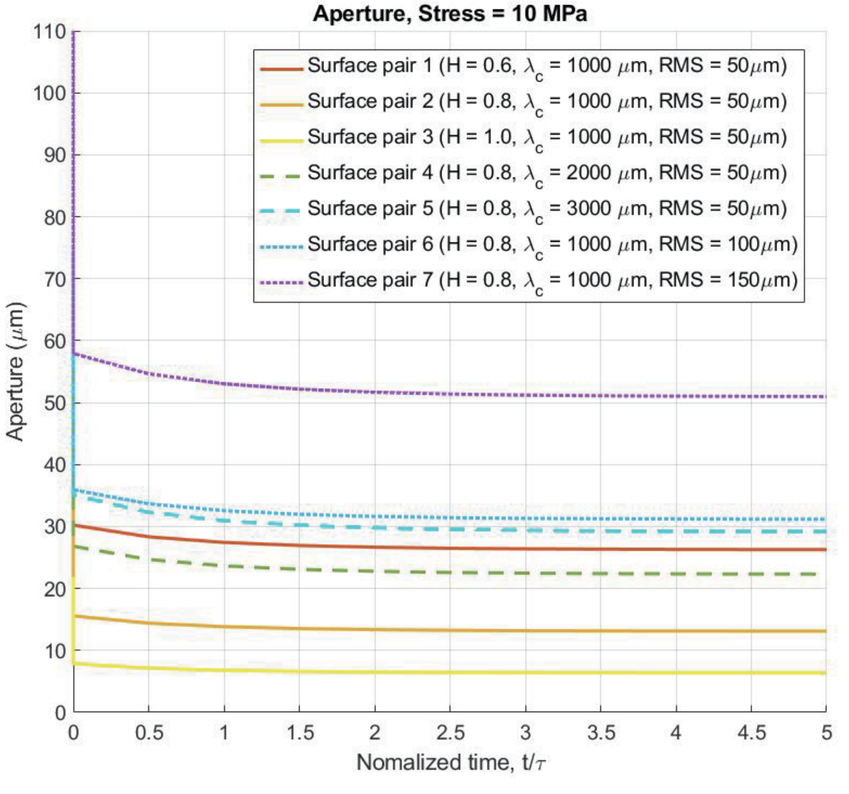

Figure 15.
Mean aperture changing with time (Kang et al. [13]). The time duration is normalized by τ.

the creep time duration is 2τ. The colored regions and white regions correspond to the contacting regions and non-contacting regions, respectively. The color bar scale is 2000 MPa. After the creep stage, the area of contacting regions becomes larger, and the local contacting stress reduces. However, even after the creep stage, the contact ratio is still less than 9.5%. Under the same time duration, if η is reduced, the contact area will increase more rapidly.

3.4 Creep simulation results for the SLS model

The author also uses the SLS model to calculate the visco-elastic deformation of seven synthetic surface pairs. The mechanical properties of Vaca Muerta Shale measured by Mighani et al. [19] are used as the input parameters, and those properties are summarized in **Table 5**.

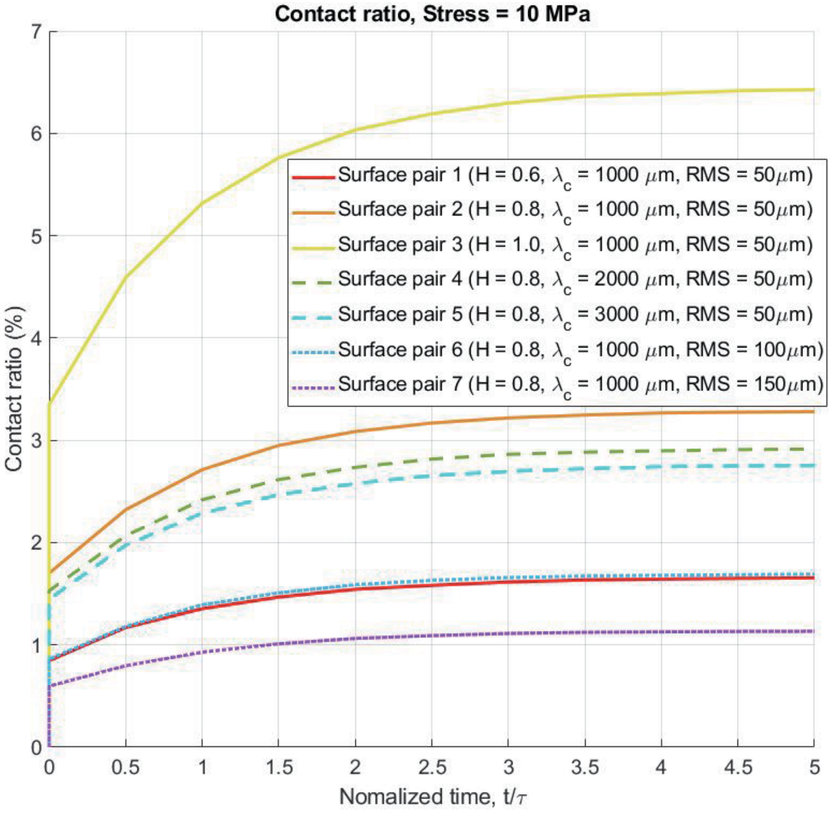

Figure 16.
Contact ratio changing with time (Kang et al. [13]). The time duration is normalized by τ.

Parameters	Average aperture		Contact ratio	
	Initial value	Decrease rate	Initial value	Increase rate
H↓	↑	↑	↓	↓
λ_c↑	↑	↑	↓	↓
RMS↑	↑	↑	↓	↓

Table 6.
Effect of surface parameters on the mean aperture and contact ratio.

Figures 15 and **16** show the mean aperture and contact ratio evolving with time for seven synthetic surface pairs, respectively. The total time duration is 5τ, and the macroscopic stress σ = 10 MPa. The total time duration is extended from 2τ to 5τ to show the time-decaying creep rate. The initial changes of the mean aperture and contact ratio correspond to fracture elastic deformation.

Based on **Figures 15** and **16**, several conclusions can be drawn:

1. As H decreases, the mean aperture increases, and the contact ratio increases slower with time;

2. As RMS increases, the mean aperture increases, and the contact ratio increases slower with time;

3. As λ_c decreases, the mean aperture increases, and the contact ratio increases slower with time.

4. Under current macroscopic stress, time duration, and surface parameters, the contact ratio is generally less than 7.0%.

5. Under current macroscopic stress, time duration, and surface parameters, the creep rate decreases significantly with time. This is mainly because the SLS model assumes an exponentially decaying creep rate.

Table 6 summarizes the effect of surface parameters on the mean aperture and contact ratio.

Figure 17 shows the contact region and local contacting stress evolution of surface pair **3** before and after the creep stage. The macroscopic stress is 10 MPa and the creep time duration is 5τ. The colored regions and white regions correspond to the contacting regions and non-contacting regions, respectively. The color bar scale is 2000 MPa. After the creep stage, the area of contacting regions becomes larger, and the local contacting stress reduces. However, even after the creep stage, the contact ratio is still less than 7.0%. Under the same time duration, if η is reduced, the contact area increase will increase more rapidly.

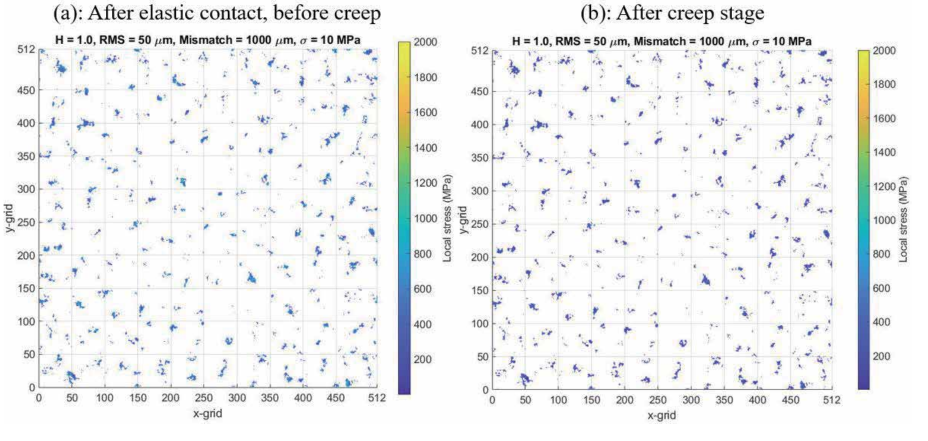

Figure 17.
Contact region and local contacting stress evolution before and after the creep stage (Kang et al. [13]). (a): before the creep stage; (b): after the creep stage. In both x- and y-directions, the number of grids is 512. The contact area increase is qualitatively shown.

4. Limitations of the method

In this numerical method, the contacting asperities deform visco-elastically, and there is no upper limit on the local contacting stress. For some synthetic surfaces, the contacting stress in a few cells exceed 1.3 GPa. In reality, under such high contacting stresses, the asperities may deform plastically. Ignoring the plastic deformation will underestimate the contact ratio and overestimate the local contacting stress. In addition, asperity breakage is ignored in this numerical method. Under high contacting stresses, asperities may break, which will further change the contacting regions and the contacting stress distribution [20]. Furthermore, the effect of shear stress on fracture visco-elastic deformations is also not considered. In engineering applications (especially in oil and gas production), fractures may subject to shear stress, which may significantly change the visco-elastic deformations.

5. Conclusions

This chapter explains how to use the boundary element method to calculate visco-elastic deformations of rough fractures. Fast numerical algorithms (CG and FFT) are implemented to further improve the efficiency. In addition, one example, which investigates the effect of surface geometry on visco-elastic deformations of rough rock fractures, is given. In this example, the author builds two in-house numerical codes: one code generates synthetic fracture surface pairs using Brown's probabilistic model [10], and the other simulates the visco-elastic deformations of the synthetic surface pairs. Seven synthetic surface pairs are generated by systematically changing the values of the root mean square roughness RMS (50 μm, 100 μm, and 150 μm), mismatch length λ_c (1000 μm, 2000 μm, and 3000 μm), and Hurst exponent H (0.6, 0.8, and 1.0). Then, the author simulates the visco-elastic deformation of the seven surface pairs by using the Standard Linear Solid (SLS) and the Maxwell model. The following key conclusions can be drawn:

1. As RMS increases, the average aperture increases, and the contact ratio increases slower with time;

2. As λ_c increases, the average aperture increases, and the contact ratio increases slower with time;

3. As H decreases, the average aperture increases, and the contact ratio increases slower with time;

4. For the macroscopic stress (10 MPa), time durations (5τ for the SLS model and 2τ for the Maxwell model), and the surface roughness parameters (RMS between 50 and 150 μm, λ_c between 1000 and 3000 μm, H between 0.6 and 1.0) used in the examples, the contact ratio is less than 9.5%.

While the results are useful, future work would be helpful. First, more surface roughness parameter values can be used so a quantitative relationship between surface parameters and contact ratio or average aperture can be obtained. In addition, other visco-elastic models, such as the Burgers model and the Power Law model, can be implemented. Furthermore, in this simulation, the plastic deformation of contacting asperities is not considered. As a result, the local contacting stress may be overestimated. The plastic deformation of contacting asperities can be considered so the results can be more realistic. Last but not least, the effect of shear stress can be simulated to make the results more applicable.

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Conflict of interest

The authors declare no conflict of interest.

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Chapter 2

Asymptotic Behavior by Krasnoselskii Fixed Point Theorem for Nonlinear Neutral Differential Equations with Variable Delays

Benhadri Mimia

Abstract

In this paper, we consider a neutral differential equation with two variable delays. We construct new conditions guaranteeing the trivial solution of this neutral differential equation is asymptotic stable. The technique of the proof based on the use of Krasnoselskii's fixed point Theorem. An asymptotic stability theorem with a necessary and sufficient condition is proved. In particular, this paper improves important and interesting works by Jin and Luo. Moreover, as an application, we also exhibit some special cases of the equation, which have been studied extensively in the literature.

Keywords: fixed points theory, stability, neutral differential equations, integral equation, variable delays

1. Introduction

For more than one hundred years, Liapunov's direct method has been very effectively used to investigate the stability problems of a wide variety of ordinary, functional, and partial differential, integro-differential equations. The success of Liapunov's direct method depends on finding a suitable Liapunov function or Liapunov functional. Nevertheless, the applications of this method to problems of stability in differential and integro-differential equations with delays have encountered serious difficulties if the delays are unbounded or if the equation has unbounded terms (see [1–3]). Therefore, new methods and techniques are needed to address those difficulties. Recently, Burton and his co-authors have applied fixed point theory to investigate the stability, which shows that some of these difficulties vanish when applying fixed point theory [1–22]. It turns out that the fixed point method is becoming a powerful technique in dealing with stability problems for indeterministic scenes (see for instance [16, 17, 21, 23]).

For example, Luo [16] studied the mean-square asymptotic stability for a class of linear scalar neutral stochastic differential equations by means of Banach's fixed point theory. The author did not use Lyapunov's method; he got interesting results for the stability even when the delay is unbounded. The author also obtained necessary and sufficient conditions for the asymptotic stability. Moreover, it possesses the advantage that it can yield the existence, uniqueness, and stability criteria of the considered system in one step.

Neutral delay differential equations are often used to describe the dynamical systems which not only depend on present and past states but also involve derivatives with delays, (see [24–28]). It has been applied to describe numerous intricate dynamical systems, such as population dynamics [18], mathematical biology [27], heat conduction, and engineering [28], etc.

In particular, qualitative analysis for neutral type equations such as stability and periodicity, oscillation theory, has been an active field of research, both in the deterministic and stochastic cases. We can refer to [6, 7, 13, 15–17, 19–21, 23, 29–31], and the references cited therein.

With this motivation, in this paper, we aim to discuss the boundedness and stability for neutral differential equations with two delays (1). It is worth noting that our research technique is based on Krasnoselskii's fixed point theory. We will give some new conditions to ensure that the zero solution is asymptotically stable. Namely, a necessary and sufficient condition ensuring the asymptotic stability is proved. Our findings generalize and improve some results that can be found in the literature. In our result, the delays can be unbounded and the coefficients in the equations can change their sign. This paper is organized as follows. In Section 1 we present some basic preliminaries and the form of the neutral functional differential equations which will be studied. In Section 2, we present the inversion of the equation and we state Krasnoselskii's fixed point theorem. The boundedness and stability of the neutral differential Eq. (1) are discussed in Section 3 via Krasnoselskii's fixed point theory. Finally, in Section 4 an example is given to illustrate our theory and our method, also to compare our result by using the fixed point theory with the known results by Ardjouni and Djoudi [6].

In this work, we consider the following class of neutral differential equations with variable delays,

$$x'(t) = -a(t)x(t-\tau_1(t)) + c(t)x'(t-\tau_1(t)) + b(t)x^{\sigma}(t-\tau_2(t)), t \geq t_0, \tag{1}$$

denote $x(t) \in \mathbb{R}$ the solution to (1) with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0], \tag{2}$$

where $\psi \in C([m(t_0), t_0], \mathbb{R})$, $\sigma \in (0,1)$ is a quotient with odd positive integer denominator. We assume that $a, b \in C(\mathbb{R}^+, \mathbb{R})$, $c \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy $t - \tau_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2$ and for each $t_0 \geq 0$,

$$m_i(t_0) = \inf\{t - \tau_i(t), t \geq t_0\}, m(t_0) = \min\{m_i(t_0), i = 1, 2\}. \tag{3}$$

Special cases of Eq. (1) have been recently considered and studied under various conditions and with several methods. Particularly, in the case $\sigma = 1/3$, and $c(t) = 0$, in [14] Jin and Luo using the fixed point theorem of Krasnoselskii obtained boundedness and asymptotic stability for the following equation:

$$x'(t) = -a(t)x(t-\tau_1(t)) + b(t)x^{\frac{1}{3}}(t-\tau_2(t)), t \geq 0. \tag{4}$$

More precisely, the following result was established.

Theorem A (Jin and Luo [14]). *Let τ_1 be differentiable and suppose that there exists $\alpha \in (0.1)$, $k_1, k_2 > 0$, and a function $h \in C([m(0), \infty), \mathbb{R}^+)$ such that for $|t_1 - t_2| \leq 1$,*

$$\left| \int_{t_1}^{t_2} |b(u)| du \right| \leq k_1 |t_1 - t_2|, \tag{5}$$

and

$$\left|\int_{t_1}^{t_2} h(u)du\right| \le k_2|t_1 - t_2|, \tag{6}$$

while for $t \ge 0$,

$$\begin{aligned}&\int_{t-\tau_1(t)}^{t} |h(u)|du + \int_0^t e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}|h(u)|du\right)ds \\ &+\int_0^t e^{-\int_s^t h(u)du}\{|h(s-\tau_1(s))(1-\tau_1'(s)) - a(s)| + |b(s)|\}ds \le \alpha.\end{aligned} \tag{7}$$

Then there is a solution $x(t, 0, \psi)$ *of* (4) *on* $\mathbb{R}^+$ *with* $|x(t, 0, \psi)| \le 1$.

Notice that when $c(t) = 0$ in the second term on the right-hand side of (1), then (1) reduces to (4). On the other hand, in the case, $\tau_1(t) = \tau_1$, a constant, Eq. (4) reduces to the one in [9]. Therefore, we considered the more general system than in [9, 14].

Very recently, by the same method of Jin and Luo [14], Ardjouni and Djoudi [6] improved the results of Jin and Luo [14] to the generalized nonlinear neutral differential equation with variable delays of the form

$$x'(t) = -a(t)x(t-\tau_1(t)) + c(t)x'(t-\tau_1(t)) + b(t)G(x^\sigma(t-\tau_2(t))), t \ge 0, \tag{8}$$

where $G : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous in x. That is, there is an $L > 0$ so that if $|x|, |y| \le 1$ then

$$|G(x) - G(y)| < |x - y| \text{ and } G(0) = 0.$$

We note that due to the presence of the term $c(t)x'(t-\tau_1(t))$, once the equation is inverted then once will face with the term $\frac{c(t)}{1-\tau_1'(t)}x(t-\tau_1(t))$, (where, $\tau_1'(t) \ne 1$ for $t \ge 0$) which produces a restrictive condition for the stability of (8) (as described in more detail below).

Theorem B (Ardjouni and Djoudi [6]). *Let* τ_1 *be twice differentiable and suppose that* $\tau_1'(t) \ne 1$ for all $t \in [m(0), \infty)$ *and suppose that there are constants* $\alpha \in (0.1)$, $k_1, k_2 > 0$, *and a function* $h \in C([m(0), \infty), \mathbb{R}^+)$ *such that for* $|t_1 - t_2| \le 1$,

$$\left|\int_{t_1}^{t_2} |b(u)|du\right| \le k_1|t_1 - t_2|, \tag{9}$$

and

$$\left|\int_{t_1}^{t_2} h(u)du\right| \le k_2|t_1 - t_2|, \tag{10}$$

while for $t \ge 0$,

$$\begin{aligned}&\left|\frac{c(t)}{1-\tau_1'(t)}\right| + \int_{t-\tau_1(t)}^{t}|h(u)|du \\ &+\int_0^t e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}|h(u)|du\right)ds \\ &+\int_0^t e^{-\int_s^t h(u)du}\{|h(s-\tau_1(s))(1-\tau'(s)) - a(s) - \mu(s)| + L|b(s)|\}ds \le \alpha,\end{aligned} \tag{11}$$

where

$$\mu(t) = \frac{(c(t)h(t) + c'(t))\left(1 - \tau_1'(t)\right) + c(t)\tau_1''(t)}{\left(1 - \tau_1'(t)\right)^2}.$$

Then there is a solution $x(t, 0, \psi)$ *of* (8) *on* $\mathbb{R}^+$ *with* $|x(t, 0, \psi)| \leq 1$.

By letting $c(t) = 0$ and $G(x^\sigma(t - \tau_2(t))) = x^\sigma(t - \tau_2(t))$ in (8), the present authors [14] have studied, the asymptotic stability and the stability by using Krasnoselskii's fixed point theorem, under appropriate conditions, of the Eq. (4) and improved the results claimed in [9]. Consequently, Theorem B improves and generalizes Theorem A. Following the technique of Jin and Luo [14], Ardjouni and Djoudi [6] studied the stability properties of (8). However, the condition (11) in Ardjouni and Djoudi [6] is restrictive. By employing two auxiliary functions p and g for constructing a fixed point mapping argument, the alternative condition (21) in Theorem 3.1 is obtained. Note that the condition

$$\left|\frac{c(t)}{1 - \tau_1'(t)}\right| < \alpha,$$

for some constant $\alpha \in (0, 1)$, is not needed in Theorem 3.1. In the present paper, we also adopt Krasnoselskii's fixed point theory to study the boundedness and stability of (1). A new criteria for asymptotic stability with a necessary and sufficient condition is given. The considered neutral differential equations, the results and assumptions to be given here are different from those that can be found in the literature and complete that one. These are the contributions of this paper to the literature and its novelty and originality. In addition, an example is provided to illustrate the effectiveness and the merits of the results introduced.

2. Inversion of equation

In this section, we use the variation of parameter formula to rewrite the equation as an integral equation suitable for the Krasnoselskii theorem. The technique for constructing a mapping for a fixed point argument comes from an idea in [21]. In our consideration we suppose that:

A1) Let τ_1 be twice differentiable and suppose that $\tau_1'(t) \neq 1$ for all $t \in [m(t_0), \infty[$.

A2) There exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ with $p(t) = 1$ for $t \in [m(t_0), t_0]$ such that $p'(t)$ exists for all $t \in [m(t_0), \infty[$.

Let $y(t) = \psi(t)$ on $t \in [m(t_0), t_0]$, and let

$$x(t) = p(t)y(t) \text{ for } t \geq t_0. \tag{12}$$

Make substitution of (12) into (1) to show

$$\begin{aligned} y'(t) = &-\frac{p'(t)}{p(t)}y(t) - \frac{a(t)p(t - \tau_1(t)) - c(t)p'(t - \tau_1(t))}{p(t)}y(t - \tau_1(t)) \\ &+\frac{c(t)p(t - \tau_1(t))}{p(t)}y'(t - \tau_1(t)) \\ &+b(t)\frac{p^\sigma(t - \tau_2(t))}{p(t)}y^\sigma(t - \tau_2(t)), t \geq t_0, \end{aligned} \tag{13}$$

then it can be verified that x satisfies (1).

We now re-write Eq. (13) in an equivalent form. To this end, we use the variation of parameter formula and rewrite the equation in an integral from which we derive a Krasnoselskii fixed point theorem. Besides, the integration by parts will be applied.

We need the following lemma in our proof of the main theorem.

Lemma 2.1. *Let* $h : [m(t_0), \infty) \to \mathbb{R}^+$ *be an arbitrary continuous function and suppose that (A1) and (A2) hold. Then y is a solution of* (13) *if and only if*

$$\begin{aligned} y(t) = &\left(\psi(t_0) - \frac{p(t_0 - \tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{\left(1 - \tau_1'(t_0)\right)} \psi(t_0 - \tau_1(t_0)) \right. \\ &\left. - \int_{t_0 - \tau_1(t_0)}^{t_0} \left(h(u) - \frac{p'(u)}{p(u)}\right) y(u) du \right) e^{-\int_{t_0}^{t} h(s) ds} \\ &+ \frac{p(t - \tau_1(t))}{p(t)} \frac{c(t)}{1 - \tau_1'(t)} y(t - \tau_1(t)) + \int_{t - \tau_1(t)}^{t} \left(h(u) - \frac{p'(u)}{p(u)}\right) y(u) du \\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u) du} \left\{ -\overline{\mu}(s) + \left(h(s - \tau_1(s)) - \frac{p'(s - \tau_1(s))}{p(s - \tau_1(s))}\right) \left(1 - \tau_1'(s)\right) - \overline{\beta}(s) \right\} \\ &\times y(s - \tau_1(s)) ds \\ &- \int_{t_0}^{t} e^{-\int_s^t h(u) du} h(s) \left(\int_{s - \tau_1(s)}^{s} \left(h(u) - \frac{p'(u)}{p(u)}\right) y(u) du \right) ds \\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u) du} b(s) \frac{p^{\sigma}(s - \tau_2(s))}{p(s)} y^{\sigma}(s - \tau_2(s)) ds, \end{aligned} \tag{14}$$

where

$$\overline{\mu}(t) = \frac{a(t)p(t - \tau_1(t)) - c(t)p'(t - \tau_1(t))}{p(t)}, C(t) = \frac{c(t)p(t - \tau_1(t))}{p(t)}. \tag{15}$$

and

$$\overline{\beta}(t) = \frac{[C(t)h(t) + C'(t)]\left(1 - \tau_1'(t)\right) + C(t)\tau_1''(t)}{\left(1 - \tau_1'(t)\right)^2}. \tag{16}$$

Proof. Let $y(t)$ be a solution of Eq. (13). Rewrite (13) as

$$\begin{aligned} y'(t) + h(t)y(t) = &\left(h(t) - \frac{p'(t)}{p(t)}\right) y(t) - \frac{a(t)p(t - \tau_1(t)) - c(t)p'(t - \tau_1(t))}{p(t)} y(t - \tau_1(t)) \\ &+ \frac{c(t)p(t - \tau_1(t))}{p(t)} y'(t - \tau_1(t)) \\ &+ b(t) \frac{p^{\sigma}(t - \tau_2(t))}{p(t)} y^{\sigma}(t - \tau_2(t)), t \geq t_0. \end{aligned} \tag{17}$$

Multiply both sides of (17) the previous equality by $e^{\int_{t_0}^{t} h(s)ds}$ and then integrate from t_0 to t, we have

$$\begin{aligned} y(t) &= \psi(t_0)e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t}\left(h(s) - \frac{p'(s)}{p(s)}\right)e^{-\int_s^t h(u)du}y(s)ds \\ &\quad -\int_{t_0}^{t} e^{-\int_s^t h(u)du}\frac{a(s)p(s-\tau_1(s)) - c(s)p'(s-\tau_1(s))}{p(s)}y(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^{t} e^{-\int_s^t h(u)du}\frac{c(s)p(s-\tau_1(s))}{p(s)}y'(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^{t} e^{-\int_s^t h(u)du}b(s)\frac{p^{\sigma}(s-\tau_2(s))}{p(s)}y^{\sigma}(s-\tau_2(s))ds. \end{aligned} \tag{18}$$

Performing an integration by parts, we can conclude, for $t \geq t_0$,

$$\begin{aligned} y(t) &= \psi(t_0)e^{-\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_s^t h(u)du} d\left(\int_{s-\tau_1(s)}^{s}\left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du\right) \\ &\quad +\int_{t_0}^{t} e^{-\int_s^t h(u)du}\left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right) \times (1-\tau_1'(s))y(s-\tau_1(s))ds \\ &\quad -\int_{t_0}^{t} e^{-\int_s^t h(u)du}\frac{a(s)p(s-\tau_1(s)) - c(s)p'(s-\tau_1(s))}{p(s)}y(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^{t} e^{-\int_s^t h(u)du}\frac{c(s)p(s-\tau_1(s))}{p(s)}y'(s-\tau_1(s))ds \\ &\quad +\int_{t_0}^{t} e^{-\int_s^t h(u)du}b(s)\frac{p^{\sigma}(s-\tau_2(s))}{p(s)}y^{\sigma}(s-\tau_2(s))ds. \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \left(\psi(t_0) - \frac{p(t_0-\tau_1(t_0))}{p(t_0)}\frac{c(t_0)}{(1-\tau_1'(t_0))}\psi(t_0-\tau_1(t_0))\right. \\ &\quad \left. -\int_{t_0-\tau_1(t_0)}^{t_0}\left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du\right)e^{-\int_{t_0}^{t} h(s)ds} \\ &\quad +\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}y(t-\tau_1(t)) + \int_{t-\tau_1(t)}^{t}\left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du \\ &\quad +\int_{t_0}^{t} e^{-\int_s^t h(u)du}\left\{-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right\} \\ &\quad \times y(s-\tau_1(s))ds \\ &\quad -\int_{t_0}^{t} e^{-\int_s^t h(u)du}h(s)\left(\int_{s-\tau_1(s)}^{s}\left(h(u) - \frac{p'(u)}{p(u)}\right)y(u)du\right)ds \\ &\quad +\int_{t_0}^{t} e^{-\int_s^t h(u)du}b(s)\frac{p^{\sigma}(s-\tau_2(s))}{p(s)}y^{\sigma}(s-\tau_2(s))ds, \end{aligned}$$

where $\overline{\mu}(s)$ and $\overline{\beta}(s)$ are defined in (15) and (16), respectively. The proof is complete.

Below we state Krasnoselskii's fixed point theorem which will enable us to establish a stability result of the trivial solution of (1) For more details on Krasnoselskii's captivating theorem, we refer to smart [20] or [3].

Theorem 2.1. (see, [Kranoselskii's fixed point theorem, [20]]). *Suppose that* $(X, \|.\|)$ *is a Banach space and* $\mathcal{M}$ *is a bounded, convex, and closed subset of* X. Suppose further that there exist, two operators, $\mathcal{A}, \mathcal{B} \to \mathcal{M}$ *into* X *such that:*

i. $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ *for all* $x, y \in \mathcal{M}$;

ii. $\mathcal{A}$ *is completely continuous*;

iii. $\mathcal{B}$ *is a contraction mapping.*

Then $\mathcal{A} + \mathcal{B}$ *has a fixed point in* $\mathcal{M}$.

3. Stability by Krasnoselskii fixed point theorem

From the existence theory, which can be found in Hale [26] or Burton [3], we conclude that for each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, a solution of (1) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \rho) \to \mathbb{R}$ for some positive constant $\rho > 0$ such that x satisfies (1) on $[t_0, t_0 + \rho)$ and $x = \psi$ on $[m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$. We define $\|\psi\| = \max\{|\psi(t)| : m(t_0) \le t \le t_0\}$.

As we mentioned previously, our results in this section extend and improve the work in [14] by considering more general classes of neutral differential equations presented by (1). Our main results in this part can be applied to the case when

$$\left|\frac{c(t)}{1 - \tau_1'(t)}\right| \ge 1,$$

which improve [14]. In other words, we will establish and prove a necessary and sufficient condition ensuring the boundedness of solutions and the asymptotic stability of the zero solution to Eq. (1). However, the mathematical analysis used in this research to construct the mapping to employ Krasnoselskii's fixed point theorem is different from that of [14].

The results of this work are news and they extend and improve previously known results. To the best of our knowledge from the literature, there are few authors who have used the fixed point theorem to prove the existence of a solution and the stability of trivial equilibrium of several special cases of (1) all at once [9, 14].

Let us know to recall the definitions of stability that will be used in the next section. For stability definitions, we refer to [3].

Definition 3.1. The zero solution of (1) is said to be:

i. stable, if for any $\varepsilon > 0$ and $t_0 \ge 0,$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\psi \in C([m(t_0), t_0], \mathbb{R})$ and $\|\psi\| < \delta$ imply $|x(t, t_0, \psi)| < \varepsilon$ for $t \ge t_0$.

ii. asymptotically stable, if the zero solution is stable and for any $\varepsilon > 0$ and $t_0 \ge 0,$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\psi \in C([m(t_0), t_0], \mathbb{R})$ and $\|\psi\| < \delta$ imply $|x(t, t_0, \psi)| \to 0$ as $t \to \infty$..

Now, we can state our main result.

Theorem 3.1. *Suppose that assumptions (A1) and (A2) hold, and that there are constants* $\alpha \in (0,1)$, $k_1, k_2 > 0$, *and an arbitrary continuous function* $h \in C([m(t_0),\infty),\mathbb{R}^+)$ *such that for* $|t_1 - t_2| \le 1$,

$$\left|\int_{t_1}^{t_2}\left|b(u)\frac{p^\sigma(u-\tau_2(u))}{p(u)}\right|du\right| \le k_1|t_1-t_2|, \tag{19}$$

and

$$\left|\int_{t_1}^{t_2} h(u)du\right| \le k_2|t_1-t_2|, \tag{20}$$

while for $t \ge t_0$

$$\begin{aligned}
&\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right| + \int_{t-\tau_1(t)}^{t}\left|h(u)-\frac{p'(u)}{p(u)}\right|du \\
&+\int_{t_0}^{t} e^{-\int_s^t h(u)du}\left\{\left|-\overline{\mu}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\overline{\beta}(s)\right|\right\}ds \\
&+\int_{t_0}^{t} e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)ds \\
&+\int_{t_0}^{t} e^{-\int_s^t h(u)du}|b(s)|\left|\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds < \alpha,
\end{aligned} \tag{21}$$

where $\overline{\mu}(s)$ *and* $\overline{\beta}(s)$ *are defined in* (15) *and* (16), *respectively. If* ψ *is a given continuous initial function which is sufficiently small, then there is a solution* $x(t,t_0,\psi)$ *of* (1) *on* $\mathbb{R}^+$ *with* $|x(t,t_0,\psi)| \le 1$.

We are now ready to prove Theorem 3.1.

Proof. We start with some preparation:

Let $\left(X, |.|_g\right)$ be the Banach space of continuous $\varphi : [m(t_0),\infty) \to \mathbb{R}$ with

$$|\varphi|_g := \sup_{t\ge m(t_0)} |\varphi(t)/g(t)| < \infty.$$

For each $t_0 \ge 0$ and $\psi \in C([m(t_0),t_0],\mathbb{R})$ fixed, we define X_ψ as the following space

$$X_\psi = \{\varphi \in X : |\varphi(t)| \le 1 \ \text{for} t \in [m(t_0),\infty) \text{and} \varphi(t) = \psi(t) \text{if} t \in [m(t_0),t_0]\}.$$

It is easy to check that X_ψ is a complete metric space with metric induced by the norm $|.|_g$.

We note that to apply Krasnoselskii's fixed point theorem we need to construct two mappings; one is contraction and the other is compact. Therefore, we use (14) to define the operator $H : X_\psi \to X_\psi$ by

$$(H\varphi)(t) := (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : X_\psi \to X_\psi$ are given by

$$(\mathcal{A}\varphi)(t) := \int_{t_0}^{t} e^{-\int_s^t h(u)du} b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}\varphi^\sigma(s-\tau_2(s))ds, \tag{22}$$

and

$$\begin{aligned}(\mathcal{B}\varphi)(t) := &\left(\psi(t_0) - \frac{p(t_0 - \tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{(1 - \tau_1'(t_0))} \psi(t_0 - \tau_1(t_0))\right.\\ &\left. - \int_{t_0-\tau_1(t_0)}^{t_0} \left(h(u) - \frac{p'(u)}{p(u)}\right)\varphi(u)du\right) e^{-\int_{t_0}^{t} h(s)ds}\\ &+ \frac{p(t-\tau_1(t))}{p(t)} \frac{c(t)}{1-\tau_1'(t)} \varphi(t-\tau_1(t)) + \int_{t-\tau_1(t)}^{t} \left(h(u) - \frac{p'(u)}{p(u)}\right)\varphi(u)du\\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left\{-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right\}\\ &\times \varphi(s-\tau_1(s))ds\\ &- \int_{t_0}^{t} e^{-\int_s^t h(u)du} h(s) \left(\int_{s-\tau_1(s)}^{s} \left(h(u) - \frac{p'(u)}{p(u)}\right)\varphi(u)du\right)ds.\end{aligned} \tag{23}$$

If we are able to prove that H possesses a fixed point φ on the set X_ψ, then $y(t, t_0, \psi) = \varphi(t)$ for $t \geq t_0$, $y(t, t_0, \psi) = \psi(t)$ on $[m(t_0), t_0]$, $y(t, t_0, \psi)$ satisfies (13) when its derivative exists and $|y(t, t_0, \psi)| < 1$ for $t \geq t_0$. That $\mathcal{A}$ maps X_ψ into itself can be deduced from condition (21).

For $\alpha \in (0, 1)$, we choose $\delta > 0$ such that

$$\begin{aligned}&\left(1 + \left|\frac{p(t_0 - \tau_1(t_0))}{p(t_0)} \frac{c(t_0)}{(1 - \tau_1'(t_0))}\right|\right.\\ &\left. + \int_{t_0-\tau_1(t_0)}^{t_0} \left|h(u) - \frac{p'(u)}{p(u)}\right|du\right) e^{-\int_{t_0}^{t} h(s)ds} \delta + \alpha \leq 1.\end{aligned} \tag{24}$$

Let $\psi : [m(t_0), t_0] \to \mathbb{R}$ be a given continuous initial function with $\|\psi\| < \delta$. Let $g : [m(t_0), \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $g(m(t_0)) = 1$, $g(s) \to \infty$ as $s \to \infty$, such that

$$\begin{aligned}&\left|\frac{p(t-\tau_1(t))}{p(t)} \frac{c(t)}{1-\tau_1'(t)}\right| + \int_{t-\tau_1(t)}^{t} \left|h(u) - \frac{p'(u)}{p(u)}\right| g(u)/g(t)du\\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^{s} \left|h(u) - \frac{p'(u)}{p(u)}\right| g(u)/g(t)du\right)ds\\ &+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left|-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right|\\ &\times g(s-\tau_1(s))/g(t)ds < \alpha.\end{aligned} \tag{25}$$

Now we split the rest of our proof into three steps.

First step: We now show that $\varphi, \phi \in X_\psi$ implies that $\mathcal{A}\varphi + \mathcal{B}\phi \in X_\psi$. Now, let $\|.\|$ be the supremum norm on $[m(t_0), \infty)$ of $\varphi \in X_\psi$ if φ is bounded. Note that if $\varphi, \phi \in X_\psi$ then

$$|(\mathcal{A}\varphi)(t) + (\mathcal{B}\phi)(t)| \leq$$

$$\|\psi\|\left(1+\left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)}\right|\left|\frac{c(t_0)}{(1-\tau_1'(t_0))}\right|+\int_{t_0-\tau_1(t_0)}^{t_0}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)e^{-\int_{t_0}^{t}h(s)ds}$$

$$+\|\phi\|\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right|$$

$$+\|\phi\|\int_{t-\tau_1(t)}^{t}\left|h(u)-\frac{p'(u)}{p(u)}\right|du$$

$$+\|\phi\|\int_{t_0}^{t}e^{-\int_s^t h(u)du}\left\{\left|-\overline{\mu}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\overline{\beta}(s)\right|\right\}ds$$

$$+\|\phi\|\int_{t_0}^{t}e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)ds$$

$$+\|\varphi^\sigma\|\int_{t_0}^{t}e^{-\int_s^t h(u)du}|b(s)|\left|\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds$$

$$\leq\left(1+\left|\frac{p(t_0-\tau_1(t_0))}{p(t_0)}\right|\left|\frac{c(t_0)}{(1-\tau_1'(t_0))}\right|\right.$$

$$\left.+\int_{t_0-\tau_1(t_0)}^{t_0}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)e^{-\int_{t_0}^{t}h(s)ds}\delta+\alpha\leq 1.$$

By applying (24), we see that $|(\mathcal{A}\varphi)(t)+(\mathcal{B}\phi)(t)|\leq 1$ for $t\in[m(t_0),\infty)$.

We see that also $\mathcal{B}$ maps X_ψ into itself by letting $\varphi=0$ in the preceding sum.

Second step: Next, we will show that $\mathcal{A}X_\psi$ is equicontinuous and $\mathcal{A}$ is continuous. We first show that $\mathcal{A}X_\psi$ is equicontinuous. If $\varphi\in X_\psi$ and if $0\leq t_1<t_2$ with $t_2-t_1<1,$ then

$$|(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)|=\left|\int_{t_0}^{t_2}e^{-\int_s^{t_2}h(u)du}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}ds\right.$$

$$\left.-\int_{t_0}^{t_1}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}e^{-\int_s^{t_1}h(u)du}ds\right|$$

$$\leq\left|\int_{t_1}^{t_2}b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}e^{-\int_s^{t_2}h(u)du}ds\right|$$

$$+\left|\int_{t_0}^{t_1}\left(e^{-\int_s^{t_2}h(u)du}-e^{-\int_s^{t_1}h(u)du}\right)b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}ds\right|$$

$$\leq\int_{t_1}^{t_2}e^{-\int_s^{t_2}h(u)du}d\left(\int_{t_1}^{s}\left|b(v)\frac{p^\sigma(v-\tau_2(v))}{p(v)}\right|dv\right)$$

$$+\left|e^{-\int_{t_1}^{t_2}h(u)du}-1\right|\int_{t_0}^{t_1}e^{-\int_s^{t_1}h(u)du}\left|b(s)\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds$$

$$\leq\int_{t_1}^{t_2}\left|b(u)\frac{p^\sigma(u-\tau_2(u))}{p(u)}\right|du\left(1+\int_{t_1}^{t_2}h(u)e^{-\int_s^{t_2}h(u)du}ds\right)+\alpha\left|e^{-\int_{t_1}^{t_2}h(u)du}-1\right|$$

$$\leq 2\int_{t_1}^{t_2}\left|b(u)\frac{p^{\sigma}(u-\tau_2(u))}{p(u)}\right|du+\alpha\left|\int_{t_1}^{t_2}h(u)du\right|\leq(2k_1+\alpha k_2)|t_2-t_1|,$$

by (19)–(21). In the space $\left(X,|.|_g\right)$, the set $\mathcal{A}X_{\psi}$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela theorem $\mathcal{A}X_{\psi}$ resides in a compact set.

Next, we need to show that $\mathcal{A}$ is continuous. Let $\varepsilon>0$ be given and let $\varphi,\phi\in X_{\psi}$. Now y^{σ}, is uniformly continuous on $[-1,+1]$ so for a fixed $T>0$ with $4/g(T)<\varepsilon$ there is an $\eta>0$ such that $|y_1-y_2|<\eta g(T)$ implies $|y_1^{\sigma}-y_2^{\sigma}|<\varepsilon/2$. Thus for $|\varphi(t)-\phi(t)|<\eta g(t)$ and for $t>T$ we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)-(\mathcal{A}\phi)(t)|/g(t)\\
&=(1/g(t))\int_{t_0}^{t}e^{-\int_s^t h(u)du}\left|b(s)\frac{p^{\sigma}(s-\tau_2(s))}{p(s)}\right||\varphi^{\sigma}(s-\tau_2(s))-\phi^{\sigma}(s-\tau_2(s))|ds\\
&\leq(1/g(t))\left\{\int_{t_0}^{T}e^{-\int_s^t h(u)du}\left|b(s)\frac{p^{\sigma}(s-\tau_2(s))}{p(s)}\right||\varphi^{\sigma}(s-\tau_2(s))-\phi^{\sigma}(s-\tau_2(s))|ds\right.\\
&\left.+2\int_{T}^{t}e^{-\int_s^t h(u)du}\left|b(s)\frac{p^{\sigma}(s-\tau_2(s))}{p(s)}\right|ds\right\}\\
&\leq\{(\alpha\varepsilon/2g(t))+(2\alpha/g(T))\}\leq\alpha\varepsilon.
\end{aligned}$$

Third step: Finally, we show that $\mathcal{B}$ is a contraction with respect to the norm $|.|_g$ with constant α. Let $\mathcal{B}$ be defined by (23). Then for $\phi_1,\phi_2\in X_{\psi}$ we have

$$\begin{aligned}
&|(\mathcal{B}\phi_1)(t)-(\mathcal{B}\phi_2)(t)|/g(t)\leq\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right||\phi_1(t-\tau_1(t))-\phi_2(t-\tau_1(t))|/g(t)\\
&+\int_{t-\tau_1(t)}^{t}\left|h(u)-\frac{p'(u)}{p(u)}\right||\phi_1(u)-\phi_2(u)|/g(t)du\\
&+\int_{t_0}^{t}e^{-\int_s^t h(u)du}\left|-\overline{\mu}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\overline{\beta}(s)\right|\\
&\times|\phi_1(s-\tau_1(s))-\phi_2(s-\tau_1(s))|/g(t)ds\\
&+\int_{t_0}^{t}e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right||\phi_1(u)-\phi_2(u)|/g(t)du\right)ds\\
&\leq|\phi_1-\phi_2|_g\left\{\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right|\right.\\
&+\int_{t-\tau_1(t)}^{t}\left|h(u)-\frac{p'(u)}{p(u)}\right|g(u)/g(t)du\\
&+\int_{t_0}^{t}e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|g(u)/g(t)du\right)ds\\
&+\int_{t_0}^{t}e^{-\int_s^t h(u)du}\\
&\left.\times\left|-\overline{\mu}(s)+\left(h(s-\tau_1(s))-\frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s))-\overline{\beta}(s)\right|g(s-\tau_1(s))/g(t)ds\right\}\\
&\leq\alpha|\phi_1-\phi_2|_g,\text{ by (22).}
\end{aligned}$$

Since $\alpha \in (0,1)$, we can conclude that $\mathcal{B}$ is a contraction on $\left(X_\psi, |.|_g\right)$.

The conditions of Krasnoselskii's theorem are satisfied with $\mathcal{M} = X_\psi$. Hence, we deduce that $H : X_\psi \to X_\psi$ has a fixed point $y(t)$, which is a solution of (13) with $y(s) = \psi(s)$ on $s \in m(t_0), t_0]$ and $|y(t, t_0, \psi)| \leq 1$ for $t \in [m(t_0), \infty)$. Since there exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ with $p(t) = 1$ for $t \in [m(t_0), t_0]$, by hypotheses (12) and from the above arguments we deduce that there exists a solution x of (1) with $x = \psi$ on $[m(t_0), t_0]$ satisfies $|x(t, t_0, \psi)| \leq 1$ for all $t \in [m(t_0), \infty)$. The proof is complete.

Letting $\sigma = 1/3,$ and $c(t) = 0$ in Theorem 3.1. Then we have the following corollary.

Corollary 3.1. *Let* (19) *and* (20) *hold, and* (21) *be replaced by*

$$\int_{t-\tau_1(t)}^{t} \left| h(u) - \frac{p'(u)}{p(u)} \right| du$$

$$+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left\{ \left| \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))} \right) (1-\tau_1'(s)) - a(s) \frac{p(s-\tau(s))}{p(s)} \right| \right\} ds$$

$$+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^{s} \left| h(u) - \frac{p'(u)}{p(u)} \right| du \right) ds \leq \alpha. \tag{26}$$

Then there is a solution $x(t, t_0, \psi)$ of (4) *on $\mathbb{R}^+$ with $|x(t, t_0, \psi)| \leq 1$.*

Remark 3.2: When $p(t) = 1,$ then Corollary 3.1 reduces to Theorem A, which was recently stated in Jin and Luo [14]. Therefore, the paper (Jin and Luo [14]) is a particular case of ours.

For the next Theorem, we manipulate function spaces defined on infinite t -intervals. So, for compactness, we need an extension of the Arzelà-Ascoli theorem. This extension is taken from ([3], Theorem 1.2.2 p. 20).

Theorem 3.2. *Let* (19)–(21) *hold and assume that*

$$\int_{t_0}^{t} e^{-\int_s^t h(u)du} \left| b(s) \frac{p^\sigma(s-\tau_2(s))}{p(s)} \right| ds \to 0 \text{ as } t \to \infty, \tag{27}$$

and

$$\liminf_{t\to\infty} \int_{t_0}^{t} h(s)ds > -\infty. \tag{28}$$

If ψ is given continuous initial function which is sufficiently small, then (1) *has a solution $x(t, t_0, \psi) \to 0$ as $t \to \infty$ if and only if*

$$\int_{t_0}^{t} h(s)ds \to \infty \text{ as } t \to \infty. \tag{29}$$

Proof. We set

$$K = \sup_{t \geq t_0} \left\{ e^{-\int_{t_0}^t h(s)ds} \right\}, \tag{30}$$

by (28), K is well defined. Suppose that (29) holds.

Since p is bounded, it remains to prove that the zero solution of (1) is asymptotically stable.

All of the calculations in the proof of Theorem 3.1 hold with $g(t) = 1$ when $|.|_g$ is replaced by the supremum norm $\|.\|$.

For

$$\varphi \in X_\psi,$$

$$|(\mathcal{A}\varphi)(t)| \leq \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left| b(s) \frac{p^\sigma(s - \tau_2(s))}{p(s)} \right| ds =: q(t), \tag{31}$$

where $q(t) \to 0$ as $t \to \infty$ by (27).

Add to X_ψ the condition that $\varphi \in X_\psi$ implies that $\varphi(t) \to 0$ as $t \to \infty$. We can see that for $\varphi \in X_\psi$ then $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$ by (31), and $(\mathcal{B}\varphi)(t) \to 0$ as $t \to \infty$ by (29).

Since $\mathcal{A}X_\psi$ has been shown to be equicontinuous, $\mathcal{A}$ maps X_ψ into a compact subset of X_ψ. By Krasnoselskii's theorem, there is $y \in X_\psi$ with $\mathcal{A}y + \mathcal{B}y = y$. As $y \in X_\psi, y(t, t_0, \psi) \to 0$ as $t \to \infty$. By condition (12), it is very easy to show that there exists a solution $x \in X_\psi$ of (1) with $x(t, t_0, \psi) \to 0$ as $t \to \infty$.

Conversely, we suppose that (29) fails. From (28) there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_{t_0}^{t_n} h(u)du = \xi$ for some $\xi \in \mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J \leq \int_{t_0}^{t_n} h(u)du \leq +J,$$

for all $n \geq 1$. To simplify the expression, we define

$$\omega(s) := \left| -\overline{\mu}(s) + \left(h(s - \tau_1(s)) - \frac{p'(s - \tau_1(s))}{p(s - \tau_1(s))} \right) (1 - \tau_1'(s)) - \overline{\beta}(s) \right|$$

$$+ |h(s)| \int_{s-\tau_1(s)}^{s} \left| h(u) - \frac{p'(u)}{p(u)} \right| du + \left| b(s) \frac{p^\sigma(s - \tau_2(s))}{p(s)} \right|,$$

for all $s \geq 0$. By (21), we have

$$\int_{t_0}^{t_n} e^{-\int_s^{t_n} h(u)du} \omega(s) ds \leq \alpha.$$

This yields

$$\int_{t_0}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds \leq \alpha e^{\int_0^{t_n} h(u)du} \leq e^J.$$

The sequence $\left\{ \int_{t_0}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds \right\}$ is bounded, hence there exists a convergent subsequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty} \int_{t_0}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds = \theta,$$

for some $\theta \in \mathbb{R}^+$. Let m be an integer such that

$$\int_{t_m}^{t_n} e^{\int_0^s h(u)du} \omega(s)ds \leq \frac{\delta_0}{4K},$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies $2\delta_0 Ke^J + \alpha \leq 1$.

We now consider the solution $y(t) = y(t, t_m, \psi)$ of (1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that $|y(t)| \leq 1$ for $t \geq t_m$ and

$$\psi(t_m) - \frac{p(t_m - \tau_1(t_m))}{p(t_m)} \frac{c(t_m)}{(1 - \tau_1'(t_m))} \psi(t_m - \tau_1(t_m))$$
$$- \int_{t_m - \tau_1(t_m)}^{t_m} \left(h(u) - \frac{p'(u)}{p(u)} \right) z(u) du \geq \frac{1}{2} \delta_0.$$

In follows from (22) and (23) with $y(t) = (\mathcal{A}y)(t) + (\mathcal{B}y)(t)$ that for $n \geq m$

$$\left| y(t_n) - \frac{p(t_n - \tau_1(t_n))}{p(t_n)} \frac{c(t_n)}{(1 - \tau_1'(t_n))} y(t_n - \tau_1(t_n)) - \int_{t_n - \tau_1(t_n)}^{t_n} \left(h(s) - \frac{p'(s)}{p(s)} \right) y(s) ds \right|$$
$$\geq \frac{1}{2} \delta_0 e^{-\int_{t_m}^{t_n} h(u)du} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} h(u)du} \omega(s) ds$$
$$= e^{-\int_{t_m}^{t_n} h(u)du} \left(\frac{1}{2} \delta_0 - e^{-\int_0^{t_m} h(u)du} \int_{t_m}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds \right)$$
$$\geq e^{-\int_{t_m}^{t_n} h(u)du} \left(\frac{1}{2} \delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s h(u)du} \omega(s) ds \right)$$
$$\geq \frac{1}{4} \delta_0 e^{-\int_{t_m}^{t_n} h(u)du} \geq \frac{1}{4} \delta_0 e^{-2J} > 0. \tag{32}$$

On the other hand, if the zero solution of (13) $y(t) = y(t, t_m, \psi) \to 0$ as $t \to \infty$, since $t_n - \tau_i(t_n) \to \infty$ as $t \to \infty$, $i = 1, 2,$ and (21) holds, we have

$$y(t_n) - \frac{p(t_n - \tau_1(t_n))}{p(t_n)} \frac{c(t_n)}{(1 - \tau_1'(t_n))} y(t_n - \tau_1(t_n)) - \int_{t_n - \tau_1(t_n)}^{t_n} \left(h(s) - \frac{p'(s)}{p(s)} \right) y(s) ds \to 0$$

as $t \to \infty$, which contradicts (32). Hence condition (29) is necessary for the asymptotic stability of the zero solution of (13), and hence the zero solution of (1) is asymptotically stable. The proof is complete.

For the special case $c(t) = 0$ and $\sigma = \frac{1}{3}$, we can get.

Corollary 3.2. *Let* (19), (20) *and* (27) *hold and* (21) *be replaced by*

$$\int_{t-\tau_1(t)}^{t} \left| h(u) - \frac{p'(u)}{p(u)} \right| du$$
$$+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} \left\{ \left| \left(h(s - \tau_1(s)) - \frac{p'(s - \tau_1(s))}{p(s - \tau_1(s))} \right) (1 - \tau_1'(s)) - a(s) \frac{p(s - \tau(s))}{p(s)} \right| \right\} ds$$
$$+ \int_{t_0}^{t} e^{-\int_s^t h(u)du} |h(s)| \left(\int_{s-\tau_1(s)}^{s} \left| h(u) - \frac{p'(u)}{p(u)} \right| du \right) ds \leq \alpha.$$

Then the zero solution $x(t, t_0, \psi)$ of (4) *with a small continuous function $\psi(t)$ is asymptotically stable if only if*

$$\int_{t_0}^{t} h(s)ds \to \infty \text{ as } t \to \infty.$$

Remark 3.3. The method in this paper can be applied to more general nonlinear neutral differential equations than Eq. (1).

Remark 3.4. Theorem 3.1 is still true if condition (21) is satisfied for $t \geq t_\rho$ with some $t_\rho \in \mathbb{R}^+$.

4. Example

In this section, we now give an example to show the applicability of Theorem 3.1.

Example. Let us consider the following neutral differential equation of first order with two variable delays, which is a special case of (1):

$$\begin{aligned} x'(t) = -a(t)x(t-\tau_1(t)) + \ln\left(\frac{0.95t+1}{4(t+1)}\right)x'(t-\tau_1(t)) \\ + \frac{0.6(0.95t+1)^{\frac{1}{3}}}{(t+1)^2}x^{\frac{1}{3}}(t-\tau_2(t)), \end{aligned} \tag{33}$$

for $t \geq 0$ where $\tau_2(t) = 0.5t,\ \tau_1(t) = 0.05t,$ and $a(t)$ satisfies

$$\left|-\overline{\mu}(t) + \left(h(t-\tau_1(t)) - \frac{p'(t-\tau_1(t))}{p(t-\tau_1(t))}\right)(1-\tau_1'(t)) - \overline{\beta}(t)\right| \leq \frac{0.03}{t+1},$$

where $\overline{\mu}(t)$ and $\overline{\beta}(t)$ are defined in (15) and (16), respectively. Choosing $h(t) = \frac{1.5}{t+1}$ and $p(t) = \frac{1}{t+1}$. By straightforward computations, we can check that condition (21) in Theorem 3.1 holds true. As $t \to \infty,$ we have

$$\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right| \leq \left|\frac{1}{4\times 0.95}\right| \leq 0.263,$$

$$\int_{t-\tau_1(t)}^{t}\left|h(u) - \frac{p'(u)}{p(u)}\right|du \leq 0.026,$$

$$\int_0^t e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u)-\frac{p'(u)}{p(u)}\right|du\right)ds \leq 0.026,$$

$$\int_0^t e^{-\int_s^t h(u)du}\left|-\overline{\mu}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right|ds$$

$$\leq \int_0^t e^{-\int_s^t \frac{1.5}{u+1}du}\frac{0.3}{s+1}ds \leq 0.2,$$

and

$\int_0^t e^{-\int_s^t h(u)du}|b(s)|\left|\frac{p^\sigma(s-\tau_2(s))}{p(s)}\right|ds \leq 0.4,$ and since $\int_0^t h(s)ds \to \infty$ as $t \to \infty,\ p(t) \leq 1$. Let $\alpha = 0.263 + 0.026 + 0.026 + 0.2 + 0.4$. It is easy to see that all the conditions of Theorem 3.1 hold for $\alpha \simeq 0.915 < 1$. Thus, Theorem 3.1 implies that the zero solution of (33) is asymptotic stable.

However, for the asymptotic stable of the zero solution of (33), the corresponding conditions used by the fixed point theory in Ardjouni and Djoudi [6] are

$$\lim \left| \frac{c(t)}{1 - \tau_1'(t)} \right| = \lim \left| \frac{1}{0.95} \ln \left(\frac{0.95t + 1}{4(t+1)} \right) \right| = 1.513\text{as}t \to \infty.$$

This implies that condition (11) does not hold. So it is clear that the reduction of the conservatism by our method is quite significant when compared to Ardjouni and Djoudi [6].

Remark 4.1. It is an open problem whether the zero solution of (1) is uniform asymptotically stable, perseverance, and so on.

5. Conclusion

This work is a new attempt at applying the fixed point theory to the stability analysis of neutral differential equations with variable delays, several special cases of which have been studied in [9, 14]. Some of the results, like Theorem B, is mainly dependent on the constraint

$$\left| \frac{c(t)}{1 - \tau_1'(t)} \right| < 1.$$

But in many environments, the constraint does not hold. So by employing two auxiliary continuous functions g and p to define an appropriate mapping, and present new criteria for asymptotic stability of Eq. (1) which makes stability conditions more feasible and the results in [14] are improved and generalized. From what has been discussed above, we see that Krasnoselskii's fixed point theorem is effective for not only the investigation of the existence of solution but also for the boundedness and the stability analysis of trivial equilibrium. We introduce an example to verify the applicability of the results established. In the future, we will continue to explore the application of other kinds of fixed point theorems to the stability research of fractional neutral systems with variable delays.

Additional classifications

AMS Subject Classifications: 34K20, 34K30, 34B40

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Chapter 3

Spectral Properties of a Non-Self-Adjoint Differential Operator with Block-Triangular Operator Coefficients

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Abstract

In this chapter, the Sturm-Liouville equation with block-triangular, increasing at infinity operator potential is considered. A fundamental system of solutions is constructed, one of which decreases at infinity, and the second increases. The asymptotic behavior at infinity was found out. The Green's function and the resolvent for a non-self-adjoint differential operator are constructed. This allows to obtain sufficient conditions under which the spectrum of this non-self-adjoint differential operator is real and discrete. For a non-self-adjoint Sturm-Liouville operator with a triangular matrix potential growing at infinity, an example of operator having spectral singularities is constructed.

Keywords: differential operators, spectrum, non-self-adjoint, block-triangular operator coefficients, Green's function, resolvent

1. Introduction

The question of the generalization of the oscillatory Sturm theorem for scalar equations of higher orders and for equations with matrix coefficients for a long time remained open. Only in recent joint papers by F. Rofe-Beketov and A. Kholkin (see [1]) a connection was established between spectral and oscillatory properties for self-adjoint operators generated by equations of arbitrary even order with operator coefficients and boundary conditions of general form. Later, a Sturm-type oscillation theorem was proved [2] for a problem on finite and infinite intervals for a second-order equation with block-triangular matrix coefficients. In the case of non-self-adjoint differential operators, oscillation theorems have not been considered earlier.

Results turning out in self-adjoint and non-self-adjoint cases differentiate substantially. The theory of non-self-adjoint singular differential operators, generated by scalar differential expressions, has been well studied. An overview on the theory of non-self-adjoint singular ordinary differential operators is provided in V.E. Lyantse's Appendix I to the monograph [3]. In the study of the connection between spectral and oscillation properties of non-self-adjoint differential operators with block-triangular operator coefficients [2, 4] the question arises of the structure of the spectrum of such operators. For scalar non- self-adjoint differential operators these questions were studied in the papers [5–8]. The theory of singular non-self-adjoint differential operators with matrix and operator coefficients is relatively new. In the context of the inverse scattering problem, for an operator with a triangular matrix potential decreasing at infinity, the first moment of which is bounded, the structure of the spectrum was established in [9, 10]. The theory of equations with block - triangular operator coefficients the first results were published in 2012 in the works of the author [11–13].

In this works we construct the fundamental system of solutions of differential equation with block-triangular operator potential that increases at infinity, one of that is decreasing at infinity, and the second growing. The asymptotics of the fundamental system of solutions of this equation is established. The Green's function is constructed for a non-self-adjoint system with a block-triangular potential, the diagonal blocks of which are self-adjoint operators. We obtained a resolvent for a non-self-adjoint differential operator, using which the structure of the operator spectrum is set. Sufficient conditions at which a spectrum of such non-self-adjoint differential operator is real and discrete are obtained. Here the rate of growth elements, not on the main diagonal, is subordinated to the rate of growth of the diagonal elements. In case of infringement of this condition, the operator can have spectral singularities [14].

2. The fundamental solutions for an non-self-adjoint differential operator with block – triangular operator coefficients.

Let us designate $H_k, k = \overline{1,r}$ as a finite-dimensional or infinite-dimensional separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Denote by $\mathbf{H} = H_1 \oplus H_2 \oplus \dots \oplus H_r$. Element $\overline{h} \in \mathbf{H}$ will be written in the form of $\overline{h} = \mathrm{col}(\overline{h}_1, \overline{h}_2, \dots, \overline{h}_r)$, where $\overline{h}_k \in H_k$, $k = 1, 2, \dots, r$, I_k, I- are identity operators in H_k and $\mathbf{H}$ accordingly.

We denote by $L_2(\mathbf{H}, (0,\infty))$ the Hilbert space of vector-valued functions $y(x)$ with values in $\mathbf{H}$ with inner product $\langle y, z\rangle = \int_0^\infty (y(x), z(x))\,dx$ and the norm $\|\cdot\|$.

Now let us consider the equation with block-triangular operator potential in $B(\mathbf{H})$

$$l[\overline{y}] = -\overline{y}'' + V(x)\overline{y} = \lambda\overline{y}, \quad 0 \le x < \infty, \tag{1}$$

where

$$V(x) = v(x)\cdot I + U(x), \quad U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \dots & U_{1r}(x) \\ 0 & U_{22}(x) & \dots & U_{2r}(x) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & U_{rr}(x) \end{pmatrix}, \tag{2}$$

$v(x)$ is a real scalar function such that $0 < v(x) \to \infty$ monotonically, as $x \to \infty$, and it has monotone absolutely continuous derivative. Also, $U(x)$ is a relatively small perturbation, e. g. as $x \to \infty$ $|U(x)| \cdot v^{-1}(x) \to 0$ or $|U|v^{-1} \in L^\infty(\mathbb{R}_+)$. The diagonal blocks $U_{kk}(x),\ k = \overline{1,r}$ are assumed to be bounded self-adjoint operators in H_k.

In case where

$$v(x) \ge Cx^{2\alpha}, C > 0, \alpha > 1, \tag{3}$$

we suppose that coefficients of the Eq. (1) satisfy relations:

$$\int_0^\infty |U(t)| \cdot v^{-\frac{1}{2}}(t)dt < \infty, \tag{4}$$

$$\int_0^\infty v'^2(t) \cdot v^{-\frac{5}{2}}(t)dt < \infty, \quad \int_0^\infty v''(t) \cdot v^{-\frac{3}{2}}(t)dt < \infty. \tag{5}$$

In case of $v(x) = x^{2\alpha}$, $0 < \alpha \le 1$, we suppose that the coefficients of the Eq. (1) satisfy the relation

$$\int_a^\infty |U(t)| \cdot t^{-\alpha} dt < \infty, \quad a > 0. \tag{6}$$

2.1 Construction of the fundamental system of solutions for an operator differential equation with a rapidly increasing at infinity potential

Consider first the case where $v(x) \ge Cx^{2\alpha}, C > 0, \alpha > 1$.

Condition (3) is performed, for example, quickly increasing functions e^x, $\exp\{e^x\}$ etc.

Rewrite the Eq. (1) in the form

$$-\bar{y}'' + (v(x) + q(x))\bar{y} = ((\lambda + q(x))I - U(x))\bar{y}, \tag{7}$$

where $q(x)$ determined by a formula (cf. with the monograph [15])

$$q(x) = \frac{5}{16}\left(\frac{v'(x)}{v(x)}\right)^2 - \frac{1}{4}\frac{v''(x)}{v(x)}. \tag{8}$$

Now let us denote.

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(-\int_0^x \sqrt{v(u)}du\right), \gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(\int_0^x \sqrt{v(u)}du\right). \tag{9}$$

It is easy to see that $\gamma_0(x,\lambda) \to 0, \gamma_\infty(x,\lambda) \to \infty$ as $x \to \infty$. These solutions constitute a fundamental system of solutions of the scalar differential equation

$$-z'' + (v(x) + q(x))z = 0, \tag{10}$$

in such a way that for all $x \in [0, \infty)$ one has.

$$W(\gamma_0, \gamma_\infty) := \gamma_0(x,\lambda) \cdot \gamma'_\infty(x,\lambda) - \gamma'_0(x,\lambda) \cdot \gamma_\infty(x,\lambda) = 1. \tag{11}$$

Theorem 2.1 *Under conditions (3), (4), (5) Eq. (1) has a unique decreasing at infinity operator solution $\Phi(x,\lambda) \in B(\mathbf{H})$, satisfying the conditions*

$$\lim_{x\to\infty} \frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I \quad and \quad \lim_{x\to\infty} \frac{\Phi'(x,\lambda)}{\gamma'_0(x,\lambda)} = I. \tag{12}$$

Also, there exists increasing at infinity operator solution $\Psi(x,\lambda) \in B(\mathbf{H})$ satisfying the conditions

$$\lim_{x\to\infty}\frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)}=I \ \ and \ \ \lim_{x\to\infty}\frac{\Psi'(x,\lambda)}{\gamma'_\infty(x,\lambda)}=I. \tag{13}$$

Proof

a. Eq. (7) equivalently to integral equation

$$\Phi(x,\lambda)=\gamma_0(x,\lambda)I+\int\limits_x^\infty K(x,t,\lambda)\cdot\Phi(t,\lambda)dt, \tag{14}$$

where

$$K(x,t,\lambda)=C(x,t,\lambda)\cdot[(\lambda+q(t))I-U(t)], \tag{15}$$

$$C(x,t,\lambda)=\gamma_\infty(x,\lambda)\cdot\gamma_0(t,\lambda)-\gamma_\infty(t,\lambda)\cdot\gamma_0(x,\lambda), \tag{16}$$

with $C(x,t,\lambda)$ being the Cauchy function that in each variable satisfies Eq. (10) and the initial conditions $C(x,t,\lambda)|_{x=t}=0, \quad C'_x(x,t,\lambda)|_{x=t}=1, \quad C'_t(x,t,\lambda)|_{x=t}=-1.$ Set $\chi(x,\lambda)=\frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)}$ to rewrite Eq. (14) in form

$$\chi(x,\lambda)=I+\int\limits_x^\infty R(x,t,\lambda)\chi(t,\lambda)dt, \tag{17}$$

where $R(x,t,\lambda)=K(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}$. Thus

$$\begin{aligned}\left|C(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right| &= \left|\gamma_0^2(t,\gamma)\cdot\frac{\gamma_\infty(x,\lambda)}{\gamma_0(x,\lambda)}-\gamma_0(t,\lambda)\cdot\gamma_\infty(t,\lambda)\right|=\\ &=\left|\frac{1}{2\sqrt{v(t)}}\cdot\exp\left(-2\int\limits_0^t\sqrt{v(u)}du\right)\cdot\exp\left(2\int\limits_0^x\sqrt{v(u)}du\right)-\frac{1}{2\sqrt{v(t)}}\right|=\\ &=\frac{1}{2\sqrt{v(t)}}\cdot\left|\exp\left(-2\int\limits_x^t\sqrt{v(u)}du\right)-1\right|\end{aligned} \tag{18}$$

and since with $x\le t$ one has $\exp\left(-2\int\limits_x^t\sqrt{v(u)}du\right)\le 1$, we deduce that

$$\left|C(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right|\le\frac{1}{\sqrt{v(t)}}. \tag{19}$$

Hence.

$$|R(x,t,\lambda)|=\left|C(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\gamma)}\cdot[(\lambda+q(t))I-U(t)]\right|\le\frac{1}{\sqrt{v(t)}}(|\lambda|+|q(t)|+|U(t)|). \tag{20}$$

By virtue of (3)–(5), (8),

$$\frac{1}{\sqrt{v(t)}}(|\lambda|+|q(t)|+|U(t)|)\in L(0,\infty), \tag{21}$$

and therefore integral equation has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)| \le$ const. By (17), one has that $\lim_{x\to\infty}\chi(x,\lambda) = I$, where the first part of formula (12) follows from.

Differentiable (14) to get $\frac{\Phi'(x,\lambda)}{\gamma_0'(x)} = I + \int_x^\infty S(x,t,\lambda)\chi(t,\lambda)dt$, where $S(x,t,\lambda) = K_x'(x,t,\lambda)\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)} = C_x'(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)}\cdot[(\lambda+q(t))I - U(t)]$. We have similarly (18), that $\left|C_x'(x,t)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0'(x,\lambda)}\right| \le \frac{1}{\sqrt{v(t)}}$, and therefore $|S(x,t,\lambda)| \le \frac{1}{\sqrt{v(t)}}\cdot[|\lambda| + |q(t)| + |U(t)|] \in L(0,\infty)$, where the second part of formula (12) follows from.

b. Denote by $\hat{\Psi}(x,\lambda) \in B(\mathbf{H})$ block-triangular operator solution of Eq. (1) that increases at infinity, $\Psi_{kk}(x,\lambda) \in B(H_k,H_k)$, $k=\overline{1,r}$ -its diagonal blocks. Now Eq. (7) is equivalent to the integral equation

$$\hat{\Psi}(x,\lambda) = \gamma_\infty(x,\lambda)\cdot I - \int_0^x K(x,t,\lambda)\cdot\hat{\Psi}(t,\lambda)dt, \tag{22}$$

where, just as in (14), the kernel $K(x,t,\lambda)$ is given by (15). Now set $\chi(x,\lambda) = \frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)}$ to rewrite Eq. (22) in form

$$\chi(x,\lambda) = I - \int_0^x R(x,t,\lambda)\cdot\chi(t,\lambda)dt, \tag{23}$$

where $R(x,t,\lambda) = C(x,t,\lambda)\cdot\frac{\gamma_\infty(t,\lambda)}{\gamma_\infty(x,\lambda)}\cdot[(q(t)+\lambda)\cdot I - U(t)]$. Similarly we can prove that the integral Eq. (23) has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)| \le$ const. Pass in (23) to a limit as $x\to\infty$ to get $\lim_{x\to\infty}\chi(x,\lambda) = I + \tilde{C}(\lambda)$ where $\tilde{C}(\lambda)$ is block-triangular operator in $\mathbf{H}$, that is

$$\lim_{x\to\infty}\frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x,\lambda)} = I + \tilde{C}(\lambda). \tag{24}$$

Now consider another block-triangular operator solution $\tilde{\Psi}(x,\lambda)$ that increases at infinity diagonal blocks which are defined by.

$$\tilde{\Psi}_{kk}(x,\lambda) = \Phi_{kk}(x,\lambda)\int_a^x \Phi_{kk}^{-1}(t,\lambda)\left(\Phi_{kk}^*(t,\lambda)\right)^{-1}dt, k=\overline{1,r}, (a\ge 0), \tag{25}$$

$\Phi_{kk}(x,\lambda)$ are the diagonal blocks of operator solution $\Phi(x,\lambda)$ as in Section a). In view (16) and the definition of the functions $\gamma_0(x), \gamma_\infty(x)$ can be proved that

$$\lim_{x\to\infty}\frac{\tilde{\Psi}_{kk}(x,\lambda)}{\gamma_\infty(x,\lambda)} = I_k, \quad k=\overline{1,r}. \tag{26}$$

Since $\hat{\Psi}(x,\lambda)$ and $\tilde{\Psi}(x,\lambda)$ are the operator solutions of Eq. (1) that increase at infinity,

$$\hat{\Psi}(x,\lambda) = \tilde{\Psi}(x,\lambda) + \Phi(x,\lambda)\cdot C_0(\lambda), \tag{27}$$

where $C_0(\lambda)$ is some block-triangular operator. Thus $\lim\limits_{x\to\infty}\frac{\hat{\Psi}(x,\lambda)}{\gamma_\infty(x)} = \lim\limits_{x\to\infty}\frac{\tilde{\Psi}(x,\lambda)}{\gamma_\infty(x)}$, hence, by virtue (26), $\lim\limits_{x\to\infty}\frac{\Psi_{kk}(x,\lambda)}{\gamma_\infty(x)} = I_k,\ k=\overline{1,r}$ and in (24) has

$$\tilde{C}(\lambda) = \begin{pmatrix} 0 & C_{12}(\lambda) & \dots & C_{1r}(\lambda) \\ 0 & 0 & \dots & C_{2r}(\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{28}$$

The solution $\Psi(x,\lambda)$ given by $\Psi(x,\lambda) = \hat{\Psi}(x,\lambda)\big(I+\tilde{C}(\lambda)\big)^{-1}$ is subject to first from condition (13). Use (12) to differentiate (27), then find the asymptotes of $\tilde{\Psi}'(x,\lambda)$ as $x\to\infty$ similarly to (21) to obtain the second part of formula (13). Theorem is proved. □

In this section, the fundamental system of solution is constructed for an operator differential equation with a rapidly increasing at infinity potential.

2.2 Asymptotic of the fundamental system solutions of equation with block-triangular potential

Now consider the case when $v(x) = x^{2\alpha}$, $0<\alpha\le 1$ and coefficients of Eq. (1) satisfy the condition (6). Rewrite Eq. (1) in the form

$$-\overline{y}'' + \big(x^{2\alpha} - \lambda + q(x,\lambda)\big)\overline{y} = (q(x,\lambda)\cdot I - U(x))\overline{y}, \tag{29}$$

where $q(x,\lambda)$ determined by a formula

$$q(x,\lambda) = \frac{5\alpha^2}{4}\left(\frac{x^{2\alpha-1}}{x^{2\alpha}-\lambda}\right)^2 - \frac{\alpha(2\alpha-1)x^{2\alpha-2}}{2(x^{2\alpha}-\lambda)}. \tag{30}$$

Denote

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha}-\lambda)}}\cdot\exp\left(-\int\limits_a^x\sqrt{u^{2\alpha}-\lambda}\,du\right), \tag{31}$$

$$\gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha}-\lambda)}}\cdot\exp\left(\int\limits_a^x\sqrt{u^{2\alpha}-\lambda}\,du\right). \tag{32}$$

There solutions constitute a fundamental system of solutions of the scalar differential equation $-z'' + (x^{2\alpha} - \lambda + q(x,\lambda))z = 0$, in such a way that for all $x\in[0,\infty)$ one has $W(\gamma_0,\gamma_\infty) := \gamma_0(x,\lambda)\cdot\gamma'_\infty(x,\lambda) - \gamma'_0(x,\lambda)\cdot\gamma_\infty(x,\lambda) = 1$.

We are about to establish the asymptotics[1] of $\gamma_0(x,\lambda)$ as $x\to\infty$:

$$\gamma_0(x,\lambda) = (2x^\alpha)^{-\frac{1}{2}}\cdot\left(1-\frac{\lambda}{x^{2\alpha}}\right)^{-\frac{1}{4}}\cdot\exp\left(-\int\limits_a^x u^\alpha\left(1-\frac{\lambda}{u^{2\alpha}}\right)^{\frac{1}{2}}du\right). \tag{33}$$

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[1] For $\alpha=1$ and $\alpha=\frac{1}{2}$, i.e., for $v(x)=x^2$ and $v(x)=x$, the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ is known.

After expanding here the integral, we obtain the exponential as follows

$$\exp\left(-\int_a^x u^\alpha \cdot \left(1-\frac{1}{2}\cdot\frac{\lambda}{u^{2\alpha}}-\sum_{k=2}^{\infty}\frac{1\cdot 3\cdot\ \dots\ \cdot(2k-3)}{k!\cdot 2^k}\cdot\left(\frac{\lambda}{u^{2\alpha}}\right)^k\right)du\right). \tag{34}$$

In case $\frac{\alpha+1}{2\alpha}=n\in N$, i.e. $\alpha=\frac{1}{2n-1}$, this expression after integration acquires the form:

$$c\cdot\ \exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \dots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\cdot\ \exp\left(\frac{1\cdot 3\cdot\ \dots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n\cdot\ \ln x+o(1)\right)=$$

$$=c\cdot\ \exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \dots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot$$

$$\cdot\, x^{\frac{1\cdot 3\cdot\ \dots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n}\cdot(1+o(1)). \tag{35}$$

The asymptotics of $\gamma_0(x,\lambda)$ as $x\to\infty$ is as follows:

$$\gamma_0(x,\lambda)=c\cdot\ \exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \dots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$
$$\cdot x^{\frac{1\cdot 3\cdot\ \dots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n-\frac{\alpha}{2}}\cdot(1+o(1)). \tag{36}$$

In particular, for $\alpha=1\ \ (n=1)$, $\gamma_0(x,\lambda)$ has the following asymptotics at infinity:

$$\gamma_0(x,\lambda)=c\cdot x^{\frac{\lambda-1}{2}}\cdot\ \exp\left(-\frac{x^2}{2}\right)(1+o(1)). \tag{37}$$

In case $\frac{\alpha+1}{2\alpha}\notin N$ we set $n=\left[\frac{\alpha+1}{2\alpha}\right]+1$, with $[\beta]$ being the integral part of β, to obtain the following asymptotics for $\gamma_0(x,\lambda)$ at infinity:

$$\gamma_0(x,\lambda)=c\cdot x^{-\frac{\alpha}{2}}\exp\left(-\frac{x^{1+\alpha}}{1+\alpha}+\frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha}+\sum_{k=2}^{n-1}\frac{1\cdot 3\cdot\ \dots\ \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot$$
$$\cdot\exp\left(-\frac{1\cdot 3\cdot\ \dots\ \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n\cdot\frac{x^{-\alpha}}{\alpha}\right)\cdot(1+o(x^{-\alpha})) \tag{38}$$

In particular, with $\alpha=\frac{1}{2}\ \ (n=2)$ one has

$$\gamma_0(x,\lambda)=cx^{-\frac{1}{4}}\cdot\ \exp\left(-\frac{2}{3}x^{\frac{3}{2}}+\lambda x^{\frac{1}{2}}-\left(\frac{\lambda}{2}\right)^2x^{-\frac{1}{2}}\right)\cdot\left(1+o\left(x^{-\frac{1}{2}}\right)\right). \tag{39}$$

A similar procedure allows to establish the asymptotics of $\gamma_\infty(x)$ as $x\to\infty$. If $\frac{\alpha+1}{2\alpha}=n\in N$, i.e. $\alpha=\frac{1}{2n-1}$, then

$$\gamma_\infty(x,\lambda) = c\cdot \exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha} - \sum_{k=2}^{n-1}\frac{1\cdot 3\cdot \dots \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot$$
$$\cdot x^{-\left(\frac{1\cdot 3\cdot \dots \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n + \frac{\alpha}{2}\right)}\cdot(1+o(1)). \tag{40}$$

With $\alpha = 1$ $(n = 1)$, this becomes

$$\gamma_\infty(x,\lambda) = c\cdot x^{-\frac{\lambda+1}{2}}\cdot \exp\left(\frac{x^2}{2}\right)(1+o(1)). \tag{41}$$

In case $\frac{\alpha+1}{2\alpha} \notin N$, we set $n = \left[\frac{\alpha+1}{2\alpha}\right] + 1$ to get the asymptotics.

$$\gamma_\infty(x,\lambda) = c\cdot x^{-\frac{\alpha}{2}}\exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1}\frac{1\cdot 3\cdot \dots \cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)\cdot$$
$$\cdot\exp\left(\frac{1\cdot 3\cdot \dots \cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n\cdot\frac{x^{-\alpha}}{\alpha}\right)\cdot(1+o(x^{-\alpha})). \tag{42}$$

In case $\alpha = \frac{1}{2}$ $(n = 2)$, one has

$$\gamma_\infty(x,\lambda) = cx^{-\frac{1}{4}}\cdot \exp\left(\frac{2}{3}x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}} + \left(\frac{\lambda}{2}\right)^2 x^{-\frac{1}{2}}\right)\cdot\left(1+o\left(x^{-\frac{1}{2}}\right)\right). \tag{43}$$

Theorem 2.2 *Under* $0<\alpha\le 1$ *and condition (6), the statement of Theorem 2.1 is also valid for Eq. (1).*

Proof is similar to Theorem 2.1. Moreover, note that

$$\left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right| = \left|\gamma_0^2(t,\lambda)\cdot\frac{\gamma_\infty(x,\lambda)}{\gamma_0(x,\lambda)} - \gamma_0(t,\lambda)\cdot\gamma_\infty(t,\lambda)\right| =$$
$$= \left|\frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\cdot \exp\left(-2\int_a^t\sqrt{u^{2\alpha}-\lambda}du\right)\cdot\exp\left(2\int_a^x\sqrt{u^{2\alpha}-\lambda}du\right) - \frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\right|$$
$$= \frac{1}{2\sqrt{t^{2\alpha}-\lambda}}\cdot\left|\exp\left(-2\int_x^t\sqrt{u^{2\alpha}-\lambda}du - 1\right)\right|. \tag{44}$$

As $x\le t$, one has $\exp\left(-2\int_x^t\sqrt{u^{2\alpha}-\lambda}du\right)\le 1$, and that is why

$$\left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right| \le \frac{1}{\sqrt{t^{2\alpha}-\lambda}}. \tag{45}$$

Hence

$$|R(x,t,\lambda)| = \left|C(x,t,\lambda)\cdot\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\cdot[q(t,\lambda)\cdot I - U(t)]\right| \le \frac{1}{\sqrt{t^{2\alpha}-\lambda}}(|q(t,\lambda)| + |U(t)|). \tag{46}$$

By virtue of (6) and (30), $\frac{1}{\sqrt{t^{2\alpha}-\lambda}}(|q(t,\lambda)| + |U(t)|)\in L(a,\infty)$ and therefore integral equation has a unique solution $\chi(x,\lambda)$ and $|\chi(x,\lambda)|\le$ const. By (17), one has that $\lim_{x\to\infty}\chi(x,\lambda) = I$, where the first part of formula (12) follows from.

The remaining statements of Theorem 2.1 are proved similarly. □

From Theorem 2.2 and the asymptotic formulas (37), (39), (41), (43) follows.

Corollary 2.1 *If $\alpha = 1$, i.e. $v(x) = x^2$, then, under condition (6), the solutions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ have common (known) asymptotics, as in the quality $\gamma_0(x, \lambda)$ and $\gamma_\infty(x, \lambda)$ you can take the following functions.*

$$\gamma_0(x, \lambda) = x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right), ..\gamma_\infty(x, \lambda) = x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right). \tag{47}$$

If $\alpha = \frac{1}{2}$, i.e. the coefficient $v(x) = x$, and the condition (6) holds, then.

$$\gamma_0(x, \lambda) = x^{-\frac{1}{4}} \cdot \exp\left(-\frac{2}{3}x^{\frac{3}{2}} + \lambda x^{\frac{1}{2}}\right), \gamma_\infty(x, \lambda) = x^{-\frac{1}{4}} \cdot \exp\left(\frac{2}{3}x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}}\right). \tag{48}$$

Remark 2.1 It is known that scalar equation

$$-\varphi'' + x^2 \cdot \varphi = \lambda\varphi \tag{49}$$

for $\lambda = 2n + 1$ has the solution $\varphi_n(x) = H_n(x) \cdot exp\left(-\frac{x^2}{2}\right)$, where $H_n(x)$ is the Chebyshev – Hermitre polynomial, that at $x \to \infty$ has next asymptotics $H_n(x) = (2x)^n(1 + o(1))$. Hence the solution $\varphi_n(x)$ of the Eq. (49) at $x \to \infty$ will have the following asymptotics at infinity: $\varphi_n(x) = (2x)^n \cdot exp\left(-\frac{x^2}{2}\right) \cdot (1 + o(1))$.

In the case of $U(x) = 0, v(x) = x^2$ in (2), the Eq. (1) is splitting into infinity system scalar equations of the form (49). The operator solution $\Phi(x, \lambda)$ will be diagonal in this case. Denote by $\varphi(x, \lambda)$ the diagonal elements of the operator $\Phi(x, \lambda)$. Then, by Corollary 2.1, the solution $\varphi(x, \lambda)$ will have the following asymptotics at infinity: $\varphi(x, \lambda) = (x)^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)(1 + o(1))$. In particular, for $\lambda = 2n + 1$, this yields the solution proportional to $\varphi_n(x)$.

In this section, the asymptotics of the fundamental system of solutions for the Sturm-Liouville equation with block-triangular operator potential, increasing at infinity is established. One of the solutions is found decreasing at infinity, the other one increasing.

3. Green's function for an operator differential equation with block – triangular coefficients

Let us suppose that at the $x = 0$ given boundary conditions

$$\cos A \cdot \bar{y}'(0) - \sin A \cdot \bar{y}(0) = 0, \tag{50}$$

where A- the block-triangular operator of the same structure as the coefficients of the differential equation, $A_{kk},\ k = \overline{1, r}$- bounded self-adjoint operators in H_k, which satisfy the conditions

$$-\frac{\pi}{2}I_k < < A_{kk} \leq \frac{\pi}{2}I_k. \tag{51}$$

Together with the problem (1), (50) we consider the separated system

$$l_k\left[y_k\right] = -y_k'' + (v(x)I_k + U_{kk}(x))y_k = \lambda y_k,\ \ k = \overline{1, r} \tag{52}$$

with the boundary conditions

$$\cos A_{kk} \cdot y_k{}'(0) - \sin A_{kk} \cdot y_k(0) = 0, \quad k = \overline{1, r}. \tag{53}$$

Let L' denote the minimal differential operator generated by differential expression $l[\overline{y}]$ (1) and the boundary condition (50), and let $L_k{}'$, $k = \overline{1, r}$ denote the minimal differential operator on $L_2(H_k, (0, \infty))$ generated by differential expression $l_k\left[y_k\right]$ and the boundary conditions (53). Taking into account the conditions on coefficients, as well as sufficient smallness of perturbations $U_{kk}(x)$ and conditions (51), we conclude that, for every symmetric operator $L_k{}'$, there is a case of limit point at infinity. Hence their self-adjoint extensions L_k are the closures of operators $L_k{}'$ respectively. The operators L_k are semi-bounded below, and their spectra are discrete.

Let L denote the operator extensions L', by requiring that $L_2(\mathbf{H}, (0, \infty))$ be the domain of operator L.

The following theorem is proved in [4].

Theorem 3.1 *Suppose that, for Eq. (1) conditions (3)-(5) are satisfied for* $\alpha > 1$ *or condition (6) for* $0 < \alpha \leq 1$. *Then the discrete spectrum of the operator L is real and coincides with the union of spectra of the self-adjoint operators* L_k, $k = \overline{1, r}$, *i.e.,* $\sigma_d(L) = \cup_{k=1}^{r} \sigma(L_k)$.

Comment 3.1 Note that this theorem contains a statement of the discrete spectrum of the non-self-adjoint operator L only and no allegations of its continuous and residual spectrum.

Along with the Eq. (1) we consider the equation

$$l_1[\overline{y}] = -\overline{y}'' + V^*(x)\overline{y} = \lambda \overline{y} \tag{54}$$

($V^*(x)$ is adjoint to the operator $V(x)$). If the space $\mathbf{H}$ is finite-dimensional, then the Eq. (54) can be rewritten as

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda \tilde{y}, \tag{55}$$

where $\tilde{y} = \left(\tilde{y}_1\ \tilde{y}_2 \ldots \tilde{y}_r\right)$ and the equation is called the left.

For operator -functions $Y(x, \lambda), Z(x, \lambda) \in B(\mathbf{H})$ let

$$W\{Z^*, Y\} = Z^{*\prime}\left(x, \overline{\lambda}\right) Y(x, \lambda) - Z^*\left(x, \overline{\lambda}\right) Y'(x, \lambda). \tag{56}$$

If $Y(x, \lambda)$ - operator solution of the Eq. (1), and $Z(x, \lambda)$ - operator solution of Eq. (54), the Wronskian does not depend on x.

Now we denote $Y(x, \lambda)$ and $Y_1(x, \lambda)$ the solutions of the Eqs. (1) and (54), respectively, satisfying the initial conditions

$$Y(0, \lambda) = \cos A, Y'(0, \lambda) = \sin A, Y_1(0, \lambda) = (\cos A)^*, Y_1{}'(0, \lambda) = (\sin A)^*, \ \lambda \in \mathbb{C}. \tag{57}$$

Because the operator function $Y_1^*\left(x, \overline{\lambda}\right)$ satisfies equation

$$-Y_1^{*\prime\prime}\left(x, \overline{\lambda}\right) + Y_1^*\left(x, \overline{\lambda}\right) \cdot V(x) = \lambda Y_1^*\left(x, \overline{\lambda}\right), \tag{58}$$

the operator function $\tilde{Y}(x, \lambda) =: Y_1^*\left(x, \overline{\lambda}\right)$ is a solution to the left of the equation

$$-\tilde{Y}''(x, \lambda) + \tilde{Y}(x, \lambda) \cdot V(x) = \lambda \tilde{Y}(x, \lambda) \tag{59}$$

and satisfies the initial conditions $\tilde{Y}(0, \lambda) = \cos A$, $\tilde{Y}'(0, \lambda) = \sin A$, $\lambda \in \mathbb{C}$.

Operator solutions of Eq. (54) decreasing and increasing at infinity will be denoted by $\Phi_1(x,\lambda)$, $\Psi_1(x,\lambda)$, and the corresponding solutions of the Eq. (59) denote by $\tilde{\Phi}(x,\lambda)$ and $\tilde{\Psi}(x,\lambda)$. For the system operator solutions $Y(x,\lambda), \tilde{\Phi}(x,\lambda)\in B(\mathbf{H})$ of the Eqs. (1) and (59), respectively, will take the form of Wronskian $W\{\tilde{\Phi}, Y\} = \tilde{\Phi}'(x,\lambda)Y(x,\lambda) - \tilde{\Phi}(x,\lambda)Y'(x,\lambda)$ and do not depend on x.

Let us designate

$$G(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(t,\lambda) & 0\le x\le t \\ -\Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(t,\lambda) & x\ge t \end{cases}. \tag{60}$$

In the following theorem it is proved that the operator function $G(x,t,\lambda)$ possesses all the classical properties of the Green's function.

Theorem 3.2 *The operator function $G(x,t,\lambda)$ is the Green's function of the differential operator L, i.e.:*

1. *The function $G(x,t,\lambda)$ is continuous for all values $x,t\in[0,\infty)$;*

2. *For any fixed t, the function $G(x,t,\lambda)$ has a continuous derivative with respect to x on each of the intervals $[0,t)$ and (t,∞), and at $x=t$ it has the jump*

$$G_x{}'(x+0,x,\lambda) - G_x{}'(x-0,x,\lambda) = -I. \tag{61}$$

3. *For a fixed t, the function $G(x,t,\lambda)$ of the variable x is an operator solution of Eq. (1) on each of the intervals $[0,t)$, (t,∞), and it satisfies the boundary condition (50), and at a fixed x function $G(x,t,\lambda)$ of the variable t is an operator solution of the Eq. (59) on each of the intervals $[0,x)$, (x,∞), and it satisfies the boundary condition $\tilde{y}'(0)\cdot\cos A - \tilde{y}(0)\cdot\sin A = 0$.*

Proof The function $G(x,t,\lambda)$ is continuous with respect to x at each of the intervals $[0,t)$ and (t,∞). Similarly to the variable t. To prove the continuity of the function $G(x,t,\lambda)$ for all $x,t\ge 0$, it is sufficient that the identity shown as

$$Y(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(x,\lambda) + \Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(x,\lambda)\equiv 0. \tag{62}$$

is satisfied for all $x\ge 0$. This identity shown as

$$Y(x,\lambda)\left(\tilde{\Phi}(x,\lambda)Y'(x,\lambda) - \tilde{\Phi}'(x,\lambda)Y(x,\lambda)\right)^{-1}\tilde{\Phi}(x,\lambda) -$$

$$-\Phi(x,\lambda)\left(\tilde{Y}'(x,\lambda)\Phi(x,\lambda) - \tilde{Y}(x,\lambda)\Phi'(x,\lambda)\right)^{-1}\tilde{Y}(x,\lambda)\equiv 0 \tag{63}$$

or

$$\left(Y'(x,\lambda)Y^{-1}(x,\lambda) - \tilde{\Phi}^{-1}(x,\lambda)\tilde{\Phi}'(x,\lambda)\right)^{-1} \equiv \left(\tilde{Y}^{-1}(x,\lambda)\tilde{Y}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda)\right)^{-1},$$

$$Y'(x,\lambda)Y^{-1}(x,\lambda) - \tilde{\Phi}^{-1}(x,\lambda)\tilde{\Phi}'(x,\lambda) \equiv \tilde{Y}^{-1}(x,\lambda)\tilde{Y}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda), \tag{64}$$

which is equivalent to

$$Y'(x,\lambda)Y^{-1}(x,\lambda) - \tilde{Y}^{-1}(x,\lambda)\tilde{Y}'(x,\lambda) \equiv \tilde{\Phi}^{-1}(x,\lambda)\tilde{\Phi}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda) \tag{65}$$

or to.

$$\tilde{Y}^{-1}(x,\lambda)\left(\tilde{Y}(x,\lambda)Y'(x,\lambda)-\tilde{Y}'(x,\lambda)Y(x,\lambda)\right)Y^{-1}(x,\lambda)\equiv$$
$$\equiv-\tilde{\Phi}^{-1}(x,\lambda)\left(\tilde{\Phi}(x,\lambda)\Phi'(x,\lambda)-\tilde{\Phi}'(x,\lambda)\Phi^{-1}(x,\lambda)\right)\Phi^{-1}(x,\lambda). \quad (66)$$

This follows from the fact that $W\{\tilde{Y},Y\}=W\{\tilde{\Phi},\Phi\}=0$.

To make sure that the jump in the first derivative at $t=x$ is equal to $(-I)$, i.e., that the equality (61) holds, it is sufficient to prove the identity

$$Y'(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(x,\lambda)+\Phi'(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}\,(x,\lambda)\equiv I. \quad (67)$$

Now we consider the function

$$C(x,t,\lambda)=Y(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(t,\lambda)+\Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}\,(t,\lambda), \quad (68)$$

which is an analogue of the Cauchy function. This function is the solution of Eq. (1) of the variable x, and it is the solution of Eq. (59) of the variable t. By (62), we have $C(x,x,\lambda)\equiv 0$. But in this case $C_{xx}''|_{t=x}=(V(x)-\lambda I)C|_{t=x}\equiv 0$, and, therefore, $C_x'(x,t,\lambda)|_{t=x}\equiv\Omega_1(\lambda)$, i.e.,

$$Y'(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(x,\lambda)+\Phi'(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}\,(x,\lambda)\equiv\Omega_1(\lambda). \quad (69)$$

It shows that $\Omega_1(\lambda)=I$, we obtain (61).

Since operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ form a fundamental system of solutions of Eq. (1), the operator solution $Y(x,\lambda)$ of Eq. (1) satisfying the initial conditions (57), can be written as $Y(x,\lambda)=\Phi(x,\lambda)A(\lambda)+\Psi(x,\lambda)B(\lambda)$, where $A(\lambda)=-W\{\tilde{\Psi},Y\},B(\lambda)=W\{\tilde{\Phi},Y\}$,

$$Y(x,\lambda)=\Psi(x,\lambda)W\{\tilde{\Phi},Y\}-\Phi(x,\lambda)W\{\tilde{\Psi},Y\}. \quad (70)$$

Similarly, operator solution $\tilde{Y}(x,\lambda)$ of Eq. (59) can be represented in the form

$$\tilde{Y}(x,\lambda)=\tilde{W}\{\tilde{\Phi},Y\}\tilde{\Psi}(x,\lambda)-\tilde{W}\{\tilde{\Psi},Y\}\tilde{\Phi}(x,\lambda), \quad (71)$$

where

$$\tilde{W}\{\tilde{\Phi},Y\}=\sin A\cdot\Phi(0,\lambda)-\cos A\cdot\Phi'(0,\lambda)=-\Omega(0,\lambda)=-W\{\tilde{Y},\Phi\}. \quad (72)$$

Similarly we get $\tilde{W}\{\tilde{\Psi},Y\}=-W\{\tilde{Y},\Psi\}$. Thus,

$$\tilde{Y}(x,\lambda)=W\{\tilde{Y},\Psi\}\tilde{\Phi}(x,\lambda)-W\{\tilde{Y},\Phi\}\tilde{\Psi}(x,\lambda). \quad (73)$$

Substituting (70) and (73) into the formula (69), using the fact that the equality (69) is performed on x identically, we obtain

$$\Omega_1(\lambda)=\lim_{x\to\infty}\left[\Psi'(x,\lambda)\tilde{\Phi}(x,\lambda)-\Phi'(x,\lambda)\tilde{\Psi}(x,\lambda)\right]. \quad (74)$$

By Theorem 2.1, on the asymptotic behavior of functions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ at infinity, we have

$$\Omega_1(\lambda) = \lim_{x\to\infty} [\gamma_0(x,\lambda)\gamma_\infty{}'(x,\lambda) - \gamma_0{}'(x,\lambda)\gamma_\infty(x,\lambda)] \cdot I = W\{\gamma_0, \gamma_\infty\} \cdot I = I. \quad (75)$$

This completes the proof of the formula (61), and with it the theorem 3.1. □

Corollary. *By the definition (60), function $G(x,t,\lambda)$ is meromorphic of the parameter λ with the poles coincide with the eigenvalues of the operator L.*

We constructed Green's function for the non-self-adjoint differential operator.

4. Resolvent for an non-self-adjoint operator differential equation with block – triangular coefficients

We consider the operator R_λ defined in $L_2(\mathbf{H},(0,\infty))$ by the relation

$$\left(R_\lambda \bar{f}\right)(x) = \int_0^\infty G(x,t,\lambda)\bar{f}(t)dt = \quad (76)$$

$$= -\int_0^x \Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(t,\lambda)\bar{f}(t)\,dt + \int_x^\infty Y(x,\lambda)\left(W\{\tilde{\Phi},Y\}\right)^{-1}\tilde{\Phi}(t,\lambda)\bar{f}(t)dt.$$

Theorem 4.1 *The operator R_λ is the resolvent of the operator$L$.*

Proof One can directly verify that, for any function $\bar{f}(x) \in L_2(\mathbf{H},(0,\infty))$, the vector-function $\bar{y}(x,\lambda) = \left(R_\lambda \bar{f}\right)(x)$ is a solution of the equation $l[\bar{y}] - \lambda\bar{y} = \bar{f}$ whenever $\lambda \notin \sigma(L)$. We will prove that $\bar{y}(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$.

Since operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ form a fundamental system of solutions of Eq. (1), the operator solution $Y(x,\lambda)$ of Eq. (1) satisfying the initial conditions (57), can be written as $Y(x,\lambda) = \Phi(x,\lambda)A(\lambda) + \Psi(x,\lambda)B(\lambda)$, where $A(\lambda) = W\{\tilde{\Psi},Y\}, B(\lambda) = -W\{\tilde{\Phi},Y\}$,

$$Y(x,\lambda) = \Phi(x,\lambda)W\{\tilde{\Psi},Y\} - \Psi(x,\lambda)W\{\tilde{\Phi},Y\}. \quad (77)$$

Similarly, the operator solution $\tilde{Y}(x,\lambda)$ of Eq. (59) can be represented in the following form

$$\tilde{Y}(x,\lambda) = W\{\tilde{Y},\Phi\}\tilde{\Psi}(x,\lambda) - W\{\tilde{Y},\Psi\}\tilde{\Phi}(x,\lambda). \quad (78)$$

By using formulas (77) and (78), we can rewrite the relation (76) as follows:

$$(R_\lambda f)(x) = -\int_0^a \Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}\tilde{Y}(t,\lambda)f(t)\,dt + \bar{\chi}_1(x,\lambda) - \bar{\chi}_2(x,\lambda) + \bar{\chi}_3(x,\lambda) - \bar{\chi}_4(x,\lambda), \quad (79)$$

where $a > 0$ and

$$\bar{\chi}_1(x,\lambda) = \Phi(x,\lambda)\left(W\{\tilde{Y},\Phi\}\right)^{-1}W\{\tilde{Y},\Psi\}\int_a^x \tilde{\Phi}(t,\lambda)\bar{f}(t)\,dt, \quad (80)$$

$$\bar{\chi}_2(x,\lambda) = \Phi(x,\lambda)\int_a^x \tilde{\Psi}(t,\lambda)\bar{f}(t)\,dt\ , \quad (81)$$

$$\overline{\chi}_3(x,\lambda) = \Phi(x,\lambda)\,W\{\tilde{\Psi},Y\}\left(W\{\tilde{\Phi},Y\}\right)^{-1}\int_x^{\infty}\tilde{\Phi}\,(t,\lambda)\overline{f}(t)\,dt, \tag{82}$$

$$\overline{\chi}_4(x,\lambda) = \Psi(x,\lambda)\int_x^{\infty}\tilde{\Phi}\,(t,\lambda)\overline{f}(t)\,dt. \tag{83}$$

Let us show that each of these vector-functions $\overline{\chi}_1(x,\lambda), \overline{\chi}_2(x,\lambda), \overline{\chi}_3(x,\lambda), \overline{\chi}_4(x,\lambda)$ belongs to $L_2(\mathbf{H},(0,\infty))$. Since the operator solution $\Phi(x,\lambda)$ decays fairly quickly as $x \to \infty$, then $|\Phi(x,\lambda)| \in L_2(0,\infty)$. It follows that

$$|\overline{\chi}_1(x,\lambda)| \le c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\int_a^x\left|\tilde{\Phi}(t,\lambda)\right|\cdot\left|\overline{f}(t)\right|dt \le$$

$$\le c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\left(\int_a^x\left|\tilde{\Phi}(t,\lambda)\right|dt\right)^{\frac{1}{2}}\cdot\left(\int_a^x\left|\overline{f}(t)\right|dt\right)^{\frac{1}{2}} <$$

$$< c(\lambda)\cdot|\Phi(x,\lambda)|\cdot\left(\int_a^{\infty}\left|\tilde{\Phi}(t,\lambda)\right|dt\right)^{\frac{1}{2}}\cdot\left(\int_a^{\infty}\left|\overline{f}(t)\right|dt\right)^{\frac{1}{2}} \le c_1(\lambda)\cdot|\Phi(x,\lambda)|, \tag{84}$$

and therefore $\overline{\chi}_1(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. Similarly we get that $\overline{\chi}_3(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. First we prove the assertion for the function $\overline{\chi}_2(x,\lambda)$, when $\alpha > 1$ and the coefficients of the Eq. (1) satisfy the conditions (3)-(5). In this case, we have $|\overline{\chi}_2(x,\lambda)| \le |\Phi(x,\lambda)|\int_a^x\left|\tilde{\Psi}\,(t,\lambda)\right|\left|\overline{f}(t)\right|dt$.

By virtue of the asymptotic formulas for the operator solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ we obtain that

$$|\overline{\chi}_2(x,\lambda)| \le c_1(\lambda)\gamma_0(x,\lambda)\int_a^x\gamma_\infty(t,\lambda)\left|\overline{f}(t)\right|dt. \tag{85}$$

Let us rewrite this relation in the following form

$$|\overline{\chi}_2(x,\lambda)| \le c_1(\lambda)\gamma_0(x,\lambda)\gamma_\infty(x,\lambda)\int_a^x\frac{\gamma_\infty(t,\lambda)}{\gamma_\infty(x,\lambda)}\left|\overline{f}(t)\right|dt. \tag{86}$$

By using the definition of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ (see (9)) and by applying the Cauchy-Bunyakovskii inequality we obtain

$$|\overline{\chi}_2(x,\lambda)| \le \frac{1}{2}c_1(\lambda)\frac{1}{\sqrt{v(x)}}\left(\int_a^x\sqrt{\frac{v(x)}{v(t)}}\exp\left(-2\int_t^x\sqrt{v(u)}du\right)dt\right)^{\frac{1}{2}}\left(\int_0^{\infty}\left|\overline{f}(t)\right|^2dt\right)^{\frac{1}{2}}. \tag{87}$$

Since $t \le x$, we get $\exp\left(-2\int_t^x\sqrt{v(u)du}\right) \le 1$, and then the latter estimate for $\chi_2(x,\lambda)$ can be rewritten as follows

$$|\overline{\chi}_2(x,\lambda)| \le c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^x \frac{1}{\sqrt{v(t)}}dt\right)^{\frac{1}{2}} \le c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^\infty \frac{1}{\sqrt{v(t)}}dt\right)^{\frac{1}{2}}. \tag{88}$$

By formula (3), we get $|\overline{\chi}_2(x,\lambda)| \le \frac{c_3(\lambda)}{\sqrt[4]{v(x)}}$, and hence, if $\alpha > 1$ and the coefficients of the Eq. (1) satisfy the conditions (3)-(5), we have $\overline{\chi}_2(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. In the case of $v(x) = x^{2\alpha}, 0 < \alpha \le 1$, the assertion can be proved similarly.

For the function $\overline{\chi}_4(x,\lambda)$ we will conduct the proof for the case when $v(x) = x^{2\alpha}, 0 < \alpha \le 1$ and the coefficients of the Eq. (1) satisfy the condition (6). As in (85) we have $|\overline{\chi}_4(x,\lambda)| \le c_1(\lambda)\gamma_\infty(x,\lambda)\int_x^\infty \gamma_0(t,\lambda)\left|\overline{f}(t)\right|dt$, which can be rewritten as follows $|\overline{\chi}_4(x,\lambda)| \le c_1(\lambda)\gamma_0(x,\lambda)\gamma_\infty(x,\lambda)\int_x^\infty \frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|\overline{f}(t)\right|dt$.

Let us use the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$, for example, in the case $\frac{\alpha+1}{2\alpha} = n \in N$, i.e. $\alpha = \frac{1}{2n-1}$ (see (36) and (40)). Setting $a(\alpha,\lambda) = \frac{1\cdot 3\cdot \dots \cdot (2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n$, we obtain

$$|\overline{\chi}_4(x,\lambda)| \le c_2(\lambda)x^{-\alpha}\int_x^\infty \frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|\overline{f}(t)\right|dt \le c_2(\lambda)x^{-\alpha}\left(\int_a^x \left(\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right)^2 dt\right)^{\frac{1}{2}}\left(\int_0^\infty \left|\overline{f}(t)\right|^2 dt\right)^{\frac{1}{2}}, \tag{89}$$

$$|\overline{\chi}_4(x,\lambda)| \le c_3(\lambda)x^{-\alpha}\left(\int_x^\infty \left(\frac{t}{x}\right)^{2a(\alpha,\lambda)-\alpha}\exp\frac{-2x^{\alpha+1}\left(\left(\frac{t}{x}\right)^{\alpha+1}-1\right)}{1+\alpha}dt\right)^{\frac{1}{2}}. \tag{90}$$

Replacing variables $t = xu$, we get

$$|\overline{\chi}_4(x,\lambda)| \le c_3(\lambda)x^{-\alpha+\frac{1}{2}}\left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha}\exp\frac{-2x^{\alpha+1}(u^{\alpha+1}-1)}{1+\alpha}du\right)^{\frac{1}{2}}. \tag{91}$$

Since the inequality $\exp\frac{-x^{\alpha+1}\left(u^{\alpha+1}-1\right)}{1+\alpha} \le x^{-2}$ holds for all $\alpha \in (0,1]$ and $u \in [1,\infty)$ with sufficiently large $u \in [1,\infty)$, we have

$$|\overline{\chi}_4(x,\lambda)| \le c_3(\lambda)x^{-\alpha-\frac{1}{2}}\left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha}\exp\frac{-x^{\alpha+1}(u^{\alpha+1}-1)}{1+\alpha}du\right)^{\frac{1}{2}}. \tag{92}$$

Hence it follows that $|\overline{\chi}_4(x,\lambda)| \le c_4(\alpha,\lambda)x^{-\alpha-\frac{1}{2}}$, therefore $\overline{\chi}_4(x,\lambda) \in L_2(\mathbf{H},(0,\infty))$. In case, where $0 < \alpha \le 1$ and $\frac{\alpha+1}{2\alpha} \notin N$ and where $\alpha > 1$, the proof is similar.

Thus, $R_\lambda\overline{f} \in L_2(\mathbf{H},(0,\infty))$ for any function $\overline{f} \in L_2(\mathbf{H},(0,\infty))$. □

Since the resolvent R_λ is a meromorphic function of λ, the poles of which coincide with the eigenvalues of the operator L, the statement of Theorem 3.1 can be refined.

Theorem 4.2 *If the conditions (3)-(5)) where*$\alpha > 1$ *or condition (6) where* $0 < \alpha \le 1$ *are satisfied for the Eq. (1), then the spectrum of the operator L is real, discrete and coincides with the union of spectra of self-adjoint operators* L_k, $k = \overline{1,m}$, *i.e.* $\sigma(L) = \cup_{k=1}^r \sigma(L_k)$.

In this section, a resolvent for a non-self-adjoint differential operator with a block-triangular operator potential, increasing at infinity, is constructed. Sufficient conditions under which the spectrum is real and discrete are obtained.

5. Spectral singularities of differential operator with triangular matrix coefficients

Remark 5.1 *If the perturbation $U(x)$ in Eq. (1) does not satisfy conditions (3)-(5) or condition (6), then the statement of Theorem 4.2 ceases to be true, which is shown by the following example.*

Example 5.1 Consider the equation:

$$l[\bar{y}] = -\bar{y}'' + \begin{pmatrix} x^2 & q(x) \\ 0 & \pi^2 x^2 \end{pmatrix}\bar{y} = \lambda\bar{y}, \quad 0 \le x < \infty, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{93}$$

with the boundary condition

$$\bar{y}(0) = 0. \tag{94}$$

Together with the problem (93), (94), consider the separated system

$$l_1[y_1] = -y_1'' + x^2 y_1 = \lambda y_1, \tag{95}$$

$$l_2[y_2] = -y_2'' + \pi^2 x^2 y_2 = \lambda y_2 \tag{96}$$

with the boundary conditions.

$$y_1(0) = 0, y_2(0) = 0. \tag{97}$$

As above, denote by L_0 the differential operator generated by the differential expression $l[\bar{y}]$ (93) and the boundary condition (94), and by L_1, L_2 denote the minimal symmetric operators on $L_2(0;\infty)$, generated by the differential expressions $l_1[y_1], l_2[y_2]$ and the boundary conditions (97). Their self-adjoint extensions $\tilde{L_1}, \tilde{L_2}$ are the closures of the operators L_1, L_2, respectively. The operators $\tilde{L_1}, \tilde{L_2}$ are semi-bounded; let us denote their spectra by $\sigma_1 = \sigma(\tilde{L_1}), \quad \sigma_2 = \sigma(\tilde{L_2})$.

The Eq. (95) (cf. (49)) has the solution $y_{1,n}(x) = H_n(x) \cdot exp\left(-\frac{x^2}{2}\right)$ for $\lambda = 2n+1$. Since $H_{2n+1}(0) = 0$, the eigenvalues of the operator $\tilde{L_1}$ are $\lambda_n = 4n+3$. The sets σ_1 and σ_2 do not intersect.

Denote by L the extension of the operator L_0 generated by the requirement on the functions from the domain of the operator L to belong to $L_2(H_2, (0;\infty))$, and by $\sigma(L)$ its spectrum.

Denote by $Y(x,\lambda) = \begin{pmatrix} y_{11}(x,\lambda) & y_{12}(x,\lambda) \\ 0 & y_{22}(x,\lambda) \end{pmatrix}$ the matrix solution of the Eq. (93), satisfying the initial conditions $Y(0,\lambda) = 0, Y'(0,\lambda) = I$.

If some $\lambda_0 \in \sigma(\tilde{L}_1)$, and $y(x,\lambda_0)$- is the corresponding eigenfunction of the operator $\tilde{L}_1$, then the vector function $\bar{y}(x,\lambda_0) = \begin{pmatrix} y(x,\lambda_0) \\ 0 \end{pmatrix}$ is the eigenfunction of the operator L, corresponding to the eigenvalue λ_0, i.e. $\lambda_0 \in \sigma(L)$. Moreover, $\lambda_0 \in \sigma(\tilde{L}_2)$ is the eigenvalue of the operator L if and only if the solution $y_{12}(x,\lambda_0)$ of the equation

$$-y_{12}{}'' + x^2 y_{12} + q(x) y_{22} = \lambda_0 y_{12}, \tag{98}$$

satisfying the initial conditions $y_{12}(0,\lambda) = y_{12}{}'(0,\lambda) = 0$, belongs to $L_2(0;\infty)$. Let $u(x,\lambda), v(x,\lambda)$ be the solutions of the Eq. (95), satisfying the initial conditions $u(0,\lambda) = 0, u'(0,\lambda) = 1, v(0,\lambda) = -1, v'(0,\lambda) = 0$, and let $C(x,t,\lambda) = u(x,\lambda)v(t,\lambda) - v(x,\lambda)u(t,\lambda)$- be the Cauchy function of the Eq. (95). Then the solution $y_{12}(x,\lambda_0)$ is given by

$$y_{12}(x,\lambda_0) = \int_0^x q(t) \cdot C(x,t,\lambda_0) \cdot y_{22}(t,\lambda_0)\, dt. \tag{99}$$

Choose the coefficient $q(x) = y_{22}(x,\lambda_0) e^{x^\mu}$, where $\mu > 2$ (for instance, $\mu = 4$), and show that the integral $\int_0^\infty y_{12}^2(x,\lambda_0)\, dx$ diverges and, consequently, $\lambda_0 \notin \sigma(L)$. Indeed, since the solution $y_{22}(x,\lambda_0)$ has finitely many zeros, we conclude that, for any $x \ge N_1 > 0$,

$$y_{22}(x,\lambda_0) \ge c_1 e^{-\alpha x^2}, \alpha > 0, \tag{100}$$

and the Cauchy function decays no faster than $e^{-(x-t)^2}$. Hence, if $|x - t| > N_2$, we have

$$C(x,t,\lambda_0) \ge c_2 e^{-(x-t)^2}. \tag{101}$$

In the case of $\frac{x}{4} \le t \le \frac{x}{2}$ and $x \ge \max(4N_1, 2N_2)$, the inequalities (100) and (101) are fulfilled simultaneously, therefore, $y_{12}(x,\lambda_0) > c_3 \int_{\frac{x}{4}}^{\frac{x}{2}} e^{t^4} \cdot e^{-2\alpha t^2} \cdot e^{-(x-t)^2}\, dt$. Since $e^{-(x-t)^2} \ge e^{-\frac{x^2}{4}}$ for $t \le \frac{x}{2}$, we get $y_{12}(x,\lambda_0) > c_3 e^{-\frac{x^2}{4}} \int_{\frac{x}{4}}^{\frac{x}{2}} e^{t^4} \cdot e^{-2\alpha t^2}\, dt$. If x is sufficiently large and $t \in \left[\frac{x}{4}, \frac{x}{2}\right]$, we have $e^{t^4 - 2\alpha t^2} > e^{\frac{1}{2}t^4} \ge e^{\frac{x^4}{32}}$, hence for $x \to \infty y_{12}(x,\lambda_0) > c_3 \frac{x}{4} e^{-\frac{x^2}{4} + \frac{x^4}{32}} \to \infty$. It follows that $y_{12}(x,\lambda_0) \notin L_2(0;\infty)$ and $\lambda_0 \notin \sigma(L)$.

There arises the question on the nature of such values λ.

Consider the equation with a triangular matrix potential:

$$l[\bar{y}] = -\bar{y}'' + \begin{pmatrix} p(x) & q(x) \\ 0 & r(x) \end{pmatrix} \bar{y} = \lambda \bar{y}, \quad 0 \le x < \infty, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{102}$$

where $p(x), q(x), r(x)$ are scalar functions, $p(x), r(x)$ are real functions and $p(x), r(x) \to \infty$ monotonically as $x \to \infty$.

Let the boundary condition is given at $x = 0$:

$$\cos A \cdot \bar{y}'(0) - \sin A \cdot \bar{y}(0) = 0, \tag{103}$$

where A is a triangular matrix, $cosA = \begin{pmatrix} \cos\alpha_{11} & cos\,\alpha_{12} \\ 0 & \cos\alpha_{22} \end{pmatrix}$.

Consider the separated system

$$l_1[y_1] = -y_1{}'' + p(x) y_1 = \lambda y_1, \tag{104}$$

$$l_2[y_2] = -y_2'' + r(x) y_2 = \lambda y_2. \tag{105}$$

with the boundary conditions

$$\cos\alpha_{11}y_1'(0) - \sin\alpha_{11}y_1(0) = 0, \tag{106}$$

$$\cos\alpha_{22}y_2'(0) - \sin\alpha_{22}y_2(0) = 0. \tag{107}$$

Let L_0 be the differential operator generated by the differential expression $l[\bar{y}]$ (103) and the boundary condition (104), and let L_1, L_2 be minimal symmetric operators on $L_2(0,\infty)$ generated by the differential expressions $l_1[y_1], l_2[y_2]$ and the boundary conditions (106), (108) respectively. Denote by $\tilde{L_1}, \tilde{L_2}$ the self-adjoint extensions of the operators L_1, L_2 respectively. The operators $\tilde{L_1}, \tilde{L_2}$ are semi-bounded; let us denote their spectra by σ_1 and σ_2 respectively. Denote by L the extension of the operator L_0 and by $\sigma(L)$ its spectrum.

Let $u(x,\lambda), v(x,\lambda)$ be the solutions of the Eq. (104) with the boundary conditions $u(0,\lambda) = 0,\ u'(0,\lambda) = 1,\ v(0,\lambda) = -1,\ v'(0,\lambda) = 0$. The general solution of the Eq. (104) has the form $\varphi(x,\lambda) = u(x,\lambda) + lv(x,\lambda)$ up to a constant. Choose an l such that the condition $\varphi(b,\lambda) = 0$ holds true. This equality is valid for $l = l(b,\lambda) = -\frac{u(b,\lambda)}{v(b,\lambda)}$ (the solution $v(x,\lambda)$ has finitely many zeros for a fixed λ, hence $v(b,\lambda) \neq 0$ whenever b is sufficiently large). Put $\varphi_{11}^{(b)}(x,\lambda) = u(x,\lambda) + l(b,\lambda)v(x,\lambda)$. Since for the operator L_1 there is the case of a limit point, then, as is known, $l(b,\lambda)$ has a unique limit $m(\lambda)$ as $b \to \infty$, and the solution of the Eq. (104) satisfies $\varphi_{11}(x,\lambda) = u(x,\lambda) + m(\lambda)v(x,\lambda) \in L_2(0,\infty)$. Similarly we obtain that the solution of the Eq. (105) satisfies $\varphi_{22}(x,\lambda) \in L_2(0,\infty)$.

Denote by $\Phi_b(x,\lambda) = \begin{pmatrix} \varphi_{11}^{(b)}(x,\lambda) & \varphi_{12}^{(b)}(x,\lambda) \\ 0 & \varphi_{22}^{(b)}(x,\lambda) \end{pmatrix}$ the matrix solution of the Eq. (103) satisfying the initial conditions $\Phi_b(b,\lambda) = 0,\ \Phi_b{}'(b,\lambda) = I$. We have $\varphi_{11}^{(b)}(x,\lambda) \to \varphi_{11}(x,\lambda) \in L_2(0,\infty)$; $\varphi_{22}^{(b)}(x,\lambda) \to \varphi_{22}(x,\lambda) \in L_2(0,\infty)$ as $b \to \infty$.

The solution $\varphi_{12}^{(b)}(x,\lambda)$ is given by $\varphi_{12}^{(b)}(x,\lambda) = \int\limits_0^x q(t)\cdot C(x,t,\lambda)\cdot \varphi_{22}^{(b)}(t,\lambda)\,dt$, where $C(x,t,\lambda) = u(x,\lambda)v(t,\lambda) - v(x,\lambda)u(t,\lambda)$ is the Cauchy function of the Eq. (104).

Further, we have $\varphi_{12}^{(b)}(x,\lambda) \to \int\limits_0^x q(t)\cdot C(x,t,\lambda)\cdot\varphi_{22}(t,\lambda)\,dt := \varphi_{12}(x,\lambda)$ as $b \to \infty$.

Put $\Phi(x,\lambda) = \begin{pmatrix} \varphi_{11}(x,\lambda) & \varphi_{12}(x,\lambda) \\ 0 & \varphi_{22}(x,\lambda) \end{pmatrix}$.

Together with the Eq. (102), we consider the left equation.

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda\tilde{y}, \quad \tilde{y} = (y_1\ y_2). \tag{108}$$

The matrix solutions of the Eq. (108) will be denoted by $\tilde{\Phi}_b(x,\lambda)$ and $\tilde{\Phi}(x,\lambda)$.

Denote by $Y(x,\lambda)$ and $\tilde{Y}(x,\lambda)$ the solutions of the Eqs. (102) and (108) respectively satisfying the initial conditions

$$Y(0,\lambda) = \cos A,\ Y'(0,\lambda) = \sin A,\ \tilde{Y}(0,\lambda) = \cos A,\ \tilde{Y}'(0,\lambda) = \sin A,\ \lambda\in\mathbb{C}. \tag{109}$$

Put

$$G_b(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W(\tilde{\Phi}_b, Y)\right)^{-1}\tilde{\Phi}_b(t,\lambda) & 0\le x\le t \\ -\Phi_b(x,\lambda)\left(W(\tilde{Y},\Phi_b)\right)^{-1}\tilde{Y}(t,\lambda) & t\le x\le b \end{cases}. \tag{110}$$

The function $G_b(x,t,\lambda)$ is the Green function of the operator L_b^0 generated by the problem (102), (103), $y(b)=0$, which spectrum coincides with the union of spectra of the operators $L_{b,1}^0, L_{b,2}^0$ generated by the problems (104), (106), $y_1(b)=0$ and (105), (107), $y_2(b)=0$ respectively. Eigenvalues of the operators $L_{b,1}^0$ and $L_{b,2}^0$ tend to ones of the operators $\tilde{L}_1$ and $\tilde{L}_2$ respectively as $b\to\infty$, $\Phi_b(x,\lambda)\to\Phi(x,\lambda)$, $\tilde{\Phi}_b(x,\lambda)\to\tilde{\Phi}(x,\lambda)$, and

$$W\left(\tilde{Y},\Phi_b\right)=\cos A\cdot{\Phi_b}'(0,\lambda)-\sin A\cdot\Phi_b(0,\lambda)\to\cos A\cdot\Phi'(0,\lambda)-\sin A\cdot\Phi(0,\lambda)=$$

$$=W\left(\tilde{Y},\Phi\right),W\left(\tilde{\Phi}_b,Y\right)\to W\left(\tilde{\Phi},Y\right), \tag{111}$$

$$G_b(x,t,\lambda)\to G(x,t,\lambda)=\begin{cases} Y(x,\lambda)\left(W\left(\tilde{\Phi},Y\right)\right)^{-1}\tilde{\Phi}(t,\lambda) & 0\le x\le t\\ -\Phi(x,\lambda)\left(W\left(\tilde{Y},\Phi\right)\right)^{-1}\tilde{Y}(t,\lambda) & t\le x\end{cases}. \tag{112}$$

Poles of the Green function $G(x,t,\lambda)$ of the operator L coincide with the zero set of the determinant $\Delta(\lambda):=\det\Omega(\lambda)$, where.

$$\Omega(\lambda)=W\left(\tilde{Y},\Phi\right)\Big|_{x=0}=\cos A\cdot\Phi'(0,\lambda)-\sin A\cdot\Phi(0,\lambda). \tag{113}$$

Since the matrices $\cos A$, $\sin A$, $\Phi(0,\lambda)$, $\Phi'(0,\lambda)$ are triangle, we have $\Delta(\lambda)=\Delta_1(\lambda)\cdot\Delta_2(\lambda)$, where $\Delta_k(\lambda)=\cos\alpha_{kk}\cdot\varphi_{kk}'(0,\lambda)-\sin\alpha_{kk}\cdot\varphi_{kk}(0,\lambda), k=1,2$. On the other hand, zeros of the function $\Delta_k(\lambda)$ are eigenvalues of the self-adjoint operator $\tilde{L}_k$. Hence the poles of the Green function $G(x,t,\lambda)$ of the operator L are situated on the real axis, and their set coincides with the union of spectra of the operators $\tilde{L}_1$ and $\tilde{L}_2$.

Consider the operator $R_{\lambda,b}$ defined on $L_2(H_2,(0;b))$ by.

$$\left(R_{\lambda,b}\overline{f}\right)(x)=\int_0^b G_b(x,t,\lambda)\overline{f}(t)dt=-\int_0^x\Phi_b(x,\lambda)\left(W\left(\tilde{Y},\Phi_b\right)\right)^{-1}\tilde{Y}(t,\lambda)\overline{f}(t)dt+$$

$$+\int_x^b Y(x,\lambda)\left(W\left(\tilde{\Phi}_b,Y\right)\right)^{-1}\tilde{\Phi}(t,\lambda)\overline{f}(t)dt. \tag{114}$$

One can directly verify that the operator $R_{\lambda,b}$ is the resolvent of the operator L_b^0.

Let $\overline{f}(x)$ be an arbitrary vector function square integrable on $[0,\infty)$. Choose a sequence of finite continuous vector functions $\left\{\overline{f}_n(x)\right\}$ $(n=1,2,\ldots)$ converging in mean square to $\overline{f}(x)$. Substituting $\overline{f}_n$ for $\overline{f}$ in (114) and letting first $b\to\infty$ and then $n\to\infty$, we obtain the following formula for the resolvent R_λ of the operator L: $\left(R_\lambda\overline{f}\right)(x)=\int_0^\infty G(x,t,\lambda)\overline{f}(t)dt$, where the Green function of the operator L is defined by the formula (112).

Theorem 5.1 *The operator R_λ is the resolvent of the operator L. The resolvent's poles coincide with the union of the spectra of the self-adjoint operators $\tilde{L}_1$ and $\tilde{L}_2$.*

Remark 5.2 *As in Example 5.1, if $\lambda_0\in\sigma\left(\tilde{L}_2\right)$ and $\varphi_{12}(x,\lambda_0)\notin L_2(0,\infty)$, then λ_0 is the pole of the resolvent R_λ of the operator L but it is not the eigenvalue of this operator, i.e., λ_0 is the* ***point of the spectral singularity of the operator*** *L.*

Theorem 5.1 implies that, if the rate of the coefficient's growth $q(x)$ of the Eq. (102) is subordinated to one of $p(x)$ and $r(x)$, then the operator L has no spectral singularities, and its spectrum is real and coincides with the union of the spectra of the operators $\tilde{L}_1$ and $\tilde{L}_2$.

For a non-self-adjoint Sturm-Liouville operator with a triangular matrix potential growing at infinity, an example of operator having spectral singularities is constructed. A special role of these points was found first by M.A. Naimark in [16]. The notion "spectral singularity" was introduced later due to J. Schwartz [17] (see also Supplement I in the monograph [3]).

6. Conclusion

We consider the Sturm-Liouville equation with block-triangular, increasing at infinity operator potential. For him, built a fundamental system of solutions, one of which is decreasing at infinity, and the second is growing. The asymptotics of these solutions at infinity is defined. For non-self-adjoint operator generated by such differential expression obtained the Green's function. A resolvent of such an operator is constructed. Sufficient conditions at which a spectrum of such non-self-adjoint differential operator is real and discrete are obtained.

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Chapter 4

Deformed Sine-Gordon Models, Solitons and Anomalous Charges

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Abstract

We study certain deformations of the integrable sine-Gordon model (DSG). It is found analytically and numerically several towers of infinite number of anomalous charges for soliton solutions possessing a special space–time symmetry. Moreover, it is uncovered exact conserved charges associated to two-solitons with a definite parity under space-reflection symmetry, i.e. kink-kink (odd parity) and kink-antikink (even parity) scatterings with equal and opposite velocities. Moreover, we provide a linear formulation of the modified SG model and a related tower of infinite number of exact non-local conservation laws. We back up our results with extensive numerical simulations for kink-kink, kink-antikink and breather configurations of the Bazeia *et al.* potential $V_q(w) = \frac{64}{q^2} \tan^2 \frac{w}{2} \left(1 - \left|\sin \frac{w}{2}\right|^q\right)^2, (q \in R)$, which contains the usual SG potential $V_2(w) = 2[1 - \cos(2w)]$.

Keywords: quasi-integrability, solitons, deformed sine-Gordon, anomalous charges, non-local charges

1. Introduction

Solitons can be regarded as isolated waves that travel without loss of energy. The solitons emerge with their velocities and shapes completely unchanged after collision to each other, the only outcome being their phase shifts. The soliton solution is the main feature of the integrable models [1–3]. However, certain non-linear models in physics, with solitary wave solutions, are not integrable. Recently, certain deformations of integrable models such as the sine-Gordon (SG), nonlinear Schrödinger (NLS), Korteweg-de Vries (KdV) and Toda models have been introduced, such that they exhibit soliton-type solutions with some properties resembling to their counterparts of the truly integrable ones. In this context the so-called quasi-integrability concept has been put forward [4]. These properties have been examined in the frameworks of the anomalous zero-curvature [4–7] and the Riccati-type pseudo-potential approaches [8–10], respectively.

The main developments have been focused on the construction of infinite number of quasi-conservation laws which give rise to asymptotically conserved charges, i.e. conserved charges, such that their values vary during the scattering of the solitons only. The main observation in the both approaches to quasi-integrability is that, in general, the conserved charges of the standard integrable systems turn out to be the so-called asymptotically conserved charges in the deformed models. In fact, the exact conservation laws of the usual integrable systems become quasi-conservation laws of the deformed integrable models. The non-homogeneous terms of the quasi-conservation laws are dubbed as anomalies such that they vanish when integrated on the space–time plane, provided that the fields satisfy a special space–time symmetry.

The properties of the soliton-like configurations in the quasi-integrable models are, so far, largely unknown. We summarize the main results. First, the one-soliton sectors exhibit infinite conserved charges. Second, the space–time integration of the anomalies vanish when one-soliton like solutions are located far away from each other. The anomalies are significant around the space–time regions of their interaction. Third, a sufficient condition for the vanishing of the space–time integrated anomalies is that the $N-$soliton possesses definite parity under a shifted parity and delayed time reversion $(\mathcal{P}_s\mathcal{T}_d)$ symmetry. When the anomaly densities possess odd parities the space–time integration of them vanish, which imply the existence of anomalous charges. Fourth, the conserved charges of the usual integrable systems turn out to be the anomalous charges upon deformation. Fifth, there exist infinite towers of infinitely many anomalous charges, different in form from the ones of the usual integrable models. New towers of anomalous charges have been uncovered in [8–10]. Remarkably, even the usual integrable models possess quasi-conservation laws with anomalous charges for analytical $N-$ soliton with $C\mathcal{P}_s\mathcal{T}_d$ symmetry [9, 10]. For the standard SG theory it has been discussed for the 2-soliton sector of the theory [8]. Sixth, there is a subset of exact conserved charges for soliton eigenstates simply of the shifted space-reflection $\mathcal{P}_s$. The deformed NLS model for two-soliton solutions [6, 7] and the deformed sine-Gordon model [11] for two-kink and breather solutions exhibit this property.

In the context of the Riccati-type method there have been shown that the deformed SG, KdV and NLS models [8–10], respectively, possess linear system formulations and that they exhibit infinite towers of exact non-local conservation laws. The NLS-type, KdV-type and SG-type models share the same importance due to their potential applications, since they are ubiquitous in all areas of nonlinear physics, such as Bose-Einsten condensation and superconductivity [12–14], soliton gas and soliton turbulence in fluid dynamics [15–20], the Alice-Bob physics [21, 22] and the understanding of a kind of triality among the gauge theories, integrable models and gravity theories [23].

Here, we discuss the previous results in the field by utilizing a deformed sine-Gordon model. We will introduce the relationship between the space–time parity and asymptotically conserved charges. Next, we clarified on the space-reflection parity related to the linear combination of the dual sets of anomalous quantities. In addition, it is focused on the space-reflection symmetry of some two-soliton solutions of deformed sine-Gordon models. Then one proceeds to construct a tower of exactly conserved charges for each solution possessing a definite space-reflection parity. Lastly, by considering linear combinations of the anomalous conserved charges it is showed, through analytical and numerical methods, that there is a subset of exactly conserved charges.

A modified SG model and the space–time symmetries are presented in the next section. In Section 3, the towers of quasi-conservation laws are presented. In Section 4 our numerical simulations are described. The linear formulation and the non-local conservation laws are discussed in the Riccati-type pseudo-potential approach in Section 5. Finally, in Section 6 we present some conclusions.

2. A deformation of the sine-Gordon model

Let us consider the relativistic field theories in $(1+1)$-dimensions with equation of motion[1]

[1] In the x and t laboratory coordinates: $\eta = \frac{t+x}{2}, \xi = \frac{t-x}{2}, \partial_\eta = \partial_t + \partial_x, \partial_\xi = \partial_t - \partial_x, \partial_\eta\partial_\xi = \partial_t^2 - \partial_x^2$

$$\partial_\xi \partial_\eta w + V^{(1)}(w) = 0, \tag{1}$$

where w is a real scalar field w, $V(w)$ is the scalar potential and $V^{(1)}(w) \equiv \frac{d}{dw}V(w)$. The family of potentials $V(w)$ will represent certain deformations of the usual SG model. The theory (1) has been studied using the techniques of integrable field theories, such as the anomalous zero-curvature [4, 11] and deformed Riccati-type pseudo-potential formulations [8], respectively. In our simulations we will consider [4, 24].

$$V(w,q) = \frac{2}{q^2} \tan^2 w[1 - |\sin w|^q]^2, \tag{2}$$

where q is a real parameter such that for $q = 2$ the potential reduces to the SG potential

$$V(w,2) = \frac{1}{16}[1 - \cos(4w)]. \tag{3}$$

So, we introduce the deformation parameter ε as $q = 2 + \varepsilon$, such that in the limit $\varepsilon = 0$ one reproduces the SG model.

The model (1) possesses several towers of anomalous charges associated to quasi-conservation laws [4, 8, 11]. In [11] it has been introduced a subset of exactly conserved charges associated to space-reflection eigenstates as kink-antikink, kink-kink and breather configurations, respectively. New types of two sets of dual towers of asymptotically conserved charges have been uncovered [8]. Remarkably, even the usual sine-Gordon models possesses anomalous charges. So far, it is attributed to the space–time symmetry properties of the solitons. Those charges can be relevant in the study of soliton gases and formation of certain structures, such as soliton turbulence, soliton gas dynamics and rogue waves [16].

The quasi-integrability has been introduced for deformed sine-Gordon models such that the field w and the potential V satisfy the symmetry [4, 8, 11].

$$\mathcal{P} : w \to -w + \text{const.}; \quad V(w) \to V(w), \tag{4}$$

under the special space–time reflection

$$\mathcal{P} \equiv \mathcal{P}_s \mathcal{T}_d, \ \ \mathcal{P}_s : \tilde{x} \to -\tilde{x}, \ \ \mathcal{T}_d : \tilde{t} \to -\tilde{\tilde{t}}, \ \ \tilde{x} \equiv x - x_\Delta, \ \tilde{t} = t - t_\Delta, \tag{5}$$

defined around a given point (x_Δ, t_Δ). Moreover, let us consider the space-reflection transformation

$$\mathcal{P}_x : x \leftrightarrow -x, \tag{6}$$

and assume that the scalar field is an eigenstate of the operator P_x

$$\mathcal{P}_x : w \to \varrho\, w, \ \ \varrho = \pm 1. \tag{7}$$

In addition, consider an even potential V under $\mathcal{P}_x$

$$\mathcal{P}_x(V) = V. \tag{8}$$

Several towers of quasi-conservation laws, with anomaly terms possessing odd parities under (6)–(8), have been found [8, 11]. Next, we consider those quasi-conservation laws and examine their anomalies in view of the symmetries (4)–(5) and (6)–(8), respectively.

3. Quasi-conservation laws of the deformed SG model

We will discuss some of the infinite towers of quasi-conservation laws of the deformed SG model (1).

3.1 First type of towers: The SG-type quasi-conservation laws

The usual SG charges turn out to be the anomalous charges of the DSG. So, one has the infinite set of quasi-conservation laws [4, 11].

$$\frac{d}{dt}q_a^{(2n+1)} = \int dx\, \beta^{(2n+1)}, \;\; n = 1, 2, 3, \ldots \tag{9}$$

where the quantities $q_a^{(2n+1)}$ define the anomalous charges, provided that the time-integrated anomalies $\int dt \int dx\, \beta^{(2n+1)}$ vanish for solitons satisfying (4) and (5). This condition, when combined with Eq. (9), implies $q_a^{(2n+1)}(t \to +\infty) = q_a^{(2n+1)}(t \to -\infty)$. So, we have that $q_a^{(2n+1)}$ are anomalous for $n = 1, 2, 3, \ldots$. The charges $q_a^{(2n+1)}$ maintain the same form as the ones of the usual SG.

In $(1+1)$-dimensional Lorentz invariant integrable field theories one has dual integrability conditions or Lax equations. Analogously, for the deformations of the SG model there exist a dual formulation for each equation as in (9) by interchanging $\xi \leftrightarrow \eta$ in the procedure to obtain the relevant quasi-conservation laws. So, one can get

$$\frac{d}{dt}\tilde{q}_a^{(2n+1)} = \int dx\, \tilde{\beta}^{(2n+1)}, \;\; n = 1, 2, 3, \ldots \tag{10}$$

where the quantities $\tilde{q}_a^{(2n+1)}$ define the dual asymptotically conserved charges, provided that the time-integrated anomalies $\int dt \int dx\, \tilde{\beta}^{(2n+1)}$ vanish. Likewise, this result implies $\tilde{q}_a^{(2n+1)}(t \to +\infty) = \tilde{q}_a^{(2n+1)}(t \to -\infty)$.

These towers of quasi-conservation laws reproduce the same polynomial form as in the usual sine-Gordon charge densities. In fact, the anomalies $\beta^{(2n+1)}$ and $\tilde{\beta}^{(2n+1)}$ vanish identically provided that the deformed potential $V(w)$ recovers the form of the standard SG potential.

The importance and the relevance of such a dual construction will become clear below when the linear combinations of the charges in (9) and (10) give rise to infinite towers of exactly conserved charges, provided that the space-integral of the linear combination of the anomaly densities $\beta^{(2n+1)}$ and $\tilde{\beta}^{(2n+1)}$ vanish for special two-soliton solutions.

3.1.1 Space-reflection parity and conserved charges

The above dual sets of quasi-conservation laws are used to construct a sequence of conserved charges and vanishing anomalies. The space-reflection symmetry of some soliton solutions of the deformed SG model will imply the existence of an infinite tower of conserved charges. So, let us examine a linear combination, at each order $n = 1, 2, \ldots$, of the above two sets of quasi-conserved charges $q_a^{(2n+1)}$ (9) and $\tilde{q}_a^{(2n+1)}$ (10). Consider the new quasi-conservation laws

$$\frac{d}{dt}q_{a,\pm}^{(2n+1)} = -\int dx\, \beta_{\pm}^{(2n+1)}, \;\; n = 1, 2, \ldots, \tag{11}$$

with the charges $q_{a,\pm}^{(2n+1)}$ and anomalies $\beta_{\pm}^{(2n+1)}$, respectively, defined as

$$q_{a,\pm}^{(2n+1)} \equiv \mp\frac{1}{16}\left(q_a^{(2n+1)} \pm \tilde{q}_a^{(2n+1)}\right), \tag{12}$$

$$\beta_{a,\pm}^{(2n+1)} \equiv \mp\frac{1}{16}\left(\beta^{(2n+1)} \pm \tilde{\beta}^{(2n+1)}\right) \tag{13}$$

in which the quantities $q^{(2n+1)}$ and $\beta^{(2n+1)}$ defined in (9) and the quantities $\tilde{q}_a^{(2n+1)}$ and $\tilde{\beta}^{(2n+1)}$ in (10) have been used, respectively.

Since the theory (1) is invariant under space–time translations one has that the energy momentum tensor is conserved. In fact, one has $\beta^{(1)} = \tilde{\beta}^{(1)} = 0$ at the zero'th order $n = 0$, and the linear combinations of the charges $q_a^{(1)}$ and $\tilde{q}_a^{(1)}$ leads to the energy and momentum, respectively [11].

$$q_+^{(1)} = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}(\partial_t w)^2 + \frac{1}{2}(\partial_x w)^2 + V\right], \tag{14}$$

$$q_-^{(1)} = \int_{-\infty}^{+\infty} dx\, \partial_x w \partial_t w, \tag{15}$$

where $E = q_+^{(1)}$ is the energy and $P = q_-^{(1)}$ is the momentum.

The first non-trivial anomalies become [11].

$$\beta_{\pm}^{(3)} = \pm\frac{1}{2}Z\left\{\partial_\xi\left[(\partial_\xi w)^2\right]\mp\partial_\eta\left[(\partial_\eta w)^2\right]\right\}, \quad Z \equiv V^{(2)} + 16V - 1. \tag{16}$$

$$\beta_{\pm}^{(5)} = \pm\frac{1}{2}Z\left[\left(24(\partial_\xi w)^2\partial_\xi^2 w + \partial_\xi^4 w\right)\partial_\xi w \pm \left(24(\partial_\eta w)^2\partial_\eta^2 w + \partial_\eta^4 w\right)\partial_\eta w\right]. \tag{17}$$

Notice that for the SG potential (3) the factor Z above vanishes identically; therefore, the anomalies vanish $\beta_{\pm}^{(3)} = 0$, and the relevant charges $q_{\pm}^{(3)}$ turn out to be the exactly conserved charges of the standard SG model at this order.

The properties of the quantities $q_{\pm}^{(2n+1)}$ and $\int dx\, \beta_{\pm}^{(2n+1)}$ in (11) will depend on the symmetry properties of the solitons, in particular on the space-reflection symmetry of $\beta_{\pm}^{(2n+1)}$, as we will see below. So, let us examine the space-reflection symmetry of them.

Let us write the anomalies in terms of the ∂_x and ∂_t derivatives. So, once the eq. of motion (1) is used to substitute $\partial_t^2 w \to \left[\partial_x^2 w - V'(w)\right]$, as well as, neglecting surface terms one has

$$\alpha_+^{(3)} \equiv -2\int dx f_+^{(3)}(x,t), \tag{18}$$

$$f_+^{(3)}(x,t) \equiv \left[V'' + 16V\right]\left\{\partial_x\left[(\partial_t w)^2\right] + \partial_x\left[(\partial_x w)^2\right]\right\}, \tag{19}$$

where we have defined the anomaly density $f_+^{(3)}$. Notice that for even parity potentials (8) and for definite parity (even or odd) fields w the density $f_+^{(3)}$ is an odd function, and thus the $x-$integrated anomaly $\alpha_+^{(3)}$ vanishes.

Following analogous procedure as above one has

$$\alpha_-^{(3)} = -4\int dx f_-^{(3)}(x,t), \tag{20}$$

$$f_{-}^{(3)}(x,t) \equiv \left[V'' + 16V\right]\left\{\partial_t w \partial_x^2 w + \partial_x w \partial_x \partial_t w\right\}, \tag{21}$$

where we have defined the anomaly density $f_{-}^{(3)}$. Notice that for even parity potentials (8) and for definite parity (even or odd) fields w the density $f_{-}^{(3)}$ is an even function, and thus the $x-$integrated anomaly $\alpha_{-}^{(3)}$ will not vanish solely by a space-reflection parity reason.

The anomalies $\alpha_{\pm}^{(3)}$ and $\int dt \alpha_{\pm}^{(3)}$ in (18) and (20), will be computed numerically for two-solitons and breather-like solutions below.

By direct construction it has been found new towers of anomalous charges in [8]. In the next subsections we will discuss those charges and anomalies in relation to the symmetry (4) and (5).

3.2 Second type of towers

The quasi-conservation laws [8].

$$\frac{d}{dt} Q_a^{(N)} = \mathrm{a}^{(N)}, \tag{22}$$

$$Q_a^{(N)} \equiv \int dx \left[\frac{1}{N}(\partial_\xi w)^N + V(\partial_\xi w)^{N-2}\right], \tag{23}$$

$$\mathrm{a}^{(N)} \equiv \int dx\,(N-2)(\partial_\xi w)^{N-3} \partial_\xi^2 w V, \quad N \geq 3, \tag{24}$$

define the asymptotically conserved charges $Q_a^{(N)}$ and the corresponding anomalies $\mathrm{a}^{(N)}$.

The dual quasi-conservation laws become

$$\frac{d}{dt} \tilde{Q}_a^{(N)} = \tilde{\mathrm{a}}^{(N)}, \tag{25}$$

$$\tilde{Q}_a^{(N)} \equiv \int dx \left[\frac{1}{N}\left(\partial_\eta w\right)^N + V\left(\partial_\eta w\right)^{N-2}\right], \tag{26}$$

$$\tilde{\mathrm{a}}^{(N)} \equiv \int dx\,(N-2)\left(\partial_\eta w\right)^{N-3} \partial_\eta^2 w V, \quad N \geq 3, \tag{27}$$

where we have introduced the dual asymptotically conserved charges $\tilde{Q}_a^{(N)}$ and the relevant anomalies $\tilde{\mathrm{a}}^{(N)}$.

The densities of the anomalies $\mathrm{a}^{(N)}$ and $\tilde{\mathrm{a}}^{(N)}$ in (24) and (27), respectively, possess odd parities under (4) and (5), so the quasi-conservation laws (22) and (25), respectively, allow the construction of asymptotically conserved charges.

3.3 Third type of towers

Let us define the quasi-conservation laws [8].

$$\frac{d}{dt} Q_a^{(N)} = \gamma^{(N)}, \tag{28}$$

$$Q_a^{(N)} \equiv \int dx \left[\frac{1}{2} V^{N-1}(\partial_\xi w)^2 + \frac{1}{N} V^N\right], \tag{29}$$

$$\gamma^{(N)} \equiv \int dx \, \frac{1}{2} (\partial_\xi w)^2 \partial_\eta V^{N-1}, \quad N \geq 2, \tag{30}$$

where we have introduced the asymptotically conserved charges $\hat{Q}_a^{(N)}$ and the corresponding anomalies $\gamma^{(N)}$.

The interchange $\eta \leftrightarrow \xi$ allows us to reproduce the dual quasi-conservation laws. So, one has

$$\frac{d}{dt}\tilde{Q}_a^{(N)} = \tilde{\gamma}^{(N)}, \tag{31}$$

$$\tilde{Q}_a^{(N)} \equiv \int dx \left[\frac{1}{2} V^{N-1} (\partial_\xi w)^2 + \frac{1}{N} V^N\right], \tag{32}$$

$$\tilde{\gamma}^{(N)} \equiv \int dx \, \frac{1}{2} (\partial_\eta w)^2 \partial_\xi V^{N-1}, \quad N \geq 2, \tag{33}$$

where we have defined the dual asymptotically conserved charges $\tilde{Q}_a^{(N)}$ and the anomalies $\tilde{\gamma}^{(N)}$.

Similarly, the densities of the anomalies $\gamma^{(N)}$ and $\tilde{\gamma}^{(N)}$ in (30) and (33), respectively, possess odd parities under (4) and (5), so the quasi-conservation laws (28) and (31), respectively, allow the construction of asymptotically conserved charges.

The relevant anomalies of the lowest order quasi-conservation laws of the above towers will be simulated below for 2-soliton interactions.

Remarkably, the above charges turn out to be anomalous even for the standard sine-Gordon model. In fact, the relevant 2-soliton solutions have been constructed analytically [4, 11] which possess a definite parity under (4)–(5), such that the odd anomaly densities vanish upon space–time integration. The usual explanation for the appearance of novel anomalous charges in the standard sine-Gordon model is the symmetry argument. The anomalous charges also appear in the standard KdV and its deformations [9].

These charges have been computed for soliton collisions in the treatment of soliton gases and formation of some structures in integrable systems, such as integrable turbulence and rogue waves. In the context of the usual KdV model it has been analyzed the behavior of the statistical moments defined by (see e.g. [16, 17]) $M_n(t) = \int_{-\infty}^{+\infty} v^n \, dx, n \geq 1$; where v is the KdV field. The $M_{1,2}$ cases are conserved charges. It is remarkable that the moments, $M_{3,4}$, respectively, in the interaction region of two-solitons, behave as the anomalous charges of the quasi-integrable KdV models [9]. In fact, in the quasi-integrable KdV models the moments $M_{2,3}$ are in fact anomalous charges [9]. So, since the two-soliton collision is an important ingredient in the formation of soliton turbulence and the dynamics of soliton gases, we can expect they will be important in the quasi-integrable counterparts. In the case of the SG soliton ensemble, to our knowledge, it is needed a further theoretical research.

4. Numerical simulations

Here we will check numerically the lowest order expressions of the various towers of quasi-conservation laws presented above. For this purpose we will numerically solve the Eq. (1) with the particular deformed potential (2). In the **Figures 1** and **2** we plot the kink-kink and kink-antikink collisions, respectively. Moreover, we show the first conserved charges, i.e. the energy and momentum for these field configurations.

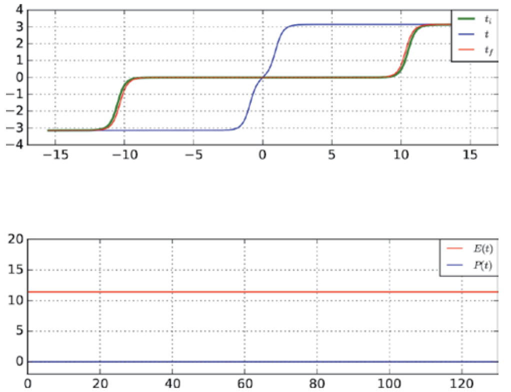

Figure 1.
Kink-kink with velocities $v_2 = -v_1 = 0.15$ and $q = 2.01$ in (2), for initial (green), collision (blue) and final (red) times. Bottom, the energy (E) and momentum (P) charges of the kink-kink.

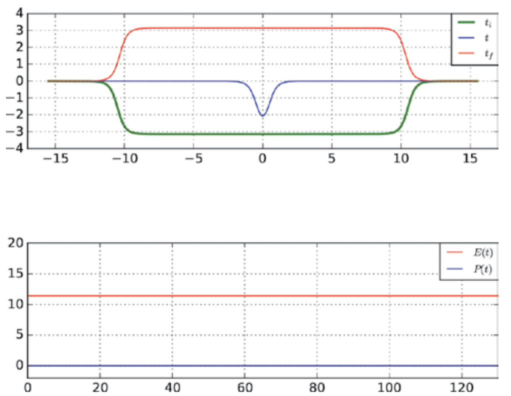

Figure 2.
Kink-antikink with velocities $v_2 = -v_1 = 0.15$ and $q = 2.01$ in (2), for initial (green), collision (blue) and final (red) times. Bottom, the energy (E) and momentum (P) charges of the kink-antikink.

4.1 First non-trivial anomalies of the SG-type quasi-conservation laws

We have checked our results by numerical simulation of the anomalies $\alpha_{\pm}^{(3)}$ (18)–(21) for kink-antikink, kink-kink and breather solutions of the model (2).

So, let us write (11) in the form

$$q_{a,\pm}^{(3)}(t) - q_{a,\pm}^{(3)}(t_0) = -\int_{t_0}^{t} dt\alpha_{\pm}^{(3)}(t), \tag{34}$$

where $\alpha_{\pm}^{(3)}(t)$ were defined in (18) and (20) and t_0 is the initial time.

The simulations of the kink-antikink, kink-kink and breather systems of the deformed SG model will consider, as the initial condition, two analytic solitary wave solutions presented in Eq. (1.2) of [4], located some distance apart and stitched together at the middle point.

4.1.1 Kink-antikink

In the **Figures 3** and **4** we show the results for kink-antikink system with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The plots of (19) and (21) as $f_{\pm}^{(3)}(x, t)\, vs\, x$ are

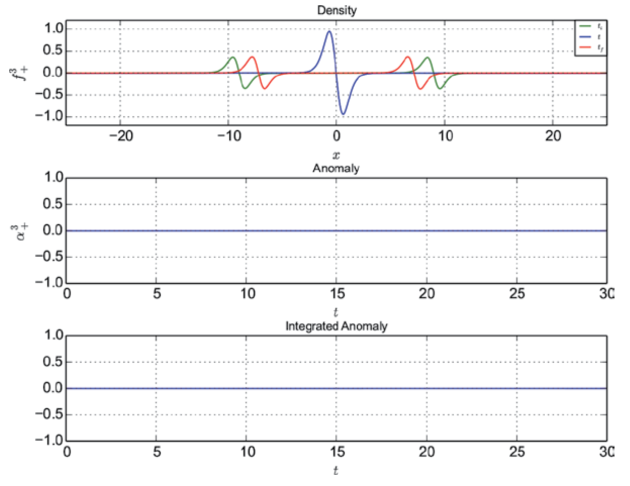

Figure 3.
$f_+^{(3)}$, $\alpha_+^{(3)}$ and $\int dt\alpha_+^{(3)}$ in (18) and (19) for kink-antikink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-antikink.

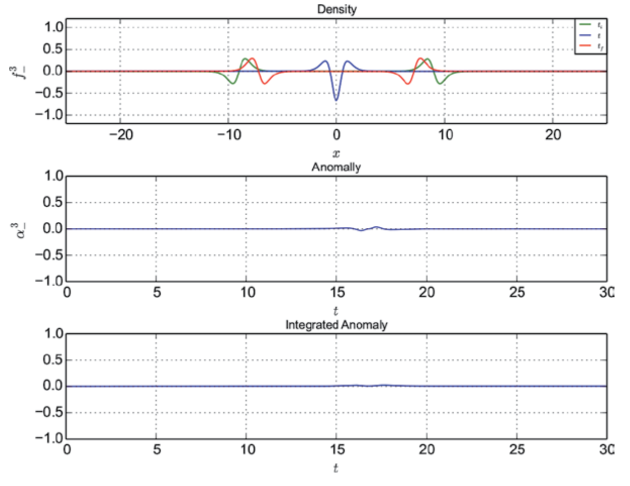

Figure 4.
$f_-^{(3)}$, $\alpha_-^{(3)}$ and $\int dt\alpha_-^{(3)}$ in (20)–(21) for kink-antikink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-antikink.

shown for three successive times (top figures). Their integration in space furnish vanishing $\alpha_+^{(3)}(t)$ and non-vanishing $\alpha_-^{(3)}(t)$ (middle figures). The bottom figures show $\int dt'\alpha_+^{(3)}(t')$, vanishing in **Figure 3** and $\int dt'\alpha_-^{(3)}(t')$ asymptotically vanishing in **Figure 4**, respectively. According to (34) our numerical simulations show the asymptotically conservation of the charge $q_{a,-}^{(3)}$ and the exact conservation of the charge $q_{a,+}^{(3)}$, within numerical accuracy.

4.1.2 kink-kink

In the **Figures 5** and **6** we show the results for kink-kink system with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The plots of (19) and (21) as $f_\pm^{(3)}(x,t) vs x$ are shown for three successive times (top figures). Their integration in space furnish vanishing $\alpha_+^{(3)}(t)$ and non-vanishing $\alpha_-^{(3)}(t)$ (middle figures). The bottom figures show

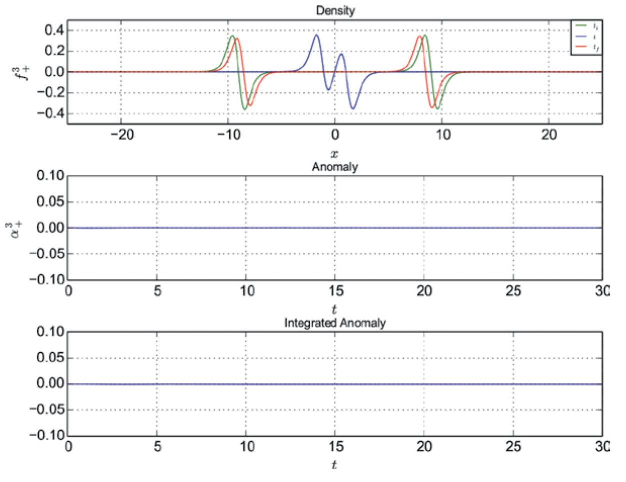

Figure 5.
$f_+^{(3)}$, $\alpha_+^{(3)}$ and $\int dt\alpha_+^{(3)}$ in (18) and (19) for kink-kink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-kink.

$\int dt'\alpha_+^{(3)}(t')$, vanishing in **Figure 5** and $\int dt'\alpha_-^{(3)}(t')$ asymptotically vanishing in **Figure 6**. According to (34) our numerical results show the asymptotically conservation of the charge $q_{a,-}^{(3)}$ and the exact conservation of the charge $q_{a,+}^{(3)}$, within numerical accuracy.

So, one can conclude that for kink-antikink (kink-kink) solution the definite parity related to the space-reflection symmetry is a necessary condition in order to achieve a conserved $q_{a,+}^{(3)}$ charge, within numerical accuracy.

The both kink-antikink and kink-kink solitons of the SG model with opposite and different velocities do not possess the required parity symmetry. However, it has been shown that in the center-of-mass reference frame (x', t') the parity symmetries are recovered, as discussed in [11]. So, the simulations performed in these reference frames, in the both kink-antikink and kink-kink cases, will provide vanishing $\alpha_+^{(3)}$ anomalies as shown above.

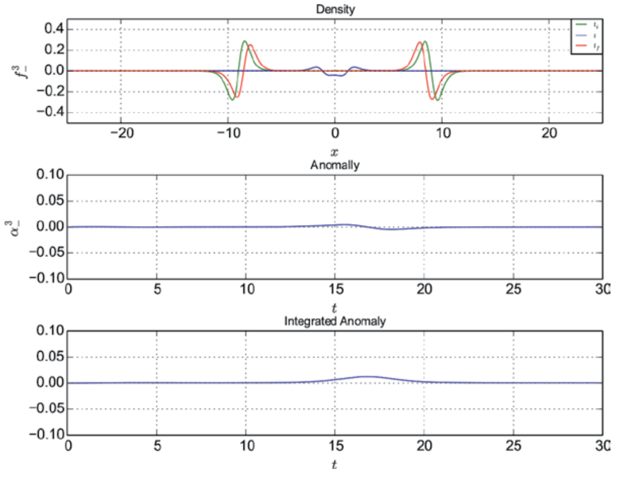

Figure 6.
$f_-^{(3)}$, $\alpha_-^{(3)}$ and $\int dt\alpha_-^{(3)}$ in (20) and (21) for kink-kink with velocities $v_2 = -v_1 = 0.5$ and $\varepsilon = 0.06$. The density figure shows initial t_i (green), collision t_c (blue) and final t_f (red) times of the kink-kink.

4.1.3 Breather: kink-antikink bound state

Figures 7 and **8** show the results for breather (kink-antikink bound state) with $\varepsilon = 0.06$. The densities $f_{\pm}^{(3)}(x,t)$ in (19) and (21), respectively, have been plotted as functions of x for three successive times (top figures). They show the vanishing $\alpha_{+}^{(3)}(t)$ and non-vanishing (periodic in time) $\alpha_{-}^{(3)}(t)$ (middle figures). The bottom figures of **Figures** 7 and **8** show the vanishing $\int dt' \alpha_{+}^{(3)}(t')$ and periodic $\int dt' \alpha_{-}^{(3)}(t')$ expressions. According to (34) our numerical results show the oscillation of the charges $q_{a,-}^{(3)}$ around a fixed value and the exact conservation of the charge $q_{a,+}^{(3)}$, within numerical accuracy.

4.2 Lowest order anomalies of the second and third types of towers

We will compute the linear combinations of the lowest order anomalies of the second and third types of towers in (22)–(27) and (28)–(33), respectively,

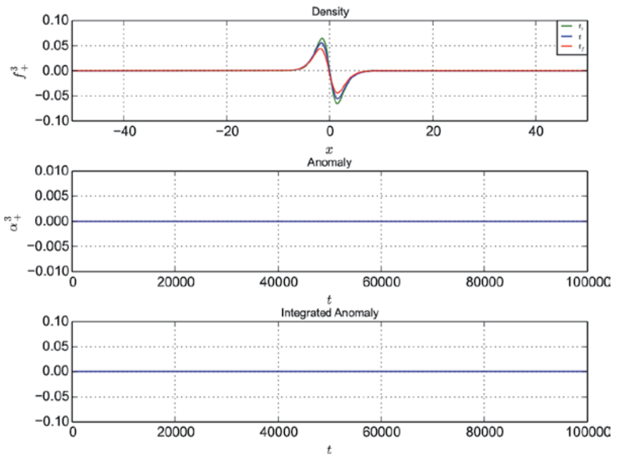

Figure 7.
$f_{+}^{(3)}$, $\alpha_{+}^{(3)}$ and $\int dt\alpha_{+}^{(3)}$ in (18) and (19) for breather with $\varepsilon = 0.06$. The density is shown for three times $t \in [t_f - T_0, t_f]$, $T_0 = 7.025$. The long-lived breather for $t_f \approx 10^5$.

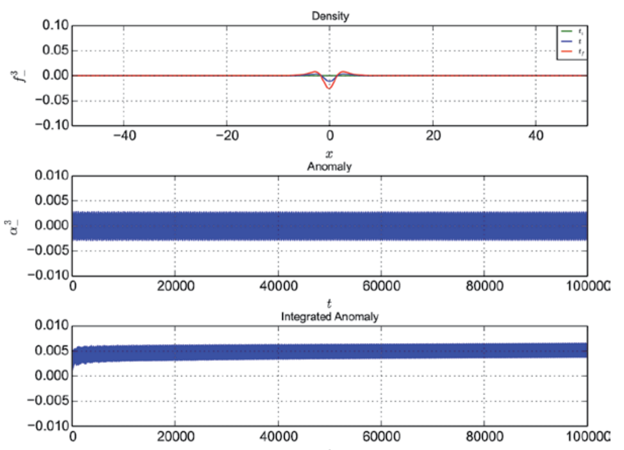

Figure 8.
$f_{-}^{(3)}$, $\alpha_{-}^{(3)}$ and $\int dt\alpha_{-}^{(3)}$ in (20) and (21) for breather with $\varepsilon = 0.06$. The density is shown for three times $t \in [t_f - T_0, t_f]$, $T_0 = 7.025$. The long-lived breather for $t_f \approx 10^5$.

$$\mathrm{a}_{\pm} \equiv \mathrm{a}^{(3)} \pm \tilde{\mathrm{a}}^{(3)}, \tag{35}$$

$$\gamma_{\pm} \equiv \gamma^{(2)} \pm \tilde{\gamma}^{(2)}. \tag{36}$$

4.2.1 Second and third types of towers and lowest order anomalies

The two anomalies in (35) can be written as

$$\mathrm{a}_{+} = \int dx\, 2\left[\partial_t^2 w + \partial_x^2 w\right] V, \tag{37}$$

$$\mathrm{a}_{-} = \int dx\, 4\left[\partial_t \partial_x w\right] V. \tag{38}$$

Similarly, the two anomalies in (36) can be written as

$$\gamma_{+} = \int dx \left[(\partial_t w)^2 - (\partial_x w)^2\right] \partial_t V, \tag{39}$$

$$\gamma_{-} = -\int dx \left[(\partial_t w)^2 - (\partial_x w)^2\right] \partial_x V. \tag{40}$$

Notice that under the space–time reflection transformation (4) and (5), the densities of the above anomalies $\mathrm{a}_{\pm}^{(3)}$ and $\gamma_{\pm}$, respectively, are odd; then they must vanish upon space–time integration. Therefore, one has asymptotically conserved charges associated to the relevant quasi-conservation laws.

Under the space-reflection symmetry (6) and (8), some of the densities of the above anomalies will present odd parities; therefore, they must vanish upon space integration. So, in such cases one can have exact conserved charges. These results will be verified for certain solutions as we will see below in the numerical simulations for the kink-kink and kink-antikink solutions.

Figures 9–12 show the anomalies $\mathrm{a}_{\pm}$ and $\gamma_{\pm}$ and their corresponding densities. The anomalies a_{-} and γ_{-} vanish as shown in the **Figures 9** and **10**, respectively, for symmetric kink-antikink soliton (see **Figure 2**), within numerical accuracy, since their densities are odd under space reflection. Similarly, for anti-symmetric kink-kink soliton (see **Figure 1**) the anomalies a_{+} and γ_{-} vanish in the **Figures 11** and **12**, respectively, since their densities are odd under space reflection.

These results suggest that the quasi-integrable models set forward in the literature [4, 6, 7], and in particular the model (1), would possess more specific integrability structures, such as an infinite set of exactly conserved charges, and some type

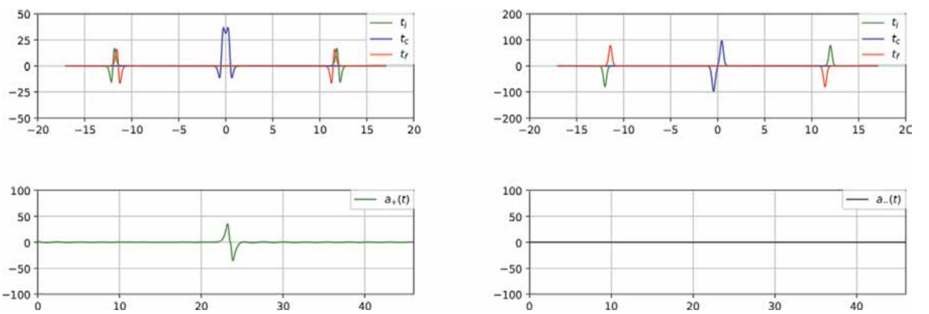

Figure 9.
*Top: The anomaly densities (37) and (38), respectively, plotted in $x-$coordinate for three times t_i (green), t_c (blue) and t_f (red). Bottom: The anomalies $a_{\pm}$ vs t, for kink-antikink collision shown in **Figure 2**.*

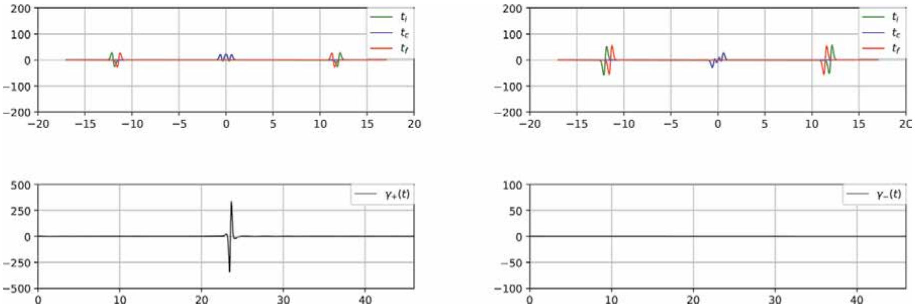

Figure 10.
*Top: Anomaly densities (39) and (40), respectively, plotted in $x-$coordinate for three times t_i(green), t_c(blue) and t_f (red). Bottom: Anomalies $\gamma_\pm$ vs t, for kink-antikink shown in **Figure 2**.*

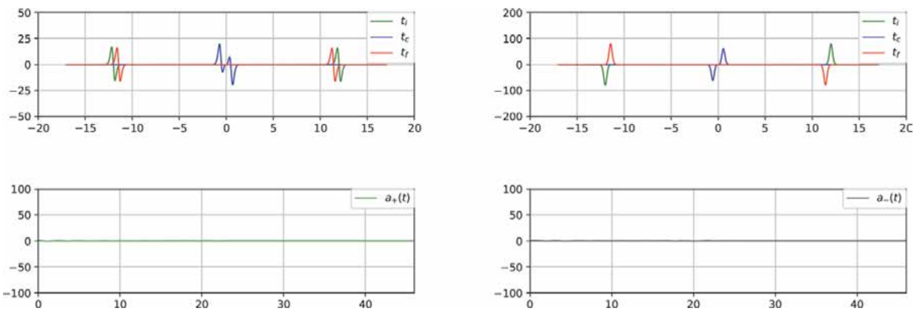

Figure 11.
*Top: Anomaly densities of (37) and (38), respectively, plotted in $x-$coordinate for three successive times t_i(green), t_c(blue) and t_f (red). Bottom figures show the relevant anomalies $a_\pm$ vs t, for kink-kink shown in **Figure 1**.*

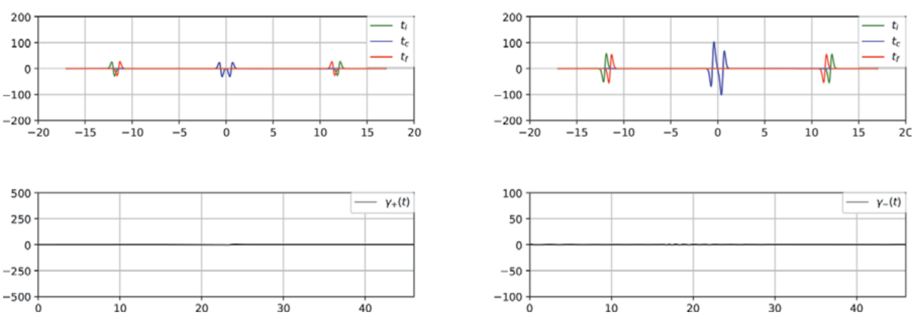

Figure 12.
*Top: Anomaly densities of (39) and (40), respectively, plotted in $x-$coordinate for three successive times t_i (green), t_c(blue) and t_f (red). Bottom: Anomalies $\gamma_\pm$ vs t, for kink-kink shown in **Figure 1**.*

of linear formulations for certain deformed potentials. So, in the next section we will tackle the problem of extending the Riccati-type pseudo-potential formalism to the deformed sine-Gordon model (1).

5. Riccati-type pseudo-potentials and non-local conservation laws

The Lax equations and Backlund transformations, as well as the conservation laws for the well-known non-linear evolution equations can be generated from the pseudo-potentials and the properties of the Riccati Equation [25–29].

So, in the next steps we consider a convenient deformation of the usual pseudo-potential approach to integrable field theories. Let us consider the system of Riccati-type equations

$$\partial_\xi u = -2\lambda^{-1} u + \partial_\xi w + \partial_\xi w \, u^2, \tag{41}$$

$$\partial_\eta u = -2\lambda (V-2) u - \frac{1}{2}\lambda V^{(1)} + \frac{1}{2}\lambda V^{(1)} u^2 + \psi, \tag{42}$$

and the next linear first order equation for ψ

$$\partial_\xi \psi + 2\lambda^{-1}\psi - 2u\partial_\xi w \psi = (2\lambda - 2u - \lambda\partial_\xi u) Z, \quad Z \equiv V^{(2)}(w) + 16V(w) - 1. \tag{43}$$

The compatibility condition $\partial_\eta(\partial_\xi u) - \partial_\xi(\partial_\eta u) = 0$ of the system (41) and (42), taking into account (43), provides the equation of motion of the DSG model (1). Moreover, the ordinary differential equation for ψ in the variable ξ can be integrated by quadratures [8]. Its expression will become highly non-local and, once inserted into (42), the system of Eqs. (41) and (42) will provide a non-local Riccati-type representation of the DSG model (1).

From the system (41) and (42) one can get a quasi-conservation law

$$\partial_\eta(u\,\partial_\xi w) + \partial_\xi\left(\lambda(V-2) - \frac{1}{2}\lambda u V^{(1)}\right) = -\lambda\partial_\xi w u Z + \partial_\xi w \psi. \tag{44}$$

This equation has been used to construct a tower of infinite number of quasi-conservation laws [8]. For the standard SG one has $Z = \psi = 0$; so the Eq. (44) can generate the well known conservation laws of the usual SG model.

5.1 Pseudo-potentials and a linear system associated to DSG

In this section we search for a linear system formulation of the DSG model. It is achieved by taking into account the Riccati Eq. (41) and the conservation law (44), as well as the Eq. (43). So, the following system of equations has been proposed as a linear formulation of the deformed SG model [8].

$$\mathcal{L}_1\Phi = 0, \qquad \mathcal{L}_2\Phi = 0, \tag{45}$$

$$\mathcal{L}_1 \equiv \partial_\xi - A_\xi, \;\; A_\xi \equiv \frac{\lambda}{2}(\partial_\xi w)^2 - 2\frac{(\partial_\xi w)^3}{\partial_\xi^2 w}, \tag{46}$$

$$\mathcal{L}_2 \equiv \partial_\eta - A_\eta, \;\; A_\eta \equiv -2\lambda - \lambda V + \zeta, \tag{47}$$

where the auxiliary non-local field ζ is defined as

$$\zeta = \int d\xi' \left[6V^{(1)} \frac{(\partial_{\xi'} w)^2}{\partial_{\xi'}^2 w} - 2V^{(2)} \frac{(\partial_{\xi'} w)^4}{\left(\partial_{\xi'}^2 w\right)^2} \right]. \tag{48}$$

In fact, taking into account the expression for the auxiliary field ζ, the compatibility condition of the linear problem (45) provides the equation

$$\Delta(\xi,\eta)\lambda - 6\frac{\partial_\xi w}{\partial_\xi^2 w}\Delta(\xi,\eta) + 2\frac{(\partial_\xi w)^2}{\left(\partial_\xi^2 w\right)^2}\partial_\xi \Delta(\xi,\eta) = 0, \tag{49}$$

with

$$\Delta(\xi,\eta) \equiv \partial_\xi \partial_\eta w + V^{(1)}(w). \tag{50}$$

In (49) the coefficient of the linear term in $\lambda\, \Delta(\xi,\eta)$ must vanish, providing the DSG equation of motion (1). The other terms in (49) must also vanish provided that $\Delta(\xi,\eta) = 0$ is imposed. So, $\mathcal{L}_1$ and $\mathcal{L}_2$ in (45) become a pair of linear operators associated to the DSG model (1).

5.2 Non-local conservation laws

For non-linear equations, not necessarily integrable, which can be derived from a compatibility condition of an associated linear system with spectral parameter, explicit expressions of local and non-local currents can be obtained (see e.g. [30, 31]). In the non-linear $\sigma-$model the non-local conserved charges imply the absence of particle production and the first non-trivial one alone fixes almost completely the on-shell dynamics of the model (see e.g. [3, 32]). These charges may be constructed through an iterative procedure [33]. Following this method one gets a set of infinite number of non-local conservation laws for the system (45). In fact, this system satisfies the properties: i) $\left(A_\xi, A_\eta\right)$ is a "pure gauge"; i.e. $A_\mu = \partial_\mu \Phi \Phi^{-1}, \mu = \xi, \eta$; ii) $J_\mu = \left(A_\xi, A_\eta\right)$ is a conserved current satisfying

$$\partial_\eta A_\xi - \partial_\xi A_\eta = 0. \tag{51}$$

So, one can construct an infinite set of non-local conserved currents through an inductive procedure. Let us define the currents

$$J_\mu^{(n)} = \partial_\mu \chi^{(n)}, \;\; \mu \equiv \xi, \eta; \;\; n = 0, 1, 2, \ldots \tag{52}$$

$$d\chi^{(1)} = A_\xi d\xi + A_\eta d\eta \equiv dI_0(\xi,\eta) + \lambda dI_1(\xi,\eta); \tag{53}$$

$$J_\mu^{(n+1)} = \partial_\mu \chi^{(n)} - A_\mu \chi^{(n)}; \;\; \chi^{(0)} = 1, \tag{54}$$

where

$$dI_0(\xi,\eta) \equiv a_0(\xi,\eta)d\xi + b_0(\xi,\eta)d\eta, \;\; dI_1(\xi,\eta) \equiv a_1(\xi,\eta)d\xi + b_1(\xi,\eta)d\eta, \tag{55}$$

where

$$a_0 \equiv -2\frac{\left(\partial_\xi w\right)^3}{\partial_\xi^2 w}; \;\; b_0 \equiv \zeta = \int d\xi' \left[6V^{(1)} \frac{\left(\partial_{\xi'} w\right)^2}{\partial_{\xi'}^2 w} - 2V^{(2)} \frac{\left(\partial_{\xi'} w\right)^4}{\left(\partial_{\xi'}^2 w\right)^2} \right]; \tag{56}$$

$$a_1 \equiv \frac{1}{2}\left(\partial_\xi w\right)^2; \quad\; b_1 \equiv -2 - V. \tag{57}$$

Then one can show by an inductive procedure that the (non-local) currents $J_\mu^{(n)}$ are conserved

$$\partial_\mu J^{(n)\mu} = 0, \;\; n = 1, 2, 3, \ldots, +\infty. \tag{58}$$

The first current conservation law $\partial_\mu J^{(1)\mu} = 0$ reduces to the Eq. (51), and then provides the first two conservation laws

$$\partial_\eta a_0 - \partial_\xi b_0 = 0, \quad \partial_\eta a_1 - \partial_\xi b_1 = 0. \tag{59}$$

The next conservation law $\partial_\mu J^{(2)\mu} = 0$, in powers of λ, furnishes

$$\partial_\eta (a_0 I_0) - \partial_\xi (b_0 I_0) = 0, \tag{60}$$

$$\partial_\eta (a_0 I_1 + a_1 I_0) - \partial_\xi (b_0 I_1 + b_1 I_0) = 0, \tag{61}$$

$$\partial_\eta (a_1 I_1) - \partial_\xi (b_1 I_1) = 0. \tag{62}$$

The construction of analogous linear systems have been performed for deformations of the KdV and NLS models [9, 10]. The construction of the classical Yangian as a Poisson-Hopf type algebra [34] for those non-local currents is worth to pursue in a future work.

6. Conclusions

Our work presents an in-depth demonstration of the quasi-integrability property of the modified sine-Gordon models and the presence of several towers of infinite number of asymptotically conserved charges for soliton configurations satisfying the space–time symmetry (4) and (5). In addition, it is observed that there exist a subset of towers of infinite number of exactly conserved charges, provided that some two-soliton configurations are eigenstates (even or odd) of the space-reflection symmetry (6)–(8).

Moreover, we have uncovered a linear system formulation (45) of the modified SG model, and an infinite set of exact non-local conservation laws (58) associated to that linear formulation.

The space–time and internal symmetries related to quasi-integrability deserve further investigations, due to their applications in several areas of non-linear science, but we hope that the results reported here have opened new lines of research in the context of the quasi-integrability phenomena.

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Chapter 5

Continuous One Step Linear Multi-Step Hybrid Block Method for the Solution of First Order Linear and Nonlinear Initial Value Problem of Ordinary Differential Equations

Kamoh Nathaniel, Kumleng Geoffrey and Sunday Joshua

Abstract

In this paper, a collocation approach for solving initial value problem of ordinary differential equations (ODEs) of the first order is presented. This approach consists of reducing the problem to a set of linear multi-step algebraic equations by approximating the ODE with a shifted Legendre polynomial basis function to determine the unknown constants. The proposed method is simple and efficient; it approximates the solutions very closely to the closed form solutions. Some problems were considered using Maple Software to illustrate the simplicity, efficiency and accuracy of the method. The results obtained revealed that the hybrid method can be suitable candidate for all forms of first order initial value problems of ordinary differential equations.

Keywords: collocation, hybrid block method, consistent, zero stable, convergent

1. Introduction

The development of mathematics parallels the human endeavor to understand our physical environment. Differential equations were discovered when the need to understand the behavior of nearly all systems undergoing change became more demanding. They are found in science and engineering as well as economics, social science, biology, business and health care. Many systems described by differential equations are so large and complex that a purely analytic solution is sometimes not traceable [1–5]. However mathematicians have studied the nature of these equations for decades of years and there are many well-developed numerical methods for the solution of different order initial value problems of ordinary differential equations. Unfortunately, in trying to achieve efficient and accurate solution, the choice of the numerical method to be adopted becomes very essential [4, 6–8].

The main goal of this paper is to derive a one step continuous hybrid block method using shifted Legendre polynomials basis function with the expectation that the numerical (proposed) method will give a solution that is close to the close form solution of the initial value problems of first order nonlinear ordinary differential equations. The paper is structured as follows. In Section 2, we derived and analyze the obtained schemes for consistency, zero stability and convergence. Some first order nonlinear problems of ordinary differential equations were solved using the derived schemes and the main results are presented in Section 3. Finally, we end with some concluding remarks in Section 4, where we compared our results with some earlier results contained in the literature.

1.1 Linear multistep methods (LMMs)

Linear Multi-step Methods (LMMs) are very popular for solving Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs). They are also applied in solving higher order ODEs. LMMs are not self-starting and therefore, need starting values from single-step methods like Euler's method and Runge–Kutta family of methods [1, 9, 10].

The general $k - step$ LMM of the discrete form as given in [11–14] is;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{1}$$

where α_j and β_j are uniquely determined and $\alpha_j + \beta_j \neq 0$. The LMM (1) generates discrete schemes which are used to solve first order ODEs. However, the continuous Linear Multi-step Methods (CLMMs) which is now being used was introduced by [15] and used by so many researchers such as [6, 7, 9, 16, 17] leading to the development of what is now called continuous Linear Multi-step Methods (CLMMs) given by;

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h \sum_{j=0}^{k} \beta_j(x) f_{n+j}, \tag{2}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are expressed as continuous functions of x and are continuously differentiable at least m times $(m \geq 1)$. According to [1, 2, 11, 12, 18], the existing methods of deriving the LMMs in discrete form include the interpolation approach, numerical integration, Taylor series expansion. While the collocation and interpolation technique is widely used for the derivation of CLMMs, this method is derived using different techniques and approaches.

The introduction of CLMMs have numerous advantages which is of great importance; better global error is estimated, it can be used to recover some standard schemes, it provides a simplified form of coefficients for further analytical work at different points and guarantee easy approximation of solutions at all interior points of the integration interval [1, 7, 16, 19, 20].

In this work, the CLMM is developed for the solution of (linear and nonlinear) first-order initial value problems of ordinary differential equations using the shifted Legendre polynomials basis function. The corresponding discrete schemes are obtained from the evaluation of the continuous scheme at some selected grid points.

1.2 Shifted Legendre polynomials

The shifted Legendre polynomials are well known family of orthogonal polynomials on the interval $[0, A]$ and are denoted by $P_i(t)$, the $P_i(t)$ can be obtained by the recurrence formula:

$$p_i(t) = \sum_{k=0}^{i} (-1)^{(i+k)} \frac{(i+k)!t^k}{(i-k)!(k!)^2 A^k}, i = 1, 2, \dots,$$

where $P_0(t) = 1$ and $P_1(t) = 2\mathrm{t} - 1$

The first few terms of the shifted Legendre polynomials on the interval [0, A] with $A = 1$ are:

$$P_0(t) = 1,$$
$$P_1(t) = 2t - 1,$$
$$P_2(t) = 6t^2 - 6t + 1,$$
$$P_3(t) = 20t^3 - 30t^2 + 12t - 1,$$
$$P_4(t) = 70t^4 - 140t^3 + 90t^2 - 20t + 1,$$
$$P_5(t) = 252t^5 - 630t^4 + 560t^3 - 210t^2 + 30t - 1,$$
$$P_6(t) = 924t^6 - 2772t^5 + 3150t^4 - 1680t^3 + 420\, t^2 - 42t + 1.$$

1.3 Collocation method

A collocation is a method which involves the determination of an approximate solution of a functional equation in a suitable set of functions called trial or basis functions. The approximate solution is required to satisfy the equation and its supplementary conditions at certain points in the range of interest called collocation points.

2. Derivation of one step hybrid block methods with shifted Legendre polynomials

We consider the first order ordinary differential equation of the form

$$y'(x) = f(x, y(x)), y(x_0) = y_0, \tag{3}$$

where $y(x)$ is the unknown function to be determined. The idea here is to approximate the exact solution $y(x)$ of (3) in the partition $I_n = [a = x_0 < x_1 < x_2 < \dots < x_n = b]$ of the integration interval $[a, b]$ with a constant step size $h = x_i - x_{i-1}, i = 1, \dots, n$ by a shifted Legendre polynomial basis function of degree $s + r - 1$ of the form;

$$y(x) = \sum_{i=0}^{s+r-1} c_i P_i(t), \tag{4}$$

where $c_i \in \mathbb{R}, y \epsilon C^1(a, b)$ and $t = (x - x_n)$. The first derivative of (4), is substituted into (3), to obtain a differential system of the form

$$y'(x) = \sum_{i=0}^{s+r-1} c_i p_i'(t) = f(x, y(x)), \tag{5}$$

Now interpolating (4) at $x_{n+s}, s = \frac{1}{2}, \frac{3}{4}$ and collocating (5) at $x_{n+r}, r = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, k$ where s and r represents the interpolation and collocation points respectively, and k is the step number, after some substitutions and manipulations the continuous scheme of the form;

$$y(x) = \alpha_{\frac{1}{2}}(x) y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(x) y_{n+\frac{3}{4}} + h\left(\sum_{\tau=0}^{k} \beta_{\tau}(x) f\left(x_{n+\tau}, y_{n+\tau}\right) + \beta_{\mu}(x) f\left(x_{n+\mu}, y_{n+\mu}\right)\right),$$
$$\mu = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}. \tag{6}$$

is obtained with the following continuous coefficients

$$\left.\begin{aligned}
\alpha_{\frac{1}{2}}(x) &= \frac{27}{11} + \frac{24576}{11h^5}t^5 - \frac{26880}{11h^4}t^4 + \frac{12800}{11h^3}t^3 - \frac{2304}{11h^2}t^2 - \frac{8192}{11h^6}t^6 \\
\alpha_{\frac{3}{4}}(x) &= -\frac{16}{11} + \frac{2304}{11h^2}t^2 - \frac{12800}{11h^3}t^3 + \frac{26880}{11h^4}t^4 - \frac{24576}{11h^5}t^5 + \frac{8192}{11h^6}t^6 \\
\beta_0(x) &= t - \frac{3}{40}h - \frac{149}{30h}t^2 + \frac{110}{9h^2}t^3 - \frac{16}{h^3}t^4 + \frac{32}{3h^4}t^5 - \frac{128}{45h^5}t^6 \\
\beta_{\frac{1}{4}}(x) &= -\frac{21}{55}h + \frac{736}{55h}t^2 - \frac{5248}{99h^2}t^3 + \frac{2864}{33h^3}t^4 - \frac{2176}{33h^4}t^5 + \frac{9472}{495h^5}t^6 \\
\beta_{\frac{1}{2}}(x) &= \frac{9}{55}h - \frac{2154}{55h}t^2 + \frac{6916}{33h^2}t^3 - \frac{4608}{11h^3}t^4 + \frac{4032}{11h^4}t^5 - \frac{19456}{165h^5}t^6 \\
\beta_{\frac{3}{4}}(x) &= \frac{9}{55}h - \frac{3712}{165h}t^2 + \frac{12608}{99h^2}t^3 - \frac{3024}{11h^3}t^4 + \frac{8576}{33h^4}t^5 - \frac{44288}{495h^5}t^6 \\
\beta_1(x) &= -\frac{3}{440}h + \frac{97}{110h}\,t^2 - \frac{518}{99h^2}t^3 + \frac{400}{33h^3}t^4 - \frac{416}{33h^4}t^5 + \frac{2432}{495h^5}t^6
\end{aligned}\right\}. \tag{7}$$

In order to obtain the block discrete scheme for ($K = 1$), Eq. (7) is evaluated at $x = x_n, x_{n+\frac{1}{8}}, x_{n+\frac{1}{4}}, x_{n+1}$ and its first derivative at $x_{n+\frac{1}{8}}$ to give the following discrete schemes;

$$\left.\begin{aligned}
y_{n+\frac{1}{8}} &= \frac{325}{352}y_{n+\frac{1}{2}} - \frac{15}{2048}hf_n + \frac{27}{352}y_{n+\frac{3}{4}} - \frac{15}{22528}hf_{n+1} - \frac{735}{5632}hf_{n+\frac{1}{2}} - \frac{2895}{11264}hf_{n+\frac{1}{4}} + \frac{15}{11264}hf_{n+\frac{3}{4}} \\
y_{n+\frac{1}{4}} &= \frac{1}{360}hf_n + y_{n+\frac{3}{4}} + \frac{1}{360}hf_{n+1} - \frac{19}{60}hf_{n+\frac{1}{2}} - \frac{17}{180}hf_{n+\frac{1}{4}} - \frac{17}{180}hf_{n+\frac{3}{4}} \\
y_{n+\frac{1}{2}} &= -\frac{11}{27}y_n + \frac{11}{360}hf_n + \frac{16}{27}y_{n+\frac{3}{4}} + \frac{1}{360}hf_{n+1} - \frac{1}{15}hf_{n+\frac{1}{2}} + \frac{7}{45}hf_{n+\frac{1}{4}} - \frac{1}{15}hf_{n+\frac{3}{4}} \\
y_{n+\frac{3}{4}} &= y_{n+\frac{1}{2}} - \frac{11}{720}hf_n - \frac{13}{3360}hf_{n+1} + \frac{283}{1440}hf_{n+\frac{1}{2}} - \frac{49}{480}hf_{n+\frac{1}{4}} + \frac{151}{1440}hf_{n+\frac{3}{4}} + \frac{22}{315}hf_{n+\frac{1}{8}} \\
y_{n+1} &= \frac{27}{11}y_{n+\frac{1}{2}} + \frac{1}{360}hf_n - \frac{16}{11}y_{n+\frac{3}{4}} + \frac{281}{3960}hf_{n+1} + \frac{49}{165}hf_{n+\frac{1}{2}} - \frac{13}{495}hf_{n+\frac{1}{4}} + \frac{257}{495}hf_{n+\frac{3}{4}}
\end{aligned}\right\} \tag{8}$$

Eq. (7) is the continuous scheme while (8) is the block discrete schemes for step number $K = 1$.

2.1 Order and error constant

Expanding (8), in Taylor's series gives;

$$C_q = \begin{bmatrix} \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left((1)^j - \frac{27}{11}\left(\frac{1}{2}\right)^j + \frac{16}{11}\left(\frac{3}{4}\right)^j\right)y_n^j - h\left(\frac{1}{360}(0)^j - \frac{13}{495}\left(\frac{1}{4}\right)^j + \frac{49}{165}\left(\frac{1}{2}\right)^j + \frac{257}{495}\left(\frac{3}{4}\right)^j + \frac{281}{3960}(1)^j\right)y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!}y_n^j\left(\left(\left(\frac{3}{4}\right)^j - \left(\frac{1}{2}\right)^j\right)y_n^j - h\left(-\frac{11}{720}(0)^j - \frac{49}{480}\left(\frac{1}{4}\right)^j + \frac{283}{1440}\left(\frac{1}{2}\right)^j + \frac{151}{1440}\left(\frac{3}{4}\right)^j + \frac{22}{315}\left(\frac{1}{8}\right)^j - \frac{13}{3360}(1)^j\right)y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left(\left(\frac{1}{2}\right)^j - \frac{16}{27}\left(\frac{1}{2}\right)^j + \frac{11}{27}(0)^j\right)y_n^j - h\left(\frac{11}{360}(0)^j + \frac{7}{45}\left(\frac{1}{4}\right)^j - \frac{1}{15}\left(\frac{1}{2}\right)^j - \frac{7}{15}\left(\frac{3}{4}\right)^j + \frac{1}{360}(1)^j\right)y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left(\left(\frac{1}{4}\right)^j - \left(\frac{3}{4}\right)^j\right)y_n^j - h\left(\frac{1}{360}(0)^j - \frac{17}{180}\left(\frac{1}{4}\right)^j - \frac{19}{60}\left(\frac{1}{2}\right)^j - \frac{17}{180}\left(\frac{3}{4}\right)^j + \frac{1}{360}(1)^j\right)y_n^{(j+1)}\right) = 0 \\ \sum_{j=0}^{q} \frac{h^j}{j!}\left(\left(\left(\frac{1}{8}\right)^j - \frac{325}{352}\left(\frac{1}{2}\right)^j - \frac{27}{352}\left(\frac{3}{4}\right)^j\right)y_n^j - h\left(-\frac{15}{2048}(0)^j - \frac{2895}{5632}\left(\frac{1}{4}\right)^j - \frac{735}{5632}\left(\frac{1}{2}\right)^j + \frac{15}{11264}\left(\frac{3}{4}\right)^j - \frac{15}{22528}(1)^j\right)y_n^{(j+1)}\right) = 0 \end{bmatrix}$$

and collecting like terms in powers of h, gives $c_0 = c_1 = c_2 = \ldots = c_6 = (0, 0, 0, 0, 0)^T$ and $c_7 = \left(-\frac{17}{85155840}, \frac{311}{1981808640}, -\frac{1}{7741440}, -\frac{1}{12386304}, \frac{135}{2583691264}\right)^T$. Hence, the method has order $p = (6, 6, 6, 6, 6)^T$ and with error constants of $c_7 = \left(-\frac{17}{85155840}, \frac{311}{1981808640}, -\frac{1}{7741440}, -\frac{1}{12386304}, \frac{135}{2583691264}\right)^T$.

2.2 Consistency

The linear multi-step method (8) is said to be consistent if the following conditions hold:

i. it has order $\check{p} \geq 1$,

ii. $\sum_{j=0}^{k} \check{\alpha}_j = 0$,

iii. $\sum_{j=0}^{k} j\check{\alpha}_j = \sum_{j=0}^{k} \check{\beta}_j$,

iv. $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$,

where $\rho(r)$ and $\sigma(r)$ are the first and the second characteristic polynomials of (8) respectively, [21]. Following [8, 14], (i) is sufficient condition for the block method (8) to be consistent since $p = (6, 6, 6, 6, 6)^T > 1$. Hence, the method is consistent.

2.3 Zero stability

The block solution (8), is said to be zero stable if the roots $z_r; r = 1, \ldots, n$ of the first characteristic polynomial $p(z)$, defined by

$$p(z) = det|zQ - T|$$

satisfies $|z_r| \leq 1$ and every root with $|z_r| = 1$ has multiplicity not exceeding the order of the differential equation in the limit as $h \to 0$.

Calculations from all available information revealed that the block method (8) has the following roots

$$z^4(z - 1) = 0 \Rightarrow z = (0, 0, 0, 0, 1)$$

Hence the block method is zero stable, since all roots with modulus one do not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$.

2.4 Convergence

According to [8, 14, 22], we can safely assert the convergence of the block method (8) since the method is consistent and zero stable.

2.5 Region of absolute stability of the block method

Reformulating the block (8) as a General Linear Method (GLM) containing a partition of matrices A and B using the stability polynomial $(Ar - B)$, where

$$A = \begin{bmatrix} 1 & \frac{2895}{11264}z & -\frac{325}{352}+\frac{735}{5632}z & -\frac{27}{352}-\frac{15}{11264}z & \frac{15}{22528}z \\ 0 & 1+\frac{17}{180}z & \frac{19}{60}z & -1+\frac{17}{180}z & -\frac{1}{360}z \\ 0 & -\frac{7}{45}z & 1+\frac{1}{15}z & -\frac{16}{27}+\frac{1}{15}z & -\frac{1}{360}z \\ -\frac{22}{315}z & -\frac{49}{480} & 1+\frac{283}{1440}z & 1+\frac{151}{1440}z & \frac{13}{3360}z \\ 0 & \frac{13}{495}z & -\frac{27}{11}-\frac{49}{165}z & \frac{16}{11}-\frac{257}{495}z & 1-\frac{281}{3960}z \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{15}{2048}z \\ 0 & 0 & 0 & 0 & \frac{1}{360}z \\ 0 & 0 & 0 & 0 & \frac{11}{27}+\frac{11}{360}z \\ 0 & 0 & 0 & 0 & \frac{11}{720}z \\ 0 & 0 & 0 & 0 & \frac{13}{3360}z \end{bmatrix}$$

we obtain the region of absolute stability shown in **Figure 1** below

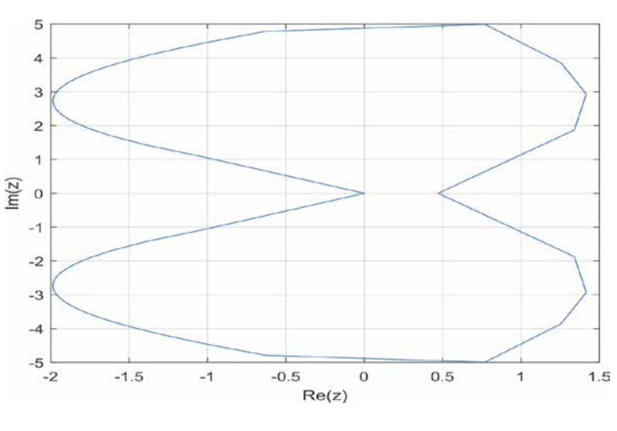

Figure 1.
Region of absolute stability.

3. Numerical experiments

This section discusses the implementation of the derived method by solving some first order nonlinear initial value problems of ordinary differential equations.

Problem 1

We consider a nonlinear first order initial value problem of ordinary differential problem which was solved by [23]. $y'(x) = -10(y-1)^2; y(0) = 2$, $h = 0.01$ With exact solution given as $y(x) = 1 + \frac{1}{1+10x}$, the result is shown in **Table 1**, while the theoretical and numerical results are presented graphically in **Figure 2**.

Problem 2

Given a nonlinear first order ordinary differential problem solved by [24] (**Table 2**). $y'(x) = 2xy, y(0) = 1,\ h = 0.1\ x \in [0, 1]$ with exact solution given by $y(x) = e^{x^2}$, the result is shown in **Table 2**, **Figure 3** shows the solution curve for problem 2.

x	Exact solution	Result of Proposed Method	Error in Proposed Method	Error in [23]
0.01	1.90909090909091	1.90909090891558	1.7533×10^{-10}	2.829001×10^{-7}
0.02	1.83333333333333	1.83333333310133	2.3200×10^{-10}	4.045782×10^{-7}
0.03	1.76923076923077	1.76923076898962	2.4115×10^{-10}	4.472541×10^{-7}
0.04	1.71428571428571	1.71428571405431	2.3140×10^{-10}	4.509027×10^{-7}
0.05	1.66666666666667	1.66666666645183	2.1484×10^{-10}	4.356251×10^{-7}
0.06	1.62500000000000	1.62499999980340	1.9660×10^{-10}	4.117637×10^{-7}
0.07	1.58823529411765	1.58823529393878	1.7887×10^{-10}	3.846989×10^{-7}
0.08	1.55555555555556	1.55555555539306	1.6250×10^{-10}	3.572176×10^{-7}
0.09	1.52631578947368	1.52631578932595	1.4773×10^{-10}	3.307245×10^{-7}
0.10	1.50000000000000	1.49999999986543	1.3457×10^{-10}	3.058785×10^{-7}

Table 1.
(Problem 1): Comparing results of proposed method with [23].

x	Exact solution	Result of Proposed Method	Error in Proposed Method	Error in [24]
0.1	1.01005016708417	1.01005016708855	4.3800×10^{-12}	1.899500×10^{-1}
0.2	1.04081077419239	1.04081077421089	1.8500×10^{-11}	1.714527×10^{-1}
0.3	1.09417428370521	1.09417428375087	4.5660×10^{-11}	1.556419×10^{-1}
0.4	1.17351087099181	1.17351087108422	9.2410×10^{-11}	1.415053×10^{-1}
0.5	1.28402541668774	1.28402541685820	1.7046×10^{-10}	1.280382×10^{-1}
0.6	1.43332941456034	1.43332941486021	2.9987×10^{-10}	1.141249×10^{-1}
0.7	1.63231621995538	1.63231622047036	5.1498×10^{-10}	9.839200×10^{-2}
0.8	1.89648087930495	1.89648088017992	8.7497×10^{-10}	7.9005900×10^{-2}
0.9	2.24790798667647	2.24790798815910	1.48263×10^{-9}	5.3376500×10^{-2}
1.0	2.71828182845905	2.71828183097715	2.51810×10^{-9}	1.7703800×10^{-2}

Table 2.
(Problem 2): Comparing results of proposed method with [24].

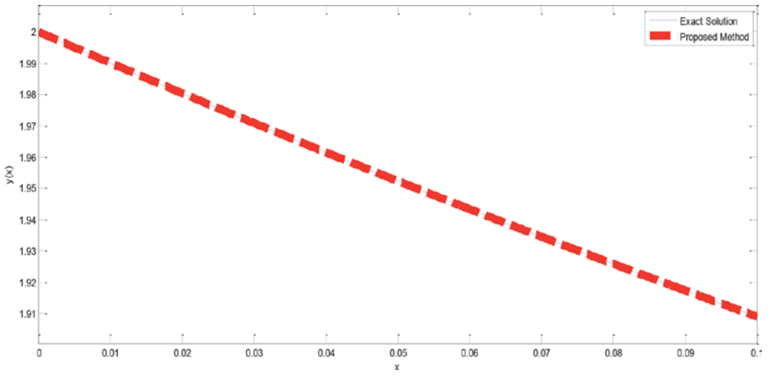

Figure 2.
Solution curve for problem 1.

Problem 3

Considering the first order initial value problem of ordinary differential problem solved by [25] (**Table 3**). $y'(x) = -y^2, y(0) = 1,\ h = 0.01\ x \in [0, 1]$ with exact

x	Exact solution	Result of Proposed Method	Error in Proposed Method	Error in [25]
0.01	0.990099009900990	0.990099009900989	1.000000×10^{-15}	2.91799×10^{-11}
0.02	0.980392156862745	0.980392156862744	1.000000×10^{-15}	3.71577×10^{-11}
0.03	0.970873786407767	0.970873786407766	1.000000×10^{-15}	3.93663×10^{-11}
0.04	0.961538461538462	0.961538461538460	2.000000×10^{-15}	3.39936×10^{-11}
0.05	0.952380952380952	0.952380952380951	1.000000×10^{-15}	2.94922×10^{-11}
0.06	0.943396226415094	0.943396226415093	1.000000×10^{-15}	2.61278×10^{-11}
0.07	0.934579439252336	0.934579439252335	1.000000×10^{-15}	2.31487×10^{-11}
0.08	0.925925925925926	0.925925925925925	1.000000×10^{-15}	6.80704×10^{-11}
0.09	0.917431192660550	0.917431192660549	1.000000×10^{-15}	8.31745×10^{-11}
0.10	0.909090909090909	0.909090909090908	1.000000×10^{-15}	7.50649×10^{-11}

Table 3.
(Problem 3): Comparing results of proposed method with [25].

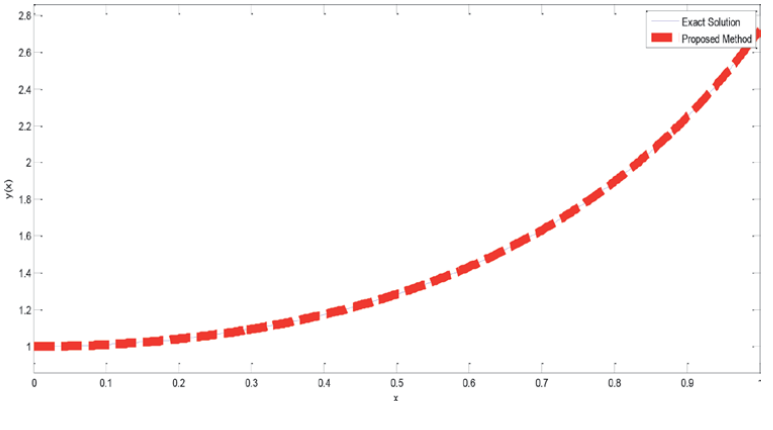

Figure 3.
Solution curve for problem 2.

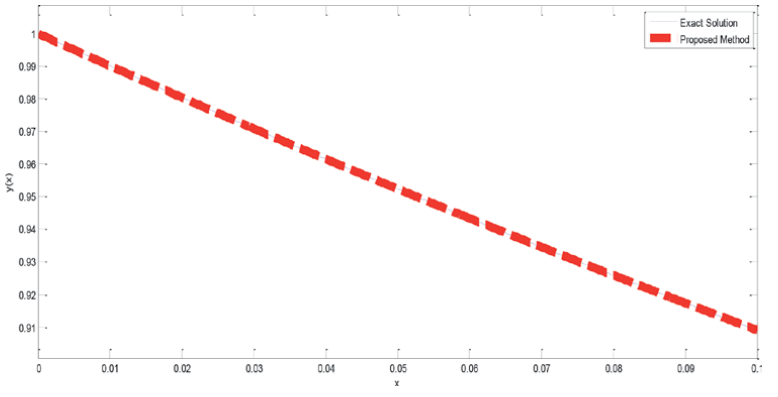

Figure 4.
Solution curve for problem 3.

solution given by $y(x) = \frac{1}{1+x}$ with the result is shown in **Table 3**, **Figure 4** compare the two results (theoretical and numerical graphically).

4. Conclusion

In this paper, we derived one step block hybrid continuous linear multi-step method for solving first order initial value problems of ordinary differential equations, the method was found to be consistent, zero stable and convergent. The method was implemented on some nonlinear initial value problems of ordinary differential equations and the numerical results were found to be accurate when compared with the exact solutions and other numerical methods as contained in **Tables 1–3** and their respective solution curves. The new hybrid block method can be suitable candidate for all forms (linear and nonlinear) of first order initial value problems of ordinary differential equations.

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Authors contributions

This work was carried out in collaboration among the authors. Author Kamoh, N.M. proposed, derived and implemented the method. Author Kumleng, G.M. analyzed the method while Author Sunday; J. presented the numerical results graphically. All the authors managed the literature searches, read and approved the final manuscript.

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Chapter 6

Existence and Asymptotic Behaviors of Nonoscillatory Solutions of Third Order Time Scale Systems

Özkan Öztürk

Abstract

Nonoscillation theory with asymptotic behaviors takes a significant role for the theory of three-dimensional (3D) systems dynamic equations on time scales in order to have information about the asymptotic properties of such solutions. Some applications of such systems in discrete and continuous cases arise in control theory, optimization theory, and robotics. We consider a third order dynamical systems on time scales and investigate the existence of nonoscillatory solutions and asymptotic behaviors of such solutions. Our main method is to use some well-known fixed point theorems and double/triple improper integrals by using the sign of solutions. We also provide examples on time scales to validate our theoretical claims.

Keywords: nonoscillation, three-dimensional time scale systems, dynamical systems, existence, fixed point theorems

1. Introduction

This chapter deals with the nonoscillatory solutions of 3D nonlinear dynamical systems on time scales. In addition, it is very critical to discuss whether or not there exist such solutions. Therefore, the existence along with limit behaviors are also studied in this chapter by using double/triple integrals and fixed point theorems. Stefan Hilger, a German mathematician, introduced a theory in his PhD thesis in 1988 [1] that unifies continuous and discrete analysis and extend it in one comprehensive theory, which is called the *time scale* theory. A *time scale*, symbolized by $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. After Hilger, the theory and its applications have been developed by many mathematicians and other researchers in Control Theory, Optimization, Population Dynamics and Economics, see [2–5]. In addition to those articles, two books were published by Bohner and Peterson in 2001 and 2003, see [6, 7].

Now we explain what we mean by continuous and discrete analysis in details. Assuming readers are all familiar with differential and difference equations; the results are valid for differential equations when $\mathbb{T} = \mathbb{R}$ (set of real numbers), while the results hold for difference equations when $\mathbb{T} = \mathbb{Z}$ (set of integers). So we might have two different proofs and maybe similar in most cases. In order to avoid repeating similarities, we combine continuous and discrete cases in one general theory and remove the duplication from both. For more details in the theory of differential and difference equations, we refer the books [8–10] to interested readers.

3D nonlinear dynamical systems on time scales have recently gotten a valuable attention because of its potential in applications of control theory, population dynamics and mathematical biology and Physics. For example, Akn, Güzey and Öztürk [3] considered a 3D dynamical system to control a wheeled mobile robots on time scales

$$\begin{cases} \alpha^{\Delta}(t) = -v(t)\cos\beta(t) \\ \beta^{\Delta}(t) = \dfrac{\sin\beta^{\sigma}(t)}{\alpha^{\sigma}(t)}v(t) - w(t) \\ \gamma^{\Delta}(t) = \dfrac{\sin\beta^{\sigma}(t)}{\alpha^{\sigma}(t)}v(t), \end{cases} \tag{1}$$

where α is the distance of the reference point from the origin, β is the angle of the pointing vector to the origin, γ is the angle with respect to the x axis, and v, w are controllers. They showed the asymptotic stability of the system above on time scales. Another example for $\mathbb{T} = \mathbb{R}$, Bernis and Peletier [11] considered an equation that can be written as the following system

$$\begin{cases} u_1' = u_2 \\ u_2' = u_3 \\ u_3' = h(u) \end{cases} \tag{2}$$

to show the existence and uniqueness and properties of solutions for flows of thin viscous films over solid surfaces, where (u_1, u_2, u_3) is the film profile in a coordinate frame moving with the fluid.

We assume that readers may not be familiar with the time scale basics, so we give an introductory section to the time scale calculus. We refer the books [6, 7] for more details and information about time scales. Structure of the rest of this chapter is as follows: In Section 3.1 and 3.2 we consider a system with different values, 1 and -1, respectively, and show the qualitative behavior of solutions. In Section 4, we give some examples for readers to comprehend our theoretical results. Finally, we give a short conclusion about the summary of our results and open problems in the last section.

2. Time scale essentials

In the introduction section, we have only mentioned the time scales $\mathbb{R}$ and $\mathbb{Z}$. However, there are some other time scales in the literature, which also have gotten too much attention because of the applications of them. For example, when $\mathbb{T} = q^{\mathbb{N}_0} = \{1, q, q^2, \cdots, \}, q > 1$, the results hold for so-called *q-difference* equations, see [12]. Another well-known time scale is $\mathbb{T} = h\mathbb{Z}, h > 0$.

Definition 2.1 Let $\mathbb{T}$ be a time scale. Then for all $t \in \mathbb{T}$,

1. $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is called *forward jump operator* $(\sigma(t) : \mathbb{T} \to \mathbb{T})$.

2. $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ is said to be *backward jump operator* $(\rho(t) : \mathbb{T} \to \mathbb{T})$.

3. $\mu(t) := \sigma(t) - t$ for all $t \in \mathbb{T}$ is called the *graininess function* $((\mu(t) : \mathbb{T} \to [0, \infty))$.

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$h\mathbb{Z}$	$t+h$	$t-h$	h
$q^{\mathbb{N}_0}$	tq	$\frac{t}{q}$	$t(q-1)$

Table 1.
Some time scales with σ, ρ and μ.

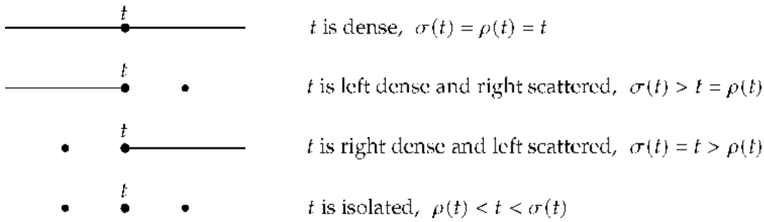

Figure 1.
Classification of points.

For the sake of the rest of the chapter, **Table 1** summarizes how σ, ρ and μ are defined for some time scales.

As we know, the set of real numbers are dense and set of integers are scattered. Now we show how we classify the points on general time scales. For any $t \in \mathbb{T}$, **Figure 1** shows the classification of points on time scales and how we represent those points by using σ, ρ and μ, see [6] for more details.

Now, let us introduce the derivative for general time scales. Note that

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if} \quad \sup\mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup\mathbb{T} = \infty. \end{cases}$$

Definition 2.2 If there exists a $\delta > 0$ such that

$$|h(\sigma(t)) - h(s) - h^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in (t-\delta, t+\delta) \cap \mathbb{T},$$

for any ε, then h is said to be *delta-differentiable* on $\mathbb{T}^\kappa$ and h^Δ is called the *delta derivative* of h.

Theorem 2.3 *Let $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ be functions with $t \in \mathbb{T}^\kappa$. Then.*

i. h_1 is said to be continuous at t if h_1 is differentiable at t.

ii. h_1 is differentiable at t and

$$h_1^\Delta(t) = \frac{h_1(\sigma(t)) - h_1(t)}{\mu(t)},$$

provided h_1 is continuous at t and t is right-scattered.

iii. Suppose t is right dense, then h_1 is differentiable at t if and only if

$$h_1^\Delta(t) = \lim_{s \to t} \frac{h_1(t) - h_1(s)}{t - s}$$

exists as a finite number.

iv. If $h_2(t)h_2(\sigma(t)) \neq 0$, then $\frac{h_1}{h_2}$ is differentiable at t with

$$\left(\frac{h_1}{h_2}\right)^{\Delta}(t) = \frac{h_1^{\Delta}(t)h_2(t) - h_1(t)h_2^{\Delta}(t)}{h_2(t)h_2(\sigma(t))}.$$

A function $h_1 : \mathbb{T} \to \mathbb{R}$ is called *right dense continuous* (rd-continuous) if it is continuous at right dense points in $\mathbb{T}$ and its left sided limits exist at left dense points in $\mathbb{T}$. We denote the set of rd-continuous functions with $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. On the other hand, the set of differentiable functions whose derivative is *rd-continuous* is denoted by $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. Finally, we use C for the set of continuous functions throughout this chapter.

After derivative and its properties, we also introduce integrals for any time scale $\mathbb{T}$. The Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for all } a, b \in \mathbb{T}.$$

Every rd-continuous function has an antiderivative. Moreover, F given by

$$F(t) = \int_{t_0}^{t} f(s)\Delta s \text{ for } t \in \mathbb{T}$$

is an antiderivative of f.

The following theorem leads us to the properties of integrals on time scales, which are similar to continuous case.

Theorem 2.4 Suppose that h_1 and h_2 are rd-continuous functions, $c, d, e \in \mathbb{T}$, and $\beta \in \mathbb{R}$,

a. h_1 is nondecreasing if $h_1^{\Delta} \geq 0$.

b. If $h_1(t) \geq 0$ for all $c \leq t \leq d$, then $\int_c^d h_1(t)\Delta t \geq 0$.

c. $\int_c^d [(\beta h_1(t)) + (\beta h_2(t))] = \beta \int_c^d h_1(t)\Delta t + \beta \int_a^b h_2(t)\Delta t$.

d. $\int_c^e h_1(t)\Delta t = \int_c^d h_1(t)\Delta t + \int_d^e h_1(t)\Delta t$.

e. $\int_c^d h_1(t)h_2^{\Delta}(t)\Delta t = (h_1h_2)(d) - (h_1h_2)(c) - \int_c^d h_1^{\Delta}(t)h_2(\sigma(t))\Delta t$

f. $\int_a^a h_1(t)\Delta t = 0$.

Table 2 shows how the derivative and integral are defined for some time scales for $a, b \in \mathbb{T}$.

$\mathbb{T}$	$f^{\Delta}(t)$	$\int_a^b f(t)\Delta t$
$\mathbb{R}$	$f'(t)$	$\int_a^b f(t)dt$
$\mathbb{Z}$	$\Delta f(t) = f(t+1) - f(t)$	$\sum_{t=a}^{b-1} f(t)$
$q^{\mathbb{N}_0}$	$\Delta_q f(t) = \frac{f(tq) - f(t)}{(q-1)t}$	$\sum_{[a,b)_{q^{\mathbb{N}_0}}} f(t)\mu(t)$

Table 2.
Derivative and integral for some time scales.

This chapter assumes that $\mathbb{T}$ is unbounded above and whenever it is written $t \geq t_1$, we mean $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. Finally, we provide Schauder's fixed point theorem, proved in 1930, see ([13], Theorem 2.A), the Knaster fixed point theorem, proved in 1928, see [14] and the following lemma, see [15], to show the existence of solutions.

Lemma 2.5 Let X be equi-continuous on $[t_0, t_1]_{\mathbb{T}}$ for any $t_1 \in [t_0, \infty)_{\mathbb{T}}$. In addition to that, let $X \subseteq BC[t_0, \infty)_{\mathbb{T}}$ be bounded and uniformly Cauchy. Then X is relatively compact.

Theorem 2.6 (Schauder's Fixed Point Theorem) Suppose that X is a Banach space and M is a nonempty, closed, bounded and convex subset of X. Also let $T : M \to M$ be a compact operator. Then, T has a fixed point such that $y = Ty$.

Theorem 2.7 (The Knaster Fixed Point Theorem) Supposing $(M, \leq)$ being a complete lattice and $F : M \to M$ is order-preserving, we have F has a fixed point so that $y = Fy$. In fact, the set of fixed points of F is a complete lattice.

3. Nonoscillatory solutions of nonlinear dynamical systems

Motivated by [16, 17], we deal with the nonlinear system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = q(t)g(z(t)) \\ z^{\Delta}(t) = \lambda r(t)h(x(t)), \end{cases} \tag{3}$$

where $p.q, r \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, $\lambda = \pm 1$, and f and g are nondecreasing functions such that $uf(u) > 0$, $ug(u) > 0$ and $uh(u) > 0$ for $u \neq 0$.

The other continuous and discrete cases of system (3) were studied in [18–20]. We first give the following definitions to help readers understand the terminology.

Definition 3.1 If (x, y, z), where $x, y, z \in \mathrm{C}^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}) T \geq t_0$, satisfies system (3) for all large $t \geq T$, then we say (x, y, z) is a solution of (3).

Definition 3.2 By a proper solution (x, y, z), we mean a solution (x, y, z) of system (3) that holds

$$\sup\{|x(s)|, |y(s)|, |z(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$$

for $t \geq t_0$.

Finally, let us define nonoscillatory solutions of system (3).

Definition 3.3 By a nonoscillatory solution (x, y, z) of system (3), we mean a proper solution and the component functions x, y and z are all nonoscillatory. In other words, (x, y, z) is either eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

For the sake of simplicity, let us set

$$P(t_0, t) = \int_{t_0}^{t} p(s)\Delta s, \quad Q(t_0, t) = \int_{t_0}^{t} q(s)\Delta s \quad \text{and} \quad R(t_0, t) = \int_{t_0}^{t} r(s)\Delta s,$$

where $s, t, t_0 \in \mathbb{T}$ and we assume that $P(t_0, \infty) = Q(t_0, \infty) = \infty$ throughout the chapter.

Suppose that N is the set of all nonoscillatory solutions (x, y, z) of system (3). Then according to the possible signs of solutions of system (3), we have the following classes:

$$N^a := \{(x, y, z) \in N: \quad \text{sgnx}(t) = \text{sgny}(t) = \text{sgnz}(t), \quad t \ge t_0\}$$
$$N^b := \{(x, y, z) \in N: \quad \text{sgnx}(t) = \text{sgnz}(t) \ne \text{sgny}(t), \quad t \ge t_0\}$$
$$N^c := \{(x, y, z) \in N: \quad \text{sgnx}(t) = \text{sgny}(t) \ne \text{sgnz}(t), \quad t \ge t_0\}.$$

It was shown in [21] that any nonoscillatory solution of system (3) for $\lambda = 1$ belongs to N^a or N^c, while it belongs to N^a or N^b for $\lambda = -1$. In the literature, solutions in N^a, N^b and N^c are also known as *Type* (a), *Type* (b) and *Type* (c) solutions, respectively.

Next, we consider system (3) for $\lambda = 1$ and $\lambda = -1$ separately in different subsections, split the classes N^a, N^b and N^c into some subclasses and show the existence of nonoscillatory solutions in those subclasses. To show the existence and limit behaviors, we use the following improper integrals:

$$Y_1 = \int_{t_0}^{\infty} r(t)h\left(\int_{t_0}^{t} p(s)f\left(k_1\int_{t_0}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_2 = \int_{t_0}^{\infty} p(t)f\left(k_2 + \int_{t}^{\infty} q(s)g\left(k_3\int_{s}^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_3 = \int_{t_0}^{\infty} q(t)g\left(k_4 + \int_{t}^{\infty} r(s)h\left(k_5\int_{t_0}^{s} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_4 = \int_{t_0}^{\infty} p(t)f\left(k_6 - \int_{t}^{\infty} q(s)g\left(k_7 + k_8\int_{s}^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_5 = \int_{t_0}^{\infty} p(t)f\left(\int_{t_0}^{t} q(s)g\left(k_9\int_{s}^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_6 = \int_{t_0}^{\infty} p(t)f\left(k_{10}\int_{t_0}^{t} q(s)\Delta s\right)\Delta t,$$
$$Y_7 = \int_{t_0}^{\infty} q(t)g\left(\int_{t}^{\infty} r(s)h\left(k_{11}\int_{s}^{\infty} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t,$$
$$Y_8 = \int_{t_0}^{\infty} q(t)g\left(k_{12} + k_{13}\int_{t}^{\infty} r(s)\Delta s\right)\Delta t,$$
$$Y_9 = \int_{t_0}^{\infty} r(t)h\left(k_{14}\int_{t_0}^{t} p(s)\Delta s\right)\Delta t,$$

for some nonnegative k_i, $i = 1, \dots, 14$.

3.1 The case $\lambda = 1$

In this section, we consider system (3) with $\lambda = 1$ and investigate the limit behaviors and the criteria for the existence of nonoscillatory solutions. The limit behaviors are characterized by Akin, Došla and Lawrence in the following lemma, see [21].

Lemma 3.4 Let (x, y, z) be any nonoscillatory solution of system (3). Then we have:

i. Nonoscillatory solutions in N^a satisfy

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty.$$

ii. Nonoscillatory solutions in N^c satisfy

$$\lim_{t\to\infty} |z(t)| = 0.$$

Therefore, for a nonoscillatory solution (x, y, z), we at least know that the components x and y tend to infinity while the other component z tends to 0 as $t \to \infty$.

3.1.1 Existence in N^a

Let (x, y, z) be a nonoscillatory solution of system (3) in N^a such that x is eventually positive. ($x < 0$ can be repeated very similarly.) Then by System (3), we have that x, y and z are positive and increasing. Hence, one can have the following cases:

$$\text{(i) } x \to c_1 \text{ or } x \to \infty,\ \text{(ii) } y \to c_2 \text{ or } y \to \infty,\ \text{(iii) } z \to c_3 \text{ or } z \to \infty, .$$

where $0 < c_1, c_2, c_3 < \infty$. But, the cases $x \to c_1$ and $y \to c_2$ are impossible due to Lemma 3.4 (i). So we have that any nonoscillatory solution (x, y, z) of system (3) in N^a must be in one of the following subclasses:

$$N^a_{\infty,\infty,B} := \left\{ (x, y, z) \in N^a : \quad \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty,\ \lim_{t\to\infty} |z(t)| = c_3 \right\}$$

$$N^a_{\infty,\infty,\infty} := \left\{ (x, y, z) \in N^a : \quad \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |z(t)| = \infty \right\}.$$

Now, we start with our first main result which shows that the existence of a nonoscillatory solution in $N^a_{\infty,\infty,B}$.

Theorem 3.5 $N^a_{\infty,\infty,B} \neq \varnothing$ if the improper integral Y_1 is finite for some $k_1 > 0$.

Proof: Suppose that $Y_1 < \infty$. Then choose $t_1 \geq t_0$, $k_1 > 0$ such that

$$\int_{t_1}^{\infty} r(t) h\left(\int_{t_1}^{t} p(s) f\left(k_1 \int_{t_1}^{s} q(\tau)\Delta\tau \right) \Delta s \right) \Delta t < \frac{1}{2}, \quad t \geq t_1, \tag{4}$$

where $k_1 = g(1)$. Suppose that Φ is the partially ordered Banach space of all real-valued continuous functions with the norm $\|z\| = \sup_{t\geq t_1} |z(t)|$ and the usual pointwise ordering $\leq$. Let ϕ be a subset of Φ so that

$$\phi := \left\{ z \in X : \quad \frac{1}{2} \leq z(t) \leq 1, \quad t \geq t_1 \right\}$$

and define an operator $Tz : \Phi \to \Phi$ by

$$(Tz)(t) = \frac{1}{2} + \int_{t_1}^{t} r(s) h\left(\int_{t_1}^{s} p(u) f\left(\int_{t_1}^{u} q(\tau) g(z(\tau))\Delta\tau \right) \Delta u \right) \Delta s \tag{5}$$

for $t \geq t_1$. First, it is trivial to show that T is increasing, hence let us prove that $Tz : \phi \to \phi$. Indeed,

$$\frac{1}{2} \leq (Tz)(t) \leq \frac{1}{2} + \int_{t_1}^{t} r(s) h\left(\int_{t_1}^{s} p(u) f\left(\int_{t_1}^{u} q(\tau) g(1)\Delta\tau \right) \Delta u \right) \Delta s \leq 1$$

by (2). Also, it is trivial to show that $\inf B \in \phi$ and $\sup B \in \phi$ for any subset B of ϕ, i.e., $(\phi, \leq)$ is a complete lattice. Therefore, by Theorem 2.7, we have that there exists $\bar{z} \in \phi$ such that $\bar{z} = T\bar{z}$, i.e.,

$$\bar{z}(t) = \frac{1}{2} + \int_{t_1}^{t} r(s)h\left(\int_{t_1}^{s} p(u)f\left(\int_{t_1}^{u} q(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u\right)\Delta s. \tag{6}$$

Then taking the derivative of (4) gives us

$$\bar{z}^{\Delta}(t) = r(t)h\left(\int_{t_1}^{t} p(u)f\left(\int_{t_1}^{u} q(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

By setting

$$\bar{x}(t) = \int_{t_1}^{t} p(u)f\left(\int_{t_1}^{u} q(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u \tag{7}$$

and taking the derivative of (5), we have

$$\bar{x}^{\Delta}(t) = p(t)f\left(\int_{t_1}^{t} q(\tau)g(\bar{z}(\tau))\Delta\tau\right), \quad t \geq t_1.$$

Finally letting

$$\bar{y}(t) = \int_{t_1}^{t} q(\tau)g(\bar{z}(\tau))\Delta\tau \tag{8}$$

and taking the derivative yield

$$\bar{y}^{\Delta}(t) = q(t)g(\bar{z}(t)), \quad t \geq t_1,$$

that leads us to $(\bar{x}, \bar{y}, \bar{z})$ is a solution of system (3). Thus, by taking the limit of (4)–(6) as $t \to \infty$, we have that $\bar{x}, \bar{y}$ tend to infinity and $\bar{z}$ tend to a finite number, i.e., $N^a_{\infty,\infty,B} \neq \varnothing$. This completes the proof.

Showing existence of a nonoscillatory solution in $N^a_{\infty,\infty,\infty}$ is not easy (left as an open problem in Conclusion section). So, we only provide the following result by assuming the existence of such solutions in N^a. We leave the proof to readers.

Theorem 3.6 Suppose that (x, y, z) is a nonoscillatory solution of system (3) in N^a with $C(t_0, \infty) = \infty$. Then any such solution belongs to $N^a_{\infty,\infty,\infty}$.

3.1.2 Existence in N^c

Similarly, for any nonoscillatory solution of system (3) in N^c with $x > 0$, we have x is positive increasing, z is negative increasing and y is positive decreasing, that results in the following cases:

(i) $x \to c_1$ or $x \to \infty$, (ii) $y \to c_2$ or $y \to 0$, (iii) $z \to c_3$ or $z \to 0$,

where $0 < c_1, c_2 < \infty$ and $-\infty < c_3 < 0$. However, the component function z cannot tend to c_3 by Lemma 3.4 (ii). Hence, any nonoscillatory solution of (3) in N^c must belong to one of the following sub-classes:

$$N^c_{B,B,0} := \left\{(x, y, z) \in N^c : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0\right\}$$

$$N^c_{B,0,0} := \left\{ (x,y,z) \in N^c : \ \lim_{t\to\infty} |x(t)| = c_1 \ \lim_{t\to\infty} |y(t)| = 0, \ \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^c_{\infty,B,0} := \left\{ (x,y,z) \in N^c : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^c_{\infty,0,0} := \left\{ (x,y,z) \in N^c : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = 0, \lim_{t\to\infty} |z(t)| = 0 \right\},$$

where $0 < c_1, c_2 < \infty$.

Next, we show the existence of nonoscillatory solutions of (3) in those subclasses by using fixed point theorems. Observe that we have some additional assumption in theorems such that g is an odd function. This assumption is very critical and cannot show the existence without it.

Theorem 3.7 Let g be an odd function. Then $N^c_{B,B,0} \neq \varnothing$ if $Y_2 < \infty$ for some $k_2, k_3 > 0$.

Proof: Supposing $Y_2 < \infty$ and g is odd lead us to that we can choose $k_2, k_3 > 0$ and $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} p(t) f\left(k_2 + \int_t^{\infty} q(s) g\left(k_3 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{4}, \tag{9}$$

where $k_2 = \frac{1}{2}$ and $k_3 = h\left(\frac{1}{2}\right)$. Suppose Φ is the space of all bounded, continuous and real-valued functions with $\|x\| = \sup_{t \geq t_1} |x(t)|$. It is easy to show that Φ is a Banach space, see [22]. Let ϕ be a subset of Φ so that

$$\phi := \left\{ x \in X : \quad \frac{1}{4} \leq x(t) \leq \frac{1}{2}, \quad t \geq t_1 \right\}.$$

Set an operator $Tx : \Phi \to \Phi$ such that

$$(Tx)(t) = \frac{1}{4} + \int_{t_1}^{t} p(s) f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

One can show that ϕ is bounded, closed and convex. So, we first prove that $Tx : \phi \to \phi$. Indeed,

$$\frac{1}{4} \leq (Tx)(t) \leq \frac{1}{4} + \int_{t_1}^{t} p(s) f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(h\left(\frac{1}{2}\right)\int_u^{\infty} r(\tau)\Delta\tau\right)\Delta u\right)\Delta s \leq \frac{1}{2}.$$

Second, we need to show T is continuous on ϕ. Supposing x_n is a sequence in ϕ such that $x_n \to x \in \phi = \overline{\phi}$ gives us

$$\begin{aligned}
&\|(Tx_n)(t) - (Tx)(t)\| \\
&\leq \int_{t_1}^{t} p(s) \left| f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x_n(\tau))\Delta\tau\right)\Delta u\right) - \right. \\
&\left. f\left(\frac{1}{2} + \int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right) \right| \Delta s.
\end{aligned}$$

So the Lebesgue dominated convergence theorem, continuity of f, g and h lead us to that T is continuous on ϕ. As a last step, we prove that T is relatively compact, i.e., equibounded and equicontinuous. Since

$$(Tx)^{\Delta}(t) = p(t)f\left(\frac{1}{2} + \int_t^{\infty} q(u)g\left(\int_u^{\infty} r(\tau)h(x(\tau))\Delta\tau\right)\Delta u\right) < \infty,$$

we have that T is relatively compact by Lemma 2.5 and the mean value theorem. So, there does exist $\overline{x} \in \phi$ such that $\overline{x} = T\overline{x}$ by Theorem 2.6. In addition to that, convergence of $\overline{x}(t)$ to a finite number as $t \to \infty$ is so easy to show. Therefore, setting

$$\overline{y}(t) = \frac{1}{2} + \int_t^{\infty} q(u)g\left(\int_u^{\infty} r(\tau)h(\overline{x}(\tau))\Delta\tau\right)\Delta u > 0, \quad t \geq t_1$$

and

$$\overline{z}(t) = -\int_t^{\infty} r(\tau)h(\overline{x}(\tau))\Delta\tau < 0, \quad t \geq t_1,$$

and by a similar discussion as in Theorem 3.5, we get $\overline{y}(t) \to \frac{1}{2}$ and $\overline{z}(t) \to 0$. So we conclude that $(\overline{x}, \overline{y}, \overline{z})$ is a nonoscillatory solution of system (3) in $N^c_{B,B,0}$..

Next, we focus on the existence of nonoscillatory solutions in $N^c_{\infty,B,0}$ and $N^c_{B,0,0}$. In other words, we will show there exists such a solution (x, y, z) such that x tend to infinity while y and z tend to a finite number. After that, we provide the fact that it is possible to have such a solution whose limit is finite for all component functions x, y and z. Since the following theorems can be proved similar to the previous theorem, the proofs are skipped.

Theorem 3.8 Let g be an odd function. Then we have the followings:

i. There does exist a nonoscillatory solution in $N^c_{\infty,B,0}$ if Y_3 is finite for $k_4 = 0$ and some $k_5 > 0$.

ii. There does exist a nonoscillatory solution in $N^c_{B,0,0}$ if $Y_2 < \infty$ for $k_2 = 0$ and $k_3 > 0$.

Finally, the last theorem in this section leads us to the fact that there must be a solution such that $x \to \infty$ while the other components converge to zero according to the convergence and divergence of the improper integrals of Y_2 and Y_3.

Theorem 3.9 Supposing the fact that g is an odd function, $N^c_{\infty,0,0} \neq \varnothing$ if $Y_2 = \infty$ and $Y_3 < \infty$ for $k_2 = k_4 = 0$ and $k_3, k_5 > 0$.

Proof: Suppose that $Y_2 = \infty$ and $Y_3 < \infty$. Then choose $t_1 \geq t_0$ and $k_3, k_5 > 0$ such that

$$\int_{t_1}^{\infty} q(t)g\left(\int_t^{\infty} r(s)h\left(k_5\int_{t_1}^{s} p(\tau)\Delta\tau\right)\Delta s\right)\Delta s < \frac{1}{2}, \quad t \geq t_1. \tag{10}$$

and

$$\int_{t_1}^{\infty} p(t)f\left(\int_t^{\infty} q(s)g\left(k_3\int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t > \frac{1}{2}, \quad t \geq t_1, \tag{11}$$

where $k_5 = \frac{1}{2}$ and $k_3 = h\left(\frac{1}{2}\right)$. Let Φ be the partially ordered Banach space of all continuous functions with the supremum norm $\|x\| = \sup_{t \geq t_1} \frac{x(t)}{P(t_1, t)}$ and usual pointwise ordering $\leq$. Define a subset ϕ of Φ such that

$$\phi := \left\{ x \in \Phi : \quad \frac{1}{2} \leq x(t) \leq \frac{1}{2}\int_{t_1}^{t} p(s)\Delta s, \quad t \geq t_1 \right\}$$

and an operator $Tx : \Phi \to \Phi$ by

$$(Tx)(t) = \int_{t_1}^{t} p(s) f\left(\int_{s}^{\infty} q(\tau) g\left(\int_{\tau}^{\infty} r(u) h(x(u))\Delta u\right)\Delta\tau\right)\Delta s.$$

One can easily show that $T : \phi \to \phi$ is an increasing mapping and $(\phi, \leq)$ is a complete lattice. So by Theorem 2.7, there does exist $\overline{x} \in \phi$ such that $\overline{x} = T\overline{x}$. So $\overline{x}(t) \to \infty$ as $t \to \infty$. By setting

$$\overline{y}(t) = \int_{t}^{\infty} q(\tau) g\left(\int_{\tau}^{\infty} r(u) h(\overline{x}(u))\Delta u\right)\Delta\tau, \quad t \geq t_1$$

and

$$\overline{z}(t) = -\int_{t}^{\infty} r(u) h(\overline{x}(u))\Delta u, \quad t \geq t_1,$$

one can have $\overline{y}(t) > 0$ and $\overline{z}(t) < 0$ for $t \geq t_1$ so that $\overline{y}(t) \to 0$ and $\overline{z}(t) \to 0$ as $t \to \infty$. This proves the assertion.

3.2 The case $\lambda = -1$

This section deals with system (3) for $\lambda = -1$. The assumptions on f, g and h are the same assumptions with the previous section. The following lemma describes the long-term behavior of two of the components of a nonoscillatory solution, see ([21], Lemma 4.2).

Lemma 3.10 Supposing (x, y, z) is a nonoscillatory solution in N^b, we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) = 0.$$

In the next section, we examine the solutions in each class N^a and N^b. We used fixed-point theorems to establish our results.

3.2.1 Existence in N^a

For any nonoscillatory solution (x, y, z) of system (3) in N^a with $x > 0$ eventually, one has the following subclasses by using the same arguments as in Section 3.1.1:

$$N^a_{B,B,B} := \left\{ (x, y, z) \in N^a : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = c_3 \right\}$$

$$N^a_{B,B,0} := \left\{ (x, y, z) \in N^a : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^a_{B,\infty,B} := \left\{ (x, y, z) \in N^a : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = \infty, \lim_{t\to\infty} |z(t)| = c_3 \right\}$$

$$N^a_{B,\infty,0} := \left\{ (x, y, z) \in N^a : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = \infty, \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^a_{\infty,B,B} := \left\{ (x,y,z) \in N^a \, : \, \lim_{t\to\infty} |x(t)| = \infty, \, \lim_{t\to\infty} |y(t)| = c_2, \, \lim_{t\to\infty} |z(t)| = c_3 \right\}$$

$$N^a_{\infty,B,0} := \left\{ (x,y,z) \in N^a \, : \, \lim_{t\to\infty} |x(t)| = \infty, \, \lim_{t\to\infty} |y(t)| = c_2, \, \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^a_{\infty,\infty,B} := \left\{ (x,y,z) \in N^a \, : \, \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty, \, \lim_{t\to\infty} |z(t)| = c_3 \right\}$$

$$N^a_{\infty,\infty,0} := \left\{ (x,y,z) \in N^a \, : \, \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty, \, \lim_{t\to\infty} |z(t)| = 0 \right\},$$

where c_1, c_2 and c_3 are positive constants. Finally, we have the following results:

Theorem 3.11 Suppose $R(t_0, \infty) < \infty$. If $Y_4 < \infty$ and $Y_8 < \infty$ for all positive constants $k_6, k_7, k_8, k_{12}, k_{13}$, then $N^a_{B,B,B} \neq \emptyset$.

Proof: Assume $Y_4 < \infty$ and $Y_8 < \infty$ for all $k_6, k_7, k_8, k_{12}, k_{13} > 0$. Choose $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} p(t) f\left(k_6 - \int_t^{\infty} q(s) g\left(k_7 + k_8 \int_s^{\infty} r(\tau) \Delta\tau \right) \Delta s \right) \Delta t < \frac{1}{2}$$

and

$$\int_{t_1}^{\infty} q(s) g\left(k_{12} + k_{13} \int_s^{\infty} r(\tau) \Delta\tau \right) \Delta s < k_6,$$

where $k_8 = k_{13} = h\left(\frac{1}{2}\right) > 0$ and $k_7 = k_{12}$ for $t \geq t_1$.

Let $\mathbb{X}$ be the set of all continuous and bounded functions with the norm $\|x\| = \sup_{t \geq t_1} |x(t)|$. Then $\mathbb{X}$ is a Banach space ([22]). Define a subset Ω of $\mathbb{X}$ such that

$$\Omega := \left\{ x \in \mathbb{X} : \quad \frac{1}{2} \leq x(t) \leq 1, \quad t \geq t_1 \right\}$$

and an operator $Fx : \mathbb{X} \to \mathbb{X}$ by

$$(Fx)(t) = \frac{1}{2} + \int_{t_1}^{t} p(s) f\left(k_6 - \int_s^{\infty} q(u) g\left(k_7 + \int_u^{\infty} r(\tau) h(x(\tau)) \Delta\tau \right) \Delta u \right) \Delta s$$

for $t \geq t_1$. First, for every $x \in \Omega$, $\|x\| = \sup\limits_{t \geq t_1} |x(t)|$, we have $\frac{1}{2} \leq \|x(t)\| \leq 1$ for $t \geq t_1$, which implies Ω is bounded. For showing that Ω is closed, it is enough to show that it includes all limit points. So let x_n be a sequence in Ω converging to x as $n \to \infty$. Then $\frac{1}{2} \leq x_n(t) \leq 1$ for $t \geq t_1$. Taking the limit of x_n as $n \to \infty$, we have $\frac{1}{2} \leq x(t) \leq 1$ for $t \geq t_1$, which implies $x \in \Omega$. Since x_n is any sequence in Ω, it follows that Ω is closed. Now let us show Ω is also convex. For $x_1, x_2 \in \Omega$ and $\alpha \in [0, 1]$, we have

$$\frac{1}{2} = \frac{\alpha}{2} + (1-\alpha)\frac{1}{2} \leq \alpha x_1 + (1-\alpha) x_2 \leq \alpha + (1-\alpha) = 1,$$

where $\frac{1}{2} \leq x_1, x_2 \leq 1$, i.e., Ω is convex. Also, because

$$\begin{aligned}\frac{1}{2} \leq (Fx)(t) &\leq \frac{1}{2} + \int_{t_1}^{t} p(s) f\left(k_6 - \int_s^{\infty} q(u) g\left(k_7 + h\left(\frac{1}{2}\right) \int_u^{\infty} r(\tau) \Delta\tau \right) \Delta u \right) \Delta s \\ &\leq 1,\end{aligned}$$

i.e., $F : \Omega \to \Omega$. Let us now show that F is continuous on Ω. Let $\{x_n\}$ be a sequence in Ω such that $x_n \to x \in \Omega$ as $n \to \infty$. Then

$$\begin{aligned} &|(Fx_n - Fx)(t)| \\ &\leq \int_{t_1}^{t} p(s) \left| f\left(k_6 - \int_s^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(x_n(\tau))\Delta\tau\right)\Delta u\right) \right. \\ &\left. - f\left(k_6 - \int_s^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\right| \Delta s. \end{aligned}$$

Then the continuity of f, g and h and Lebesgue Dominated Convergence theorem imply that F is continuous on Ω. Finally, since

$$(Fx)^\Delta(t) = p(t) f\left(k_6 - \int_t^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right) < \infty,$$

we have F is relatively compact by the Mean Value theorem and Arzelà-Ascoli theorem. So, by Theorem 2.6, we have there exists $\overline{x} \in \Omega$ such that $\overline{x} = F\overline{x}$. Then by taking the derivative of $\overline{x}$, we obtain

$$\overline{x}^\Delta(t) = p(t) f\left(k_6 - \int_t^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(\overline{x}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

Setting

$$\overline{y}(t) := k_6 - \int_t^\infty q(u) g\left(k_7 + \int_u^\infty r(\tau) h(\overline{x}(\tau))\Delta\tau\right)\Delta u$$

for $k_6 > 0$ and taking its derivative yields

$$\overline{y}^\Delta(t) = q(t) g\left(k_7 + \int_t^\infty r(\tau) h(\overline{x}(\tau))\Delta\tau\right), \quad t \geq t_1.$$

Finally, differentiating

$$\overline{z}(t) := k_7 + \int_t^\infty r(\tau) h(\overline{x}(\tau))\Delta\tau$$

gives

$$\overline{z}^\Delta(t) = -r(t) h(\overline{x}(t)), \quad t \geq t_1.$$

Consequently $(\overline{x}, \overline{y}, \overline{z})$ is a solution of system (3) such that $\overline{x}(t) \to \alpha$, $\overline{y}(t) \to k_6$ and $\overline{z}(t) \to k_7$, where $0 < \alpha < \infty$, i.e., $N^a_{B,B,B} \neq \emptyset$.

The following theorems can be proven very similarly to Theorem 3.11 with appropriate operators. Therefore, the proof is left to the reader, see [17].

Theorem 3.12 We have the following results:

i. Suppose $R(t_0, \infty) < \infty$. If $Y_4 < \infty$ and $Y_8 < \infty$ for $k_7 = k_{12} = 0$ and for all $k_6, k_8, k_{13} > 0$, then $N^a_{B,B,0} \neq \emptyset$.

ii. If both Y_3 and Y_9 are finite for $k_4 = 0$ and for all $k_5, k_{14} > 0$, then $N^+_{\infty,B,0} \neq \emptyset$.

iii. If $Y_3 < \infty$ and $Y_9 < \infty$ for all $k_4, k_5, k_{14} > 0$, then $N^a_{\infty,B,B} \neq \varnothing$.

iv. If $Y_1 < \infty$ and $Y_6 = \infty$ for all $k_1, k_{10} > 0$, then $N^a_{\infty,\infty,B} \neq \varnothing$.

We continue with the case when $z(t)$ converges to 0 while other components $x(t)$ and $y(t)$ of solution (x, y, z) tend to infinity as $t \to \infty$.

Theorem 3.13 Suppose $R(t_0, \infty) < \infty$. If $Y_1 < \infty$ and $Y_5 = Y_8 = \infty$ for all positive constants k_1, k_9, k_{13} and $k_{12} = 0$, then $N^a_{\infty,\infty,0} \neq \varnothing$..

Proof: Suppose $Y_1 < \infty$ and $Y_5 = Y_8 = \infty$ for $k_1, k_9, k_{13} > 0, k_{12} = 0$. Then choose a $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} r(t)h\left(\int_{t_1}^{t} p(s)f\left(k_1\int_{t_0}^{s} q(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2}$$

and

$$\int_{t_1}^{\infty} p(s)f\left(\int_{t_1}^{s} q(\tau)g\left(k_9\int_{\tau}^{\infty} r(v)\Delta v\right)\Delta\tau\right)\Delta s > 1, \quad t \geq t_1,$$

where $k_1 = g\left(\frac{1}{2}\right)$ and $k_9 = k_{13} = h(1)$. Suppose that Φ is a space of real-valued continuous functions and partially ordered Banach space with $\|y\| = \sup_{t \geq t_1} |y(t)|$ and the usual pointwise ordering $\leq$. Let ϕ be a subset of Φ such that

$$\phi := \left\{ z \in \Phi : \quad h(1)\int_{t}^{\infty} r(s)\Delta s \leq z(t) \leq \frac{d_1}{2}, \quad t \geq t_1 \right\}.$$

and set an operator $F : \Phi \to \Phi$ such that

$$(Fz)(t) = \int_{t}^{\infty} r(s)h\left(\int_{t_1}^{s} p(u)f\left(\int_{t_1}^{u} q(\tau)g(z(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

The rest of the proof can be done as in proofs of the previous theorems by using the fact $Y_5 = Y_8 = \infty$, and therefore, $N^a_{\infty,\infty,0} \neq \varnothing$.

3.2.2 Existence in N^b

Assuming (x, y, z) is a nonoscillatory solution of system (3) in N^b such that $x > 0$ eventually and by a similar discussion as in the previous section, and by Lemma 3.10, we have the following subclasses:

$$N^b_{B,0,0} := \left\{ (x,y,z) \in N^b : \ \lim_{t\to\infty} |x(t)| = c_1 \ \lim_{t\to\infty} |y(t)| = 0, \ \lim_{t\to\infty} |z(t)| = 0 \right\}$$

$$N^b_{0,0,0} := \left\{ (x,y,z) \in N^b : \lim_{t\to\infty} |x(t)| = 0, \lim_{t\to\infty} |y(t)| = 0, \lim_{t\to\infty} |z(t)| = 0 \right\},$$

where $0 < c_1 < \infty$.

The first result of this section considers the case when each of the component solutions converges.

Theorem 3.14 Suppose $R(t_0, \infty) < \infty$ and f is odd. Then $N^b_{B,0,0} \neq \varnothing$ if $Y_2 < \infty$ and $Y_8 < \infty$ for all $k_3 = k_{13} > 0$ and $k_{12} = 0$.

Proof: Suppose that $Y_2 < \infty$ and $Y_8 < \infty$ for all $k_3 = k_{13} > 0$ and $k_{12} = 0$. Then choose $k_3, k_{13} > 0$ and $t_1 \geq t_0$ sufficiently large such that

$$\int_{t_1}^{\infty} p(t) f\left(\int_t^{\infty} q(s) g\left(k_3 \int_s^{\infty} r(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2},$$

where $k_3 = h\left(\frac{3}{2}\right)$. Let Φ be a partially ordered Banach space of real-valued continuous functions with $\|x\| = \sup_{t\geq t_1}|x(t)|$ and the usual pointwise ordering $\leq$. Let us set a subset ϕ of Φ such that

$$\phi := \left\{x \in \Phi : \quad 1 \leq x(t) \leq \frac{3}{2}, \quad t \geq t_1\right\}$$

and an operator $Fx : \Phi \to \Phi$ by

$$(Fx)(t) = 1 + \int_t^{\infty} p(s) f\left(\int_s^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

One can prove that F is an increasing mapping into itself and $(\Omega, \leq)$ is a complete lattice. Therefore, by Theorem 2.7, there does exist $\overline{x} \in \Omega$ such that $\overline{x} = F\overline{x}$. It follows that $\overline{x}(t) > 0$ for $t \geq t_1$ and converges to 1 as t approaches infinity. Also,

$$\overline{x}^{\Delta}(t) = -p(t) f\left(\int_t^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(\overline{x}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

Now for $t \geq t_1$, set

$$\overline{y}(t) = -\int_t^{\infty} q(u) g\left(\int_u^{\infty} r(\tau) h(\overline{x}(\tau))\Delta\tau\right)\Delta u$$

and

$$\overline{z}(t) = \int_t^{\infty} r(\tau) h(\overline{x}(\tau))\Delta\tau.$$

Then, since f is odd, we have

$$\begin{aligned} \overline{x}^{\Delta}(t) &= p(t) f(\overline{y}(t)) \\ \overline{y}^{\Delta}(t) &= q(t) g(\overline{z}(t)) \\ \overline{z}^{\Delta}(t) &= -r(t) h(\overline{x}(t)). \end{aligned}$$

Consequently $(\overline{x}, \overline{y}, \overline{z})$ is a solution of system (3). Since both $\overline{y}(t)$ and $\overline{z}(t)$ converge to 0 as t approaches infinity, $N^b_{B,0,0} \neq \emptyset$..

4. Examples

In this section, we provide some examples to highlight our theoretical claims. The following theorem help us evaluate the integrals on a specific time scale, see ([6] Theorem 1.79 (ii)).

Theorem 4.1 Suppose that $[a, b]$ has only isolated points with $a < b$. Then

$$\int_a^b f(t)\Delta t = \sum_{t \in [a,b)} \mu(t) f(t).$$

Example 4.2 Let $\mathbb{T} = 3^{\mathbb{N}}, k_5 = 1 = k_{14}$ and consider the following system

$$\begin{cases} \Delta_3 x(t) = \left(\frac{t}{t-1}\right)^{\frac{1}{3}} y^{\frac{1}{3}}(t) \\ \Delta_3 y(t) = \dfrac{1}{3t^{\frac{1}{5}}} z^{\frac{3}{5}}(t) \\ \Delta_3 z(t) = -\dfrac{26}{54 t^{\frac{21}{5}}} x^{\frac{1}{5}}(t), \end{cases} \tag{12}$$

where

$$\Delta_3 k(t) = \frac{k(\sigma(t)) - k(t)}{\mu(t)} \quad \text{for} \quad \sigma(t) = 3t \quad \text{and} \quad \mu(t) = 2t, \quad t \in \mathbb{T}.$$

First we show $P(t_0, \infty) = Q(t_0, \infty) = \infty$. If $s = 3^m$ and $t = 3^n$, $m, n \in \mathbb{N}$, we have

$$\int_3^\infty p(s)\Delta s = \lim_{t\to\infty} \int_3^t p(s)\Delta s = 2\lim_{n\to\infty} \sum_{s=3}^{\rho(3^n)} \left(\frac{s^4}{s-1}\right)^{\frac{1}{3}} > 2\lim_{n\to\infty} \sum_{m=1}^{n-1} 3^m = \infty.$$

Similarly one can obtain $\int_3^\infty q(s)\Delta s = \infty$.
Now we consider Y_3. With $\tau = 3^m$ and $s = 3^n$, $m, n \in \mathbb{N}$, we have

$$\int_3^s \left(\frac{\tau}{\tau-1}\right)^{\frac{1}{3}} \Delta\tau = 2\sum_{m=1}^{n-1} \left(\frac{3^{4m}}{3^m - 1}\right)^{\frac{1}{3}} < 2\sum_{m=1}^{n-1} (3^m)^{\frac{4}{3}}$$

since $3^m - 1 > 1$ on $\mathbb{N}$. We claim that

$$\sum_{m=1}^{n-1} (3^m)^{\frac{4}{3}} < (3^n)^{\frac{4}{3}}.$$

The sum formula for a finite geometric series, $1 - 3^{\frac{4}{3}} < 0$, and.
$\left(3^{\frac{4}{3}}\right)^{1-n} - 1 < 1$ for $n \in \mathbb{N}$ yield

$$0 \leq \frac{\left(3^{\frac{4}{3}}\right)^{1-n} - 1}{1 - 3^{\frac{4}{3}}} < 1.$$

So the claim indeed holds, and consequently we have

$$\int_3^s \left(\frac{\tau}{\tau - 1}\right)^{\frac{1}{3}} \Delta\tau < 2s^{\frac{4}{3}}. \tag{13}$$

Also, we obtain

$$\int_t^T r(s) h\left(\int_3^s p(\tau)\Delta\tau\right)\Delta s < \int_t^T \frac{26}{54} \frac{1}{s^{\frac{21}{5}}} \left(2s^{\frac{4}{3}}\right)^{\frac{1}{5}} \Delta s = \frac{26 \cdot 2^{\frac{6}{5}}}{54} \sum_{s \in [t,T)_{3^{\mathbb{N}}}} \frac{1}{s^{\frac{44}{15}}} < 2 \sum_{s \in [t,T)_{3^{\mathbb{N}}}} \frac{1}{s^{\frac{44}{15}}}$$

by (11). Therefore, as $T \to \infty$, we obtain

$$\sum_{s\in[t,\infty)_{3^{\mathbb{N}}}} \frac{1}{s^{\frac{44}{15}}} = \alpha \cdot \frac{1}{t^{\frac{44}{15}}}, \tag{14}$$

where $\alpha = 1 - \frac{1}{3^{\frac{44}{15}}}$. Finally, with $t = 3^m$ and $T = 3^n$, $m, n \in \mathbb{N}$, we have

$$\int_{t_0}^{T} q(t) g\left(\int_t^\infty r(s) h\left(\int_{t_0}^{s} p(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{(2\alpha)^{\frac{3}{5}}}{3}\int_3^T \frac{1}{t^{\frac{1}{5}}}\left(\frac{1}{t^{\frac{44}{15}}}\right)^{\frac{3}{5}}\Delta t = \frac{(2\alpha)^{\frac{3}{5}}}{3}\int_3^T \frac{1}{t^{\frac{49}{25}}}\Delta t$$
$$= \frac{(2\alpha)^{\frac{3}{5}}}{3}\sum_{m=1}^{n-1} 2\frac{1}{(3^m)^{\frac{49}{25}}}3^m = \frac{2(2\alpha)^{\frac{3}{5}}}{3}\sum_{m=1}^{n-1}\left(\frac{1}{3^{\frac{24}{25}}}\right)^m$$

by (12). Since the above integral converges as T approaches infinity, we have $Y_3 < \infty$. By using a similar discussion and (12), it is shown $Y_9 < \infty$. One can also show that $\left(t, 1 - \frac{1}{t}, \frac{1}{t^3}\right)$ is a nonoscillatory solution of system (10). Hence $N^a_{\infty,B,0} \neq \varnothing$ by Theorem 3.12 (ii).

Example 4.3 Let $\mathbb{T} = q^{\mathbb{N}_0}$. Consider the system

$$\begin{cases} x^\Delta(t) = \dfrac{1}{(1+t)^{\frac{1}{3}}} y^{\frac{1}{3}}(t) \\ y^\Delta(t) = \left(\frac{t}{2t-1}\right)^{\frac{1}{5}} z^{\frac{1}{5}}(t) \\ z^\Delta(t) = \dfrac{1}{qt^3} x(t). \end{cases} \tag{15}$$

We show that $N^a_{\infty,\infty,B} \neq \varnothing$ by Theorem 3.5 for $s = q^m, t = q^n, k_1 = 1$ and $t_0 = 1$. So we need to show $P(t_0, \infty) = Q(t_0, \infty) = \infty$ and $Y_1 < \infty$. Indeed,

$$\int_1^T p(t)\Delta t = \sum_{t\in[1,\rho(T))_{q^{\mathbb{N}_0}}} \frac{1}{(1+t)^{\frac{1}{3}}} \cdot (q-1)t.$$

So as $T \to \infty$, we have

$$P(1,\infty) = (q-1)\sum_{n=0}^{\infty} \frac{q^n}{(1+q^n)^{\frac{1}{3}}} = \infty$$

by the ratio test. We can also easily show $Q(1, \infty) = \infty$. As the final step, let us show $Y_1 < \infty$ holds. Indeed,

$$\int_1^T r(t) h\left(\int_1^t p(s) f\left(\int_1^s q(\tau)\Delta\tau\right)\Delta s\right)\Delta t$$
$$= \int_1^T r(t) h\left(\int_1^t p(s)\left(\sum_{\tau\in[1,\rho(s)]_{q^{\mathbb{N}_0}}}\left(\frac{t}{2t-1}\right)^{\frac{1}{5}} \cdot (q-1)t\right)^{\frac{1}{3}}\Delta s\right)\Delta t$$
$$\leq \int_1^T r(t) h\left(\int_1^t p(s)\cdot s^{\frac{1}{3}}\right)\Delta t = (q-1)\int_1^T r(t)\left(\sum_{s\in[1,\rho(t)]_{q^{\mathbb{N}_0}}} \frac{1}{(1+s)^{\frac{1}{3}}}\cdot s^{\frac{1}{3}}\cdot s\right)\Delta t$$
$$\leq (q-1)\sum_{t\in[1,\rho(T)]_{q^{\mathbb{N}_0}}} \frac{1}{t}.$$

Hence, by the geometric series, and taking the limit of the latter inequality as $T \to \infty$ yield us

$$\sum_{n=0}^{\infty} \frac{1}{q^n} < \infty.$$

Therefore, we have $Y_1 < \infty$. One can also show that $\left(t, 1+t, 2-\frac{1}{t}\right)$ is a solution of system (13) in $N^a_{\infty,\infty,B}$..

Exercise 4.4 Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Show that $\left(1+t, \frac{3t+1}{t}, \frac{-1}{t^2}\right)$ is a solution of

$$\begin{cases} x^{\Delta}(t) = \left(\frac{t}{3t+1}\right)^{\frac{1}{3}} y^{\frac{1}{3}}(t) \\ y^{\Delta}(t) = \dfrac{1}{2} z(t) \\ z^{\Delta}(t) = \dfrac{3}{4(1+t)t^3} x(t) \end{cases} \tag{16}$$

in N^c such that $x(t) \to \infty, y(t) \to 3$ and $z(t) \to 0$ as $t \to \infty$, i.e., $N^c_{\infty,B,0} \neq \emptyset$ by Theorem 3.8 (i).

5. Conclusion and open problems

In this chapter, we consider a 3D time scale system and show the asymptotic properties of the nonoscillatory solutions along with the existence of such solutions. We are able to show the existence of solutions in most subclasses. On the other hand, it is still an open problem to show the existence in $N^a_{\infty,\infty,\infty}$ for system (3), where $\lambda = 1$. In addition to that, there is one more open problem that also can be considered as a future work, which is to find the criteria for the existence of a nonoscillatory solution in $N^b_{0,0,0}$ of system (3), where $\lambda = -1$.

Another significance of our system that we consider in this chapter is the following system

$$\begin{cases} x^{\Delta}(t) = p(t)|y(t)|^{\alpha} \operatorname{sgn} y(t) \\ y^{\Delta}(t) = q(t)|z(t)|^{\beta} \operatorname{sgn} z(t) \\ z^{\Delta}(t) = -r(t)|x^{\sigma}(t)|^{\gamma} \operatorname{sgn} x^{\sigma}(t), \end{cases} \tag{17}$$

which is known as the third order Emden-Fowler system. Here, p, q and r have the same properties as System (3) and α, β, γ are positive constants. Emden-Fowler equation has a lot of applications in fluid mechanics, astrophysics and gas dynamics. It would be very interesting to investigate the characteristics of solutions because of its potential in applications.

Notes/thanks/other declarations

I would like to dedicate this chapter to my beloved friend Dr. Serdar Çağlak, who always will be remembered as a fighter for his life. Also, I would like to thank to my wife for her tremendous support for writing this chapter.

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Chapter 7

Global Existence of Solutions to a Class of Reaction–Diffusion Systems on $\mathbb{R}^n$

Salah Badraoui

Abstract

We prove in this work the existence of a unique global nonnegative classical solution to the class of reaction–diffusion systems

$$u_t(t,x) = a\Delta u(t,x) - g(u)v^m,$$
$$v_t(t,x) = d\Delta v(t,x) + \lambda(t,x)g(u)v^m,$$

where $a > 0, d > 0, t > 0,\ x \in \mathbb{R}^n, n, m \in \mathbb{N}^*$, λ is a nonnegative bounded function with $\lambda(t,.) \in BUC(\mathbb{R}^n)$ for all $t \in \mathbb{R}_+$, the initial data $u_0, v_0 \in BUC(\mathbb{R}^n), g : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ is a of class C^1, $\frac{dg(u)}{du} \in L^\infty(\mathbb{R}), g(0) = 0$ and $g(u) \geq 0$ for all $u \geq 0$. The ideas of the proof is inspired from the work of Collet and Xin who proved the same result in the particular case $d > a = 1, \lambda = 1, g(u) = u$. Moreover, they showed that the L^∞-norm of v can not grow faster than $O(\ln \ln t)$ for any space dimension.

Keywords: reaction–diffusion systems, local existence, positivity, comparison principle, global existence

1. Introduction

In the sequel, we use the notations.

$\mathbb{R}_+ = [0, \infty[,\ \mathbb{R}_+^* =]0, \infty[$.

$\mathbb{N} = \{0, 1, \dots\}$ the set of natural numbers and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

For $p \in \mathbb{R} : [p]$ the integer part of p.

For $n \in \mathbb{N}^*$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x|^2 = \sum_{j=1}^n x_j^2$.

$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$ the set of integers.

For $x^{(0)} \in \mathbb{R}^n$ and $\rho \in \mathbb{R}_+^*,$:

$B'\left(x^{(0)}, \rho\right) = \left\{x \in \mathbb{R}^n : \left|x - x^{(0)}\right| \leq \rho\right\}$ the closed ball of center $x^{(0)}$ and radius ρ.

$S\left(x^{(0)}, \rho\right) = \left\{x \in \mathbb{R}^n : \left|x - x^{(0)}\right| = \rho\right\}$ the boundary of $B'\left(x^{(0)}, \rho\right)$.

Let $Q \subset \mathbb{R}^n$ $(n \in \mathbb{N}^*)$ a subset. ∂Q denote the boundary of Q.

ln: the natural logaritm function.

$\omega_n(\rho) = \frac{2\pi^{n/2}\rho^{n-1}}{\Gamma(n/2)}$ the surface area of $S(0, \rho)$, where $\Gamma(x) = \int_0^\infty e^{-t}t^{-x}dt$ $\left(x \in \mathbb{R}_+^*\right)$ is the Gamma function.

$BUC(\mathbb{R}^n)$ the Banach space of bounded and uniformly continuous functions on $\mathbb{R}^n$ with the supremum norm $\|u\|_\infty = \sup_{x\in\mathbb{R}^n}|u(x)|$.

$X = BUC(\mathbb{R}^n) \times BUC(\mathbb{R}^n)$ which is a Banach space endowed with the norm $\|(u,v)\|_X = \|u\|_\infty + \|v\|_\infty$.

For $u \in L^p(\mathbb{R}^n)$ $(p \in [1,\infty[)$, we denote by $\|u\|_p^p = \int_{\mathbb{R}^n}|u|^p dx$.

For $u, v : \mathbb{R}^n \to \mathbb{R}$ two regular functions, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$ and $\nabla u.\nabla v = \sum_{j=1}^n \frac{\partial u}{\partial x_j} \cdot \frac{\partial v}{\partial x_j}$.

Reaction-Diffuison equations are nonlinear parabolic partial differential equations arises in many fields of sciences like chemistry, physics, biology, ecology and even medicine. It appears usually as coupled systems.

The somewhat general form of these systems of two equations is

$$\begin{cases} u_t(t,x) = a\Delta u(t,x) + f_1(t,x,u,v), \\ v_t(t,x) = d\Delta v(t,x) + f_2(t,x,u,v), \end{cases}$$

where $t > 0$, $x \in \Omega$ with $\Omega \subset \mathbb{R}^n (n \in \mathbb{N}^*)$ is an open set, Δ is the Laplacian operator, a, d are two real positive constants called the coefficients of the diffusion. For a chemical reaction where two substances S_1 and S_2, u and v represent their concentrations at time t and position x respectively, and f_1 and f_2 represent the rate of production of these substances in the given order. For more details see [1, 2].

In this chapter, we are concerned with the existence of global solutions to the reaction–diffusion system

$$u_t(t,x) = a\Delta u(t,x) - g(u)v^m, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \tag{1}$$

$$v_t(t,x) = d\Delta v(t,x) + \lambda(t,x)g(u)v^m, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \tag{2}$$

with initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}^n. \tag{3}$$

Whe assume that.

(H1) The constants a, d are such that $a,\ d \in \mathbb{R}_+^*$.

(H2) $\lambda : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a non-null, nonnegative and bounded function on $\mathbb{R}^+ \times \mathbb{R}^n$ such that $\lambda(t,.) \in BUC(\mathbb{R}^n)$ for all $t \in \mathbb{R}_+$. We denote $\lambda_\infty = \sup_{t\ge 0}\left(\|\lambda(t)\|_\infty\right)$.

(H3) n and m are positive integers, i.e. $n, m \in \mathbb{N}^*$.

(H4) $g : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ is a function defined on $BUC(\mathbb{R}^n)$ such that:

i. $g(0) = 0$ and $g(u) \ge 0$ pour $u \ge 0$.

ii. g is of class C^1 and $\frac{dg(u)}{du}$ is bounded on $\mathbb{R}$.

(H5) The initial data u_0, v_0 are nonnegative and are in $BUC(\mathbb{R}^n)$.

One of the essential questions for (1)–(3) is the existence of global solutions and possibly bounds uniform in time. Recently, Collet and Xin in their paper [3] published in 1996 have studied the system (1)–(3) but with $a = \lambda = 1$, $d > 1$ and $\varphi(u) = u$. In this particular case, this system describes the evolution of u the mass fraction of reactant A and that v of the product B for the autocatalytic chemical reaction of the form $A + mB \to (m+1)B$. They proved the existence of global solutions and showed that the L^∞ norm of v can not grow faster than $O(\ln \ln t)$ for any space dimension.

If we replace $g(u)v^m$ by $u \exp\{-E/v\}$ where $E > 0$ is a constant and take $\lambda = 1$, there are many works on global solutions, see Avrin [4], Larrouturou [5] for results in one space dimension, among others.

It is worth mentioning here the result of S. Badraoui [6] who studied the system

$$u_t = a\Delta u - uv^m,$$

$$v_t = b\Delta u + d\Delta v + uv^m,$$

where $a > 0, d > 0, b \neq 0, x \in \mathbb{R}^n, n \in \mathbb{N}^*,\ m \in 2\mathbb{N}^*$ is an even positive integer. He has proved that if u_0, v_0 are nonnegative and are in $BUC(\mathbb{R}^n)$ that:

If $a > d,\ b > 0,\ v_0 \geq \frac{b}{a-d} u_0$ on $\mathbb{R}^n$, then the solution is global and uniformly bounded.

If $a < d,\ b < 0,\ v_0 \geq \frac{b}{a-d} u_0$ on $\mathbb{R}^n$, then the solution is global.

Our work here is a continuation of the work of Collet and Xin [3]. We treat the same question in a slightly general case. Inspired by the same ideas in [3] we prove that the system (1)–(3) under the assumptions (H1) to (H5) has a unique global nonnegative classical solution.

The chapter is organized as follows: In section 2, we treat the existence of local solution and reveal its positivity using the maximum principle.

In section 3, firstly, we prove by a simple comparison argument that if $a \geq d$, the solution is uniformly bounded and we give an upper bound of it. Afterwards, we attack the hard case in which $a < d$ where we used the Lyapunov functional $L(u, v) = [\alpha + 2u - \ln(1+u)]e^{\varepsilon v}$ $(\alpha,\ \varepsilon > 0)$ and the cut-off function $\varphi(x) = \left(1 + |x|^2\right)^{-n}$. We show that for α sufficiently large and ε small enough we can control the L^p-norms of v $(p > \max\{1, n/2\})$ on every unit spacial cub in $\mathbb{R}^n$ from which we deduce the L^∞-norm of v at any time $t > 0$.

We emphazise here that I have engaged to calculate the constants encountered in all equations and inequalities exactly.

2. Existence of a local solution and its positivity

We convert the system (1)–(3) to an abstract first order system in the Banach space $X := BUC(\mathbb{R}^n) \times BUC(\mathbb{R}^n)$ of the form

$$\begin{cases} w'(t) = Aw(t) + F(w(t)), & t > 0, \\ w(0) = w_0 \in X. \end{cases} \tag{4}$$

Here $w(t) = (u(t), v(t))$; the operator A is defined as

$$Aw := \begin{pmatrix} a\Delta & 0 \\ 0 & d\Delta \end{pmatrix} w = (a\Delta u, d\Delta v),$$

where $D(A) := \{w = (u, v) \in X : (\Delta u, \Delta v) \in X\}$. The function F is defined as $F(w(t)) = (-\varphi(u(t))v^m(t), \lambda(t)\varphi(u(t))v^m(t))$.

It is known that for $c > 0$ the operator $c\Delta$ generates an analytic semigroup $G(t)$ in the space $BUC(\mathbb{R}^n)$:

$$G(t)u = (4\pi ct)^{-n/2} \int_{\mathbb{R}^n} \exp\left\{-\frac{|x-y|^2}{4ct}\right\} u(y) dy. \tag{5}$$

Hence, the operator A generates an analytic semigroup defined by

$$S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}, \tag{6}$$

where $S_1(t)$ is the semigroup generated by the operator $a\Delta$, and $S_2(t)$ is the semigroup generated by the operator $d\Delta$.

Since the map F is locally Lipschitz in w in the space X, then proving the existence of a loacl classical solution on $[0, t_1]$ where $t_1 \in \mathbb{R}_+^*$ is standard [7, 8].

For the positivity, let $w(t) = (u(t), v(t))$ is a local solution of the problem (1)–(3) under the assumptions $\{Hj\}_{j=1}^5$ on the interval $[0, t_1]$.

We can write the first equation as

$$u_t - a\Delta u + \left[v^m \frac{d}{du} g(\xi) \right] u = 0, (t, x) \in]0, t_1] \times \mathbb{R}^n, \tag{7}$$

for some $\xi \in \mathbb{R}$. Thanks to the assumption (H4)-ii we deduce that $v^m \frac{\partial}{\partial u} g(\xi)$ is bounded on $[0, t_1] \times \mathbb{R}^n$. Whence, by the theorem 9 on page 43 in [9], we obtain that

$$u(t, x) \geq 0, \text{for all } (t, x) \in [0, t_1] \times \mathbb{R}^n, \tag{8}$$

The second equation can be written as

$$v_t - d\Delta v + \left[-\lambda g(u) v^{m-1} \right] v, \quad (t, x) \in]0, t_1] \times \mathbb{R}^n. \tag{9}$$

By the same theorem we get

$$v(t, x) \geq 0, \text{for all } (t, x) \in [0, t_1] \times \mathbb{R}^n. \tag{10}$$

For the existence of a global solution, we use the contraposed of the characterization of the maximal existence time $t_{\max}$ ([8] on page 193) as follows

$$\begin{bmatrix} \text{there exists a map}\, C : \mathbb{R}_+ \to \mathbb{R}_+ \text{such that :} \\ \|u(t)\|_\infty + \|v(t)\|_\infty \leq C(t) \,\text{for all}\, t \in \mathbb{R}_+ \end{bmatrix} \Rightarrow t_{\max} = +\infty. \tag{11}$$

3. Existence of a global solution

For this task we will use the fact that the solution is nonnegative.

Theorem 3.1. Let (u, v) be the solution of the problem (1)–(3) under the assumptions $\{Hj\}_{j=1}^5$ and such that

$$a \geq d. \tag{12}$$

Then, the solution is global and uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^n$. More precisely, we have the estimates

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \text{for all } t \in \mathbb{R}_+, \tag{13}$$

$$\|v(t)\|_\infty \leq \|v_0\| + \lambda_\infty \left(\frac{a}{d} \right)^{n/2} \|u_0\|_\infty, \quad \text{for all } t \in \mathbb{R}_+. \tag{14}$$

Proof. By the comparison principle we get (13).

The solution (u, v) satisfies the integral equations

$$u(t,x) = S_1(t)u_0 - \int_0^t S_1(t-\tau)g(u(\tau))v^m(\tau)d\tau, \tag{15}$$

$$v(t,x) = S_2(t)v_0 + \int_0^t S_2(t-\tau)\lambda(\tau)g(u(\tau))v^m(\tau)d\tau. \tag{16}$$

Here $S_1(t)$ and $S_2(t)$ are the semigroups generated by the operators $a\Delta$ and $d\Delta$ in the space $BUC(\mathbb{R}^n)$ respectively. As u is nonnegative, then from (15) we get

$$\int_0^t S_1(t-\tau)g(u(\tau))v^m(\tau)d\tau \leq S_1(t)u_0. \tag{17}$$

Since $a \geq d$, using the explicit expression of $S_1(t-\tau)g(u(\tau))v^m(\tau)$ and $S_2(t-\tau)g(u(\tau))v^m(\tau)$, one can observe that (see [10])

$$\begin{aligned}\int_0^t S_2(t-\tau)\lambda(\tau)g(u(\tau))v^m(\tau)d\tau &\leq \left(\frac{a}{d}\right)^{n/2}\int_0^t S_1(t-\tau)\lambda(\tau)g(u(\tau))v^m(\tau)d\tau \\ &\leq \lambda_\infty\left(\frac{a}{d}\right)^{n/2}\int_0^t S_1(t-\tau)g(u(\tau))v^m(\tau)d\tau.\end{aligned} \tag{18}$$

From (17) and (18) into (16) we get

$$v(t) \leq S_2(t)v_0 + \lambda_\infty\left(\frac{a}{d}\right)^{n/2} S_1(t)u_0. \tag{19}$$

This last inequality leads to the veracity of (14).

Thus, from (13) and (14), we deduce that the solution (u, v) is global and uniformly bounded on $\mathbb{R}_+ \times \mathbb{R}^n$. ♦

In the case where $d > a$, it seems that the idea of comparison cannot be applied. Nevertheless, we can prove the existence of global classical solutions; but it appears that their boundedness is not assured.

Theorem 3.2. Let (u, v) be the solution of the problem (1)–(3) with the assumptions $\{Hj\}_{j=1}^5$. If

$$a < d, \tag{20}$$

the solution (u, v) is global. More precisely we have the estimates (13) and (83).

Proof. In this case, it is not easy to prove global existence. But can derive estimates of solutions independent of t_1 by using the same method used in [3] and the same form of the functional used in [6] but with different coefficients.

We need some lemmas.

Lemma 3.3. Let (u, v) be the solution of the problem (1)–(3) under the assumptions $\{Hj\}_{j=1}^5$ on the local interval time $[0, t_1]$. Define the functional

$$L(u,v) = [\alpha + 2u - \ln(1+u)]e^{\varepsilon v} \quad \text{with } \alpha, \varepsilon \in \mathbb{R}_+^*. \tag{21}$$

Then for any $\varphi = \varphi(x)$ $(x \in \mathbb{R}^n)$ a smooth nonnegative function with exponential spacial decay at infinity, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx = d\int_{\mathbb{R}^n}\Delta\varphi L dx + (d-a)\int_{\mathbb{R}^n} L_1\nabla\varphi.\nabla u dx$$
$$-\int_{\mathbb{R}^n}\varphi\left[aL_{11}|\nabla u|^2 + (a+d)L_{12}\nabla u\nabla v + dL_{22}|\nabla v|^2\right]dx \tag{22}$$
$$+\int_{\mathbb{R}^n}\varphi(\lambda L_2 - L_1)g(u)v^m dx,$$

where

$$L_1 \equiv \frac{\partial L}{\partial u} = \left(2-\frac{1}{1+u}\right)e^{\varepsilon v}, L_2 \equiv \frac{\partial L}{\partial v} = \varepsilon[\alpha + 2u - \ln(1+u)]e^{\varepsilon v},$$
$$L_{11} \equiv \frac{\partial^2 L}{\partial u^2} = \frac{1}{(1+u)^2}e^{\varepsilon v}, L_{12} \equiv \frac{\partial^2 L}{\partial u \partial v} = \varepsilon\left(2-\frac{1}{1+u}\right)e^{\varepsilon v}, \tag{23}$$
$$L_{22} \equiv \frac{\partial^2 L}{\partial v^2} = \varepsilon^2[\alpha + 2u - \ln(1+u)]e^{\varepsilon v}.$$

Proof. Note that $L>0$, $L_1>0$, $L_2>0$, $L_{11}>0$, $L_{12}>0$ and $L_{22}>0$. We can differentiate under the integral symbol

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx = a\int_{\mathbb{R}^n}\varphi L_1 u dx + d\int_{\mathbb{R}^n}\varphi L_2\Delta v dx + \int_{\mathbb{R}^n}\varphi(\lambda L_2 - L_1)g(u)v^m dx. \tag{24}$$

Using integration by parts, we get

$$\int_{\mathbb{R}^n}\varphi L_1\Delta u dx = \int_{\mathbb{R}^n}(\varphi L_1)\Delta u dx = -\int_{\mathbb{R}^n}\nabla(\varphi L_1)\nabla u dx = -\int_{\mathbb{R}^n}L_1\nabla\varphi\nabla u dx$$
$$-\int_{\mathbb{R}^n}\varphi L_{11}|\nabla u|^2 dx - \int_{\mathbb{R}^n}\varphi L_{12}\nabla u\nabla v dx, \tag{25}$$

In fact, let $\rho\in\mathbb{R}_+^*$, then we have by the Geen theorem

$$\int_{B'(0,\rho)}\varphi L_1\Delta u dx = \int_{B'(0,\rho)}(\varphi L_1)\Delta u dx$$
$$= -\int_{B'(0,\rho)}\nabla(\varphi L_1).\nabla u dx + \int_{S(0,\rho)}(\varphi L_1)\frac{\partial u}{\partial \nu}dx, \tag{26}$$

where $\frac{\partial u}{\partial \nu}$ is the derivative of u with respect to the unit outer normal ν to the boundary $S(0,\rho)$.

We have

$$\left|\int_{S(0,\rho)}(\varphi L_1(t))\frac{\partial u(t)}{\partial \nu}dx\right| \le 2e^{\varepsilon\|v(t)\|_\infty}\left\|\frac{\partial u(t)}{\partial \nu}\right\|_\infty\int_{S(0,\rho)}\varphi dx$$
$$\le 2e^{\varepsilon\|v(t)\|_\infty}\left\|\frac{\partial u(t)}{\partial \nu}\right\|_\infty\frac{1}{(1+\rho^2)^n}\frac{2\pi^{n/2}\rho^{n-1}}{\Gamma(n/2)}. \tag{27}$$

From (27) we obtain

$$\lim_{\rho\to\infty}\int_{S(x_0,\rho)}[\varphi L_1(t)]\frac{\partial u(t)}{\partial \nu}dx = 0. \tag{28}$$

We pass to the limit for $\rho \to \infty$ in (26) taking into account (28) we obtain (25).
By the same way we get

$$\int_{\mathbb{R}^n} \varphi L_2 \Delta v dx = -\int_{\mathbb{R}^n} L_2 \nabla\varphi . \nabla v dx - \int_{\mathbb{R}^n} \varphi L_{22} |\nabla v|^2 dx - \int_{\mathbb{R}^n} \varphi L_{12} \nabla u \nabla v dx, \tag{29}$$

$$\int_{\mathbb{R}^n} L \Delta\varphi dx = -\int_{\mathbb{R}^n} L_1 \nabla\varphi . \nabla u dx - \int_{\mathbb{R}^n} L_2 \nabla\varphi . \nabla v dx. \tag{30}$$

From (30) we find that

$$\int_{\mathbb{R}^n} L_2 \nabla\varphi . \nabla v dx = -\int_{\mathbb{R}^n} L_1 \nabla\varphi . \nabla u dx - \int_{\mathbb{R}^n} L \Delta\varphi dx. \tag{31}$$

From (25), (29) and (31) into (24) we get our basic identity (22). ♦

Lemma 3.4. There exist two positive real constants $\alpha = \alpha(a, d, \gamma_1, \|u_0\|_\infty)$ and $\varepsilon = \varepsilon(a, d, \gamma_1, \gamma_2, \lambda_\infty, \|u_0\|_\infty)$ such that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^n} \varphi L dx \leq & d\int_{\mathbb{R}^n} L\Delta\varphi dx + (d-a)\int_{\mathbb{R}^n} L_1 \nabla\varphi . \nabla u dx \\ & -\gamma_1 \int_{\mathbb{R}^n} \varphi\left[aL_{11}|\nabla u|^2 + dL_{22}|\nabla v|^2\right] dx - \gamma_2 \int_{\mathbb{R}^n} \varphi L_1 g(u) v^m dx, \end{aligned} \tag{32}$$

where $\gamma_1, \gamma_2 \in]0, 1[$ are two arbitrary constants.

Proof. We seek L such that

$$aL_{11}|\nabla u|^2 + (a+d)L_{12}\nabla u \nabla v + dL_{22}|\nabla v|^2 \geq \gamma_1\left[aL_{11}|\nabla u|^2 + dL_{22}|\nabla v|^2\right] \tag{33}$$

and

$$\lambda L_2 - L_1 \leq -\gamma_2 L_1 \tag{34}$$

for $\gamma_1, \gamma_2 \in]0, 1[$.
The inequality (33) is satisfied if

$$\frac{(a+d)^2 L_{12}^2}{4ad(1-\gamma_1)^2 L_{11} L_{12}} \leq 1. \tag{35}$$

From (23); (35), then (33) is satisfied if

$$\alpha \geq \frac{(a+d)^2\left[1 + 2\|u_0\|_\infty\right]^2}{4ad(1-\gamma_1)^2}. \tag{36}$$

Also, (34) is satisfied if $\frac{\varepsilon\lambda_\infty(\alpha + 2\|u_0\|_\infty)}{1-\gamma_2} \leq 1$, i.e. $\varepsilon \leq \frac{1-\gamma_2}{\lambda_\infty(\alpha+2\|u_0\|_\infty)}$, and from (36) we get

$$0 < \varepsilon \leq \frac{1-\gamma_2}{\lambda_\infty} \frac{4ad(1-\gamma_1)^2}{(a+d)^2\left[1+2\|u_0\|_\infty\right]^2 + 8ad(1-\gamma_1)^2\|u_0\|_\infty}. \tag{37}$$

Whence, if α satisfies (36) and ε satisfies (37), we obtain (32). ♦

As a consequence of (33) we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \le d\int_{\mathbb{R}^n} L\Delta\varphi dx + (d-a)\int_{\mathbb{R}^n} L_1\nabla\varphi.\nabla u dx - \gamma_1 a\int_{\mathbb{R}^n}\varphi L_{11}|\nabla u|^2 dx. \tag{38}$$

Lemma 3.5. With the functional L defined in (21) and α, ε defined in (36) and (37) respectively and with the truncation function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{\left(1+|x-x_0|^2\right)^n}. \tag{39}$$

We have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \le dk_1(n)\int_{\mathbb{R}^n}\varphi L dx + \frac{1}{4\gamma_1 a}(d-a)^2 k_2^2(n)\int_{\mathbb{R}^n}\varphi\frac{L_1^2}{L_{11}}dx, \tag{40}$$

where

$$k_1(n) = 2n(3n+2), k_2(n) = 2n. \tag{41}$$

Proof. Calulate $\Delta\varphi$ and estimate it

$$\Delta\varphi = -\frac{2n^2}{\left(1+|x-x_0|^2\right)^{n+1}} - \frac{4n(n+1)|x-x_0|^2}{\left(1+|x-x_0|^2\right)^{n+2}};$$

whence

$$|\Delta\varphi| \le 2n(3n+2)\varphi. \tag{42}$$

Calulate $\nabla\varphi$ and estimate it

$$|\nabla\varphi|^2 = 4n^2\frac{|x-x_0|^2}{\left(1+|x-x_0|^2\right)^{2(n+2)}};$$

whence

$$|\nabla\varphi| \le 2n\varphi. \tag{43}$$

Using the Cauchy-Schwarz inequality $\nabla\varphi.\nabla u \le |\nabla\varphi||\nabla u|$ and the inequalities (42) and (43) into (38) we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \le dk_1(n)\int_{\mathbb{R}^n}\varphi L dx + (d-a)k_2(n)\int_{\mathbb{R}^n}\varphi L_1|\nabla\varphi| dx - \gamma_1 a\int_{\mathbb{R}^n}\varphi L_{11}|\nabla u|^2 dx. \tag{44}$$

We pove that

$$(d-a)k_2(n)\varphi L_1|\nabla\varphi| - \gamma_1 a\varphi L_{11}|\nabla u|^2 \le \frac{1}{4\gamma_1}\frac{(d-a)^2}{a}k_2^2(n)\varphi\frac{L_1^2}{L_{11}}. \tag{45}$$

To do this, it sufficies to compute the discriminant of the trinoma in $|\nabla\varphi|$

$$\Delta = -\gamma_1 a\varphi L_{11}|\nabla u|^2 + (d-a)k_2(n)\varphi L_1|\nabla\varphi| - \frac{1}{4\gamma_1}\frac{(d-a)^2}{a}k_2^2(n)\varphi\frac{L_1^2}{L_{11}}.$$

From (45) into (42) we find the desired result (40). ◆

Lemma 3.6. For α and ε defined in (36) and (37) respectively and for all real constant γ

$$\gamma \geq \max\left\{\frac{1}{a}, 8\|u_0\|_\infty + 4\right\}, \tag{46}$$

we have

$$\int_{\mathbb{R}^n}\varphi L dx \leq \beta e^{\sigma t}, \text{ for all } t \in \mathbb{R}_+; \tag{47}$$

where

$$\beta = \frac{2}{n}\left(\alpha + 2\|u_0\|_\infty\right)\omega_n e^{\varepsilon\|v_0\|_\infty}, \tag{48}$$

and

$$\sigma = dk_1(n) + \frac{\gamma}{4\gamma_1 a}(d-a)^2 k_2^2(n). \tag{49}$$

Proof. We seek a constant $\gamma \in \mathbb{R}_+^*$ such that

$$\frac{L_1^2}{L_{11}} \leq \gamma L, \text{ for all } u \in \left[0, \|u_0\|_\infty\right]. \tag{50}$$

The inequality (50) is equivalent to $(2u+1)^2 e^{\varepsilon v} \leq \gamma[\alpha + 2u - \ln(1+u)]$. We prove that if γ satisfies (46) then (50) follows.

Whence, from (50) into (40) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n}\varphi L dx \leq \left[dk_1(n) + \frac{\gamma}{4\gamma_1 a}(d-a)^2 k_2^2(n)\right]\int_{\mathbb{R}^n}\varphi L dx, \text{ for all } t \in \mathbb{R}_+. \tag{51}$$

As

$$\int_{\mathbb{R}^n}\varphi L(t=0)dx = \int_{\mathbb{R}^n}\varphi[\alpha + 2u_0 - \ln(1+u_0)]e^{\varepsilon v_0}dx; \tag{52}$$

then, from (51) and (52) we get

$$\int_{\mathbb{R}^n}\varphi L dx \leq \left(\alpha + 2\|u_0\|_\infty\right)\|\varphi\|_1\left[\exp\left(\varepsilon\|v_0\|_\infty\right)\right]e^{\sigma t}, \text{ for all } t \in \mathbb{R}_+, \tag{53}$$

where σ is defined by (49).

Now, let us estimate $\|\varphi\|_1$. We have ([11] on page 485)

$$\|\varphi\|_1 = \int_{\mathbb{R}^n}\varphi dx = \int_{\mathbb{R}^n}\frac{1}{\left(1+|x|^2\right)^n}dx = \omega_n\int_0^\infty r^{n-1}\frac{1}{(1+r^2)^n}dr.$$

As

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^n} dr = \int_0^1 \frac{r^{n-1}}{(1+r^2)^n} dr + \int_1^\infty \frac{r^{n-1}}{(1+r^2)^n} dr$$
$$\le \int_0^1 r^{n-1} dr + \int_1^\infty \frac{1}{r^{n+1}} dr \le \frac{2}{n},$$

then

$$\|\varphi\|_1 \le \frac{2}{n}\omega_n. \tag{54}$$

Thus, from (54) in (53) we get the estimate (47) with β and σ given by (48) and (49). ◆

In the following step we trie to control the second component v of the solution on any unit spacial cube in the L^p- norms with $p \in [1, \infty[$.

Let $x^{(0)} = \left(x_1^{(0)}, \dots, x_n^{(0)}\right) \in \mathbb{R}^n$ be an arbitrary fixed point and

$$Q = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \left| x_k - x_k^{(0)} \right| \le \frac{1}{2}, \text{for all} \, k = 1, \dots, n \right\}. \tag{55}$$

Lemma 3.7. Let (u, v) be the solution of the problem in consideration. For α and ε satisfying (36) above and (63) below respectively, then for any unit cube Q of $\mathbb{R}^n$ of the form (55) we have

$$\int_Q v^p dx \le \frac{\beta(p+1)^{p+1}}{\alpha\varepsilon^{[p]+1}} \left(\frac{4+n}{4}\right)^n e^{\sigma t}, \text{for all} (p, t) \in [1, \infty[\times \mathbb{R}_+. \tag{56}$$

Proof. It's obvious that

$$\varphi(x) \ge \left(\frac{4}{4+n}\right)^n, \text{for all} \, x \in \mathbb{R}^n, \tag{57}$$

and

$$e^{\varepsilon v} \ge \frac{\varepsilon^k}{k!} v^k, \text{for all} \, k \in \mathbb{N}^*. \tag{58}$$

Then

$$\int_{\mathbb{R}^n} \varphi L dx \ge \frac{\alpha \varepsilon^k}{k!} \left(\frac{4}{4+n}\right)^n \int_Q v^k dx. \tag{59}$$

Let us combine (47) and (59)

$$\int_Q v^k dx \le \frac{\beta k!}{\alpha \varepsilon^k} \left(\frac{4+n}{4}\right)^n e^{\sigma t}, \text{for all} (k, t) \in \mathbb{N}^* \times \mathbb{R}_+. \tag{60}$$

By induction we prove that

$$k! \le p^p, \text{for all} \, k \in \mathbb{N}^* \text{ and} \, p \ge k \ . \tag{61}$$

Let $p \geq 1$ and $k = [p] + 1$, then we have by the imbedding theorem for L^p −spaces

$$\int_Q v^p dx \leq \left(\int_Q v^k dx \right)^{p/k}. \tag{62}$$

Taking ε enough small such that $\frac{\beta k!}{\alpha \varepsilon^k} \left(\frac{4+n}{4}\right)^n \geq 1$. Combining this with (37)

$$0 < \varepsilon \leq \min \left\{ \begin{array}{c} \frac{1-\gamma_2}{\lambda_\infty} \frac{4ad(1-\gamma_1)^2}{(a+d)^2 \left[1 + 2\|u_0\|_\infty\right]^2 + 8ad(1-\gamma_1)^2 \|u_0\|_\infty}, \\ \left[\frac{1}{\alpha} \beta k! \left(\frac{4+n}{4}\right)^n\right]^{1/k} \end{array} \right\}. \tag{63}$$

From (60), (61) and (63) into (62) we get (56). ♦

Lemma 3.8. Let Q_i et Q_j be two different unit cubes of center $x^{(i)} = \left(x_1^{(i)}, \dots, x_n^{(i)}\right)$ and $x^{(j)} = \left(x_1^{(j)}, \dots, x_n^{(j)}\right)$ respectively of the form

$$\begin{aligned} Q_i &= \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \left| x_k - x_k^{(i)} \right| \leq 1/2 \right\}, \text{for all} k = 1, \dots, n, \\ Q_j &= \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \left| x_k - x_k^{(j)} \right| \leq 1/2 \right\}, \text{for all} k = 1, \dots, n, \end{aligned} \tag{64}$$

with $x^{(j)} = x^{(i)} + l$, where $l = (l_1, \dots, l_n) \in \mathbb{Z}^n \backslash 0_{\mathbb{Z}^n}$. Then, there exists a positive constant

$$\delta(n) = \left(2 + \sqrt{n}\right)^2, \tag{65}$$

such that

$$dist\left(x^{(i)}, Q_j\right)^2 \leq \left|x^{(i)} - y\right|^2 \leq \delta(n) dist\left(x^{(i)}, Q_j\right)^2, \text{for all } y \in Q_j. \tag{66}$$

Proof. By Pythagorean theorem we have

$$\left|x^{(j)} - y\right| \leq \frac{\sqrt{n}}{2}. \tag{67}$$

As $\left|x^{(i)} - x^{(j)}\right| \geq 1$, then from (67)

$$\left|x^{(j)} - y\right| \leq \frac{\sqrt{n}}{2} \left|x^{(i)} - x^{(j)}\right|. \tag{68}$$

Also, it's clear that $dist\left(x^{(i)}, Q_j\right) = dist\left(x^{(i)}, \partial Q_j\right)$, but every point $z = (z_1, \dots, z_n) \in \partial Q_j$ is of the form

$$z = x^{(j)} + s, \tag{69}$$

where $s = (s_1, \dots, s_n) \neq 0$ and $s_k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, for all $k = 1, \dots, n$ with at least one of the $s_k \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

It's easy to prove that

$$\left|x_k^{(j)} - x_k^{(i)}\right| \leq 2 \left|x_k^{(j)} - x_k^{(i)} + s_k\right|, \text{for all } k = 1, \dots, n. \tag{70}$$

Then

$$\left|x^{(j)} - x^{(i)}\right| \le 2dist\left(x^{(i)}, Q_j\right). \tag{71}$$

As $\left|x^{(i)} - y\right| \le \left|x^{(i)} - x^{(j)}\right| + \left|x^{(j)} - y\right|$ we get from (68) and (71) the estimate

$$\begin{aligned}\left|x^{(i)} - y\right| &\le 2dist\left(x^{(i)}, Q_j\right) + \frac{\sqrt{n}}{2}\left|x^{(i)} - x^{(j)}\right| \\ &\le \left(2 + \sqrt{n}\right)dist\left(x^{(i)}, Q_j\right).\end{aligned} \tag{72}$$

We have obviously

$$\left|x^{(i)} - y\right| \ge dist\left(x^{(i)}, Q_j\right). \tag{73}$$

From (71) and (73) we get (66). ◆

Proof of theorem 3.2.

Let $x \in \mathbb{R}^n$ an arbitrary point and $\left\{Q_j\right\}_{j \in \mathbb{N}}$ be the family of pairwise disjoint measurable cubes of the form (64) covering $\mathbb{R}^n$ such that the center of Q_0 is $x^{(0)} = x$.

Firstly, using the fact that $\mathbb{R}^n = \cup_{j=0}^{\infty} Q_j$ and applying the left-hand inequality in (66)

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4d(t-s)}} \lambda g(u) v^m dy &= \sum_{j=0}^{\infty} \int_{Q_j} e^{-\frac{|x-y|^2}{8d(t-s)}} e^{-\frac{|x-y|^2}{8d(t-s)}} \lambda g(u) v^m dy \\ &\le \sum_{j=0}^{\infty} \left\{ e^{-\frac{dist\left(x, Q_j\right)^2}{8d(t-s)}} \int_{Q_j}^{-\frac{|x-y|^2}{8d(t-s)}} \lambda g(u) v^m dy \right\}.\end{aligned} \tag{74}$$

By Hölder ineguality with $p >$ max $\left\{1, \frac{n}{2}\right\}$ and $q = 1 - \frac{1}{p}$

$$\int_{Q_j}^{-\frac{|x-y|^2}{8d(t-s)}} \lambda g(u) v^m dy \le \left[\int_{Q_j}^{-\frac{q|x-y|^2}{8d(t-s)}} dy\right]^{1/q} \left[\int_{Q_j} \lambda^p g^p(u) v^{pm} dy\right]^{1/p}. \tag{75}$$

As

$$\int_{Q_j} e^{-\frac{q|x-y|^2}{8d(t-s)}} dy \le \int_{\mathbb{R}^n}^{-\frac{q|x-y|^2}{8d(t-s)}} dy = \left(\frac{8\pi d}{q}\right)^{n/2} (t-s)^{n/2} \tag{76}$$

and by (56) we have

$$\int_{Q_j} \lambda^p g^p(u) v^{pm} dy \le \lambda_\infty^p g_\infty^p \beta \frac{(pm+1)^{pm+1}}{\alpha \varepsilon^{[pm]+1}} \left(\frac{4+n}{4}\right)^n e^{\sigma t}, \tag{77}$$

where

$$g_\infty = \sup_{u \in \left[0, \|u_0\|_\infty\right]} g(u). \tag{78}$$

Then, from (76) and (77) into (75)

$$\int_{Q_j} {}^{-\frac{|x-y|^2}{8d(t-s)}} \lambda g(u) v^m dy \leq K \left(\frac{8\pi d}{q}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} (t-s)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} \lambda_\infty g_\infty e^{(\sigma/p)t}, \tag{79}$$

where

$$K = K(p, m, n, \alpha, \varepsilon) = \left[\beta \frac{(pm+1)^{pm+1}}{\alpha \varepsilon^{[pm]+1}} \left(\frac{4+n}{4}\right)^n\right]^{1/p}. \tag{80}$$

On the other hand, we deduce from the right-hand inequality in (66) that

$$\int_{Q_j} e^{-\frac{|x-y|^2}{8d\delta(n)(t-s)}} dy \geq e^{-\frac{dist\left(x, Q_j\right)^2}{8d(t-s)}}, \text{ for all } j \in \mathbb{N}^*. \tag{81}$$

Then

$$\begin{aligned} \sum_{j=0}^{\infty} e^{-\frac{dist\left(x, Q_j\right)^2}{8d(t-s)}} &\leq 1 + \sum_{j=1}^{\infty} e^{-\frac{dist\left(x, Q_j\right)^2}{8d(t-s)}} \leq 1 + \sum_{j=1}^{\infty} \int_{Q_j} e^{-\frac{|x-y|^2}{8d\delta(n)(t-s)}} dy \\ &\leq 1 + \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{8d\delta(n)(t-s)}} dy \leq 1 + [8\pi d\delta(n)]^{n/2} (t-s)^{n/2}. \end{aligned} \tag{82}$$

We have from (79) and (82) into (74)

$$\begin{aligned} &\frac{1}{[4\pi d(t-s)]^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4d(t-s)}} \lambda g(u) v^m dy \\ &\leq 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} (t-s)^{-\frac{n}{2p}} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left\{1 + [8\pi d\delta(n)]^{n/2} (t-s)^{n/2}\right\} \\ &\leq 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left\{(t-s)^{-\frac{n}{2p}} + [8\pi d\delta(n)]^{n/2} (t-s)^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\right\}. \end{aligned}$$

Whence

$$\begin{aligned} &\int_0^t S_2(t-s) \lambda g(u) v^m ds \\ &\leq 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left[\frac{2p}{2p-n} t^{1-\frac{n}{2p}} + [8\pi d\delta(n)]^{n/2} \frac{2p}{p(n+2)+2p} t^{\frac{n}{2}\left(1-\frac{1}{p}\right)+1}\right] \end{aligned}$$

and finally we have for all $t \in \mathbb{R}_+$

$$\begin{aligned} \|v(t)\|_\infty &\leq \|v_0\|_\infty \\ &+ 2^{n/2} \left(1 - \frac{1}{p}\right)^{\frac{n}{2}\left(1-\frac{1}{p}\right)} K \lambda_\infty g_\infty e^{(\sigma/p)t} \left[\begin{array}{c} \frac{2p}{2p-n} t^{1-\frac{n}{2p}} \\ +[8\pi d\delta(n)]^{n/2} \frac{2p}{p(n+2)-n} t^{\frac{n}{2}\left(1-\frac{1}{p}\right)+1} \end{array}\right] \end{aligned} \tag{83}$$

As $p > \max\left\{1, \frac{n}{2}\right\}$, the function in t on the right-hand side of the estimate (83) is continuous on $\mathbb{R}_+$. As $\|u(t)\|_\infty \le \|u_0\|_\infty$ on $[0, t_{\max}[$ and v satisfied (83), we conclude from (11) that $t_{\max} = +\infty$. Whence, the solution is global. ♦

Remark. We can extend the system to the case where instead of v^m we put $vh(v)$ provided that.

i. $h : BUC(\mathbb{R}) \to BUC(\mathbb{R})$ is a locally continuous Lipschitz function, namely: for all constant $\rho \in \mathbb{R}_+$, there exists a constant $c(\rho) \in \mathbb{R}_+^*$ such that for all $u, v \in BUC(\mathbb{R}^n)$ with $\|u\|_\infty \le \rho$ and $\|v\|_\infty \le \rho$ we have

$$\|h(u) - g(v)\|_\infty \le c(\rho)\|u - v\|_\infty.$$

ii. There exist two constants $M \in \mathbb{R}_+^*$ and $r \in \mathbb{N}$ such that:

$$0 \le h(v) \le Mv^r, \text{for all}\, v \in \mathbb{R}_+.$$

In this more general case, by examining the proof of the theorem 3.2; we see that under the same assumptions above, the system has also a global nonnegative classical solution. ♦

4. Illustrative example

To illustrate the previous study about global existence, we give the following reaction–diffusion system

$$\begin{cases} u_t(t,x) = a\Delta u(t,x) - \dfrac{c_1 u^3}{c_2 + c_3 u^2} v^m, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ v_t(t,x) = d\Delta v(t,x) + c_4 e^{-c_5 t|x|^2} \dfrac{u^3}{c_2 + c_3 u^2} v^m, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n \quad , \\ u(0,x) = u_0(x), v(0,x) = v_0(x), x \in \mathbb{R}^n, \end{cases} \tag{84}$$

where $c_k,\ k = 1, \dots, 4$ are real positive constants and c_5 is a real nonnegative constant. If $a, b \in \mathbb{R}_+,\ n, m \in \mathbb{N}^*,\ u_0, v_0 \in BUC(\mathbb{R}^n)$ and are nonnegative; the system (84) admits a unique global nonnegative classical solution $(u, v) \in C(\mathbb{R}_+; X) \cap C^1\left(\mathbb{R}_+^*; X\right)$. ♦

5. Conclusion and perspectives

We have prouved in the case where $a < d$ that the solution is global, but it remains an interesting question that if it is uniformly bounded or not.

As perspectives, we will replace the function $g = g(u)$ satisfying the hypothesis (H4) by the function $g(u) = u^r$ with $r \ge 1$ is a real constant and replace the term v^m by $e^{\alpha v}$ with $\alpha > 0$; namely that reaction term is of exponential growth. The system was studied on bounded domain by J. I. Kanel and M. Mokhtar in [12]. ♦

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Chapter 8

The Fourier Transform Method for Second-Order Integro-Dynamic Equations on Time Scales

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Abstract

In this chapter we introduce the Fourier transform on arbitrary time scales and deduct some of its properties. In the chapter are given some applications for second-order integro-dynamic equations on time scales.

Keywords: time scale, Fourier transform, generalized shift problem, integro-dynamic equation

1. Introduction

Starting with the pioneering work of Hilger [1], the measure chains and in particular, the time scales have gained a great attention in the last decades. Especially, theoretical studies on dynamic equations on general time scales, which can be regarded as generalization of the differential equations, achieved big progress [2, 3].

The main aim of this chapter is to introduce the Fourier transform on arbitrary time scales and to deduct some of its properties. We give applications for solving of second-order integro-dynamic equations on time scales.

The chapter is organized as follows. In the next section we give some basic definitions and facts from time scale calculus, Laplace, bilateral Laplace transform. In Section 3 we define the Fourier transform and deduct some of its properties. In Section 4 we give applications for second-order integro-dynamic equations on time scales.

2. Preliminaries and auxiliary results

2.1 Time scales

Throughout this paper, we will assume that the reader is familiar with the basics of the time scale calculus. A detailed introduction to the time scale calculus is given in [2, 3]. Here, we collect the definitions and theorems that will be most useful in this paper.

Definition 2.1. *A time scale, denoted by* $\mathbb{T}$, *is a nonempty, closed subset of* $\mathbb{R}$. *For* $a, b \in \mathbb{T}$, *we let* $[a, b]$ *denote the set* $[a, b] \cap \mathbb{T}$.

Definition 2.2. *Let* $\mathbb{T}$ *be a time scale. For* $t \in \mathbb{T}$, *we define the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{R}$ *by* $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, *and the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is given by* $\rho(t) = \sup\{x \in \mathbb{T} : s < t\}$.

By convention, we take $\inf \emptyset = \sup\mathbb{T}$, $\sup\emptyset = \inf \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, we will use the notation $f^{\sigma}(t)$ for the composition $f(\sigma(t))$.

Definition 2.3. *The graininess function* $\mu : \mathbb{T} \to [0, \infty)$ *is defined by* $\mu(t) = \sigma(t) - t$, $t \in \mathbb{T}$.

Definition 2.4. *Let* $t \in \mathbb{T}$. *If* $\sigma(t) = t$ *and* $t < \sup\mathbb{T}$, *then t is right-dense. If* $\sigma(t) > t$, *then t is right-scattered. Similarly, if* $\rho(t) = t$ *and* $t > \inf \mathbb{T}$, *then t is left-dense. If* $\rho(t) < t$, *then t is left-scattered.*

Definition 2.5. *If* $\sup\mathbb{T} = m$ *such that m is left-scattered, then define* $\mathbb{T}^{\kappa} = \mathbb{T}\backslash\{m\}$, *otherwise, define* $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 2.6. *A function* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous provided it is continuous at right-dense points in* $\mathbb{T}$ *and its left-sided limits exist and are finite at all left-dense points in* $\mathbb{T}$. *A function* $p : \mathbb{T} \to \mathbb{R}$ *is regressive provided* $1 + \mu(t)p(t) \neq 0$, $t \in \mathbb{T}^{\kappa}$. *The set of all regressive and rd-continuous functions on a time scale* $\mathbb{T}$ *is denoted by* $\mathcal{R} = \mathcal{R}(\mathbb{T})$. *We use the notation* $\mathcal{R}^{+}$ *to denote the subgroup of those* $p \in \mathcal{R}$ *for which* $1 + \mu(t)p(t) > 0$ *for all* $t \in \mathbb{T}^{\kappa}$.

Definition 2.7. *The delta derivative of* $f : \mathbb{T} \to \mathbb{R}$ *at* $t \in \mathbb{T}^{\kappa}$, *is defined to be*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \tag{1}$$

provided this limit exists.

Definition 2.8. *For* $p \in \mathcal{R}$, *the generalized exponential function* $e_p : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ *is defined by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \tag{2}$$

for $s, t \in \mathbb{T}$, *where the cylinder transformation*, $\xi_h(z)$, *is defined by*

$$\xi_h(z) = \begin{cases} \dfrac{1}{h} Log(1 + zh), & h > 0, \\ z, \quad h = 0. \end{cases} \tag{3}$$

Definition 2.9. *For* $p, q \in \mathcal{R}$, *we define the operation* $\oplus$ *and* $\ominus$ *as follows*

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t), \quad (\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)}. \tag{4}$$

The proof of the next theorem is given in [2, 3].

Theorem 2.1. *If* $p, q \in \mathcal{R}$ *and* $t, s, r \in \mathbb{T}$, *then*

1.$e_0(t, s) = 1$, $e_p(t, t) = 1$.

2.$e_p^{\sigma}(t, s) = (1 + \mu(t)p(t))e_p(t, s)$.

3.$e_p(s, t) = \frac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$.

4. $e_p(t,s)e_p(s,r) = e_p(t,r)$.

5. $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$.

6. $e_p(t,t_0) > 0$ *for any* $t_0, t \in \mathbb{T}$ *if* $p \in \mathcal{R}$ *and* $1+\mu(t)p(t) > 0$ *for any* $t \in \mathbb{T}^\kappa$.

Definition 2.10. *For* $h > 0$*, the Hilger complex plane is defined by* $\mathbb{C}_h = \mathbb{C}\backslash\{-\frac{1}{h}\}$ *and we take* $\mathbb{C}_0 = \mathbb{C}$ *and* $\mathbb{C}_\infty = \mathbb{C}\backslash\{0\}$.

Definition 2.11. *For given* $h \in [0,\infty)$*, the Hilger real part of a number* $z \in \mathbb{C}$ *is given by the formula*

$$Re_h(z) = \begin{cases} Re\,(z), & h = 0, \\ \dfrac{|1+hz|-1}{h}, & 0 < h < \infty, \\ |z|, & h = \infty. \end{cases} \tag{5}$$

It is known, see [4], that for a fixed z and $0 < h < \infty$, $\mathrm{Re}_h(z)$ is a nondecreasing function of h. This relationship extends to $h = \infty$ because for any $0 < h < \infty$,

$$\mathrm{Re}_h(z) = \frac{|1+hz|-1}{h} \le \frac{1+h|z|-1}{h} = |z| = \mathrm{Re}_\infty(z). \tag{6}$$

2.2 The Laplace transform

Here we suppose that $\sup\mathbb{T} = \infty$ and $s \in \mathbb{T}$.

Definition 2.12. *For* $0 \le h \le \infty$ *and* $\lambda \in \mathbb{R}$*, we define*

$$\mathbb{C}_h(\lambda) = \{z \in \mathbb{C}_h : Re_h(z) > \lambda\} \tag{7}$$

and

$$\bar{\mathbb{C}}_h(\lambda) = \{z \in \mathbb{C}_h : 0 < Re_h(z) < \lambda\}. \tag{8}$$

Definition 2.13. *Define minimal graininess as follows* $\mu_*(s) = \inf_{t\in[s,\infty)}\mu(t)$.

If λ is positively regressive, then for any $z \in \mathbb{C}_{\mu_*(s)}(\lambda)$, it is known (see [4]) that

$$\begin{gathered} |e_{\lambda\ominus z}(t,s)| \le e_{\lambda\ominus \mathrm{Re}_{\mu_*(s)}(z)}(t,s), \quad t \in [s,\infty), \\ \lim_{t\to\infty} \mathrm{Re}_{\mu_*(s)(z)}(t,s) = 0 \quad \text{and} \quad \lim_{t\to\infty} e_{\lambda\ominus z}(t,s) = 0. \end{gathered} \tag{9}$$

Definition 2.14. *If* $X \subset \mathbb{T}$ *and* $\alpha \in \mathcal{R}^+$ *is a constant, then we say that* $f \in C_{rd}(\mathbb{T})$ *is of exponential order* α *on* X *if there exists a constant* K *such that for all* $t \in X$*, the bound* $|f(t)| \le Ke_\alpha(t,s)$ *holds.*

If $f \in C_{rd}([s,\infty))$ is of exponential order α, then for any $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$ (see [4]) $\lim_{t\to\infty} f(t)e_{\ominus z}(t,s) = 0$.

Definition 2.15. *If* $f : \mathbb{T} \to \mathbb{C}$ *and* $z \in \mathbb{C}$ *is a complex number such that for all* $t \in [s,\infty)$ *we have* $1+\mu(t)z \ne 0$*, then the Laplace transform is defined by the improper integral*

$$\mathcal{L}(f)(z,s) = \int_s^\infty f(\tau)e_{\ominus z}(\sigma(\tau),s)\Delta\tau, \tag{10}$$

whenever the integral exists.

Significant work has been conducted in [4, 5] and references therein to understand the analytical properties of the Laplace transform.

2.3 The bilateral Laplace transform

Here we suppose that $\sup\mathbb{T} = \infty$, $\inf \mathbb{T} = -\infty$ and $s \in \mathbb{T}$. Denote $\mu^*(s) = \sup_{t\in(-\infty,s]}\mu(t)$, $\bar{\mu}(s) = \inf_{t\in(-\infty,s]}\mu(t)$. For $\lambda \in \mathcal{R}$, define

$$M_\lambda(t,s) = \int_t^s \frac{1}{1+\lambda\mu(\tau)}\Delta\tau. \tag{11}$$

For $\lambda \in \mathcal{R}^+((-\infty,s])$, $\lambda \in \mathbb{R}$, it is known (see [6])

1. $M_\lambda^\Delta(t,s) < 0$ for all $t \in (-\infty,s)$, where the differentiation is with respect to t.

2. $\lim_{t\to-\infty} M_\lambda(t,s) = \infty$.

3. $|e_{\lambda\ominus z}(t,s)| \le e_{\lambda\ominus \mathrm{Re}_{\mu^*(s)}(z)}(t,s)$.

4. $\lim_{t\to-\infty} e_{\lambda\ominus \mathrm{Re}_{\mu^*(s)}(z)}(t,s) = 0$.

5. $\lim_{t\to-\infty} e_{\lambda\ominus z}(t,s) = 0$.

Definition 2.16. *Suppose that $f : \mathbb{T} \to \mathbb{R}$ is regulated. Then the bilateral Laplace transform of f is defined by*

$$\mathcal{L}^b(f)(z,s) = \int_{-\infty}^{\infty} f(t)e_{\ominus z}(\sigma(t),s)\Delta t, \tag{12}$$

for regressive $z \in \mathbb{C}$ where the improper integral exists.

Definition 2.17. *Let $\alpha, \gamma \in \mathbb{R}$. We say that a function $f \in C_{rd}(\mathbb{T})$ has double exponential order (α,γ) on $\mathbb{T}$ if the restrictions $f|_{(-\infty,s]}$ and $f|_{[s,\infty)}$ are of exponential order α and γ, respectively.*

If $f \in C_{rd}(\mathbb{T})$ is of double exponential order (α,γ), in [6], they are proved the following properties

1. for any $z \in \mathbb{C}_{\mu_*(s)}(\gamma)$, $\lim_{t\to\infty} f(t)e_{\ominus z}(t,s) = 0$.

2. for any $z \in \bar{\mathbb{C}}_{\mu^*(s)}(\alpha)$, $\lim_{t\to-\infty} f(t)e_{\ominus z}(t,s) = 0$.

For $z \in \mathbb{C}$, we define

$$\bar{\bar{\mu}}(s,z) = \begin{cases} \mu^*(s), & \mathrm{Re}_{\bar{\mu}(s)}(z) \le 0, \\ \bar{\mu}(s), & \mathrm{Re}_{\bar{\mu}(s)}(z) > 0. \end{cases} \tag{13}$$

Definition 2.18. *Let $\alpha \in \mathcal{R}^+((-\infty,s])$ and $\gamma \in \mathcal{R}^+([s,\infty))$, α, $\gamma \in \mathbb{R}$. We say that (s,α,γ) is an admissible triple if*

$$\mathbb{C}_{s,\alpha,\gamma} = \left\{ z \in \mathbb{C} : \mathit{Re}_{\mu^*(s)}(z) < \alpha, \quad \mathit{Re}_{\mu_*(s)}(z) > \gamma, \quad 1 + \bar{\bar{\mu}}(s,z)\,\mathit{Re}_{\bar{\mu}(s)}(z) \ne 0 \right\} \ne \varnothing. \tag{14}$$

If (s, α, γ) is an admissible triple and if $f \in \mathcal{C}_{rd}(\mathbb{T})$ is of double exponential order (α, γ), then in [6] it is proved that $\mathcal{L}^b(\cdot, s)$ exists on $\mathbb{C}_{s,\alpha,\gamma}$, converges absolutely and uniformly, and

$$\lim_{|z|\to\infty} \mathcal{L}^b(f)(z, s) = 0. \tag{15}$$

3. The Fourier transform

Suppose that $\mathbb{T}$ is a time scale so that $\inf \mathbb{T} = -\infty$, $\sup\mathbb{T} = \infty$ and $s \in \mathbb{T}$.

Definition 3.1. *Suppose that $f : \mathbb{T} \to \mathbb{R}$ is regulated. Then the Fourier transform of the function f is defined by*

$$\mathcal{F}(f)(x, s) = \int_{-\infty}^{\infty} f(t) e^{\sigma}_{\ominus ix}(t, s)\Delta t \tag{16}$$

for $x \in \mathbb{R}$ for which $1 + ix\mu(t) \neq 0$ for any $t \in \mathbb{T}^\kappa$ and the improper integral exists.

Definition 3.2. *Let $\alpha \in \mathcal{R}^+([s, \infty))$, $\gamma \in \mathcal{R}^+((-\infty, s])$. We say that (s, γ, α) is a real admissible triple if*

$$\begin{aligned} R_{s,\gamma,\alpha} = \{x \in \mathbb{R} :\ & Re_{\mu^*(s)}(ix) < \gamma, \quad Re_{\mu_*(s)}(ix) > \alpha, \\ & 1 + \bar{\bar{\mu}}(s)\, Re_{\bar{\mu}(s)}(ix) \neq 0\} \neq \varnothing. \end{aligned} \tag{17}$$

If $f \in \mathcal{C}_{rd}(\mathbb{T})$, then the triple (s, γ, α) is a real admissible triple and f is of double exponential order (α, γ), then $\mathcal{F}(f)(\cdot, s)$ exists on $R_{s,\gamma,\alpha}$ and converges absolutely and uniformly on $R_{s,\gamma,\alpha}$. Below we will list some of the properties of the Fourier transform.

Theorem 3.1. *Let $f, g : \mathbb{T} \to \mathbb{R}$, $\alpha, \beta \in \mathbb{C}$. Then*

$$\mathcal{F}(\alpha f + \beta g)(x, s) = \alpha\mathcal{F}(f)(x, s) + \beta\mathcal{F}(g)(x, s) \tag{18}$$

for those $x \in \mathbb{R}$ for which $1 + x\mu(t) \neq 0$, $t \in \mathbb{T}^\kappa$, and the respective integrals exist.

Proof. We have

$$\begin{aligned} \mathcal{F}(\alpha f + \beta g)(x, s) &= \int_{-\infty}^{\infty} (\alpha f + \beta g)(t) e^{\sigma}_{\ominus ix}(t, s)\Delta t \\ = \alpha \int_{-\infty}^{\infty} f(t) e^{\sigma}_{\ominus ix}(t, s)\Delta t &+ \beta \int_{-\infty}^{\infty} g(t) e^{\sigma}_{\ominus ix}(t, s)\Delta t = \alpha\mathcal{F}(f)(x, s) + \beta\mathcal{F}(g)(x, s). \end{aligned} \tag{19}$$

This completes the proof. □

Theorem 3.2. *Let $f : \mathbb{T} \to \mathbb{R}$ be enough times Δ-differentiable. For any $k \in \mathbb{N}$, we have*

$$\mathcal{F}\left(f^{\Delta^k}\right)(x, s) = (ix)^k \mathcal{F}(f)(x, s) \tag{20}$$

for those $x \in \mathbb{R}$ for which $1 + x\mu(t) \neq 0$, $t \in \mathbb{T}^\kappa$, and the respective integrals exist and

$$\lim_{t\to\pm\infty} f^{\Delta^l}(t) e_{\ominus ix}(t, s) = 0, \quad l \in \{0, \dots, k - 1\}. \tag{21}$$

Proof. We will use the principle of mathematical induction.

1. For $k = 1$, we have

$$\begin{aligned}\mathcal{F}(f^{\Delta})(x,s) &= \int_{-\infty}^{\infty} f^{\Delta}(t)e^{\sigma}_{\ominus ix}(t,s)\Delta t = \lim_{t\to\infty} f(t)e_{\ominus ix}(t,s) - \lim_{t\to-\infty} f(t)e_{\ominus ix}(t,s)\\ &\quad -\int_{-\infty}^{\infty} (\ominus ix)(t)f(t)e_{\ominus ix}(t,s)\Delta t = ix\int_{-\infty}^{\infty} f(t)e^{\sigma}_{\ominus ix}(t,s)\Delta t\\ &= ix\mathcal{F}(f)(x,s).\end{aligned} \tag{22}$$

2. Assume that

$$\mathcal{F}\left(f^{\Delta^k}\right)(x,s) = (ix)^k\mathcal{F}(f)(x,s) \tag{23}$$

for some $k \in \mathbb{N}$.

3. We will prove that

$$\mathcal{F}\left(f^{\Delta^{k+1}}\right)(x,s) = (ix)^{k+1}\mathcal{F}(f)(x,s). \tag{24}$$

Really, we have

$$\mathcal{F}\left(f^{\Delta^{k+1}}\right)(x,s) = ix\mathcal{F}\left(f^{\Delta^k}\right)(x,s) = (ix)^{k+1}\mathcal{F}(f)(x,s). \tag{25}$$

This completes the proof. □

Theorem 3.3. *Let $f : \mathbb{T} \to \mathbb{R}$. Then*

$$\overline{\mathcal{F}(f)(x,s)} = \mathcal{F}(f)(-x,s) \tag{26}$$

for those $x \in \mathbb{R}$ for which $1 \pm x\mu(t) \neq 0$, $t \in \mathbb{T}^{\kappa}$, and the respective integrals exist.

Proof. From the definition of the Fourier transform, we have

$$\begin{aligned}\overline{\mathcal{F}(f)(x,s)} &= \overline{\int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)}\mathrm{Log}(1+\mu(\tau)(\ominus(ix))(\tau))\Delta\tau} f(t)\Delta t}\\ &= \int_{-\infty}^{\infty} \overline{e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)}\mathrm{Log}(1+\mu(\tau)(\ominus(ix))(\tau))\Delta\tau} f(t)}\,\Delta t\\ &= \int_{-\infty}^{\infty} e^{\int_s^{\sigma(t)} \frac{1}{\mu(\tau)}\mathrm{Log}(1+\mu(\tau)(\ominus(i(-x)))(\tau))\Delta\tau} f(t)\Delta t\\ &= \mathcal{F}(f)(-x,s).\end{aligned} \tag{27}$$

This completes the proof. □

Theorem 3.4. *Let $f : \mathbb{T} \to \mathbb{R}$ be regulated and*

$$F(t) = \int_a^t f(\tau)\Delta\tau, \quad t \in \mathbb{T}, \tag{28}$$

for some fixed $a \in \mathbb{T}$. Then

$$\mathcal{F}(F)(x,s) = -\frac{i}{x}\mathcal{F}(f)(x,s) \tag{29}$$

for those $x \in \mathbb{R}$, $x \neq 0$, for which

$$\lim_{t\to\pm\infty} F(t)e_{\ominus ix}(t,s) = 0. \tag{30}$$

Proof. We have

$$\begin{aligned}\mathcal{F}(F)(x,s) &= \int_{-\infty}^{\infty} F(t)e^{\sigma}_{\ominus ix}(t,s)\Delta t = \int_{-\infty}^{\infty} F(t)(1+\mu(t)(\ominus(ix))(t))e_{\ominus ix}(t,s)\Delta t \\ &= \int_{-\infty}^{\infty} F(t)\frac{1}{1+i\mu(t)x}e_{\ominus ix}(t,s)\Delta t = -\frac{1}{ix}\int_{-\infty}^{\infty} F(t)\frac{-ix}{1+i\mu(t)x}e_{\ominus ix}(t,s)\Delta t \\ &= \frac{i}{x}\int_{-\infty}^{\infty} F(t)(\ominus ix)(t)e_{\ominus ix}(t,s)\Delta t = \frac{i}{x}\int_{-\infty}^{\infty} F(t)e^{\Delta}_{\ominus ix}(t,s)\Delta t \\ &= \frac{i}{x}\left(\lim_{t\to\infty} F(t)e_{\ominus ix}(t,s) - \lim_{t\to-\infty} F(t)e_{\ominus ix}(t,s)\right) - \frac{i}{x}\int_{-\infty}^{\infty} f(t)e^{\sigma}_{\ominus ix}(t,s)\Delta t \\ &= -\frac{i}{x}\mathcal{F}(f)(x,s)\end{aligned} \tag{31}$$

for those $x \in \mathbb{R}$, $x \neq 0$, for which

$$\lim_{t\to\pm\infty} F(t)e_{\ominus ix}(t,s) = 0. \tag{32}$$

This completes the proof. □

4. Applications to second-order integro-dynamic equations

Consider the equation

$$y^{\Delta^2} + a_1y^{\Delta} + a_2y = \int_a^t f(s)\Delta s, \tag{33}$$

where $a_1, a_2 \in \mathbb{R}, f \in \mathcal{C}_{rd}(\mathbb{T}), f : \mathbb{T} \to \mathbb{R}$. Let $s \in \mathbb{T}$ be fixed. Let also, $x \in \mathbb{R}$ be such that

$$x^2 - ia_1x - a_2 \neq 0 \tag{34}$$

and

$$\lim_{t\to\pm\infty} y^{\Delta^l}(t)e_{\ominus ix}(t,s) = 0, \quad l = 0, 1, \tag{35}$$

and

$$\lim_{t\to\pm\infty} F(t)e_{\ominus ix}(t,s) = 0, \tag{36}$$

where

$$F(t) = \int_a^t f(s)\Delta s, \quad t \in \mathbb{T}. \tag{37}$$

Here $a \in \mathbb{T}$ is a fixed constant. Set

$$Y(x) = \mathcal{F}(y)(x,s). \tag{38}$$

Then

$$\begin{aligned} \mathcal{F}\left(y^{\Delta}\right)(x,s) &= ix\mathcal{F}(y)(x,s) \\ &= ixY(x), \\ \mathcal{F}\left(y^{\Delta^2}\right)(x,s) &= (ix)^2\mathcal{F}(y)(x,s) \\ &= -x^2Y(x) \end{aligned} \tag{39}$$

and

$$\mathcal{F}(f)(x,s) = -\frac{i}{x}\mathcal{F}(x,s). \tag{40}$$

Then the Eq. (33) takes the form

$$-x^2Y(x) + ia_1xY(x) + a_2Y(x) = -\frac{i}{x}\mathcal{F}(x,s), \tag{41}$$

or

$$\left(x^2 - ia_1x - a_2\right)Y(x) = \frac{i}{x}\mathcal{F}(f)(x,s), \tag{42}$$

or

$$Y(x) = \frac{i}{x(x^2 - ia_1x - a_2)}\mathcal{F}(f)(x,s). \tag{43}$$

Consequently

$$y(t) = \mathcal{F}^{-1}\left(\frac{i}{x(x^2 - ia_1x - a_2)}\mathcal{F}(f)(\cdot,s)\right)(t), \quad t \in \mathbb{T}, \tag{44}$$

provided that $\mathcal{F}^{-1}$ exists.

Additional classifications

AMS Subject Classification: 39A10, 39A11, 39A12

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Chapter 9

Effect of Additive Perturbations on the Solution of Reflected Backward Stochastic Differential Equations

Jasmina Đorđević

Abstract

This chapter has as a topic large class of general, nonlinear reflected backward stochastic differential equations with a lower barrier, whose generator, final condition as well as barrier process arbitrarily depend on a small parameter. The solutions of these equations which are obtained by additive perturbations, named the perturbed equations, are compared in the L^p-sense, $p \in]1, 2[$, with the solutions of the appropriate equations of the equal type, independent of a small parameter and named the unperturbed equations. Conditions under which the solution of the unperturbed equation is L^p-stable are given. It is shown that for an arbitrary $a > 0$ there exists $t(a) \leq T$, such that the L^p-difference between the solutions of both the perturbed and unperturbed equations is less than a for every $t \in [t(a), T]$.

Keywords: reflected, backward, stochastic, perturbation, estimate

1. Introduction

This chapter is dedicated to the problem of additive perturbations of reflected backward stochastic differential equations (shorter RBSDEs) with one lower barrier. Motivation for the topic comes from a large application of perturbation problems in real life problems from one side, and reflected backward stochastic differential equations in finance from another. Perturbed stochastic differential equations are widely applied in theory and in applications. Randomness from the environment can be introduced via stochastic models with perturbations. In such manner, complex phenomena under perturbations in analytical mechanics, control theory, population dynamics or financial models, can be compared and approximated by appropriate unperturbed models of a simpler structure, i.e. the problems are translated on more simple and familiar cases which are easier to solve and investigate (see [1–3] for example). Problem of additively perturbed backward stochastic differential equations is analysed by Janković, M. Jovanović, J. Đorđević in [4], while generally perturbed reflected backward stochastic differential equations are already observed by Đorđević and Janković in [5]. Topic of this chapter is additive type of perturbations for reflected backward stochastic differential equations as a special type of mention general problem for reflected backward stochastic differential equations, and a more general one than the additive perturbation problem for simple backward stochastic differential equations. Finer and more precise estimates are deduced and generalizations emphasised.

Backward stochastic differential equations (BSDEs for short) was introduced and developed by Pardoux and Peng [6–8] in the 90s. Notation of nonlinear BSDE and proof of the existence and uniqueness of adapted solutions is given in their fundamental paper [6]. After that, many applications incited to introduction various types of BSDEs, in mathematical problems in finance (see [9]), stochastic control and stochastic games (see [10, 11]), stochastic partial differential equations, semi-linear parabolic partial differential equations (PDEs) (see [8, 12]) etc. (for further reading see also [13–17]).

Type of RBSDEs which is observed in this chapter have been first introduced in literature by El-Karoui *et al.* in [18]. Introduced RBSDEs with one lower barrier has following form,

$$\begin{aligned} &Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dBs, \quad 0 \leq t \leq T, \\ &Y_t \geqslant L_t, \quad t \leq T \; and \int_0^T (Y_s - L_s)dK_s = 0 \quad P - a.s., \end{aligned} \tag{1}$$

where one of the components of the solution is forced to stay above a given barrier/obstacle process $L = \{L_t, t \in [0, T]\}$. The solution is a triple of adapted processes $\{(Y_t, Z_t, K_t), t \in [0, T]\}$ which satisfies Eq. (1). The process $K = \{K_t, t \in [0, T]\}$ is nondecreasing and its purpose is to push upward the state process $Y = \{Y_t, t \in [0, T]\}$ in order to keep it above the obstacle L.

As it was already mentioned, RBSDEs are connected with a wide range of applications within which, the pricing of American options (constrained or not) in markets is most famous one. Further, the important applications of RBSDEs are in mixed control problems, partial differential variational inequalities, real options (see [9, 18–21] and the references therein) etc. El-Karoui *et al.* proved in [18] the existence and uniqueness of the solution to Eq. (1) under conditions of square integrability of the data and Lipschitz property for the coefficient (also called driver) f. Field of RBSDEs is developing in two directions, some authors deal with the issue of the existence and uniqueness results for RBSDEs under weaker assumptions (than the ones in [6] which are for the general BSDEs), while others are introducing some new types of those equations by adding jumps, introducing second barrier etc.

Systematization of the papers which are done in the framework of RBSDEs can be found in paper [5] by Đorđević and Janković.

Recently, Hamèdene and Popier in [22] proved that if ξ, $\sup_{t\in[0,T]}(L_t^+)$ and $\int_0^T |f(t, 0, 0)|dt$ belong to L^p for some $p \in]1, 2[$, then RBSDE (1) with one reflecting barrier associated with (f, ξ, L) has a unique solution. Aman gave [23] a similar result for a class of generalized RBSDEs with Lipschitz condition on the coefficients, and he extended these results under non-Lipschitz condition in his paper [24]. There are several papers by Hamadène [25] and Hamadène and Ouknine [26], Matoussi [27], Lepeltier and Xu [28] and Ren *et al.* [29, 30] in which authors emphasise the significance of the case when the data are from L^p for some $p \in]1, 2[$.

The aim of this chapter is a study Eq. (1) if the terminal condition ξ and generator f are p-integrable, $p \in]1, 2[$. Regarding that in several applications such as in finance, control, games, PDEs, etc., data are not square integrable, and the influence of some random external factors on the system can be seen as perturbations of the solution of Eq.(1), it is natural to introduced additive perturbation in the parameters of equation ξ, f and barrier process L, in order to better describe the change of the system and find some measurement for the change.

This chapter is organized in following way; In Section 2 elementary notations, definitions and preliminary results regarding RBSDEs are introduced. Next section is dedicated to the formulation of the main problem, i.e. problem of additively perturbed RBSDEs with one lower obstacle is stated. Together with the set up for the problem, some auxiliary estimates are proved in this section. In Section 4, conditions under which the solutions are stable are given, and estimates for the stability are derived. Section 5 contains the most interesting result, i.e. the estimation of a time interval for a given closeness of the solutions. The chapter is finished with the Section 6, Conclusions remarks, where the highlights of the chapter are emphasised and ideas and open problems for the future research are stated.

2. Preliminaries

All random variables and processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $\{\mathcal{F}_t, t \in [0, T]\}$ is a natural filtration of a standard d-dimensional Brownian motion $B = \{B_t, t \in [0, T]\}$, that is, it is right continuous and complete. Also, all stochastic processes are defined for $t \in [0, T]$, where T is a positive, fixed, real constant, and they take values in $\mathbb{R}^n$ for some positive integer n. For any $k \in \mathbb{N}$ and $x \in \mathbb{R}^k$, $|x|$ denotes the Euclidean norm of x.

Further, for any real constant $p \in]1, 2[$, we recall on standard notations which will be used:

i. $\mathcal{S}^p(\mathbb{R})$ is the set of $\mathbb{R}$-valued, adapted and continuous processes $\{X_t, t \in [0, T]\}$ such that

$$\|X\|_{\mathcal{S}^p} = E\left[\sup_{t \in [0,T]} |X_t|^p\right]^{\frac{1}{p}} < \infty.$$

The space $\mathcal{S}^p(\mathbb{R})$ endowed with the norm $\|\cdot\|_{\mathcal{S}^p}$ is a Banach type.

ii. $\mathcal{M}^p$ is the set of predictable processes $\{Z_t, t \in [0, T]\}$ with values in $\mathbb{R}^d$ such that

$$\|X\|_{\mathcal{M}^p} = E\left[\left(\int_0^T |Z_t|^2 dt\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty.$$

Likewise, $\mathcal{M}^p(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{\mathcal{M}^p}$ is a Banach space.

iii. The space $\mathcal{S}^p \times \mathcal{M}^p$ will be denoted by B^p.

Let ξ be an $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variable and let a random function $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{P}$ denotes the σ-field of progressive subsets of $[0, T] \times \Omega$, while $L := \{L_t, t \in [0, T]\}$ is a continuous progressively measurable $\mathbb{R}$-valued process.

The following hypothesis are introduced for ξ, f and L:

$(\mathbf{H_1})$ $\xi \in L^p(\Omega)$.

$(\mathbf{H_2})$

i. The process $\{ f(t, 0, 0), t \in [0, T]\}$ satisfies $E\left(\int_0^T |f(t, 0, 0)| dt\right)^p < \infty$;

ii. (ii) (Lipschitz condition) there exists a constant $k > 0$ such that for all $t \in [0, T]$, $(y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^d$,

$$|f(t,y,z) - f(t,y',z')| \leq k(|y-y'| + |z-z'|).$$

($\mathbf{H_3}$) The barrier process L satisfies:

i. $L_T \leq \xi$;

ii. $L^+ := L \vee 0 \in \mathcal{S}^p(\mathbb{R})$.

The definition of the unique solution to Eq. (1), associated with the triple (ξ, f, L), and the existence and uniqueness theorem under Lipschitz condition are given in [22].

Definition 1

I. (Existence of the solution.) The triple $\{(Y_t, Z_t, K_t), t \in [0,T]\}$ is an L^p-solution to RBSDE (1) with a continuous lower reflecting barrier L, terminal condition ξ and drift/generator/driver f if:

1. $\{(Y_t, Z_t), t \in [0,T]\}$ belongs to $\mathcal{B}^p$;

2. $K = \{K_t, t \in [0,T]\}$ is an adapted continuous nondecreasing process such that $K_0 = 0$ and $K_T \in L^p(\Omega)$;

3. $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s \, a.s., t \in [0,T]$;

4. $Y_t \geq L_t, t \in [0,T]$;

5. $\int_0^T (Y_s - L_s) dK_s = 0 \quad P - a.s.$

II. (Uniqueness of the solution.) The triple $\{(Y_t, Z_t, K_t), t \in [0,T]\}$ is a unique L^p-solution to RBSDE (1) if for any other solution $\{(\overline{Y}_t, \overline{Z}_t, \overline{K}_t), t \in [0,T]\}$, the following holds,

$$\left\|Y_t - \overline{Y}_t\right\|_{\mathcal{S}^p} = 0, \quad \left\|Z_t - \overline{Z}_t\right\|_{\mathcal{M}^p} = 0, \quad \left\|K_t - \overline{K}_t\right\|_{\mathcal{S}^p} = 0. \tag{2}$$

Proposition 1 [Hamèdene, Popier [22]] Let $(\mathbf{H_1}) - (\mathbf{H_3})$ hold for ξ, f and L. Then, RBSDE (1) with one continuous lower reflecting barrier L associated with (ξ, f, L) has a unique L^p-solution, $p \in]1,2[$, i.e. there exists a triple of processes $\{(Y_t, Z_t, K_t), t \in [0,T]\}$ satisfying Definition 1.

The following lemma is well known result and it is widely used in stability estimates.

Lemma 1 [Hamèdene, Popier [22]] Assume that $(Y,Z) \in \mathcal{B}^p$ is a solution of the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + A_T - A_t - \int_t^T Z_s dB_s, \quad t \in [0,T],$$

where:

i. f is a function satisfying the previous assumptions;

ii. The process $\{A_t, t \in [0,T]\}$ is $P - a.s.$ of bounded variation.

Then, for any $0 \le t \le u \le T$ it follows that

$$
\begin{aligned}
|Y_t|^p + c(p)\int_t^u |Y_s|^{p-2} 1_{Y_s \neq 0} |Z_s|^2 ds \\
\le |Y_u|^p + p\int_t^u |Y_s|^{p-1} \hat{Y}_s dA_s \\
+p\int_t^u |Y_s|^{p-1} \hat{Y}_s f(s, Y_s, Z_s) ds - p\int_t^u |Y_s|^{p-1} \hat{Y}_s Z_s dB_s,
\end{aligned}
$$

where $c(p) = \frac{p(p-1)}{2}$ and $\hat{y} = \frac{y}{|y|} 1_{y\neq 0}$.

When the model of some phenomenon is described by RBSDE, than, some change of the system can be treated as additive perturbation of the initial equation. The size of the change could be estimated as the difference between the solutions of the initial equation and the perturbed one. In view of this direction, together with Eq. (1), we study the following perturbed RBSDE,

$$
\begin{aligned}
&Y_t = \xi^\varepsilon + \int_t^T f^\varepsilon\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds + K_T^\varepsilon - K_t^\varepsilon - \int_t^T Z_s^\varepsilon dB_s, \quad 0 \;\; t \in [0, T], \\
&Y_t^\varepsilon \ge L_t^\varepsilon, \quad t \le T \; and \int_0^T \left(Y_s^\varepsilon - L_s^\varepsilon\right) dK_s^\varepsilon = 0, \quad P - a.s,
\end{aligned}
\tag{3}
$$

where $\xi^\varepsilon, f^\varepsilon$ and the barrier L^ε are defined as ξ, f and L, respectively, they depend on a small parameter $\varepsilon \in (0, 1)$, and they are of a special additive form

$$
\begin{aligned}
\xi^\varepsilon &= \xi + \beta(T, \varepsilon), \\
f^\varepsilon(t, y, z, \varepsilon) &= f(t, y, z) + \alpha(t, y, z, \varepsilon), \\
L_t^\varepsilon &= L_t + l_t^\varepsilon.
\end{aligned}
$$

For a given $(f^\varepsilon, \xi^\varepsilon, L^\varepsilon)$, a triple of adapted processes $\left\{\left(Y_t^\varepsilon, Z_t^\varepsilon, K_t^\varepsilon\right), t \in [0, T]\right\}$ is a solution to Eq. (3). In the sequel Eq. (1) will be named the *unperturbed* equation, while Eq. (3) a *additively perturbed* one. It is usually expected that the additively perturbed Eq. (3) is more general and more complexed than the unperturbed one. Furthermore, it is obvious that in case when $\beta(\varepsilon) \equiv \alpha(t, y, z, \varepsilon) \equiv l_t^\varepsilon \equiv 0$, additively perturbed equation reduces to unperturbed equation. This fact is a basic motivation for us to introduce conditions guaranteeing the closeness of the solutions of the additively perturbed and unperturbed equations in the L^p-sense, and to estimate the conditions for the additive parameters in order for the solutions of these equations to stay close in the L^p-sense in some way.

After basic notations, definitions and results are present, the formulation of the main problem is given in following section.

3. Formulation of the problem of additively perturbed RBSDEs with one lower obstacle & and auxiliary results

In order to deduce estimates for the closeness of the solutions of additively perturbed and unperturbed equations, following assumptions are introduced;

($\mathcal{A}\mathbf{0}$) For the additional part in final condition of perturbed equation $\beta(T, \varepsilon)$, such that $\xi^\varepsilon = \xi + \beta(T, \varepsilon)$, while $\xi^\varepsilon, \xi \in L^p(\Omega)$, there exists a non-random function $\beta_1(\varepsilon), \varepsilon \in (0, 1)$, such that

$$
E|\beta(T, \varepsilon)|^p \le \beta_1(\varepsilon).
$$

$(\mathcal{A}\mathbf{1})$ For the additional part in the generator/driver integral, $\alpha(t, y, z, \varepsilon)$, there exists a non-random function $\alpha_1(\varepsilon), \varepsilon \in (0, 1)$, such that

$$\sup_{(t,y,z) \in [0,T] \times B^p} |\alpha(t, y, z, \varepsilon)| \leq \alpha_1(\varepsilon) \ \ a.s.$$

$(\mathcal{A}\mathbf{2})$ For the additional part in barrier processes l_t^ε, there exists a non-random function $l_1(\varepsilon), \varepsilon \in (0, 1)$, such that

$$E \sup_{t \in [0,T]} \left|l_t^\varepsilon\right|^p \leq l_1(\varepsilon).$$

We give first an auxiliary result for the stability of the solutions which we will use to prove main result.

Proposition 2 Let $p \in]1, 2[$ and let $\{(Y_t, Z_t, K_t), t \in [0, T]\}$ and $\left\{\left(Y_t^\varepsilon, Z_t^\varepsilon, K_t^\varepsilon\right), t \in [0, T]\right\}$ be the solutions to additively unperturbed and perturbed Eqs. (1) and (3), respectively. Let also assumptions $(\mathcal{A}0) - (\mathcal{A}2)$ and conditions $(H_1) - (H_3)$ be satisfied. Then,

$$E\left|Y_t^\varepsilon - Y_t\right|^p \leq C_1 e^{c_1(T-t)}, \quad t \in [0, T], \tag{4}$$

where $c_1 = p - 1 + pk + \frac{pk^2}{p-1}$ and $C_1 = \beta_1(\varepsilon) + \alpha_1^p(\varepsilon)T + l_1^{\frac{p-1}{p}}(\varepsilon)\left(E\left|\hat{K}_T\right|^p\right)^{\frac{1}{p}}$.

Proof: Let us denote for $t \in [0, T]$ the differences of the processes of the solutions,

$$\hat{Y}_t = Y_t^\varepsilon - Y_t, \quad \hat{Z}_t = Z_t^\varepsilon - Z_t, \ \ \hat{K}_t = K_t^\varepsilon - K_t.$$

If we subtract Eqs. (1) and (3), we obtain

$$\hat{Y}_t = \beta(T, \varepsilon) + \int_t^T \alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds + \hat{K}_T - \hat{K}_t - \int_t^T \hat{Z}_s dB_s, \ \ t \in [0, T]. \tag{5}$$

Applying Lemma 1 on $\left|\hat{Y}_t\right|^p$, we have

$$\begin{aligned}
&\left|\hat{Y}_t\right|^p + c(p)\int_t^T \left|\hat{Y}_s\right|^{p-2} 1_{\hat{Y}_s \neq 0}\left|\hat{Z}_s\right|^2 ds \\
&\leq |\beta(T, \varepsilon)|^p + p\int_t^T \left|\hat{Y}_s\right|^{p-1} sgn\left(\hat{Y}_s\right)\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds \\
&+p\int_t^T \left|\hat{Y}_s\right|^{p-1} sgn\left(\hat{Y}_s\right) d\left(\Delta\hat{K}_s\right) - p\int_t^T \left|\hat{Y}_s\right|^{p-1} sgn\left(\hat{Y}_s\right)\hat{Z}_s dB_s \\
&\leq |\beta(T, \varepsilon)|^p + p\int_t^T \left|\hat{Y}_s\right|^{p-1} |\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right)| ds \\
&+pk\int_t^T \left|\hat{Y}_s\right|^{p-1} |Y_s^\varepsilon - Y_s| ds + pk\int_t^T \left|\hat{Y}_s\right|^{p-1} |Z_s^\varepsilon - Z_s| ds \\
&+p\int_t^T \left|\hat{Y}_s\right|^{p-1} sgn\left(\hat{Y}_s\right) d\left(\Delta\hat{K}_s\right) - p\int_t^T \left|\hat{Y}_s\right|^{p-1} sgn\left(\hat{Y}_s\right)\hat{Z}_s dB_s \\
&:= |\beta(T, \varepsilon)|^p + I_1(t) + pk\int_t^T \left|\hat{Y}_s\right|^p ds + I_2(t) + I_3(t) + I_4(t),
\end{aligned} \tag{6}$$

where $I_i(t), i = 1, 2, 3, 4$ are the appropriate integrals. In order to estimate $I_1(t)$, we apply the elementary inequality $a^{p-1}b \leq \frac{p-1}{p}a^p + \frac{1}{p}b^p, a, b \geq 0$ and assumption $(\mathcal{A}1)$. Then,

$$\begin{aligned} I_1(t) &= p\int_t^T \left|\hat{Y}_s\right|^{p-1} |\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right)| ds \\ &\le (p-1)\int_t^T \left|\hat{Y}_s\right|^p ds + \alpha_1^p(\varepsilon)(T-t). \end{aligned} \tag{7}$$

In order to estimate $I_2(t)$, we use the elementary inequality $2ab \le \frac{a^2}{2} + 2b^2$,

$$I_2(t) \le \frac{pk^2}{p-1}\int_t^T \left|\hat{Y}_s\right|^p ds + \frac{c(p)}{2}\int_t^T \left|\hat{Y}_s\right|^{p-2} I_{\left\{\hat{Y}_s \ne 0\right\}} \left|\hat{Z}_s\right|^2 ds,$$

where $c(p) = p(p-1)/2$.

For estimation of member $I_3(t)$, we will use mapping $(x,a) \to \tilde{\theta}(x,a) = |x-a|^{p-2} 1_{x \ne a}(x-a)$, $(x,a) \in \mathbb{R} \times \mathbb{R}$. Function $x \to \tilde{\theta}(x,a)$ is non-decreasing, while the function $a \to \tilde{\theta}(x,a)$ is non-increasing. As it is known, $l_s^\varepsilon = L_s^\varepsilon - L_s$ and since $Y_s^\varepsilon \ge L_s^\varepsilon$, $Y_s \ge L_s$, then

$$\begin{aligned} I_3(t) &= p\int_t^T \left|\hat{Y}_s\right|^{p-1} sgn\left(\hat{Y}_s\right) d\left(\Delta\hat{K}_s\right) \\ &= p\int_t^T \tilde{\theta}\left(Y_s^\varepsilon, Y_s\right) I_{\left\{Y_s^\varepsilon = L_s^\varepsilon\right\}} dK_s^\varepsilon - p\int_t^T \tilde{\theta}\left(Y_s^\varepsilon, Y_s\right) I_{\{Y_s = L_s\}} dK_s \\ &\le p\int_t^T \left|L_s^\varepsilon - L_s\right|^{p-2} I_{\left\{L_s^\varepsilon - L_s \ne 0\right\}} \left(L_s^\varepsilon - L_s\right) dK_s^\varepsilon - p\int_t^T \left|L_s^\varepsilon - L_s\right|^{p-2} I_{\left\{L_s^\varepsilon - L_s \ne 0\right\}} \left(L_s^\varepsilon - L_s\right) dK_s \\ &= p\int_t^T |l_s^\varepsilon|^{p-1} d\left(\hat{K}_s\right). \end{aligned} \tag{8}$$

Substituting estimates for $I_i(t), i = 1,2,3$ in in (6), we obtain

$$\begin{aligned} &\left|\hat{Y}_t\right|^p + \frac{c(p)}{2}\int_t^T \left|\hat{Y}_s\right|^{p-2} I_{\left\{\hat{Y}_s \ne 0\right\}} \left|\hat{Z}_s\right|^2 ds \\ &\quad \le |\beta(T,\varepsilon)|^p + \left(p - 1 + pk + \frac{pk^2}{p-1}\right)\int_t^T \left|\hat{Y}_s\right|^p ds + p\int_t^T |l_s^\varepsilon|^{p-1} d\left(\hat{K}_s\right), \\ &\quad +\alpha_1^p(\varepsilon)(T-t) + I_4(t). \end{aligned}$$

Taking expectation on last inequality, and taking into account that expectation of I_4 is 0, we have

$$\begin{aligned} E\left|\hat{Y}_t\right|^p &\le \beta_1(\varepsilon) + \alpha_1^p(\varepsilon)T + l_1^{\frac{p-1}{p}}(\varepsilon)\left(E\left|\hat{K}_T\right|^p\right)^{\frac{1}{p}} \\ &\quad + \left(p - 1 + pk + \frac{pk^2}{p-1}\right)\int_t^T E\left|\hat{Y}_s\right|^p ds. \end{aligned}$$

As $K_T, K_T^\varepsilon \in L^p(\Omega)$, it follows that $E\left|\hat{K}_T\right|^p < \infty$. So (4) holds straightforwardly by applying the Gronwall-Bellman inequality ([31], Theorem 1.5):

Let $u(t)$ be a continuous function in $[a,b], f(t)$ be Riemann integrable function in $[a,b]$ and $c = const > 0$. If $u(t) = f(t) + c\int_t^b u(s)ds, t \in [a,b]$, then $u(t) \le f(t) + c\int_t^b f(s)e^{c(s-t)} ds, t \in [a,b]$.

The theorem is proved.

★

For the introduced problem, conditions for the stability of the solutions and estimates for the stability of the solutions are derived In next section.

4. Stability estimates for additive perturbations

For the estimate of L^p-difference between the solutions to Eqs. (1) and (3), the L^p-stability of the solution to Eq. (1) is necessary.

Following theorem provides the result, that in case of small additive perturbations case, we can expect that the difference of the solutions of perturbed and unperturbed equations tends to zero, when the perturbations are sufficiently small.

Theorem 1 Let all the conditions of Proposition 2 be satisfied and let the functions $\beta_1(\varepsilon), \alpha_1(\varepsilon), l_1(\varepsilon)$ tend to zero as ε tends to zero, uniformly in $t \in [0, T]$. Then it follows that

$$E \sup_{t\in[0,T]} \left|Y_t^\varepsilon - Y_t\right|^p \to 0,\ \varepsilon \to 0,$$

$$E\left(\int_0^T \left|Z_s^\varepsilon - Z_s\right|^2 ds\right)^{\frac{p}{2}} \to 0, \varepsilon \to 0,$$

$$E \sup_{t\in[0,T]} E\left|K_t^\varepsilon - K_t\right|^p \to 0,\quad \varepsilon \to 0.$$

Proof: Let us define

$$\phi(\varepsilon) := \max\left\{\beta_1, (\varepsilon), \alpha_1^p(\varepsilon), l_1^{\frac{p-1}{p}}(\varepsilon)\right\}. \tag{9}$$

From Proposition 2, we have that $C_1 \le \phi(\varepsilon)\tilde{C}$, where $\tilde{C} = 1 + T + \left(E\left|\hat{K}_T\right|^p\right)^{\frac{1}{p}}$ and, therefore,

$$E\left|\hat{Y}_t\right|^p \le \phi(\varepsilon)\tilde{C}e^{c_1(T-t)},\quad t\in[0,T].$$

Since $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$, then for every $t_0 \in [0, T]$,

$$\sup_{t\in[t_0,T]} E\left|\hat{Y}_t\right|^p \le \phi(\varepsilon)\tilde{C}e^{c_1(T-t_0)} \to 0,\quad \varepsilon \to 0. \tag{10}$$

In order to estimate the L^p-closeness between the processes Z_t and $Z^\varepsilon(t)$, as well as K_t and $K^\varepsilon(T)$, we need estimate $E\sup_{t\in[0,T]}\left|\hat{Y}_t\right|^p$, that is to estimate $I_4(t)$. By applying the Burkholder-Davis-Guandy inequality [32] and Young inequality, $u^\alpha v^{1-\alpha} \le \alpha u + (1-\alpha)v, v \ge 0, \alpha \in [0,1]$, we have

$$\begin{aligned}
E\sup_{t\in[t_0,T]} I_4(t) &\le 4\sqrt{2}pE\left(\int_{t_0}^T \left|\hat{Y}_s\right|^{2p-2}\left|\hat{Z}_s\right|^2 ds\right)^{\frac{1}{2}} \\
&\le 4\sqrt{2}pE\left(\sup_{t\in[t_0,T]}\left|\hat{Y}_t\right|^p \int_{t_0}^T \left|\hat{Y}_s\right|^{p-2}\left|\hat{Z}_s\right|^2 ds\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}E\sup_{t\in[t_0,T]}\left|\hat{Y}_t\right|^p + 16p^2E\int_{t_0}^T \left|\hat{Y}_s\right|^{p-2}\left|\hat{Z}_s\right|^2 ds.
\end{aligned}$$

We can conclude that

$$E \sup_{t\in[t_0,T]} I_4(t) \le \frac{1}{2} E \sup_{t\in[t_0,T]} \left|\hat{Y}_t\right|^p + \frac{32p^2}{c(p)} \phi(\varepsilon)\tilde{C} e^{c_1(T-t_0)}. \tag{11}$$

It follows that

$$E \sup_{t\in[t_0,T]} \left|\hat{Y}_t\right|^p \le \frac{1}{2} E \sup_{t\in[t_0,T]} \left|\hat{Y}_s\right|^p + \frac{32p^2}{c(p)} \phi(\varepsilon)\tilde{C} e^{c_1(T-t_0)} + c_1 \int_{t_0}^{T} E\left|\hat{Y}_s\right|^p ds + \phi(\varepsilon)\tilde{C}.$$

Hence,

$$E \sup_{t\in[t_0,T]} \left|\hat{Y}_t\right|^p \le 2\phi(\varepsilon)\tilde{C}\left[e^{c_1(T-t_0)}\left(1+\frac{32p^2}{c(p)}\right)+1\right] \equiv \phi(\varepsilon)A_1(t_0), \tag{12}$$

where $A_1(t_0)$ is a generic positive constant. By the assumption of the theorem, $\phi(\varepsilon) \to$ as $\varepsilon \to 0$, then $E\sup_{t\in[t_0,T]}\left|\hat{Y}_t\right|^p \to 0, as \ \ \varepsilon \to 0$. The desired estimate holds if we take $t_0 = 0$.

Now we can estimate the other two parts.

For every $i \in \{0, 1, 2, \ldots\}$ and arbitrary $t_0 \in [0, T]$, let us define stopping times

$$\tau_i = \inf\left\{t \in [0,T], \int_{t_0}^{t} \left\|\hat{Z}_s\right\|^2 ds \ge i\right\} \wedge T.$$

Clearly, $\tau_i \uparrow T \quad a.s.$ when $i \to \infty$. If we apply the Ito formula to $e^{kt}\left|\hat{Y}_t\right|^2, t \in [t_0, \tau_i]$, we find that

$$\begin{aligned}
&\left|\hat{Y}_{t_0}\right|^2 + \int_{t_0}^{\tau_i} e^{ks}\left|\hat{Z}_s\right|^2 ds \\
&\quad = e^{k\tau_i}\left|\hat{Y}_{\tau_i}\right|^2 + \int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\left[2\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) - k\hat{Y}_s\right] + 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s d\hat{K}_s - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s \\
&\quad := e^{k\tau_i}\left|\hat{Y}_{\tau_i}\right|^2 + J_1 + J_2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s,
\end{aligned} \tag{13}$$

where estimates J_1 and J_2 are the appropriate integrals. For $\lambda_1 > 0$ that

$$\begin{aligned}
J_1 &= 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds - k\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s^2 ds \\
&\le 2\sup_{s\in[t_0,\tau_i]} e^{ks}|\hat{Y}_s|\alpha_1(\varepsilon)(T-t_0) - k\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s^2 ds \\
&\le \frac{1}{\lambda_1}\sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 + \lambda_1(T-t_0)^2\alpha_1^2(\varepsilon) - k\int_{t_0}^{\tau_i} e^{ks}\left|\hat{Y}_s\right|^2 ds.
\end{aligned} \tag{14}$$

Similarly, for $\lambda_2 > 0$,

$$
\begin{aligned}
J_2 &= 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s d\hat{K}_s \le 2 \sup_{s\in[t_0,\tau_i]} e^{ks}|\hat{Y}_s| \int_{t_0}^{\tau_i} d\hat{K}_s \\
&\le \frac{1}{\lambda_2} \sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 + \lambda_2\left(\int_{t_0}^{\tau_i} d\hat{K}_s\right)^2 \\
&= \frac{1}{\lambda_2} \sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 + \lambda_2\left(\hat{K}_{\tau_i} - \hat{K}_{t_0}\right)^2.
\end{aligned}
\tag{15}
$$

Also,

$$
\begin{aligned}
\left(\hat{K}_{\tau_i} - \hat{K}_{t_0}\right)^2 &= \left(\hat{Y}_{\tau_i} - \hat{Y}_{t_0} - \int_{t_0}^{\tau_i} \alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds + \int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right)^2 \\
&\le 4\left[\left|\hat{Y}_{\tau_i}\right|^2 + \left|\hat{Y}_{t_0}\right|^2 + \left|\int_{t_0}^{T} \alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) 1_{\{s=T\wedge\tau_i\}} ds\right|^2 + \left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^2\right] \\
&\le 4\left[\left|\hat{Y}_{\tau_i}\right|^2 + \left|\hat{Y}_{t_0}\right|^2 + 2(T-t_0)^2\alpha_1^2(\varepsilon) + \left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^2\right].
\end{aligned}
\tag{16}
$$

Substituting (14), (15) and (16) in (13) yields

$$
\begin{aligned}
&(1-4\lambda_2)\left|\hat{Y}_0\right|^2 + \int_{t_0}^{\tau_i} e^{ks}\left|\hat{Z}_s\right|^2 ds \\
&\quad\le \left(e^{k\tau_i} + 4\lambda_2\right)\left|\hat{Y}_{\tau_i}\right|^2 + \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 \\
&\qquad + 4\lambda_2\left|\int_{t_0}^{\tau_i} \hat{Z}_s dBs\right|^2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s - k\int_{t_0}^{\tau_i} e^{ks}\left|\hat{Y}_s\right|^2 ds + (\lambda_1 + 8\lambda_2)(T-t_0)^2\alpha_1^2(\varepsilon).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds &\le \left(e^{k\tau_i} + 4\lambda_2 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \sup_{s\in[t_0,\tau_i]} e^{2ks}\left|\hat{Y}_s\right|^2 \\
&\quad + 4\lambda_2\left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s \\
&\quad - k\int_{t_0}^{\tau_i} e^{ks}\left|\hat{Y}_s\right|^2 ds + (\lambda_1 + 8\lambda_2)(T-t_0)^2\alpha_1^2(\varepsilon).
\end{aligned}
\tag{17}
$$

The last inequality can be written as

$$
\begin{aligned}
\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds \le c_2(t_0) \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_s\right|^2 + 4\lambda_3\left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^2 - 2\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s \\
+ (\lambda_1 + 8\lambda_2)(T-t_0)^2\alpha_1^2(\varepsilon),
\end{aligned}
\tag{18}
$$

where

$$
c_2(t_0) = \left[e^{kT} + 4\lambda_2 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - k(T-t_0)\right]e^{2kT}.
$$

By applying the inequality $\left(\sum_{i=1}^m a_i\right)^k \le \left(m^{k-1} \vee 1\right)\sum_{i=1}^m a_i^k, a_i \ge 0, k \ge 0$ on (18) and by taking expectation, we obtain

$$E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \le c_2^{\frac{p}{2}}(t_0) E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_s\right|^p + 4^{\frac{p}{2}}\lambda_2^{\frac{p}{2}} E\left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^p$$
$$+2^{\frac{p}{2}} E\left|\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s\right|^{\frac{p}{2}} + (\lambda_1 + 8\lambda_2)^{\frac{p}{2}}(T-t_0)^p \phi(\varepsilon). \tag{19}$$

It is left to estimate two integrals with respect to Brownian motion, which will be done by applying the Burkholder–Davis–Guandy inequality,

$$E\left|\int_{t_0}^{\tau_i} \hat{Z}_s dB_s\right|^p \le C_p E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}},$$
$$2^{\frac{p}{2}} E\left|\int_{t_0}^{\tau_i} e^{ks}\hat{Y}_s\hat{Z}_s dB_s\right|^{\frac{p}{2}} \le C_{\frac{p}{2}} 2^{\frac{p}{2}} e^{\frac{pkT}{2}} E\left[\left(\int_{t_0}^{\tau_i} \left|\hat{Y}_s\right|^2 \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{4}}\right]$$
$$\le c_3 E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_t\right|^p + \lambda_3 \left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}},$$

where $\lambda_3 > 0$, $c_3 = \frac{1}{\lambda_3} C_{\frac{p}{2}}^2 2^{p-2} e^{pkT}$, and $C_p = (32/p)^{p/2}$ and $C_{\frac{p}{2}} = (64/p)^{p/4}$ are the universal constants. Substituting previous estimates in (19), it follows that

$$E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \le c_2^{\frac{p}{2}}(t_0) E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_s\right|^p + 4^{\frac{p}{2}}\lambda_2^{\frac{p}{2}} C_p E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}}$$
$$+c_3 E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_t\right|^p + \lambda_3 E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}}$$
$$+(\lambda_1 + 8\lambda_2)^{\frac{p}{2}}(T-t_0)^p \phi(\varepsilon),$$

i.e.

$$\left(1 - 4^{\frac{p}{2}}\lambda_2^{\frac{p}{2}} C_p - \lambda_3\right) E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}}$$
$$\le \left(c_2^{\frac{p}{2}}(t_0) + c_3\right) E \sup_{s\in[t_0,\tau_i]} \left|\hat{Y}_s\right|^p + (\lambda_1 + 8\lambda_2)^{\frac{p}{2}}(T-t_0)^p \phi(\varepsilon). \tag{20}$$

The constants λ_2, λ_3 can be chosen such that $1 - 4^{\frac{p}{2}}\lambda_2^{\frac{p}{2}} C_p - \lambda_3 > 0$, then, from (12) and (20) it follows that

$$E\left(\int_{t_0}^{\tau_i} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \le \frac{\left(c_2^{\frac{p}{2}}(t_0) + c_3\right) A_1(t_0) + (\lambda_1 + 8\lambda_2)^{\frac{p}{2}}(T-t_0)^p}{1 - 4^{\frac{p}{2}}\lambda_2^{\frac{p}{2}} C_p - \lambda_3} \phi(\varepsilon)$$
$$\equiv A_2(t_0)\phi(\varepsilon), \tag{21}$$

where $A_2(t_0)$ is a positive generic constant. By the Fatou's Lemma,

$$E\left(\int_{t_0}^{T} \left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \to 0, \quad \varepsilon \to 0.$$

Then, second estimate holds if we take $t_0 = 0$.

It is left to estimate the difference between the processes K and K^ε. From (5), we have

$$\hat{K}_t = \beta(T,\varepsilon) - \hat{Y}_t + \int_t^T \alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds - \int_t^T \hat{Z}_s dB_s + \hat{K}_T.$$

In view of (12) and (21), we derive that

$$\begin{aligned}
&E \sup_{t\in[t_0,T]} \left|\hat{K}_t\right|^p \\
&\leq 5^{p-1}\left\{ E|\beta(T,\varepsilon)|^p + E \sup_{t\in[t_0,T]} \left|\hat{Y}_t\right|^p + \left|\int_{t_0}^T \alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds\right|^p \right. \\
&\quad \left. + E \sup_{t\in[t_0,T]} \left|\int_t^T \hat{Z}_s dB_s\right|^p + E\left|\hat{K}_T\right|^p \right\} \\
&\leq 5^{p-1}\left\{1 + A_1(t_0) + (T-t_0)^{\frac{p}{2}} 2^{\frac{p}{2}} + C_p A_2(t_0)\right\}\phi(\varepsilon) + 5^{p-1} E\left|\hat{K}_T\right|^p.
\end{aligned} \tag{22}$$

Since

$$\hat{K}_T = \hat{Y}_0 - \beta(T,\varepsilon) - \int_0^T \alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds + \int_0^T \hat{Z}_s dB_s,$$

in accordance with the last estimate, we have that

$$\begin{aligned}
E\left|\hat{K}_T\right|^p &\leq 4^{p-1}\left[E\left|\hat{Y}_0\right|^p + E|\beta(T,\varepsilon)|^p + E\left|\int_0^T \alpha\left(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon\right) ds\right|^p + E\left|\int_0^T \hat{Z}_s dB_s\right|^p \right] \\
&\leq 4^{p-1}\left[\tilde{C}e^{c_1 T} + 1 + T^{\frac{p}{2}} 2^{\frac{p}{2}} + A_2(0)\right]\phi(\varepsilon).
\end{aligned}$$

Hence, it follows from that there exists a generic constant $A_3(t_0) > 0$ such that

$$E \sup_{t\in[0,T]} \left|\hat{K}_t\right|^p \leq A_3(t_0)\phi(\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{23}$$

Then, the last estimate of the theorem holds if we take $t_0 = 0$, which completes the proof.

★

In this section complete proof for the stability of the solutions is given, which as a strong result and it enable us to estimate the time interval for a given closeness of the solutions. This result is proved in next section.

5. Time interval for a given closeness of the solutions

Theorem 1 provides that the state processes Y_t^ε and Y_t, the control processes Z_t^ε and Z_t, as well as K^ε and K could be arbitrarily close for ε sufficiently small. I.e., if perturbations are small enough, closeness of the solutions can be provided. But, from the perspective of applications and modelling, it is usually important to study the closeness between Y_t^ε and Y_t near to the terminal values ξ^ε and ξ. Per example, for the application in pricing American options, an agent would be interested how will the price behave near the exercise time. It is interesting and useful to find the time interval on which we could preserve the wanted closeness, i.e. that for some permissible $a > 0$ and ε sufficiently small, find $\bar{t}(a) = \bar{t} \in [0, T]$ so that the rate of the closeness between Y_t^ε and Y_t does not exceed a on $[\bar{t}, T]$. Even-more, estimate of the closeness between the control processes Z_t^ε and Z_t on $[\bar{t}, T]$ can be estimated.

Theorem 2 Let all the conditions of Theorem 1 hold. Also, let the function $\phi(\varepsilon), \varepsilon \in (0, 1)$ defined with (9) be continuous and monotone increasing. Then, for an arbitrary constant $a > 0$ and $\varepsilon \in \left(0, \Phi^{-1}(a)\right]$, there exists $\bar{t} \in [0, T]$, where

$$\bar{t} = \max\left\{0, T - \frac{1}{c_1}\ln\frac{\eta}{\phi(\varepsilon)\tilde{C}}\right\},$$

such that

$$\sup_{t \in [\bar{t}, T]} E\left|Y_t^\varepsilon - Y_t\right|^p \leq a, \tag{24}$$

$$E\left(\int_{\bar{t}}^{T}\left|\hat{Z}_s\right|^2 ds\right)^{\frac{p}{2}} \leq A_2(\bar{t})\,\phi(\varepsilon), \tag{25}$$

$$E\sup_{t \in [\bar{t}, T]}\left|\hat{K}_t\right|^p \leq A_3(\bar{t})\,\phi(\varepsilon), \tag{26}$$

and $A_2(\bar{t})$ and $A_3(\bar{t})$ are constants defined in (21) and (23), respectively.

Proof: Let us introduce function $S(\varepsilon, T - t), t \in [0, T]$, such that

$$S(\varepsilon, T - t) = \phi(\varepsilon)\,\tilde{C}e^{c_1(T-t)},$$

where c_1 is given in Proposition 1 and $\tilde{C}$ in Theorem 1. For an arbitrary $a > 0$, it must be

$$S(\varepsilon, 0) \leq a \leq S(\varepsilon, T),$$

that is,

$$\phi(\varepsilon)\tilde{C} \leq a \leq \phi(\varepsilon)\tilde{C}e^{c_1 T}.$$

Since $\phi(\varepsilon)$ decreases if ε decreases, it follows that

$$\varepsilon_1 = \phi^{-1}\left(\frac{a}{\tilde{C}e^{c_1 T}}\right) \leq \varepsilon \leq \phi^{-1}\left(\frac{a}{\tilde{C}}\right) = \varepsilon_2,$$

where ϕ^{-1} is the inverse function of ϕ. For every $\varepsilon \in [\varepsilon_1, \varepsilon_2]$, it is now easy to determine $\hat{t}$ from the relation $S(\varepsilon, T - \hat{t}) = a$, that is,

$$\hat{t} = T - \frac{1}{c_1}\ln\frac{\eta}{\phi(\varepsilon)\tilde{C}}.$$

If $\varepsilon \in (0, \varepsilon_1)$, then $a > S(\varepsilon, T)$. If $\varepsilon \in (0, \varepsilon_2)$, let us take

$$\bar{t} = \max\{0, \hat{t}\} = \max\left\{0, T - \frac{1}{c_1}\ln\frac{\eta}{\phi(\varepsilon)\tilde{C}}\right\}.$$

Hence, for every $\varepsilon \in (0, \varepsilon_2)$, it is easy to see that

$$\sup_{t\in[\bar{t},T]} E\left|Y_s^\varepsilon - Y_s\right|^p \leq S(\varepsilon, T-\bar{t}) = a.$$

Clearly, $\bar{t}\uparrow T$ as $\varepsilon\uparrow\varepsilon_2$ and $\bar{t}\downarrow 0$ as $\varepsilon\downarrow\varepsilon_1$, that is, $\bar{t}\downarrow 0$ as $\varepsilon\downarrow 0$.

⋆

This section illustrates the most important result of the chapter. Indeed, estimate of a time interval, for the given, precise closeness of the solutions is very important in the applications. Per example, if some random observation is modelled by RBSDE, and its behaviour (value) on fixed time T is familiar, as well as its change up to some other value in capital moment, and if the driver of the model is supposed to linearly change, it is interesting to estimate the time interval on which we could. "control" the observations, i.e. under which our change under linearisation of final value and the drift will remain within the boundaries we impose.

6. Conclusions and remarks

It should be noted that this is a special case of generally perturbed problem observed by Đorđević and Janković in [5], but we have provided and explicit, concrete estimates for the additive type of perturbations. Interesting in this case also is, that even-though we introduce the hypothesis (H2), i.e. Lipschitz condition for the drift/driver/generator function, this hypothesis is not explicitly used in the estimates for perturbations. It is necessary to have it in order to have the existence of the solutions for perturbed and unperturbed equations, but it is not necessary for the perturbation estimates with the given assumptions $(\mathcal{A}\mathbf{0})$–$(\mathcal{A}\mathbf{2})$. It follows that results from this chapter can be generalized in several ways:

I. assumption $(\mathcal{A}\mathbf{1})$ can be weaken in the sense that it can be per example of the form:

i. Lipschitz condition

$$\left|\alpha(t,y,z,\varepsilon) - \alpha(t,y_1,z_1,\varepsilon)\right|^2 \leq L\left(|y-y_1|^2 + \|z-z_1\|^2\right) + \alpha_1(t,\varepsilon), \quad a.s.$$

for some Lipschitz constant L and nonrandom function $\alpha_1(t,\varepsilon)$.

ii. Non-Lipschitz condition.

There exist constants $C > 0$ such that for any $(\omega, t) \in \Omega \times [0, T]$ and $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\left|\alpha(t,y_1,z_1,\varepsilon) - \alpha(t,y_2,z_2,\varepsilon)\right|^2 \leq \rho\left(t, |y_1-y_2|^2\right) + C\|z_1-z_2\|^2 + \alpha_1(t,\varepsilon),$$

where $\rho : [0,T] \times R^+ \to R^+$ satisfies: For fixed $t \in [0,T]$, $\rho(t,\cdot)$ is: a concave and non-decreasing function with $\rho(t,0) \equiv 0$;

- for fixed u, $\int_0^T \rho(t,u)dt < \infty$;
- for any $M > 0$, the ODE

$$u' = -M\rho(t,u), \quad u(T) = 0$$

has a unique solution $u(t) \equiv 0, t \in [0, T]$.

iii. Linear growth condition

$$|\alpha(t,y,z,\varepsilon)| \le K(|y|+\|z\|) + \alpha_1(t,\varepsilon), \ \ a.s.$$

for some constant K and nonrandom function $\alpha_1(t,\varepsilon)$.

In all alternatives, further assumption is that there exist nonrandom function $\overline{\alpha}(\varepsilon)$ such that

$$\sup_{t\in[0,T]} \alpha_1(t,\varepsilon) = \overline{\alpha}(\varepsilon).$$

II. Conditions of existence and uniqueness of the solutions of perturbed and unperturbed equations can be generalized in a sense for the driver f, f_ε of Eqs. (1) and (3) to satisfy some of mentioned conditions: non-Lipschitz or linear growth one. In this manner, these assumptions would hold for the additional function α in the perturbed driver also.

In the case when we change the initial conditions and assumptions, the steps will be similar, while the main inequality at the end of the estimates will be established by applying Bihari inequality and not Gronwall-Bellman one.

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Chapter 10

On Some Important Ordinary Differential Equations of Dynamic Economics

Anastasios Tsoularis

Abstract

Mathematical modeling in economics became central to economic theory during the decade of the Second World War. The leading figure in that period was Paul Anthony Samuelson whose 1947 book, *Foundations of Economic Analysis*, formalized the problem of dynamic analysis in economics. In this brief chapter some seminal applications of differential equations in economic growth, capital and business trade cycles are outlined in deterministic setting. Chaos and bifurcations in economic dynamics are not considered. Explicit analytical solutions are presented only in relatively straightforward cases and in more complicated cases a path to the solution is outlined. Differential equations in modern dynamic economic modeling are extensions and modifications of these classical works. Finally we would like to stress that the differential equations presented in this chapter are of the "stand-alone" type in that they were solely introduced to model economic growth and trade cycles. Partial differential equations such as those which arise in related fields, like Bioeconomics and Differential Games, from optimizing the Hamiltonian of the problem, and stochastic differential equations of Finance and Macroeconomics are not considered here.

Keywords: Walrassian condition, Marshallian condition, homogeneous function, Cobb–Douglas form, endogenous growth

1. Introduction

Ordinary differential equations are ubiquitous in the physical sciences and are fundamental for the understanding of complex engineering systems [1]. In economics they are used to model for instance, economic growth, gross domestic product, consumption, income and investment whereas in finance stochastic differential equations are indispensable in modeling asset price dynamics and option pricing. The vast majority of the ordinary differential equations in economic are autonomous differential equations or difference equations, where time is an implicit variable, whereas the more difficult to solve delay (differential-difference) equations have received much less attention. Difference equations seem a more natural choice of modeling economic processes as key economic variables are monitored at discrete time units but they can present significant complications in their asymptotic behavior and are thus more difficult to analyse. Differential equations on the other hand, can be more amenable to asymptotic stability analysis. Partial differential equations, usually of the second order, for functions of at least two variables arise naturally in modern macroeconomics from solving an optimization problem formulated in a stochastic setting and in optimal control theory. Two books that are recommended for delving deeper into the- economic applications of differential equations are the introductory one by Gandolfo [2] and the more advanced by Brock and Malliaris [3]. Both books are excellent sources for ordinary differential equations in economic dynamics. A more recent book which requires strong mathematical background is by Acemoglu [4].

2. Some differential equations of neoclassical growth theory and business cycles

Some of the most important differential equations developed by economists during a period spanning over sixty years are presented in this section. Most of them beginning with Solow's development of a growth model, which was partly motivated by the works of Harrod and Domar, are models from Neoclassical Growth Theory. The main postulate of Neoclassical Growth Theory is that *economic growth* is driven by three elements: labour, capital, and technology. Economic growth is an important topic in economics and Solow's growth model is the first topic taught in undergraduate economics because of its underlying simplicity and importance as argued by Acemoglu [5]. The differential equation by Samuelson is concerned with demand and supply scenarios. Phillips' work is the earliest attempt to employ classical feedback control theory in order to steer a national economy towards a desired target. The remaining works are differential equations with time lags inherently present in production and capital accumulation. Due to space limitations, the exposition is somewhat uneven with full mathematical analyses of most models and cursory treatments of those with time lags. The choice of the differential equations presented in this chapter is a judicious one, the list is by no means exhaustive, but is meant to afford a glimpse into how the mathematical thinking of some famous economists has influenced the economic growth theory in the twentieth century.

2.1 Harrod-Domar

The Harrod-Domar model was developed independently by Roy Harrod [6] and Evsey Domar [7] to analyze business cycles originally but later was used to explain an economy's growth rate through savings and capital productivity. Output, Y, is a function of capital stock, K, $Y = F(K)$, and the marginal productivity, $\frac{dY}{dK} = c = constant$. The model postulates that the output growth rate is given by

$$\frac{1}{Y}\frac{dY}{dt} = sc - \delta,$$

where s is the savings rate, and δ the capital depreciation rate. The straightforward solution,

$$Y(t) = Y_0 e^{(sc-\delta)t}.$$

clearly demonstrates that increasing investment through savings and productivity boosts economic growth but does not take into account labour input and population size.

2.2 Samuelson

In his 1941 Paul Samuelson [8] paper employed simple differential equations to investigate the stability of equilibrium for several demand–supply scenarios.

The simplest stability analysis was carried out under the Walrasian and Marshallian assumptions. In the former price increases (decreases) if excess demand is positive (negative), whereas in the latter quantity increases (decreases) if excess demand price is positive (negative). Excess demand is the difference between the quantity that buyers are willing to buy and the quantity that suppliers are willing to supply at the same price. Excess demand price is the difference between the price that buyers are willing to pay for a given quantity and the price required by the suppliers.

Let $D(p, \alpha)$ and $S(p)$ denote the demand and supply functions of price, p, respectively with α a shift parameter representing "taste". At equilibrium, price, p^*, and quantity, q^*, are given by

$$q^* = D(p^*, \alpha) = S(p^*)$$

$$\frac{\partial D}{\partial \alpha} > 0, \frac{\partial D}{\partial p} < 0.$$

It is the task of comparative statics to show the determination of the equilibrium values of price and quantity and their sensitivity on the "taste" parameter, α.

The dynamic formulation of the Walrasian assumption is

$$\frac{dp}{dt} = f(D(p) - S(p)), f(0) = 0, f'(0) > 0.$$

Retaining the first order term in a Taylor series expansion near the equilibrium, p^*, we obtain the following linear differential equation

$$\frac{dp}{dt} = a_0 \left(\frac{dD}{dp} - \frac{dS}{dp}\right)_{p^*} (p - p^*),$$

with solution for an initial price, p_0

$$p(t) = p^* + (p^* - p_0) e^{a_0 t \left(\frac{dD}{dp} - \frac{dS}{dp}\right)_{p^*}}.$$

The equilibrium is stable if $\left(\frac{dD}{dp}\right)_{p^*} < \left(\frac{dS}{dp}\right)_{p^*}$. Price must rise when demand increases.

The dynamic formulation of the Marshallian assumption is

$$\frac{dq}{dt} = g(p_D(q) - p_S(q)), g(0) = 0, g'(0) > 0.$$

Neglecting high order terms and using the trivial elementary calculus result, $\frac{dp_D}{dq} = \frac{1}{\frac{dD}{dp}}, \frac{dp_S}{dq} = \frac{1}{\frac{dS}{dp}}$, we obtain

$$q(t) = q^* + (q^* - q_0) \exp \left[b_0 t \left(\frac{1}{\frac{dD}{dp}} - \frac{1}{\frac{dS}{dp}} \right)_{q^*} \right].$$

The equilibrium is stable if $\left(\frac{1}{\frac{dD}{dp}}\right)_{q^*} < \left(\frac{1}{\frac{dS}{dp}}\right)_{q^*}$. Quantity supplied must rise when demand increases, while the change in price is dependent upon the algebraic sign of the supply curve's slope.

2.3 Solow

Robert Solow [9] proposed a growth equation incorporating production, capital growth and growth in the labour force absent from the Harrod-Domar model.

i. Production function: $= F(K, L)$, the quantity of goods by K units of capital and L units of labour at time t. In a closed economy where all output is invested or consumed,

$$Y(t) = C(t) + I(t),$$

where $C(t)$ and $I(t)$ are the consumption and investment functions respectively. An important assumption of the model are the Inada conditions [10]

$$\frac{\partial F}{\partial K} > 0, \frac{\partial F}{\partial L} > 0, \frac{\partial^2 F}{\partial K^2} < 0, \frac{\partial^2 F}{\partial L^2} < 0.$$

In the limits.

$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \infty, \lim_{L\to 0} \frac{\partial F}{\partial L} = \infty, \lim_{K\to \infty} \frac{\partial F}{\partial K} = 0, \lim_{L\to \infty} \frac{\partial F}{\partial L} = 0.$$

The Inada conditions ensure that F is strictly concave with slope decreasing from infinity to zero.

The function F is linearly homogeneous of degree 1 in K and L (in economic terms this is known as constant returns to scale, increasing capital and labour by a certain amount, results in a proportional rise of production) if

$$Y = F(\alpha K, aL) = \alpha F(K, L), \forall \alpha > 0.$$

In particular, choosing $\alpha = \frac{1}{L}$ and set $y = \frac{Y}{L}, k = \frac{K}{L}$, representing the output and capital per worker respectively

$$\frac{Y}{L} = y = F\left(\frac{K}{L}, 1\right) = f(k).$$

The production function is expressed in terms of a unit of labour and the capital to labour ratio. The assumption of constant returns to scale allows the simplified function, $f(k)$.

ii. Growth of Capital in Economy: The growth of the capital stock, K, is equivalent to growth in investment, I, which is used to increase capital subject to depreciation. Depreciation of capital stock will be accounted for so that I is essentially

$$\text{investment} = \text{rate of change of capital} + \text{capital depreciation rate}$$

or

$$I(t) = \frac{dK}{dt} + \delta K(t),$$

where δ is the constant capital depreciation rate.

Letting $c(t)$ and $i(t)$ denote the consumption and investment per labour unit

$$c(t) = \frac{C}{L}, i(t) = \frac{I}{L},$$
$$y(t) = c(t) + i(t) = c(t) + \frac{1}{L}\frac{dK}{dt} + \delta k = c(t) + \frac{dk}{dt} + \left(\delta + \frac{1}{L}\frac{dL}{dt}\right)k.$$

iii. Growth of the Labour Force with full employment: The assumption in the labour market is that the labour supply is equivalent to the population. There is no unemployment and the growth of labour as function of time follows an exponential growth pattern:

$$L = L_0 e^{nt}.$$

The fundamental differential equation of economic growth is then

$$\frac{dk}{dt} = f(k) - (\delta + n)k - c(t).$$

The differential equations and production functions outlined in these three assumptions are the fundamental elements for Solow's basic differential equation. In Solow's paper, a constant fraction of income is allocated to savings, in particular, $= y(t) - c(t) = f(k) - (1-s)f(k) = sf(k)$, so that

$$\frac{dk}{dt} = sf(k) - (\delta + n)k.$$

The equilibrium solution to the basic differential equation is found from $sf(k) = (\delta + n)k$. A well-known function is the Cobb–Douglas production function, $Y(K, L) = \alpha K^{\beta} L^{1-\beta}$, $0 < \beta < 1$, where β is the elasticity of output, $\frac{K}{Y}\frac{\partial Y}{\partial K}$, with respect to capital. The use of the Cobb–Douglas production function is justified because it exhibits constant returns to scale: If capital and labour are both increased by the same factor, $\lambda > 1$, output will be increased by exactly the same proportion, $Y(K, L) = \lambda\left(\alpha K^{\beta} L^{1-\beta}\right)$. Also the marginal product, $\frac{\partial Y}{\partial K}$, $\frac{\partial Y}{\partial L}$, diminishes as either K or L increases since $\frac{\partial^2 Y}{\partial K^2} < 0$, $\frac{\partial^2 Y}{\partial L^2} < 0$. Introduce $(k) = \alpha\left(\frac{K}{L}\right)^{\beta} = \alpha k^{\beta}$, so the differential equation becomes

$$\frac{dk}{dt} = s\alpha k^{\beta} - (\delta + n)k.$$

From $\frac{dk}{dt} = 0$, $k^* = \left(\frac{s\alpha}{\delta + n}\right)^{\frac{1}{1-\beta}}$. Substituting $k^* = \left(\frac{s\alpha}{\delta + n}\right)^{\frac{1}{1-\beta}}$ into $y = \alpha k^{\beta}$, the steady state level of per capita income is

$$y^* = a^{\frac{1}{1-\beta}}\left(\frac{s}{\delta + n}\right)^{\frac{\beta}{1-\beta}}.$$

The output per unit growth converges to n:

$$\frac{1}{Y}\frac{dY}{dt} = \frac{\beta}{k}\frac{dk}{dt} + n \to n.$$

A multiplicative factor in the form of technological progress, $(t) = A_0 e^{gt}$, can be introduced in the production function, so that, $Y(t) = aK(t)^{\beta}(A(t)L(t))^{1-\beta}$ and $k(t) = \frac{K(t)}{A(t)L(t)}$, leading to

$$\frac{dk}{dt} = sak^{\beta} - (\delta + n + g)k.$$

The first order nonlinear differential equation has solution

$$k(t) = \left[\frac{s\alpha}{\delta + n + g} + \left(k_0^{1-\beta} - \frac{s\alpha}{\delta + n + g}\right)e^{-(\delta+n+g)(1-\beta)t}\right]^{\frac{1}{1-\beta}}.$$

This solution includes the solution to the labour growth only model, $n = 0$. The steady state is

$$k^* = \left(\frac{s\alpha}{\delta + n + g}\right)^{\frac{1}{1-\beta}}.$$

Differentiation of $\frac{dk}{dt} = sak^{\beta} - (\delta + n + g)k$ with respect to k at k^* gives $(\beta - 1)(\delta + n + g) < 0$, the equilibrium is stable. The steady state level of per capita income is

$$y^* = a^{\frac{1}{1-\beta}}\left(\frac{s}{\delta + n + g}\right)^{\frac{\beta}{1-\beta}},$$

a constant, since s, δ, n, g are all constant.

$Y(t) = \alpha K^{\beta}\left(A_0 L_0 e^{\left(\frac{g}{1-\beta}+n\right)t}\right)^{1-\beta} = ak^{\beta} A_0 L_0 e^{\left(\frac{g}{1-\beta}+n\right)t}$. The output per unit growth, $\frac{1}{Y}\frac{dY}{dt}$, converges to $\frac{g}{1-\beta} + n$.

The Solow residual is the part of growth unexplained by changes in capital and labour. For $Y(t) = aK(t)^{\beta}(A(t)L(t))^{1-\beta}$

$$\frac{\partial Y}{\partial t} = a\beta K(t)^{\beta-1}(A(t)L(t))^{1-\beta}\frac{dK}{dt} + aK(t)^{\beta}(1-\beta)(A(t)L(t))^{-\beta}\left[\frac{dA}{dt}L(t) + \frac{dL}{dt}A(t)\right].$$

The growth rate per unit output is

$$\frac{1}{Y}\frac{\partial Y}{\partial t} = \frac{\beta}{K}\frac{dK}{dt} + (1-\beta)\frac{1}{L}\frac{dL}{dt} + (1-\beta)\frac{1}{A}\frac{dA}{dt},$$

$$\textit{Solow residual} = \frac{1}{Y}\frac{\partial Y}{\partial t} - \left[\frac{\beta}{K}\frac{dK}{dt} + (1-\beta)\frac{1}{L}\frac{dL}{dt}\right].$$

A positive Solow residual would indicate a faster output growth than that of capital and labour.

2.4 Phelps

Phelps [11] used the neoclassical growth model to address the consumption per unit of labour at equilibrium in the so-called "golden rule". At equilibrium with labour force growth rate, n, only the consumption per unit of labour is

$$c(t) = f(k) - nk.$$

For a maximum consumption per unit of labour

$$\frac{dc}{dk} = \frac{df}{dk} - n = 0.$$

Since $\frac{d^2f}{dk^2} < 0$, the turning point is a maximum given by $\frac{df}{dk} = n$. The “golden rule” concludes that the marginal output per worker must equal the growth rate of the labour force at maximum per capita consumption.

2.5 RCK

The Ramsey–Cass–Koopmans model, or RCK model, is a neoclassical model of economic growth which differs from Solow’s model in its inclusion of consumption, based primarily on the work of Ramsey [12], with later significant extensions by Cass [13] and Koopmans [14].

$$\frac{dk}{dt} = f(k) - (\delta + n)k - c(t).$$

A steady state is when $c(t) = f(k) - (\delta + n)k$.

There is a second equation of the RCK model, the social planner‘s problem of maximizing a social welfare function expressed by the integral

$$\int_0^\infty e^{-\rho t} L(t) u(c(t)) dt = \int_0^\infty e^{(n-\rho)t} u(c(t)) dt,$$

where $\rho > 0$ is the discount rate and $u(c(t))$ is a strictly increasing concave utility function of consumption. The objective is formally stated thus

$$\mathrm{u}^* = \max_{\mathrm{c(t)}} \int_0^\infty \mathrm{e}^{(\mathrm{n}-\rho)\mathrm{t}} \mathrm{u}(\mathrm{c}(\mathrm{t}))\mathrm{dt}$$

subject to

$$\frac{\mathrm{dk}}{\mathrm{dt}} = \mathrm{f(k)} - (\delta + \mathrm{n})\mathrm{k} - \mathrm{c(t)}$$

$$k_0 = k(0).$$

The Hamiltonian is

$$\mathcal{H}(c) = e^{(n-\rho)t}\left[u(c) + \lambda e^{(\rho-n)t}(f(k) - (\delta + n)k - c(t))\right],$$

where λ is the costate variable (Lagrange multiplier). From

$$\frac{\partial \mathcal{H}}{\partial c} = e^{(n-\rho)t}\frac{\partial u}{\partial c} - \lambda = 0,$$

$$\lambda = e^{(n-\rho)t}\frac{\partial u}{\partial c}.$$

Also for the costate variable

$$\frac{d\lambda}{dt} = -\frac{\partial \mathcal{H}}{\partial k} = -\lambda\left[\frac{\partial f}{\partial k} - (\delta + n)\right],$$

and

$$\frac{d\lambda}{dt} = (n-\rho)\lambda + \frac{\frac{\partial^2 u}{\partial c^2}}{\frac{\partial u}{\partial c}}\frac{dc}{dt}\lambda.$$

Hence

$$(n-\rho) + \frac{\frac{\partial^2 u}{\partial c^2}}{\frac{\partial u}{\partial c}}\frac{dc}{dt} = -\frac{\partial f}{\partial k} + (\delta + n),$$

whence

$$\frac{dc}{dt} = \frac{\frac{\partial u}{\partial c}}{\frac{\partial^2 u}{\partial c^2}}\left[-\frac{\partial f}{\partial k} + \delta + \rho\right].$$

This is a nonlinear differential equation that describes the optimal evolution of consumption, known as the Keynes-Ramsey rule. Along with the differential equation, $\frac{dk}{dt} = f(k) - (\delta + n)k - c(t)$, form the RCK dynamical system which does not admit an analytical solution. At equilibrium,

$$\left(\frac{\partial f}{\partial k}\right)_{k^*} = \delta + \rho,$$

$$c^* = f(k^*) - (\delta + n)k^*.$$

The Jacobian matrix at equilibrium,

$$J = \begin{bmatrix} \rho - n & -1 \\ -\frac{\frac{\partial u}{\partial c}}{\frac{\partial^2 u}{\partial c^2}}\left(\frac{\partial^2 f}{\partial k^2}\right)_{k^*} & 0 \end{bmatrix}$$

has eigenvalues real and opposite in sign as its determinant is $-\frac{\frac{\partial u}{\partial c}}{\frac{\partial^2 u}{\partial c^2}}\left(\frac{\partial^2 f}{\partial k^2}\right)_{k^*} < 0$ ($f(k)$ and $u(c)$ are both concave), therefore the equilibrium is a saddle point.

2.6 Romer

The growth in the Solow model is exogenous, the steady state depends on the exogenous parameters, ,g, which are due to outside trends. In the absence of $A(t)L(t)$ growth cannot be maintained. The marginal product of capital, $\frac{\partial Y}{\partial K} = a\beta A(t)^{1-\beta}\left(\frac{L}{K}\right)^{1-\beta} = \frac{a\beta A(t)^{1-\beta}}{\left(\frac{K}{L}\right)^{1-\beta}}$, is inversely proportional to the capital per labour, $\frac{K}{L}$. In countries with lower capital per labour the marginal product of capital should be higher which is not the case. The disparity could be attributed to the different g values in $A(t)$, which is treated as an exogenously given parameter in the Solow model, so an explanation is lacking.

Romer [15] proposed a mathematical theory of endogenous growth based on the following three assumptions:

i. The production function, $Y = F(K, A, L)$ offers increasing returns to scale, that is $F(\lambda K, \lambda A, \lambda L) > \lambda F(K, A, L)$.

ii. The change in capital is identical to Solow's model, $\frac{dK}{dt} = sY - \delta K$, where s is the fraction in savings, δ is the exogenous capital depreciation rate. Labour, L, is also exogenous, $\frac{dL}{dt} = nL$, and is comprises labour involved in research technology, L_A, and labour involved in the production of the final goods, $L_Y, L = L_A + L_Y$.

iii. Technology is exogenous and evolves in time, $\frac{dA}{dt} = \gamma L_A^\theta A^\varphi, 0 < \theta < 1, \varphi < 1$.

As is evident from the three assumptions, Romer's growth model consists of three sectors: the research sector of ideas, the intermediate goods sector which implements the ideas of the research sector and the final goods sector which produces the final output.

Let g_A be the technology growth rate, taken to be constant along the stable path,

$$g_A = \frac{1}{A}\frac{dA}{dt} = \gamma L_A^\theta A^{\varphi-1},$$

$$\frac{dg_A}{dt} = \gamma \theta L_A^{\theta-1}\frac{dL_A}{dt}A^{\varphi-1} + \gamma(\varphi - 1)L_A^\theta A^{\varphi-2}\frac{dA}{dt} = 0,$$

$$\theta \frac{1}{L_A}\frac{dL_A}{dt} + (\varphi - 1)\frac{1}{A}\frac{dA}{dt} = 0,$$

$$\theta n + (\varphi - 1)g_A = 0,$$

$$g_A = \frac{\theta n}{1-\varphi}.$$

In Romer's model, the output production function is given by

$$y = k^\beta \left(\frac{L_Y}{L}\right)^{1-\beta},$$

and the capital dynamics is

$$\frac{dk}{dt} = sk^\beta \left(\frac{L_Y}{L}\right)^{1-\beta} - (n + g_A + \delta)k.$$

The respective stable equilibria are

$$k^* = \frac{L_Y}{L}\left(\frac{s}{n + g_A + \delta}\right)^{\frac{1}{1-\beta}},$$

$$y^* = \frac{L_Y}{L}\left(\frac{s}{n + g_A + \delta}\right)^{\frac{\beta}{1-\beta}}.$$

The labour involved in the production of the final goods, L_Y, is determined in Romer [15] by maximizing the net profit for the final goods sector and obtaining the closed form expression for $\frac{L_Y}{L} = \frac{r-n}{r-n+\beta g_A}$, where r is the interest rate, and all parameters are exogenous except for g_A which is derived endogenously.

A nice accessible exposition of both Solow's and Romer's growth models is Chu [16]. Jones [17] argued that the predicted scale effects of Romer's theory of growth is inconsistent with the time-series evidence from industrialized economies and that long-term growth depends on exogenous parameters including the rate of population growth.

2.7 Mankiw, Romer and Weil

Mankiw, Romer and Weil [18] argued that the marginal product of capital, $\frac{\partial Y}{\partial K}$, is lower in poorer countries is due to their deficiency in human capital. Human capital is the accumulation of knowledge and skills achieved through training and education, which are essential ingredients in adding economic value. The production function is of the Cobb–Douglas type

$$Y(t) = H(t)^{\alpha} K(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta} = \left(\frac{H(t)}{A(t)L(t)}\right)^{\alpha} \left(\frac{K(t)}{A(t)L(t)}\right)^{\beta} A(t)L(t),$$

$$y(t) = \frac{Y(t)}{A(t)L(t)} = \left(\frac{H(t)}{A(t)L(t)}\right)^{\alpha} \left(\frac{K(t)}{A(t)L(t)}\right)^{\beta} = h^{\alpha} k^{\beta},$$

where $H(t)$ is the human capital stock which depreciates at the same rate, δ, as $K(t)$. As in Solow's model, a fraction of the output, $sY(t)$, is saved but in this model, it is split between human and capital stock, $s = s_H + s_K$. The evolution of the economy is determined by

$$\frac{dk}{dt} = s_K h^{\alpha} k^{\beta} - (n + g + \delta)k,$$

$$\frac{dh}{dt} = s_H h^{\alpha} k^{\beta} - (n + g + \delta)h.$$

The equilibrium is

$$k^* = \left(\frac{n + g + \delta}{s_K^{1-\alpha} s_H^{\alpha}}\right)^{\frac{1}{\alpha+\beta-1}},$$

$$h^* = \left(\frac{n + g + \delta}{s_K^{\beta} s_H^{1-\beta}}\right)^{\frac{1}{\alpha+\beta-1}}.$$

In the steady state,

$$y^* = (n + g + \delta)^{\frac{\alpha+\beta}{\alpha+\beta-1}} s_K^{\frac{-\beta}{\alpha+\beta-1}} s_H^{\frac{-\alpha}{\alpha+\beta-1}}.$$

Introduce the transformations, $x_1 = \frac{k}{k^*}, x_2 = \frac{h}{h^*}$, so that the equilibrium shifts to $(1, 1)$. Then

$$\frac{dx_1}{dt} = (n + g + \delta)\left(x_1^{\beta} x_2^{\alpha} - x_1\right),$$

$$\frac{dx_2}{dt} = (n + g + \delta)\left(x_1^{\beta} x_2^{\alpha} - x_2\right).$$

For small deviations, ξ_1, ξ_2, from the equilibrium the linear system

$$\frac{d\xi_1}{dt} = (n + g + \delta)[(\beta - 1)\xi_1 + \alpha\xi_2],$$
$$\frac{d\xi_2}{dt} = (n + g + \delta)[\beta\xi_1 + (\alpha - 1)\xi_2].$$

The eigenvalues of the Jacobian matrix,

$$(n + g + \delta)\begin{bmatrix} \beta - 1 & \alpha \\ \beta & \alpha - 1 \end{bmatrix},$$

are given by the roots of the quadratic

$$\lambda^2 + (2 - \alpha - \beta)\lambda + (1 - \alpha - \beta) = 0.$$

From the production function, $1 - \alpha - \beta > 0$. Since the sum of the eigenvalues is $\alpha + \beta - 2 < 0$, and the product is $1 - \alpha - \beta > 0$, both roots have negative real parts and the equilibrium point is stable.

2.8 Kaldor

Kaldor [19] presented a model of the trade cycle involving non-linear investment and saving functions that shift over time in response to capital accumulation or decumulation so that the system moves from stable equilibrium to unstable equilibrium to stable equilibrium again. In Kaldor's model investment, I, and savings, S, functions are non-linear with respect to the level of activity, X, measured in terms of employment.

Kaldor used a differential equation system with general non-linear forms. Net investment, I, and savings, S, are functions of national income, Y, and capital stock, K:

$$I = I(Y, K),$$
$$S = S(Y, K),$$
$$\frac{\partial I}{\partial Y} > 0, \frac{\partial I}{\partial K} < 0, \frac{\partial S}{\partial Y} > 0, \frac{\partial S}{\partial K} < 0,$$
$$\frac{\partial I}{\partial K} < \frac{\partial S}{\partial K}.$$

Also growth in capital determines investment is given by

$$\frac{dK}{dt} = I(Y, K).$$

Since income will rise if investment is greater than savings, the dynamics of the national income is captured by the differential equation

$$\frac{dY}{dt} = \alpha[I(Y, K) - S(Y, K)], \alpha > 0.$$

The necessary and sufficient assumptions for the generation of a perpetual cyclical movement are:

i. For normal income levels,

$$\frac{\partial I}{\partial Y} > \frac{\partial S}{\partial Y}.$$

ii. For extreme income levels, either low or high,

$$\frac{\partial I}{\partial Y} < \frac{\partial S}{\partial Y}.$$

iii. At equilibrium, where $\frac{dK}{dt} = 0$, income levels are normal.

2.9 Phillips

National governments design their expenditure policies to steer the national economy towards a desired income. The theory of feedback control or servomechanisms provides the mathematical methodology of correcting deviations of the controlled variables from their target values. Feedback policies applied to economic stability were implemented by Phillips [20].

If Y is national income and D_a is the aggregate demand then for some adjustment coefficient, $a > 0$,

$$\frac{dY}{dt} = a(D_a - Y).$$

A similar differential equation holds for the actual, D_g and target government demand, D_g^*, with $b > 0$, namely,

$$\frac{dD_g}{dt} = b\left(D_g^* - D_g\right).$$

Aggregate and government demand are related by

$$D_a = mY + D_g,$$

where m is the private sector's marginal propensity to spend.

Eliminate D_a to obtain

$$\frac{dY}{dt} = a(m-1)Y + aD_g.$$

Differentiate the above to obtain

$$\frac{d^2Y}{dt^2} = a(m-1)\frac{dY}{dt} + ab\left(D_g^* - D_g\right) = a(m-1)\frac{dY}{dt} + abD_g^* + ab(m-1)Y - b\frac{dY}{dt}$$

or

$$\frac{d^2Y}{dt^2} + [b + a(1-m)]\frac{dY}{dt} + ab(1-m)Y - abD_g^* = 0.$$

Phillips' model is thus described by the linear second-order differential equation where Y is the target variable and D_g^* is the control variable. Investigated three types of feedback policy:

i. Proportional, $D_g^* = -k_P Y$, where $k_P > 0$. This policy does not prevent income reduction and induces oscillations.

ii. Derivative, $D_g^* = -k_D \frac{dY}{dt}$, where $k_D > 0$. This policy does not prevent income reduction but avoids oscillations.

iii. Integral, $D_g^* = -k_I \int_0^t Y dt$, where $k_I > 0$. This policy prevents income reduction but can induce unstable movement.

2.10 Kalecki

Kalecki [21] was the first economist to investigate the relationship between production lags and endogenous business cycles by considering a closed economic system over a short period of time without trend. $A(t)$ is the gross capital accumulation (unconsumed goods). There is a "gestation period", θ, for any investment $I(t)$. Deliveries $L(t)$ are equal to investment orders, $I(t-\theta)$ at time, $t-\theta$:

$$L(t) = I(t-\theta).$$

Any orders placed during the "gestation period", $(t-\theta, t)$, remain unfulfilled, $A(t)$ is equal to the average of investment orders $I(t)$ allocated during the period $(t-\theta, t)$:

$$A(t) = \frac{1}{\theta}\int_{t-\vartheta}^{t} I(\tau)d\tau.$$

If $K(t)$ is the capital stock, and U its physical depreciation

$$\frac{dK}{dt} = L(t) - U = I(t-\theta) - U.$$

The rate of change in investment is for some constants, $m > 0, n > 0$:

$$\frac{dI}{dt} = m\frac{dA}{dt} - n\frac{dK}{dt} = \frac{m}{\theta}[I(t) - I(t-\theta)] - n[I(t-\theta) - U].$$

Denoting the deviation of $I(t)$ from the constant demand for restoration of the depreciated industrial equipment U by $J(t) = I(t) - U$, and differentiating $J(t)$

$$\frac{\mathrm{dJ}}{\mathrm{dt}} = \frac{\mathrm{m}}{\theta}[\mathrm{J(t)} - \mathrm{J(t-\theta)}] - \mathrm{nJ(t-\theta)}$$

or

$$\theta\frac{dJ}{dt} + (n\theta + m)J(t-\theta) - mJ(t) = 0.$$

During the interval $t \in [-\theta, 0]$ Kalecki assumed that $J(t) = 0$. A standard way to solve this differential equation with delay is to assume a solution of the form, $De^{\alpha t}$, with D and α (where α is a complex number), to be determined. The general solution of the differential equation for some constants, c_1 and c_2 is

$$J(t) = e^{bt}[c_1 \cos(\omega t) + c_2 \sin(\omega t)].$$

The sign of the real parameter, b, classifies the behavior of the model as explosive for $b > 0$, cyclical for $b = 0$, and damped for $b < 0$.

2.11 A Solow model with lags

Zak [22] considered a version of the Solow model with delay. Capital can be used τ periods later, so at time t, the capital to be put into productive use is $k(t-\tau)$. If $f(k)$ is the production function, $s \in (0,1)$ is the constant savings rate and $\delta \in [0,1]$ is the constant capital depreciation rate, Zak's model is

$$\frac{dk}{dt} = sf(k(t-\tau)) - \delta k(t-\tau).$$

At equilibrium,

$$sf(k^*) = \delta k^*.$$

Deviations of the form, e^t, from equilibrium are governed by

$$\frac{dk}{dt} = \left(s\frac{df}{dk} - \delta\right)e^{-\tau},$$

with characteristic equation

$$\lambda - \left(s\frac{df}{dk} - \delta\right)e^{-\lambda\tau} = 0.$$

In many cases depending on the initial conditions, the roots of the characteristic equation have real parts with opposite signs, indicating the presence of a saddle point unlike Solow's stable model. The model exhibits endogenous cycles when the roots are purely imaginary.

2.12 Goodwin

Goodwin [23] presented a nonlinear model of nonlinear business cycles with time lags between decisions to invest and the corresponding outlays. Changes at time, t, in income, $y(t)$, induce investment outlays, $O_i(t+\theta)$, at a later time, $t+\theta$. Therefore

$$O_i(t+\theta) = \varphi\left(\frac{dy}{dt}\right) = \varphi(\dot{y}).$$

Hence the nonlinear delay differential equation modeling the evolution of income is

$$\epsilon\frac{dy(t+\theta)}{dt} + (1-\alpha)y(t+\theta) = O(t) + \varphi(\dot{y}),$$

where $O(t)$ is autonomous investment outlay and ϵ, α are constants. The derivative, $\frac{d\varphi(\dot{y})}{d\dot{y}}$, measures the rate of growth in investment with relative to the income growth, termed as acceleration coefficient. Expanding the two leading terms in

Taylor series and neglecting higher order terms, Goodwin obtained the nonlinear delay differential equation

$$\varepsilon\theta\frac{d^2y}{dt^2} + [(1-\alpha)\theta + \epsilon]\frac{dy}{dt} + (1-\alpha)y(t) - \dot{\varphi(y)} = O(t).$$

Goodwin assumed that $O(t)$ is constant, $O(t) = O^*$, and introduced a new variable

$$z(t) = y(t) - \frac{O^*}{1-\alpha},$$

where $\frac{O^*}{1-\alpha}$ is the income at equilibrium. The transformed differential equation is then

$$\varepsilon\theta\frac{d^2z}{dt^2} + [(1-\alpha)\theta + \epsilon]\frac{dz}{dt} - \varphi(\dot{z}) + (1-\alpha)z = 0.$$

The asymptotic behavior of the transformed equilibrium, $z = 0$, is determined by the eigenvalue solutions of the characteristic equation

$$\varepsilon\theta\lambda^2 + [(1-\alpha)\theta + \epsilon - \dot{\varphi}(0)]\lambda + (1-\alpha) = 0,$$

with characteristic roots,

$$\lambda_{1,2} = \frac{\dot{\varphi}(0) - [(1-\alpha)\theta + \epsilon] \pm \sqrt{[(1-\alpha)\theta + \epsilon - \dot{\varphi}(0)]^2 - 4\varepsilon\theta(1-\alpha)}}{2\varepsilon\theta}.$$

Since

$$\lambda_1\lambda_2 = \frac{1-\alpha}{\varepsilon\theta} > 0$$

and

$$\lambda_1 + \lambda_2 = \frac{\dot{\varphi}(0) - [(1-\alpha)\theta + \epsilon]}{\varepsilon\theta},$$

can be either positive or negative, both eigenvalues have positive or negative real parts. So if $\dot{\varphi}(0) < (1-\alpha)\theta + \epsilon$ the deviations from equilibrium are damped oscillatory motions, but if $\dot{\varphi}(0) > (1-\alpha)\theta + \epsilon$ the system is unstable and drifts away from the locally linearized region of stability.

2.13 A brief literature survey of current research

We close this chapter by providing a very brief snapshot of the current state of the art in theories of economic growth. Most of the very recent works cited are predominantly mathematical in nature. There is an enormous literature, not touched upon here, which employs Econometrics methods, like for instance panel data regression to estimate economic growth based on explanatory variables such as income, investment, policy indicators, education and others over several decades.

In a short article Zhao [24] discusses how technology was integrated into economic growth by Romer.

Boyko et [25] use least squares linear regression to determine the values of the coefficients at which the production functions of Cobb–Douglas in Solow's growth model provide the best fit for available statistical data. Borges et al. [26] examine the dynamics of Solow's economic growth model assuming that the labour force growth rate function is a solution of a delay differential equation thereby avoiding the use of exponential growth, $L(t) = L_0 e^{nt}$, often criticized as a rather unrealistic choice. Their approach is motivated by the fact that there are delays in entering and retiring an individual from the labour force, relative to their birth date.

Zhang et al. [27] base their analysis of how the redistribution of emission quotas would impact short-run equilibrium in a specific market of interest and long-run growth on the Solow growth model with endogenous dynamics and exogenous technological shocks.

Zhang [28] develops an endogenous growth model based on modifications of both Solow's model by introducing endogenous knowledge. and Romer's by allowing knowledge to be gained from learning as well as from research.

The paper by Caraballo et al. [29] is devoted to analysis of the stability of the economy according to an extended version of Kaldor's economic growth model. They consider the role of the government's monetary and fiscal policies and we study whether or not a time delay in implementing and the fiscal policy can affect the economic stability.

Dayal [30] considers long run historical data and uses difference equation simulation to explore the Solow growth model to assess the growth changes in the recent decade.

Perez-Trujillo et al. [31] investigate the impact of improvement in accessing innovation and knowledge on economic growth and convergence among countries using an augmented Solow-Swan growth model on data from 138 countries.

Turnovsky [32] discusses contemporary aspects of stabilization policy in reference to Phillips' contributions in a lengthy paper of substantial mathematical control theory content.

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Chapter 11

Multiple Solutions for Some Classes Integro-Dynamic Equations on Time Scales

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Abstract

In this chapter we study a class of second-order integro-dynamic equations on time scales. A new topological approach is applied to prove the existence of at least two non-negative solutions. The arguments are based upon a recent theoretical result.

Keywords: integro-dynamic equations, time scale, BVP, existence, positive solution, fixed point, cone, sum of operators

1. Introduction

Many problems arising in applied mathematics and mathematical physics can be modeled as differential equations, integral equations and integro-differential equations.

Integral and integro-differential equations can be solved using the Adomian decomposition method (ADM) [1, 2], Galerkin method [3], rationalized Haar functions method [4], homotopy perturbation method (HPM) [5, 6] and variational iteration method (VIM) [7]. ADM can be applied for linear and nonlinear problems and it is a method that represents the solution of the considered problems in the form of Adomian polynomials. Rationalized Haar functions and Galerkin methods are numerical methods that can be applied in different ways for the solutions of integral and integro-differential equations. VIM is an analytical method and can be used for different classes linear and nonlinear problems. HPM is a semi-analytical method for solving of linear and nonlinear differential, integral and integro-differential equations.

In recent years, time scales and time scale analogous of some well-known differential equations, integral equations and integro-differential equations have taken prominent attention. The new derivative, proposed by Stefan Hilger in [8], gives the ordinary derivative if the time scale is the set of the real numbers and the forward difference operator if the time scale is the set of the integers. Thus, the need for obtaining separate results for discrete and continuous cases is avoided by using the time scales calculus.

This chapter outlines an application of a new approach for investigations of integro-differential equations and integro-dynamic equations on time scales. The approach is based on a new theoretical result. Let $\mathbb{T}$ be a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively. Let also, $a, b \in \mathbb{T}$, $a < b$. In this chapter we will investigate the following second-order integro-dynamic equation

$$x^{\Delta^2}(t) = \int_a^t k(t,s) f\left(s, x(s), x^{\Delta}(s)\right) \Delta s, \quad t \in [a, b], \tag{1}$$

subject to the boundary conditions

$$x(a) = \alpha, \quad x\left(\sigma^2(b)\right) = \beta, \tag{2}$$

where
(H1) $k \in \mathcal{C}_{rd}([a, \sigma^2(b)] \times [a, \sigma^2(b)])$, $\alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$.
(H2) $f \in \mathcal{C}\left([a, \sigma^2(b)] \times \mathbb{R}^2\right)$ and

$$|f(s, u, v)| \leq a_1(s)|u|^{p_1} + a_2(s)|v|^{p_2} + a_3(s), \quad s \in \left[a, \sigma^2(b)\right], \quad u, v \in \mathbb{R}, \tag{3}$$

$a_j \in \mathcal{C}_{rd}([a, \sigma^2(b)])$, $j \in \{1, 2, 3\}$, are non-negative functions, $p_1, p_2 \geq 0$.
We will investigate the BVP (1), (2) for existence of non-negative solutions. Our main result in this chapter is as follows.
Theorem 1.1. *Suppose* (*H*1)-(*H*2). *Then the BVP* (1), (2) *has at least two non-negative solutions.*
Linear integro-dynamic equations of arbitrary order on time scales are investigated in [9] using ADM. Nonlinear integro-dynamic equations of second order on time scales are studied in [10] using the series solution method. Asymptotic behavior of non-oscillatory solutions of a class of nonlinear second order integro-dynamic equations on time scales is considered in [11].
The chapter is organized as follows. In the next section, we will give some basic definitions and facts by time scale calculus. In Section 3, we give some auxiliary results which will be used for the proof of our main result. In Section 4, we will prove our main result. In Section 5, we will give an example. Conclusion is given in Section 6.

2. Time scales revisited

Time scales calculus originates from the pioneering work of Hilger [8] in which the author aimed to unify discrete and continuous analysis. Time scales have gained much attention recently. This section is devoted to a brief introduction of some basic notions and concepts on time scales. For detailed introduction to time scale calculus we refer the reader to the books [12, 13].
Definition 2.1. *A time scale* $\mathbb{T}$ *is an arbitrary nonempty closed subset of the real numbers.*
Definition 2.2.

1. *The operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *given by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ for } t \in \mathbb{T} \tag{4}$$

will be called the forward jump operator.

2. The operator $\rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{for} \quad t \in \mathbb{T} \tag{5}$$

will be called the backward jump operator.

3. The function $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t \quad \text{for} \quad t \in \mathbb{T} \tag{6}$$

will be called the graininess function.

We set

$$\inf \emptyset = \sup \mathbb{T}, \quad \sup \emptyset = \inf \mathbb{T}. \tag{7}$$

Observe that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$ and $\rho(t) \leq t$ for any $t \in \mathbb{T}$. Below, suppose that $\mathbb{T}$ is a time scale with forward jump operator and backward jump operator σ and ρ, respectively.

Definition 2.3. *We define the set*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{otherwise.} \end{cases} \tag{8}$$

Using the forward and backward jump operators, one can classify the elements of a time scale.

Definition 2.4. *The point $t \in \mathbb{T}$ is said to be*

1. right-scattered if $\sigma(t) > t$.

2. right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$.

3. left-scattered if $\rho(t) < t$.

4. left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

5. isolated if it is left-scattered and right-scattered at the same time.

6. dense if it is left-dense and right-dense at the same time.

Definition 2.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a given function and $t \in \mathbb{T}^\kappa$. The delta or Hilger derivative of f at t will be called the number $f^\Delta(t)$, provided that it exists, if for any $\varepsilon > 0$ there is a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon \ |\sigma(t) - s| \quad \text{for} \quad \text{all} \quad s \in U. \tag{9}$$

If $f^\Delta(t)$ exists for any $t \in \mathbb{T}^\kappa$, then we say that f is delta or Hilger differentiable in $\mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T} \to \mathbb{R}$ will be called the delta derivative or Hilger derivative, shortly derivative, of f in $\mathbb{T}^\kappa$.

Remark 2.6. *The delta derivative coincides with the classical derivative in the case when $\mathbb{T} = \mathbb{R}$.*

Note that the delta derivative is well defined.

Theorem 2.7. *Let $f : \mathbb{T} \to \mathbb{R}$ be a given function and $t \in \mathbb{T}^\kappa$.*

1. The function f is continuous at t, if it is differentiable at t.

2. The function f is differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \tag{10}$$

if f is continuous at t and t is tight-scattered.

3. Let t is right-dense. Then the function f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s} \tag{11}$$

exists as a finite number. In this case, we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{12}$$

4. We have

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t), \tag{13}$$

if f is differentiable at t.

Definition 2.8. *Let $f : \mathbb{T}|to\mathbb{R}$ is a given function.*

1. *We say that f is regulated if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

2. *We say that f is pre-differentiable with region of differentiation D if*

   a. *it is continuous,*

   b. $D \subset \mathbb{T}^{\kappa}$,

   c. $\mathbb{T}^{\kappa} \backslash D$ *is countable and contains no right-scattered elements of* $\mathbb{T}$,

   d. f *is differentiable at each* $t \in D$.

To define indefinite integral and Cauchy integral on time scale we have a need of the following basic result.

Theorem 2.9. *Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}$, $f : \mathbb{T}^{\kappa} \to \mathbb{R}$ be a given regulated function. Then there exists unique function F that is pre-differentiable and*

$$F^{\Delta}(t) = f(t) \text{ for any } t \in D, \quad F(t_0) = x_0. \tag{14}$$

Definition 2.10.

1. *Let $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Then any function F in Theorem 2.9. is said to be a pre-antiderivative of the function f and the indefinite integral of the regulated function f is defined by*

$$\int f(t) \Delta t = F(t) + c. \tag{15}$$

Here c is an arbitrary constant. Define the Cauchy integral as follows

$$\int_{\tau}^{s} f(t) \Delta t = F(s) - F(\tau) \text{ for all } \tau, s \in \mathbb{T}. \tag{16}$$

2. A function $F : \mathbb{T} \to \mathbb{R}$ is said to be an antiderivative of the function $f : \mathbb{T} \to \mathbb{R}$ if

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^{\kappa}. \tag{17}$$

Definition 2.11. *Let $f : \mathbb{T} \to \mathbb{R}$ be a given function. If it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$, then we say that f is rd-continuous. With $C_{rd}(\mathbb{T})$ we will denote the set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ and with $C^1_{rd}(\mathbb{T})$ we will denote the set of all functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative are rd-continuous.*

We will note that if f is rd-continuous, then it is regulated Below, we will list some of the properties of the Cauchy integral.

Theorem 2.12. *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$. Then we have the following.*

i. $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$

ii. $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t,$

iii. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$

iv. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$

v. $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t,$

vi. $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t,$

vii. $\int_a^a f(t)\Delta t = 0,$

viii. If $|f(t)| \le g(t)$ on $[a, b)$, then

$$\left| \int_a^b f(t)\Delta t \right| \le \int_a^b g(t)\Delta t, \tag{18}$$

ix. If $f(t) \ge 0$ for all $a \le t < b$, then $\int_a^b f(t)\Delta t \ge 0$.

Let

$$G(t,s) = \begin{cases} -\dfrac{(\sigma(s) - a)(\sigma^2(b) - t)}{\sigma^2(b) - a}, & \sigma(s) \le t, \\[2ex] -\dfrac{(t - a)(\sigma^2(b) - \sigma(s))}{\sigma^2(b) - a}, & t \le s, \quad t \in \left[a, \sigma^2(b)\right], \quad s \in [a, \sigma(b)]. \end{cases} \tag{19}$$

We have

$$|G(t,s)| \le \sigma^2(b) - a, \quad t \in \left[a, \sigma^2(b)\right], \quad s \in [a, \sigma(b)]. \tag{20}$$

In [12], it is proved that G is the Green function for the BVP

$$x^{\Delta^2} = 0, \quad x(a) = x\left(\sigma^2(b)\right) = 0. \tag{21}$$

3. Auxiliary results

Let X be a real Banach space.

Definition 3.1. *A mapping $K : X \to X$ that is continuous and maps bounded sets into relatively compact sets will be called completely continuous.*

The concept for k-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 3.2. *Suppose that Ω_X is the class of all bounded sets of X. The function $\alpha : \Omega_X \to [0, \infty)$ that is defined in the following manner*

$$\alpha(Y) = \inf\left\{ \delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad and \quad diam\left(Y_j\right) \le \delta, \quad j \in \{1, \dots, m\} \right\}, \tag{22}$$

where $\mathrm{diam}\left(Y_j\right) = \sup\left\{ \|x - y\|_X : x, y \in Y_j \right\}$ is the diameter of $Y_j, j \in \{1, \dots, m\}$, is said to be Kuratowski measure of noncompactness.

For the main properties of measure of noncompactness we refer the reader to [14].

Definition 3.3. *If the mapping $K : X \to X$ is continuous and bounded and there exists a nonnegative constant k such that*

$$\alpha(K(Y)) \le k\alpha(Y), \tag{23}$$

for any bounded set $Y \subset X$, then we say that it is a k-set contraction.

Note that any completely continuous mapping $K : X \to X$ is a 0-set contraction (see [15]).

Definition 3.4. *Suppose that X and Y are real Banach spaces. Then the map $K : X \to Y$ is called expansive if there exists a constant $h > 1$ for which*

$$\|Kx - Ky\|_Y \ge h\|x - y\|_X \tag{24}$$

for any $x, y \in X$.

Definition 3.5. *A closed, convex set $\mathcal{P}$ in X is said to be cone if.*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \ge 0$ and for any $x \in \mathcal{P}$,*

2. *$x, -x \in \mathcal{P}$ implies $x = 0$.*

Denote $\mathcal{P}^* = \mathcal{P}\backslash\{0\}$,

$$\begin{aligned} \mathcal{P}_{r_1} &= \{u \in \mathcal{P} : \|u\| < r_1\}, \\ \mathcal{P}_{r_1,r_2} &= \{u \in \mathcal{P} : r_1 < \|u\| < r_2\} \end{aligned} \tag{25}$$

for positive constants r_1, r_2 such that $0 < r_1 \le r_2$. The following result will be used to prove our main result. We refer the reader to [16, 17] for more details.

Theorem 3.6. *Let $\mathcal{P}$ be a cone in a Banach space $(E, \|\cdot\|)$. Let Ω be a subset of $\mathcal{P}$, $0 \in \Omega$ and $0 < r < L < R$ are real constants. Let also, $T : \Omega \to E$ is an expansive operator with a constant $h > 1$, $S : \overline{\mathcal{P}_R} \to E$ is a k-set contraction with $0 \le k < h - 1$ and $S\left(\overline{\mathcal{P}_R}\right) \subset (I - T)(\Omega)$. Assume that $\mathcal{P}_{r,L} \cap \Omega \ne \emptyset$, $\mathcal{P}_{L,R} \cap \Omega \ne \emptyset$ and there exist an $u_0 \in \mathcal{P}^*$ such that $T(x - \lambda u_0) \in \mathcal{P}$ for all $\lambda \ge 0$ and $x \in \partial\mathcal{P}_r \cap (\Omega + \lambda u_0)$ and the following conditions hold:*

a. $Sx \ne x - \lambda u_0, x \in \partial\mathcal{P}_r, \lambda \ge 0,$

b. $\|Sx + T0\| \le (h - 1)\|x\|$ and $Tx + Fx \ne x, x \in \partial\mathcal{P}_L \cap \Omega,$

c. $Sx \ne x - \lambda u_0, x \in \mathcal{P}_R, \lambda \ge 0.$

Then $T + S$ has at least two fixed points $x_1 \in \mathcal{P}_{r,L} \cap \Omega$, $x_2 \in \mathcal{P}_{L,R} \cap \Omega$, i.e.,

$$r < \|x_1\| < L < \|x_2\| < R. \tag{26}$$

Let

$$
\begin{aligned}
A_1 &= \frac{\alpha|\sigma^2(b)| + (\alpha + 2|\beta|)\max\{|a|, |\sigma^2(b)|\}}{\sigma^2(b) - a}, \\
A_2 &= \max_{(t,s)\in[a,\sigma^2(b)]\times[a,\sigma^2(b)]} |k(t,s)|, \\
A_3 &= \max\left\{\max_{s\in[a,\sigma^2(b)]} a_j(s), \quad j = 1,2,3\right\}, \\
A_4 &= \max\left\{\left(\sigma^2(b) - a\right)^2, \sigma^2(b) - a\right\},
\end{aligned}
\tag{27}
$$

and

$$
\phi(t) = \frac{\alpha\sigma^2(b) - \beta a + (\beta - \alpha)t}{\sigma^2(b) - a}, \quad t \in \left[a, \sigma^2(b)\right]. \tag{28}
$$

Then

$$
\begin{aligned}
|\phi(t)| &\leq \frac{\alpha|\sigma^2(b)| + |\beta|\max\{|a|, |\sigma^2(b)|\} + (|\beta|+\alpha)\max\{|a|, |\sigma^2(b)|\}}{\sigma^2(b) - a} \\
&= A_1, \quad t \in \left[a, \sigma^2(b)\right].
\end{aligned}
\tag{29}
$$

Suppose that $E = \mathcal{C}^1_{rd}([a, \sigma^2(b)])$ is endowed with the norm

$$
\|x\| = \max\left\{\max_{t\in[a,\sigma^2(b)]} |x(t)|, \quad \max_{t\in[a,\sigma^2(b)]} |x^\Delta(t)|\right\}, \tag{30}
$$

provided it exists. Next two lemmas give integral representations of the solutions of the BVP (1), (2).

Lemma 3.7. *If $x \in E$ is a solution to the integral equation*

$$
x(t) = \int_a^{\sigma(b)} G(t,s) \int_a^s k(s,s_1) f\left(s_1, x(s_1), x^\Delta(s_1)\right)\Delta s_1 \Delta s + \phi(t), \quad t \in \left[a, \sigma^2(b)\right], \tag{31}
$$

then x is a solution to the BVP (1), (2).

Proof. Since G is the Green function of the BVP (3) and $\phi^{\Delta^2}(t) = 0, t \in [a, \sigma^2(b)]$, we get

$$
x^{\Delta^2}(t) = \int_a^t k(t,s) f\left(s, x(s), x^\Delta(s)\right)\Delta s, \quad t \in \left[a, \sigma^2(b)\right], \tag{32}
$$

and

$$
\begin{aligned}
x(a) = \phi(a) &= \frac{\alpha\sigma^2(b) - \beta a + (\beta - \alpha)a}{\sigma^2(b) - a} = \alpha, \\
x\left(\sigma^2(b)\right) = \phi\left(\sigma^2(b)\right) &= \frac{\alpha\sigma^2(b) - \beta a + (\beta - \alpha)\sigma^2(b)}{\sigma^2(b) - a} = \beta.
\end{aligned}
\tag{33}
$$

Thus, x is a solution to the BVP (1), (2). This completes the proof. □

For $x \in E$, define the operator

$$
\begin{aligned}
F_1x(t) = \int_a^t & (t-\sigma(s))(x(s) \\
& - \int_a^{\sigma(b)} G(s,s_1) \int_a^{s_1} k(s_1,s_2) f\left(s_2, x(s_2), x^{\Delta}(s_2)\right) \Delta s_2 \Delta s_1 - \phi(s))\Delta s,
\end{aligned} \tag{34}
$$

$t \in \left[a, \sigma^2(b)\right]$.

Lemma 3.8. *If $x \in E$ is a solution to the integral equation*

$$
F_1x(t) = 0, \quad t \in \left[a, \sigma^2(b)\right], \tag{35}
$$

then x is a solution to the BVP (1), (2).
Proof. We have

$$
\begin{aligned}
0 &= (F_1x)^{\Delta}(t) \\
&= \int_a^t \left(x(s) - \int_a^{\sigma(b)} G(s,s_1) \int_a^{s_1} k(s_1,s_2) f\left(s_2, x(s_2), x^{\Delta}(s_2)\right) \Delta s_2 \Delta s_1 - \phi(s)\right)\Delta s, \ t \in \left[a, \sigma^2(b)\right],
\end{aligned} \tag{36}
$$

whereupon

$$
\begin{aligned}
0 &= (F_1x)^{\Delta^2}(t) \\
&= x(t) - \int_a^{\sigma(b)} G(t,s) \int_a^{s} k(s,s_1) f\left(s_1, x(s_1), x^{\Delta}(s_1)\right) \Delta s_1 - \phi(t),
\end{aligned} \tag{37}
$$

$t \in [a, \sigma^2(b)]$. Hence and Lemma 3.7, we conclude that x is a solution to the BVP (1), (2). This completes the proof. □

Now, we will give an estimate of the norm of the operator F_1.

Lemma 3.9. *If $x \in E$ and $\|x\| \le c$ for some positive constant c, then*

$$
\|F_1x\| \le A_4\left(c + \left(\sigma^2(b) - a\right)(\sigma(b)-a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1\right). \tag{38}
$$

Proof. We have

$$
\begin{aligned}
|F_1x(t)| \le \int_a^t & (t-\sigma(s))(|x(s)| \\
& + \int_a^{\sigma(b)} |G(s,s_1)| \int_a^{s_1} |k(s_1,s_2)| \left| f\left(s_2, x(s_2), x^{\Delta}(s_2)\right)\right| \Delta s_2 \Delta s_1 \\
& + |\phi(s)|)\Delta s \\
\le & \int_a^t (t-\sigma(s))(c \\
& + \left(\sigma^2(b) - a\right) \int_a^{\sigma(b)} \int_a^{s_1} A_2\left(a_1(s_2)|x(s_2)|^{p_1} + a_2(s_2)\left|x^{\Delta}(s_2)\right|^{p_2}\right. \\
& \left. + a_3(s_2)\right)\Delta s_2 \Delta s_1 + A_1)\Delta s \\
\le & A_4\left(c + \left(\sigma^2(b) - a\right)(\sigma(b)-a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1\right),
\end{aligned} \tag{39}
$$

$t \in [a, \sigma^2(b)]$, and

$$\begin{aligned}
\left|(F_1x)^\Delta(t)\right| \le & \int_a^t (|x(s)| \\
& + \int_a^{\sigma(b)} |G(s,s_1)| \int_a^{s_1} |k(s_1,s_2)| |f(s_2, x(s_2), x^\Delta(s_2))| \Delta s_2 \Delta s_1 \\
& + |\phi(s)|) \Delta s \\
& \le \int_a^t (c \\
& + (\sigma^2(b) - a) \int_a^{\sigma(b)} \int_a^{s_1} A_2 (a_1(s_2)|x(s_2)|^{p_1} + a_2(s_2)|x^\Delta(s_2)|^{p_2} \\
& + a_3(s_2)) \Delta s_2 \Delta s_1 + A_1) \Delta s \\
& \le A_4 \left(c + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1 \right),
\end{aligned} \tag{40}$$

$t \in [a, \sigma^2(b)]$. Thus,

$$\|F_1x\| \le A_4 \left(c + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1 \right). \tag{41}$$

This completes the proof. □

Below, suppose

(H3) Suppose that the positive constants A, m, ε, r_1, L_1, R_1 and R satisfy the following conditions

$$r_1 < \frac{L_1}{20} < L_1 < R_1, \quad m \in \left(0, \frac{4}{5}\right), \quad \varepsilon > 1, \quad R_1 < \varepsilon \frac{L_1}{20}, \tag{42}$$

$$AA_4 \left(R_1 + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (R_1^{p_1} + R_1^{p_2} + 1) + A_1 \right) < \frac{L_1}{20}, \tag{43}$$

$$AA_4 \left(L_1 + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (L_1^{p_1} + L_1^{p_2} + 1) + A_1 \right) < \left(\frac{4}{5} - m\right) L_1. \tag{44}$$

In the next section, we will give an example for constants A, m, ε, r_1, L_1, R_1 and R that satisfy $(H3)$. For $x \in E$, define the operator

$$Fx(t) = AF_1x(t), \quad t \in \left[a, \sigma^2(b)\right]. \tag{45}$$

By Lemma 3.9, we get the following result.

Lemma 3.10. *If $x \in E$ and $\|x\| \le c$ for some positive constant c, then*

$$\|Fx\| \le AA_4 \left(c + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 (c^{p_1} + c^{p_2} + 1) + A_1 \right). \tag{46}$$

Lemma 3.11. *If $x \in E$ is a solution to the integral equation*

$$0 = \frac{L_1}{5} + Fx(t), \quad t \in \left[a, \sigma^2(b)\right], \tag{47}$$

then it is a solution to the BVP (1), (2).

Proof. We have

$$0 = (Fx)^{\Delta^2}(t) = A(F_1x)^{\Delta^2}(t), \quad t \in \left[a, \sigma^2(b)\right], \tag{48}$$

whereupon

$$0 = A\left(x(t) - \int_a^{\sigma(b)} G(t,s)\int_a^s k(s,s_1)f(s_1, x(s_1), x^\Delta(s_1))\Delta s_1 - \phi(t)\right), \tag{49}$$

$t \in [a, \sigma^2(b)]$, and

$$x(t) = \int_a^{\sigma(b)} G(t,s)\int_a^s k(s,s_1)f\left(s_1, x(s_1), x^\Delta(s_1)\right)\Delta s_1 + \phi(t), \tag{50}$$

$t \in [a, \sigma^2(b)]$. Now, the assertion follows from Lemma 3.7. This completes the proof. □

4. Proof of the Main result

Let

$$\tilde{P} = \{u \in E : u \geq 0 \text{ on } [t_0, \infty)\}. \tag{51}$$

With $\mathcal{P}$ we will denote the set of all equi-continuous families in $\tilde{P}$. For $v \in E$, define the operators

$$\begin{aligned} Tv(t) &= (1 + m\varepsilon)v(t) - \varepsilon\frac{L_1}{10}, \\ Sv(t) &= -\varepsilon Fv(t) - m\varepsilon v(t) - \varepsilon\frac{L_1}{10}, \end{aligned} \tag{52}$$

$t \in [t_0, \infty)$. Note that any fixed point $v \in E$ of the operator $T + S$ is a solution to the IVP (1). Define

$$\begin{aligned} \mathcal{P}_{r_1} &= \{v \in \mathcal{P} : \|v\| < r_1\}, \\ \mathcal{P}_{L_1} &= \{v \in \mathcal{P} : \|v\| < L_1\}, \\ \mathcal{P}_{R_1} &= \{v \in \mathcal{P} : \|v\| < R_1\}, \\ \mathcal{P}_{r_1,L_1} &= \{v \in \mathcal{P} : r_1 < \|v\| < L_1\}, \\ \mathcal{P}_{L_1,R_1} &= \{v \in \mathcal{P} : L_1 < \|v\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m}A_4\left(R_1 + \left(\sigma^2(b) - a\right)(\sigma(b) - a)^2 A_2A_3\left(R_1^{p_1} + R_1^{p_2} + 1\right) + A_1\right) \\ &\quad + \frac{L_1}{5m}, \\ \Omega &= \mathcal{P}_{R_2} = \{v \in \mathcal{P} : \|v\| < R_2\}. \end{aligned} \tag{53}$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|Tv_1 - Tv_2\| = (1 + m\varepsilon)\|v_1 - v_2\|, \tag{54}$$

whereupon $T : \Omega \to E$ is an expansive operator with a constant $1 + m\varepsilon > 1$.

2. For $v \in \overline{\mathcal{P}}_{R_1}$, we get

$$\begin{aligned} \|Sv\| &\leq \varepsilon \|Fv\| + m\varepsilon \|v\| + \varepsilon \frac{L_1}{10} \\ &\leq \varepsilon \Big(AA_4 \Big(R_1 + \big(\sigma^2(b) - a\big)(\sigma(b) - a)^2 A_2 A_3 \big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1 \Big) \\ &\quad + mR_1 + \frac{L_1}{10} \Big). \end{aligned} \tag{55}$$

Therefore $S(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Since $S : \overline{\mathcal{P}}_{R_1} \to E$ is continuous, we have that $S(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Consequently $S : \overline{\mathcal{P}}_{R_1} \to E$ is a 0-set contraction.

3. Let $v_1 \in \overline{\mathcal{P}}_{R_1}$. Set

$$v_2 = v_1 + \frac{1}{m} F v_1 + \frac{L_1}{5m}. \tag{56}$$

Note that by the second inequality of $(H3)$ and by Lemma 3.10, it follows that $\varepsilon F v_1 + \varepsilon \frac{L_1}{5} \geq 0$ on $[t_0, \infty)$. We have $v_2 \geq 0$ on $[t_0, \infty)$ and

$$\begin{aligned} \|v_2\| &\leq \|v_1\| + \frac{1}{m}\|Fv_1\| + \frac{L_1}{5m} \\ &\leq R_1 + \frac{A}{m} A_4 \Big(R_1 + \big(\sigma^2(b) - a\big)(\sigma(b) - a)^2 A_2 A_3 \big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1 \Big) \\ &\quad + \frac{L_1}{5m} = R_2. \end{aligned} \tag{57}$$

Therefore $v_2 \in \Omega$ and

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon F v_1 - \varepsilon \frac{L_1}{10} - \varepsilon \frac{L_1}{10} \tag{58}$$

or

$$(I - T) v_2 = -\varepsilon m v_2 + \varepsilon \frac{L_1}{10} = S v_1. \tag{59}$$

Consequently $S(\overline{\mathcal{P}}_{R_1}) \subset (I - T)(\Omega)$.

4. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v - \lambda v_0) \in \mathcal{P}$ for all $\lambda \geq 0$, $v \in \partial \mathcal{P}_{r_1} \cap (\Omega + \lambda v_0)$ and $Sv = v - \lambda v_0$ for some $\lambda \geq 0$ and for some $v \in \mathcal{P}_{r_1}$. Then

$$\begin{aligned} r_1 &\geq \|v - \lambda v_0\| = \|Sv\| \geq -Sv(t) \\ &= \varepsilon F v(t) + \varepsilon m v(t) + \varepsilon \frac{L_1}{10} \geq \varepsilon \frac{L_1}{20}, \quad t \in [t_0, \infty), \end{aligned} \tag{60}$$

because by the second inequality of $(H3)$ and by Lemma 3.10, it follows that $\varepsilon F v + \varepsilon \frac{L_1}{20} \geq 0$ on $[t_0, \infty)$.

5. Let $v \in \partial \mathcal{P}_{L_1}$. Then

$$\|Sv + T0\| = \|\varepsilon Fv + m\varepsilon v + \varepsilon \frac{L_1}{5}\| \leq \varepsilon \left(\|Fv\| + m\|v\| + \frac{L_1}{5} \right)$$
$$\leq \varepsilon \big(AA_4 \big(L_1 + (\sigma^2(b) - a)(\sigma(b) - a)^2 A_2 A_3 \big(L_1^{p_1} + L_1^{p_2} + 1 \big) \tag{61}$$
$$+ A_1 \big) + \left(m + \frac{1}{5} \right) L_1 \Big) \leq \varepsilon L_1 = \varepsilon \|v\|.$$

Note that in the last inequality we have used the third inequality of $(H3)$.

6. Now, assume that $v \in \partial \mathcal{P}_{L_1} \cap \Omega$ is such that

$$v = Tv + Sv, \tag{62}$$

whereupon

$$Fv + \frac{L_1}{5} \equiv 0 \quad \text{on} \quad [t_0, \infty). \tag{63}$$

Since $v \in \partial \mathcal{P}_L$, we have that $v \not\equiv 0$ on $[t_0, \infty)$ and by the second inequality of $(H3)$ and by Lemma 3.10, it follows that $Fv + \frac{L_1}{5} > Fv + \frac{L_1}{20} \geq 0$ on $[t_0, \infty)$. This is a contradiction.

7. Suppose that there exists an $v_0 \in \mathcal{P}^*$ such that $T(v - \lambda v_0) \in \mathcal{P}$ for all $\lambda \geq 0$, $v \in \partial \mathcal{P}_{R_1}$, $v \in \partial \mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ and $Sv = v - \lambda v_0$ for some $\lambda \geq 0$ and for some $v \in \mathcal{P}_{R_1}$. Then

$$\begin{aligned} R_1 &\geq \|v - \lambda v_0\| = \|Sv\| \geq -Sv(t) \\ &= \varepsilon Fv(t) + \varepsilon m v(t) + \varepsilon \frac{L_1}{10} \geq \varepsilon \frac{L_1}{20}, \quad t \in [t_0, \infty), \end{aligned} \tag{64}$$

which is a contradiction.

Therefore all conditions of Theorem 3.6 hold. Hence, the IVP (1) has at least two solutions u_1 and u_2 so that

$$r_1 < \|u_1\| < L_1 < \|u_2\| < R_1. \tag{65}$$

5. An example

In this section we will illustrate our main result with an example. Firstly, we will give an example for the constants A, m, ε, r_1, L_1 and R_1 that satisfy the hypothesis $(H3)$. Let $\mathbb{T} = \mathbb{Z}$, $a = 0$, $b = 10$, $\alpha = \beta = 1$,

$$a_1(s) = a_2(s) = a_3(s) = \frac{1}{3}, \quad f(s, u, v) = \frac{1}{1 + v^4}, \quad k(s, s_1) = s_1^2, \tag{66}$$

$s \in [0, 12]$, $s_1 \in [0, 11]$, and

$$\begin{aligned} &r_1 = 1, \quad L_1 = 100, \quad R_1 = 200, \quad \varepsilon = 10^{10}, \\ &p_1 = p_2 = 0, \quad m = \frac{1}{2}, \quad A = \frac{1}{10^{50}}. \end{aligned} \tag{67}$$

Then

$$A_1 = \frac{12 + 12 \cdot 3}{12} = 4, \quad A_2 = 144, \quad A_3 = \frac{1}{3}, \quad A_4 = 144 \tag{68}$$

and

$$\begin{aligned}
&AA_4\Big(R_1 + \big(\sigma^2(b) - a\big)\big(\sigma(b) - a\big)^2 A_2 A_3 \big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1\Big) \\
&= \frac{1}{10^{50}} \cdot 144 \left(200 + 12^3 \cdot 144 \cdot \frac{1}{3} \cdot 3 + 4\right) \\
&< 5 = \frac{L_1}{20}, \\
&AA_4\Big(L_1 + \big(\sigma^2(b) - a\big)\big(\sigma(b) - a\big)^2 A_2 A_3 \big(R_1^{p_1} + R_1^{p_2} + 1\big) + A_1\Big) \\
&= \frac{1}{10^{50}} \cdot 144 \left(100 + 12^3 \cdot 144 \cdot \frac{1}{3} \cdot 3 + 4\right) \\
&< \frac{3}{10} \cdot 5 = \left(\frac{4}{5} - m\right) \frac{L_1}{20},
\end{aligned} \tag{69}$$

i.e., $(H1)$-$(H3)$ hold. Consequently the BVP

$$\begin{aligned}
x^{\Delta^2}(t) &= \int_0^t s^2 \frac{1}{1 + \left(x^\Delta(s)\right)^4} \Delta s, \quad t \in [0, 10], \\
x(0) &= x(12) = 1,
\end{aligned} \tag{70}$$

has at least two non-negative solutions.

6. Conclusion

In this chapter we introduce a class of BVPs for a class second-order integro-dynamic equations on time scales. We give some integral representations of the solutions of the considered BVP. We apply a new multiple fixed point theorem and we prove that the considered BVP has at least two nontrivial solutions. The approach in this chapter can be applied for investigations of IVPs and BVPs for dynamic equations and integro-dynamic equations of arbitrary order on time scales.

Additional classifications

AMS Subject Classification: 39 A 10, 39 A 99

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Chapter 12

Solution of Nonlinear Partial Differential Equations by Mixture Adomian Decomposition Method and Sumudu Transform

Tarig M. Elzaki and Shams E. Ahmed

Abstract

This chapter is fundamentally centering on the application of the Adomian decomposition method and Sumudu transform for solving the nonlinear partial differential equations. It has instituted some theorems, definitions, and properties of Adomian decomposition and Sumudu transform. This chapter is an elegant combination of the Adomian decomposition method and Sumudu transform. Consequently, it provides the solution in the form of convergent series, then, it is applied to solve nonlinear partial differential equations.

Keywords: adomian decomposition method, sumudu transform, nonlinear partial differential equations

1. Introduction

Many of nonlinear phenomena are a necessary part in applied science and engineering fields [1]. The wide use of nonlinear partial differential equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically. In the past, active study attempts were given a large amount of attention to the study of getting exact or approximate solutions of this kind of equations.

Therefore, it becomes increasingly important to be familiar with all traditional and recently developed methods for solving partial differential equations. For some examples of the traditional methods, such as, the separation of variables method, the method of characteristics, the σ-expansion method, the integral transforms and Hirota bilinear method [2–5]. Moreover, the recently developed methods like, Adomian decomposition method (ADM) [1, 6–9], He's semi – inverse method, the tanh method, the sinh – cosh method, the homotopy perturbation method (HPM) [3, 4, 10, 11], the differential transform method (DTM), the variational iteration method (VIM) [1, 5, 12], and the weighted finite difference.

In this chapter, our presentation will be based on applying the new method, namely the Adomian Decomposition Sumudu Transform Method (ADSTM) for solving the nonlinear partial differential equations. This method is an elegant combination of the Sumudu transform method and decomposition method.

2. Sumudu Transform

A long time ago, differential equations wared a necessary part in all aspects of applied sciences and engineering fields. In this chapter we need to develop a new technique for help us to obtain the exact and approximate solutions of these differential equations.

Watugula [13] introduced a new integral transform and called it as Sumudu transform, which is defined as:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u}\, e^{\left(-\frac{t}{u}\right)} f(t)\, dt; \tag{1}$$

Watugula [13] applied this transforms to the solution of ordinary differential equations. Because of its useful properties, the Sumudu transforms helps in solving complex problems in applied sciences and engineering mathematics. Henceforward, is the definition of the Sumudu transforms and properties describing the simplicity of the transform.

Definition 1: The Sumudu transform of the function $f(t)$ is defined by:

$$F(u) = S[f(t)] = \int_0^\infty \frac{1}{u}\, e^{\left(-\frac{t}{u}\right)} f(t)\, dt \tag{2}$$

Or,

$$F(u) = S[f(t)] = \int_0^\infty f(ut)\, e^{-t}\, dt \tag{3}$$

For any function $f(t)$ and $-\tau_1 < u < \tau_2$.

3. The relation between Sumudu and Laplace transform

The Sumudu transform $F_s(u)$ of a function $f(t)$ defined for all real numbers $t \geq 0$. The Sumudu transform is essentially identical with the Laplace transform.

Given an initial $f(t)$ its Laplace transform $G(u)$ can be translated into the Sumudu transform $F_s(u)$ of f by means of the relation;

$$F(u) = \frac{G\left(\frac{1}{u}\right)}{u}, \text{ and it's inverse, } G(s) = \frac{F_s\left(\frac{1}{s}\right)}{s}$$

Theorem 1: Let $f(t)$ with Laplace transform $G(s)$, then, the Sumudu transform $F(u)$ of $f(t)$ is given by, $F(u) = \frac{G\left(\frac{1}{u}\right)}{u}$.

Proof:

Form definition (1.1.1) we get:

$F(u) = \int_0^\infty e^{-t} f(ut)\, dt$, If we set $w = ut$ and $dt = \frac{dw}{u}$ then:

$$F(u) = \int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w)\, \frac{dw}{u} = \frac{1}{u}\int_0^\infty e^{\left(-\frac{w}{u}\right)} f(w)\, dw$$

By definition of the Laplace transform we get: $F(u) = \frac{G\left(\frac{1}{u}\right)}{u}$.

Theorem 2: It deals with the effect of the differentiation of the function $f(t)$, k times on the Sumudu transform $F(u)$ if $S[f(t)] = F(u)$ then:

i. $S[f'(t)] = \frac{1}{u}[F(u) - f(0)]$

ii. $S[f''(t)] = \frac{1}{u^2}[F(u)] - \frac{1}{u^2}f(0) - \frac{1}{u}f'(u)$

iii. $S\left[f^{(n)}(t)\right] = \frac{1}{u^n}[F(u)] - \frac{1}{u^n}\sum_{k=0}^{n-1} u^k f^{(k)}(0) = \; u^{-n}\left[F(u) - \; \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right]$

Where $f^{(0)}(0) = f(0)$, $f^{(k)}(0)$, $k = 1, 2, 3, \cdots, n-1$ are the nth-order derivatives of the function $f(t)$ evaluated at, $t = 0$.

Proof:

i. Using integration by parts,

ii. $$S[f'(t)] = \left[\frac{1}{u}\exp\left(-\frac{t}{u}f(t)\right)\right]_0^\infty + \frac{1}{u}\int_0^\infty \frac{1}{u}\exp\left(-\frac{t}{u}\right)f(t)\,dt = -\frac{1}{u}f(0) + \frac{1}{u}F(u).$$

$$S[f'(t)] = \frac{1}{u}[F(u) - f(0)]$$

Using integration by parts;

$$S[f''(t)] = \left[\frac{1}{u}e^{\left(-\frac{t}{u}\right)}f'(t)\right]_0^\infty + \frac{1}{u}\int_0^\infty \frac{1}{u}e^{\left(-\frac{t}{u}\right)}f'(t)\,dt$$

$$= -\frac{1}{u}f'(0) + \frac{1}{u}S[f'(t)]$$

From (i)

$$S[f''(t)] = \frac{1}{u^2}[F(u)] - \frac{1}{u^2}f(0) - \frac{1}{u}f'(0).$$

iii. By definition the Laplace transform for $f^{(n)}(t)$ is given by

$$G_n(s) = s^n G(s) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^{(k)}(0)$$

By using the relation between Sumudu and Laplace transform;

$$G_n\left(\frac{1}{u}\right) = \frac{G\left(\frac{1}{u}\right)}{u^n} - \sum_{k=\;0}^{n-1}\frac{f^{(k)}(0)}{u^{n-(k+1)}}$$

Since $F_n(u) = \frac{G_n\left(\frac{1}{u}\right)}{u^n}$, we get:

$$u\,F_n(u) = \frac{u\,F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}\,u^{-1}}$$

$$F_n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{u^{n-k}}$$

$$F_n(u) = u^{-n}F(u) - \sum_{k=0}^{n-1} u^{-n}u^k f^{(k)}(0)$$

$$S\left[f^{(n)}(t)\right] = F(u) = \; u^{-n}\left[F(u) - \; \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right]$$

4. Adomian decomposition method

Many of nonlinear phenomena are a necessary part in applied science and engineering fields. Nonlinear equations are noticed in a different type of physical problems [1], such as fluid dynamics, plasma physics, solid mechanics, and quantum field theory.

The wide use of these equations is the most important reason why they have drawn mathematician's attention. Despite this, they are not easy to find an answer, either numerically or theoretically.

In the past, active study attempts were given a large amount of attention to the study of getting exact or approximate solutions of this kind of equations. It is significant to note that several powerful methods have been advanced for this purpose.

The Adomian decomposition method will be used in this chapter and in other chapters to deal with nonlinear equations. The Adomian decomposition method proves to be powerful, effective and successfully used to handle most types of linear or nonlinear ordinary or partial differential equations, and linear or nonlinear integral equations.

In the following, the Adomian scheme for calculating a wide variety of forms of nonlinearity.

5. Calculation of Adomian polynomials

It is well known that the Adomian decomposition method suggests the unknown linear function u may be represented by the decomposition series;

$$u = \sum_{n=0}^{\infty} u_n, \tag{4}$$

Where the components $u_n, n \geq 0$ can be elegantly computed in a recursive way. However, the nonlinear term $F(u)$, such as $u^2, u^3, u^4, \sin u, e^u, u u_x, {u_x}^2,$ etc., can be expressed by an infinite series of the so- called Adomian polynomials A_n given in the form;

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n). \tag{5}$$

The Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by using the following expression;

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0, 1, 2, \dots \tag{6}$$

Assuming that the nonlinear function is $F(u)$, therefore, by using (6), Adomian polynomials are given by;

$$\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{2!} {u_1}^2 F''(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} {u_1}^3 F'''(u_0).
\end{aligned} \tag{7}$$

Other polynomials can be generated in a similar manner.

Substituting (7) into (5) gives;

$$F(u) = A_0 + A_1 + A_2 + A_3 + \dots = F(u_0) + (u_1 + u_2 + u_3 + \dots)F'(u_0)$$
$$+\frac{1}{2!}\left(u_1{}^2 + 2u_1u_2 + u_2{}^2 + \dots\right)F''(u_0) + \frac{1}{3!}\left(u_1{}^3 + 3u_1{}^2u_2 + 3u_1{}^2u_3 + \dots\right)F'''(u_0) + \dots$$
$$= F(u_0) + (u - u_0)F'(u_0) + \frac{1}{2!}(u - u_0)^2 F''(u_0) + \dots.$$

The last expansion confirms a fact that the series in A_n polynomials is a Taylor series about a function u_0 and not about a point as is usually used.

In the following, we will calculate Adomian polynomials for several forms of nonlinearity.

5.1 Nonlinear polynomials

$$\text{If } F(u) = u^2$$

The polynomials can be found as follows:

$$A_0 = F(u_0) = u_0{}^2, \quad A_1 = u_1F'(u_0) = 2u_0u_1, \quad A_2 = u_2F'(u_0) + \frac{1}{2!}u_1{}^2F''(u_0) = 2u_0u_2 + u_1{}^2,$$
$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1{}^3F'''(u_0) = 2u_0u_3 + 2u_1u_2.$$

And so on. Proceeding as before, we find $u^3, u^4, u^5, \dots$, etc.

5.2 Nonlinear derivatives

Case1. $F(u) = (u_x)^2$

$$A_0 = u_{0x}{}^2, \quad A_1 = 2u_{0x}u_{1_x}, \quad A_2 = 2u_{0x}u_{2x} + u_{1x}{}^2, \quad A_3 = 2u_{0x}u_{3x} + 2u_{1x}u_{2x}.$$

And so on. In a similar, we get $u_x{}^3, u_x{}^4, u_x{}^5, \dots$, etc.

Case 2. $F(u) = uu_x = \frac{1}{2}L_x(u^2)$

The A_n polynomials in this case given by;

$$A_0 = F(u_0) = u_0u_{0_x}, \quad A_1 = \frac{1}{2}L_x(2u_0u_1) = u_{0x}u_1 + u_0u_{1_x},$$
$$A_2 = \frac{1}{2}L_x\left(2u_0u_2 + u_1{}^2\right) = u_{0_x}u_2 + u_0u_{2_x}u_0 + u_1u_{1_x},$$
$$A_3 = \frac{1}{2}L_x(2u_0u_3 + 2u_1u_2) = u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0.$$

And so on.

5.3 Trigonometric nonlinearity

$$\text{If } F(u) = \sin u$$

The Adomian polynomials for this form nonlinearity are given by;

$$A_0 = \sin u_0, \quad A_1 = u_1 \cos u_0, \ A_2 = u_2 \cos u_0 - \frac{1}{2!} u_1{}^2 \sin u_0, \ A_3$$
$$= u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1{}^3 \cos u_0.$$

And so on. In a similar way, we find $F(u) = \cos u$.

5.4 Hyperbolic nonlinearity

$$\text{If } F(u) = \sinh u$$

The A_n polynomials in this case are given by;

$$A_0 = \sinh u_0, \ A_1 = u_1 \cosh u_0, \ A_2 = u_2 \cosh u_0 + \frac{1}{2!} u_1{}^2 \sinh u_0, \ A_3$$
$$= u_3 \cosh u_0 + u_1 u_2 \sinh u_0 + \frac{1}{3!} u_1{}^3 \cosh u_0.$$

And so on. In a parallel manner, Adomian polynomials can be calculated for $F(u) = \cosh u$.

5.5 Exponential nonlinearity

$$\text{If } F(u) = e^u$$

The Adomian polynomials in this form of nonlinearity are given by;

$$A_0 = e^{u_0}, \ A_1 = u_1 e^{u_0}, \ A_2 = \left(u_2 + \frac{1}{2!} u_1{}^2\right) e^{u_0}, \ A_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!} u_1{}^3\right) e^{u_0}.$$

And so on. Proceeding as a before, we find $F(u) = e^{-u}$.

5.6 Logarithmic nonlinearity

$$\text{If } F(u) = \ln u, u > 0$$

The A_n polynomials for logarithmic nonlinearity are given by;

$$A_0 = \ln u_0, \ A_1 = \frac{u_1}{u_0}, \quad A_2 = \frac{u_2}{u_0} - \frac{1}{2} \frac{u_1{}^2}{u_0{}^2}, \quad A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{u_0{}^2} + \frac{1}{3} \frac{u_1{}^3}{u_0{}^3}.$$

And so on. In a similar way, we find $F(u) = \ln(1+u), \ -1 < u \leq 1$.

6. A New algorithm for calculating Adomian polynomials (The alternative algorithm for calculating Adomian polynomials)

It is well known about the main disadvantage of the calculating Adomian polynomials A_n, that it is a difficult method to perform calculation so called nonlinear terms. There is an alternative algorithm to reduce the demerits of formula introduced before, which depends mainly on algebraic, trigonometric identities and on Taylor expansions.

In the alternative algorithm which is a simple and reliable may be employed to calculate Adomian Polynomials A_n.

The new algorithm will be clarified by discussing the following suitable forms of nonlinearity.

6.1 Nonlinear polynomials

$$\text{If } F(u) = u^2$$

We first set,

$$u = \sum_{n=0}^{\infty} u_n. \tag{8}$$

Substituting (8) into $F(u) = u^2$ gives;

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + \dots)^2. \tag{9}$$

Expanding the expression at the right- hand side gives;

$$F(u) = {u_0}^2 + 2u_0 u_1 + 2u_0 u_2 + {u_1}^2 + 2u_0 u_3 + 2u_1 u_2 + \dots \tag{10}$$

The expansion in (10) can be rearranged by grouping all terms with the sum of the subscripts of the components is the same. This means that we can rewrite (10) as;

$$F(u) = \underbrace{{u_0}^2}_{A_0} + \underbrace{2u_0 u_1}_{A_1} + \underbrace{2u_0 u_2 + {u_1}^2}_{A_2} + \underbrace{2u_0 u_3 + 2u_1 u_2}_{A_3} + \dots \tag{11}$$

This gives Adomian polynomials for $F(u) = u^2$ by;

$$A_0 = {u_0}^2, \quad A_1 = 2u_0 u_1, \quad A_2 = 2u_0 u_2 + {u_1}^2, \quad A_3 = 2u_0 u_3 + 2u_1 u_2.$$

And so on. Proceeding as before, we get $u^3, u^4, u^5, \dots$, etc.

6.2 Nonlinear derivatives

Case 1. If $F(u) = {u_x}^2$.
We first set;

$$u_x = \sum_{n=0}^{\infty} u_{n\,x}. \tag{12}$$

Substituting (12) into $F(u) = {u_x}^2$ giving;

$$F(u) = (u_{0x} + u_{1x} + u_{2x} + u_{3x} + u_{4x} + \dots)^2. \tag{13}$$

Squaring the right – hand side gives;

$$F(u) = {u_{0x}}^2 + 2u_{0x} u_{1x} + 2u_{0x} u_{2x} + {u_{1x}}^2 + 2u_{0x} u_{3x} + 2u_{1x} u_{2x} + \dots \tag{14}$$

Grouping the terms as discussed above, we find;

$$F(u) = \underbrace{{u_{0x}}^2}_{A_0} + \underbrace{2u_{0x}u_{1x}}_{A_1} + \underbrace{2u_{0x}u_{2x} + {u_{1x}}^2}_{A_2} + \underbrace{2u_{0x}u_{3x} + 2u_{1x}u_{2x}}_{A_3} + \ldots \tag{15}$$

Adomian polynomials are given by;

$$A_0 = {u_{0x}}^2, \quad A_1 = 2u_{0x}u_{1_x}, \; A_2 = 2u_{0x}u_{2x} + {u_{1x}}^2, \quad A_3 = 2u_{0x}u_{3x} + 2u_{1x}u_{2x}.$$

Case 2. $F(u) = u u_x$

Note that this form of nonlinearity appears in advection problems and in nonlinear Burgers equations. We first set;

$$u = \sum_{n=0}^{\infty} u_n, u_x = \sum_{n=0}^{\infty} u_{nx}. \tag{16}$$

Substituting (16) into $F(u) = u u_x$ yields;

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + \ldots) \times (u_{0x} + u_{1x} + u_{2x} + u_{3x} + u_{4x} + \ldots); \tag{17}$$

Multiplying the two factors gives;

$$\begin{aligned} F(u) = & u_0u_{0x} + u_{0x}u_1 + u_0u_{1x} + u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0 + u_{0x}u_3 + u_{1x}u_2 + \\ & + u_{2x}u_1 + u_{3x}u_0 + u_{0x}u_4 + u_0u_{4x} + u_{1x}u_3 + u_1u_{3x} + u_2u_{2x} + \ldots \end{aligned}. \tag{18}$$

Proceeding with grouping the terms are obtained;

$$\begin{aligned} F(u) = & \underbrace{u_0u_{0x}}_{A_0} + \underbrace{u_{0x}u_1 + u_0u_{1x}}_{A_1} + \underbrace{u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0}_{A_2} \\ & + \underbrace{u_{0x}u_3 + u_{1x}u_2 + u_{2x}u_1 + u_{3x}u_0}_{A_3} \ldots \end{aligned} \tag{19}$$

Consequently, the Adomian polynomials are given by;

$$\begin{aligned} & A_0 = u_0u_{0_x}, \quad A_1 = u_{0x}u_1 + u_0u_{1_x}, \quad A_2 = u_{0_x}u_2 + u_0u_{2_x}u_0 + u_1u_{1_x}, \\ & A_3 = u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0. \end{aligned}$$

Proceeding as before, we find $F(u) = u^2 u_x$.

6.3 Trigonometric nonlinearity

$$\text{If } F(u) = \sin u$$

First, we should be separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute;

$$u = \sum_{n=0}^{\infty} u_n; \tag{20}$$

Into $F(u) = \sin u$ to obtain;

$$F(u) = \sin\left[u_0 + (u_1 + u_2 + u_3 + u_4 + \ldots)\right]. \tag{21}$$

To separate A_0, recall the trigonometric identity;

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi. \tag{22}$$

This means that;

$$F(u) = \sin u_0 \cos(u_1 + u_2 + u_3 + u_4 + \ldots) + \cos u_0 \sin(u_1 + u_2 + u_3 + u_4 + \ldots). \tag{23}$$

Separating $F(u_0) = \sin u_0$ from other factors and using Taylor expansion for, $\cos(u_1 + u_2 + u_3 + u_4 + \ldots)$ and, $\sin(u_1 + u_2 + u_3 + u_4 + \ldots)$ gives;

$$\begin{aligned} F(u) = {} & \sin u_0 \left(1 - \frac{1}{2!}(u_1 + u_2 + \ldots)^2 + \frac{1}{4!}(u_1 + u_2 + \ldots)^4 - \ldots\right) + \\ & + \cos u_0 \left((u_1 + u_2 + \ldots) - \frac{1}{3!}(u_1 + u_2 + \ldots)^3 + \ldots\right) \end{aligned}, \tag{24}$$

So that;

$$\begin{aligned} F(u) = {} & \sin u_0 \left(1 - \frac{1}{2!}\left({u_1}^2 + 2u_1u_2 + \ldots\right) + \ldots\right) \\ & + \cos u_0 \left((u_1 + u_2 + \ldots) - \frac{1}{3!}{u_1}^3 + \ldots\right). \end{aligned} \tag{25}$$

The last expansion can be rearranged by grouping all terms with the same sum of subscripts. This leads to;

$$\begin{aligned} F(u) = {} & \underbrace{\sin u_0}_{A_0} + \underbrace{u_1 \cos u_0}_{A_1} + \underbrace{u_2 \cos u_0 - \frac{1}{2!}{u_1}^2 \sin u_0}_{A_2} + \\ & \underbrace{+ u_3 \cos u_0 - u_1u_2 \sin u_0 - \frac{1}{3!}{u_1}^3 \cos u_0}_{A_3} + \ldots \end{aligned} \tag{26}$$

This completes the calculation of the Adomian polynomials for nonlinear operator $F(u) = \sin u$, therefore we write;

$$\begin{aligned} & A_0 = \sin u_0, \;\; A_1 = u_1 \cos u_0, \;\; A_2 = u_2 \cos u_0 - \frac{1}{2!}{u_1}^2 \sin u_0, \\ & A_3 = u_3 \cos u_0 - u_1u_2 \sin u_0 - \frac{1}{3!}{u_1}^3 \cos u_0. \end{aligned}$$

And so on. In a similar way, we find $F(u) = \cos u$.

6.4 Hyperbolic nonlinearity

If $F(u) = \sinh u$ then, we first substitute

$$u = \sum_{n=0}^{\infty} u_n; \tag{27}$$

Into $F(u) = \sinh u$ to obtain;

$$F(u) = \sinh\left[u_0 + (u_1 + u_2 + u_3 + u_4 + \ldots)\right]. \tag{28}$$

To calculate A_0, recall the hyperbolic identity;

$$\sinh(\theta + \phi) = \sinh\theta \cosh\phi + \cosh\theta \sinh\phi. \tag{29}$$

Accordingly, Eq. (29) becomes;

$$\begin{aligned} F(u) = {} & \sinh u_0 \cosh(u_1 + u_2 + u_3 + u_4 + \ldots) \\ & + \cosh u_0 \sinh(u_1 + u_2 + u_3 + u_4 + \ldots). \end{aligned} \tag{30}$$

Separating $F(u_0) = \sinh u_0$ from other factors and using Taylor expansion for $\cosh(u_1 + u_2 + u_3 + u_4 + \ldots)$ and $\sinh(u_1 + u_2 + u_3 + u_4 + \ldots)$ gives;

$$\begin{aligned} F(u) = {} & \sinh u_0 \left(1 + \frac{1}{2!}(u_1 + u_2 + \ldots)^2 + \frac{1}{4!}(u_1 + u_2 + \ldots)^4 + \ldots\right) \\ & + \cosh u_0 \left((u_1 + u_2 + \ldots) + \frac{1}{3!}(u_1 + u_2 + \ldots)^3 + \ldots\right) \\ = {} & \sinh u_0 \left(1 + \frac{1}{2!}({u_1}^2 + 2u_1u_2 + \ldots) + \ldots\right) + \cosh u_0 \left((u_1 + u_2 + \ldots) + \frac{1}{3!}{u_1}^3 + \ldots\right) \end{aligned}$$

By grouping all terms with the same sum of subscripts we find

$$\begin{aligned} F(u) = {} & \underbrace{\sinh u_0}_{A_0} + \underbrace{u_1 \cosh u_0}_{A_1} + \underbrace{u_2 \cosh u_0 + \frac{1}{2!}{u_1}^2 \sinh u_0}_{A_2} \\ & + \underbrace{u_3 \cosh u_0 + u_1u_2 \sinh u_0 - \frac{1}{3!}{u_1}^3 \cosh u_0}_{A_3} + \ldots \end{aligned}$$

Consequently, the Adomian polynomials for $F(u) = \sinh u$ are given by;

$$A_0 = \sinh u_0,\ \ A_1 = u_1 \cosh u_0,\ \ A_2 = u_2 \cosh u_0 + \frac{1}{2!}{u_1}^2 \sinh u_0,$$

$$A_3 = u_3 \cosh u_0 + u_1u_2 \sinh u_0 + \frac{1}{3!}{u_1}^3 \cosh u_0.$$

Similarly as before, we find $F(u) = \cosh u$.

6.5 Exponential nonlinearity

If $F(u) = e^u$.

Substituting

$$u = \sum_{n=0}^{\infty} u_n; \tag{31}$$

Into $F(u) = e^u$ gives;

$$F(u) = e^{(u_0+u_1+u_2+u_3+u_4+\dots)}. \tag{32}$$

Or equivalently;

$$F(u) = e^{u_0} \times e^{(u_1+u_2+u_3+u_4+\dots)}. \tag{33}$$

Keeping the term $F(u_0) = e^{u_0}$ and using Taylor expansion for the other factors we obtain;

$$F(u) = e^{u_0} \times \left(1 + (u_1 + u_2 + u_3 + \dots) + \frac{1}{2!}(u_1 + u_2 + u_3 + \dots)^2 + \dots\right). \tag{34}$$

By grouping all terms with an identical sum of subscripts we find

$$F(u) = \underbrace{e^{u_0}}_{A_0} + \underbrace{u_1 e^{u_0}}_{A_1} + \underbrace{\left(u_2 + \frac{1}{2!}{u_1}^2\right)e^{u_0}}_{A_2} + \underbrace{\left(u_3 + u_1 u_2 + \frac{1}{3!}{u_1}^3\right)e^{u_0}}_{A_3} + \dots \tag{35}$$

It then follows that;

$$A_0 = e^{u_0}, \quad A_1 = u_1 e^{u_0}, \quad A_2 = \left(u_2 + \frac{1}{2!}{u_1}^2\right)e^{u_0}, \quad A_3 = \left(u_3 + u_1 u_2 + \frac{1}{3!}{u_1}^3\right)e^{u_0}.$$

And so on. Proceeding as a before, we find $F(u) = e^{-u}$.

6.6 Logarithmic nonlinearity

$$\text{If } F(u) = \ln u, u > 0$$

Substituting

$$u = \sum_{n=0}^{\infty} u_n; \tag{36}$$

Into $F(u) = \ln u$ gives;

$$F(u) = \ln(u_0 + u_1 + u_2 + u_3 + u_4 + \dots). \tag{37}$$

Eq. (34) can be written as;

$$F(u) = \ln\left(u_0\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right)\right). \tag{38}$$

Using the identity $\ln(\alpha\beta) = \ln\alpha + \ln\beta$, Eq. (38) becomes;

$$F(u) = \ln(u_0) + \ln\left(1 + \frac{u_1}{u_0} + \frac{u_2}{u_0} + \frac{u_3}{u_0} + \dots\right). \tag{39}$$

Separating $F(u_0) = \ln(u_0)$ and using Taylor expansion of the remaining term, we obtain;

$$F(u) = \ln(u_0) + \left\{ \begin{array}{l} \left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right) - \frac{1}{2}\left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right)^2 + \frac{1}{3}\left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right)^3 \\ -\frac{1}{4}\left(\frac{u_1}{u_0}+\frac{u_2}{u_0}+\frac{u_3}{u_0}+\ldots\right)^4 + \ldots \end{array} \right\} \tag{40}$$

Proceeding as before, Eq. (40) can be rewritten as;

$$F(u) = \underbrace{\ln(u_0)}_{A_0} + \underbrace{\frac{u_1}{u_0}}_{A_1} + \underbrace{\frac{u_2}{u_0} - \frac{1}{2}\frac{{u_1}^2}{{u_0}^2}}_{A_2} + \underbrace{\frac{u_3}{u_0} - \frac{u_1 u_2}{{u_0}^2} + \frac{1}{3}\frac{{u_1}^3}{{u_0}^3}}_{A_3} + \ldots \tag{41}$$

Based on this, the Adomian polynomials are given by;

$$A_0 = \ln(u_0), \quad A_1 = \frac{u_1}{u_0}, \quad A_2 = \frac{u_2}{u_0} - \frac{1}{2}\frac{{u_1}^2}{{u_0}^2}, \quad A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{{u_0}^2} + \frac{1}{3}\frac{{u_1}^3}{{u_0}^3}.$$

And so on. In a like manner, we obtain $F(u) = \ln(1+u), \ -1 < u \leq 1$.

7. Adomian decomposition Sumudu transform method for solving nonlinear partial differential equations

In this section, we will concentrate our study on the nonlinear PDEs. There are many nonlinear partial differential equations which are quite useful and applicable in engineering and physics.

The nonlinear phenomena that appear in the many scientific fields' such as solid state physics, plasma physics, fluid mechanics and quantum field theory can be modeled by nonlinear differential equations. The significance of obtaining exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is yet an important problem that needs new methods to develop new techniques for obtaining analytical solutions. Several powerful mathematical methods are used for this purpose. We, propose a new method, namely Adomian Decomposition Sumudu Transform Method (ADSTM) for solving nonlinear equations. This method is a combination of Sumudu transform and decomposition method which was introduced by D. Kumar, J. Singh and S. Rathore.

(ADSTM) provides the solution for nonlinear equations in the form of convergent series. This forms the motivation for us to apply (ADSTM) for solving nonlinear equations in understanding different physical phenomena.

To illustrate the basic idea of this method, we consider a general non-homogeneous partial differential equation with the initial conditions of the form:

$$\begin{array}{c} DU(x,t) + RU(x,t) + NU(x,t) = g(x,t) \\ U(x,0) = h(x), U_t(x,0) = f(x). \end{array}; \tag{42}$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D, N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (42), we get:

$$S[DU(x,t)] + S[RU(x,t)] + S[N(x,t)] = S[g(x,t)]; \tag{43}$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have:

$$S[U(x,t)] = u^2 S[g(x,t)] + h(x) + uf(x) - u^2 S[RU(x,t) + NU(x,t)]. \tag{44}$$

If we apply the inverse operator S^{-1} to both sides of Eq. (44), we obtain:

$$U(x,t) = G(x,t) - S^{-1}\left[u^2 S[RU(x,t) + NU(x,t)]\right]. \tag{45}$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, apply the Adomain decomposition method:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t); \tag{46}$$

The nonlinear term can be decomposed as:

$$NU(x,t) = \sum_{n=0}^{\infty} A_n(U); \tag{47}$$

For some Adomain polynomials $A_n(U)$ that are given by:

$$A_n(U_0, U_1, U_2, \ldots, U_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} \lambda^n U_n\right)\right]_{\lambda=0}, n = 0, 1, 2, \ldots$$

Substituting Eq. (46) and Eq. (47) in Eq. (45), we get:

$$\sum_{n=0}^{\infty} U_n(x,t) = G(x,t) - S^{-1}\left[u^2 S\left[R\sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} A_n(U)\right]\right]. \tag{48}$$

Accordingly, the formal recursive relation is defined by:

$$\begin{aligned} U_0(x,t) &= G(x,t), \\ U_{k+1}(x,t) &= -S^{-1}\left[u^2 S[RU_k + A_k]\right].k \geq 0. \end{aligned} \tag{49}$$

The Adomian decomposition Sumudu transform method will be illustrated by discussing the following examples.

Example 1: Consider the following nonlinear partial differential equation:

$$U_t + UU_x = 0; \tag{50}$$

With the initial condition:

$$U(x,0) = x. \tag{51}$$

Taking the Sumudu transform of both sides of Eq. (50) and using the initial condition, we have:

$$S[U(x,t)] = x - uS[UU_x]. \tag{52}$$

Applying S^{-1} to both sides of Eq. (52) implies that:

$$U(x,t) = x - S^{-1}\left[u\,S[UU_x]\right]; \tag{53}$$

Following the technique, if we assume an infinite series of the form (53), we obtain:

$$\sum_{n=0}^{\infty} U_n(x,t) = x - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n(U)\right]\right]. \tag{54}$$

Where $A_n(U)$ are Adomian polynomials that represent the nonlinear terms. The first few components of the Adomian polynomials are given by;

$$\begin{aligned} A_0(U) &= U_0\,U_{0_x}, \\ A_1(U) &= U_0\,U_{1_x} + U_1\,U_{0_x}, \\ &. \quad . \quad . \quad . \end{aligned} \tag{55}$$

This gives the recursive relation:

$$\begin{aligned} U_0(x,t) &= x, \\ U_{k+1}(x,t) &= -S^{-1}\left[u\,S[A_k]\right], k \geq 0. \end{aligned} \tag{56}$$

The first few components are given by:

$$\begin{aligned} U_0(x,t) &= x, \\ U_1(x,t) &= -S^{-1}\left[u\,S[A_0]\right] = -xt, \\ U_2(x,t) &= -S^{-1}\left[u\,S[A_1]\right] = xt^2, \\ U_3(x,t) &= -S^{-1}\left[u\,S[A_2]\right] = -xt^3. \end{aligned} \tag{57}$$

And so on. The solution in a series form is given by:

$$U(x,t) = x\left(1 - t + t^2 - t^3 + \ldots\right); \tag{58}$$

And in a closed form of:

$$U(x,t) = \frac{x}{1+t}. \tag{59}$$

Example 2: Consider the following nonlinear partial differential equation:

$$U_t + UU_x = x + xt^2; \tag{60}$$

With the initial condition:

$$U(x,0) = 0. \tag{61}$$

Proceeding as in **Example 1**, Eq. (60) becomes:

$$\sum_{n=0}^{\infty} U_n(x,t) = xt + \frac{xt^3}{3} - S^{-1}\left[u\,S\left[\sum_{n=0}^{\infty} A_n(U)\right]\right]. \tag{62}$$

The modified decomposition method admits the of a modified recursive relation given by:

$$U_0(x,t) = xt,$$
$$U_1(x,t) = \frac{xt^3}{3} - S^{-1}[uS[A_0]] \tag{63}$$
$$U_{k+1}(x,t) = \ -S^{-1}[uS[A_k]], k \geq 1.$$

Consequently, we obtain:

$$U_0(x,t) = xt,$$
$$U_1(x,t) = \frac{xt^3}{3} - S^{-1}\left[uS\left[xt^2\right]\right] = 0 \tag{64}$$
$$U_{k+1}(x,t) = \ 0, k \geq 1.$$

In few of Eq. (64), the exact solution is given by:

$$U(x,t) = \ xt. \tag{65}$$

Example 3: Consider the nonlinear partial differential equation:

$$U_{tt} + {U_x}^2 + U - U^2 = te^{-x}; \tag{66}$$

With the initial condition

$$U(x,0) = 0, U_t(x,0) = e^{-x}. \tag{67}$$

By taking Sumudu transform for (66) and using (67) we obtain:

$$S[U(x,t)] = u^3e^{-x} + ue^{-x} - u^2 S\left[{U_x}^2 - U^2 + U\right]. \tag{68}$$

Applying S^{-1} to both sides of (68) we obtain;

$$U(x,t) = te^{-x} + \frac{1}{6}t^3e^{-x} - S^{-1}\left[u^2 S\left[{U_x}^2 - U^2 + U\right]\right]. \tag{69}$$

Substituting;

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t); \tag{70}$$

And the nonlinear terms of;

$${U_x}^2 = \sum_{n=0}^{\infty} A_n \ , U^2 = \sum_{n=0}^{\infty} B_n. \tag{71}$$

Into (69) gives;

$$\sum_{n=0}^{\infty} U_n(x,t) = te^{-x} + \frac{1}{6}t^3e^{-x} - \ S^{-1}\left[u^2 S\left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} U_n(x,t) - \sum_{n=0}^{\infty} B_n\right)\right] \tag{72}$$

This gives the modified recursive relation;

$$\begin{aligned}
&U_0(x,t) = te^{-x}, \\
&U_1(x,t) = \frac{1}{6}t^3e^{-x} - \ L_t{}^{-1}(A_0 + U_0 - B_0) \\
&U_{k+1}(x,t) = - \ L_t{}^{-1}(A_k + U_k - B_k), k \geq 1.
\end{aligned} \tag{73}$$

The first few of the components are given by;

$$\begin{aligned}
&U_0(x,t) = te^{-x}, \\
&U_1(x,t) = \frac{1}{6}t^3e^{-x} - \ L_t{}^{-1}(A_0 + U_0 - B_0) = 0, \\
&U_{k+1}(x,t) = 0, k \geq 1.
\end{aligned} \tag{74}$$

The solution in a closed form is given by;

$$U(x,t) = te^{-x}. \tag{75}$$

Example 4: Consider the following nonlinear partial differential equation,

$$U_{tt} + U^2 - U_x{}^2 = 0; \tag{76}$$

With the initial conditions

$$U(x,0) = 0 \ , U_t(x,0) = e^x. \tag{77}$$

By taking Sumudu transform for (76) and using (77) we obtain:

$$S[U(x,t)] = ue^x + u^2 S\left[U_x{}^2 - U^2\right]. \tag{78}$$

By applying the inverse Sumudu transform of (78), we get:

$$U(x,t) = te^x + S^{-1}\left[u^2 S\left[U_x{}^2 - U^2\right]\right]; \tag{79}$$

This assumes a series solution of the function $U(x,t)$is given by:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t); \tag{80}$$

Using (80) into (79), we get:

$$\sum_{n=0}^{\infty} U_n(x,t) = \ te^x + \ S^{-1}\left[u^2 S\left[\sum_{n=0}^{\infty} A_n(U) - \sum_{n=0}^{\infty} B_n(U)\right]\right]. \tag{81}$$

Where $A_n(U)$ and $B_n(U)$ are Adomian polynomials that represents nonlinear terms.

The few components of the Adomian polynomials are given as follows:

$$\begin{aligned}
&A_0(U) = U_{0_x}{}^2 \ , \ A_1(U) = 2U_{0_x}U_{1_x}, \\
&B_0(U) = U_0{}^2 \ \ , \ \ B_1(U) = 2U_0U_1 \ ,
\end{aligned} \tag{82}$$

And so on. From the above equations we obtain:

$$\begin{aligned} U_0(x,t) &= te^x, \\ U_{k+1}(x,t) &= S^{-1}\left[u^2 S[A_k - B_k]\right], k \geq 0. \end{aligned} \tag{83}$$

The first few terms of $U_n(x,t)$ follows immediately upon setting:

$$\begin{aligned} U_1(x,t) &= S^{-1}\left[u^2 S[A_0 - B_0]\right] = S^{-1}\left[u^2 S\left[{U_{0_x}}^2 - {U^2}_0\right]\right] = 0 \\ U_{k+1}(x,t) &= 0, k \geq 1. \end{aligned} \tag{84}$$

Therefore the solution obtained by ADSTM is given as follows:

$$U(x,t) = te^x.$$

8. Nonlinear physical models

Now we will, concentrate our study on the linear and nonlinear particular applications that appear in applied science. The wide use of these physical significant problems is the most important reason why they have drawn mathematician's attention in recent years.

Nonlinear partial differential equations have witnessed remarkable improvement. Nonlinear problems appear in the many scientific fields' such as gravitation, chemical reaction, fluid dynamics, dispersion, nonlinear optics, plasma physics, acoustics, and others. Several approaches have been used such as the Adomian decomposition method, the variational iteration method, and the characteristics method and perturbation techniques to examine these problems.

(ADSTM) gives the solution of nonlinear equations in the form of convergent series. The main advantage of this method is its potentiality of combining two powerful methods for deriving exact and approximate solution of nonlinear equations. This forms the motivation for us to apply (ADSTM) for solving nonlinear equations in understanding different physical phenomena.

The following section offers the effectiveness of the Adomian decomposition Sumudu transform method (ADSTM) in solving nonlinear physical models.

Example 5: Consider the following inhomogeneous advection problem:

$$U_t + UU_x = 2t + x + t^3 + xt^2; \tag{85}$$

With the initial condition:

$$U(x,0) = 0. \tag{86}$$

Following discussion presented above, we obtain the recursive relation;

$$\begin{aligned} U_0(x,t) &= t^2 + xt + \frac{t^4}{4} + \frac{xt}{3}, \\ U_{k+1}(x,t) &= -S^{-1}[uS[A_k]], k \geq 0. \end{aligned} \tag{87}$$

This gives;

$$\begin{aligned} U_0(x,t) &= t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\ U_1(x,t) &= -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8. \end{aligned} \tag{88}$$

It is easily observed that two noise term appears in the components $U_0(x,t)$ and $U_1(x,t)$. By canceling these terms from $U_0(x,t)$, the remaining non-canceled term of $U_0(x,t)$ may provide the exact solution.

The exact solution is given by;

$$U(x,t) = t^2 + xt.$$

Example 6: Consider the following nonlinear Klein – Gordon equation:

$$U_{tt} - U_{xx} + U^2 = x^2 t^2; \tag{89}$$

Subject to the initial conditions:

$$U(x,0) = 0 \ , \ U_t(x,t) = x. \tag{90}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} U_0(x,t) = & \ xt + \frac{1}{12} x^2 t^4, \\ U_{k+1}(x,t) = & \ S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_k]\right], k \geq 0. \end{aligned} \tag{91}$$

So the Adomian polynomials A_n are given as follows;

$$\begin{aligned} A_0 = & \ {U_0}^2, \\ A_1 = & \ 2U_0 U_1, \\ A_2 = & \ 2U_0 U_2 + {U_1}^2. \end{aligned}$$

And so on. Using modified recursive relation Eq. (91) can be rewritten in the scheme;

$$\begin{aligned} U_0(x,t) = & \ xt, \\ U_1(x,t) = & \ \frac{1}{12} x^2 t^4 + S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_0]\right] \ , \\ U_{k+1}(x,t) = & \ S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_k]\right], k \geq 1. \end{aligned} \tag{92}$$

This lead to;

$$\begin{aligned} U_0(x,t) = & \ xt, \\ U_1(x,t) = & \ \frac{1}{12} x^2 t^4 + S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] - S^{-1}\left[u^2 S[A_0]\right] \ = 0, \\ U_{k+1}(x,t) = & \ 0, k \geq 1. \end{aligned} \tag{93}$$

Therefore, the exact solution is given by;

$$U(x,t) = xt.$$

Example 7: Consider the following Sine-Gordon equation with the given initial conditions:

$$U_{tt}(x,t) - \ U_{xx}(x,t) = \ \sin U; \tag{94}$$

Subject to the initial conditions;

$$U(x,0) = \frac{\pi}{2} \ , \ U_t(x,t) = 0. \tag{95}$$

Using the recursive scheme yields;

$$\begin{aligned} &U_0(x,t) = \frac{\pi}{2}, \\ &U_{k+1}(x,t) = S^{-1}\left[u^2 S\left[(U_k)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_k]\right], k \geq 0. \end{aligned} \tag{96}$$

The first few the Adomian polynomials for $\sin U$ are given as;

$$\begin{aligned} A_0 &= \sin U_0, \\ A_1 &= U_1 \cos U_0, \\ A_2 &= U_2 \cos U_0 - \frac{1}{2!} U_1{}^2 \sin U_0, \\ A_3 &= U_3 \cos U_0 - U_1 U_2 \sin U_0 - \frac{1}{3!} U_1{}^3 \cos U_0. \end{aligned} \tag{97}$$

Combining (96) and (97) leads to;

$$\begin{aligned} &U_0(x,t) = \frac{\pi}{2}, \\ &U_1(x,t) = S^{-1}\left[u^2 S\left[(U_0)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_0]\right] = \frac{t^2}{2!}, \\ &U_2(x,t) = S^{-1}\left[u^2 S\left[(U_1)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_1]\right] = 0 , \\ &U_3(x,t) = S^{-1}\left[u^2 S\left[(U_2)_{xx}\right]\right] + S^{-1}\left[u^2 S[A_2]\right] = -\frac{1}{240} t^6. \end{aligned} \tag{98}$$

And so on. The series solution is;

$$U(x,t) = \frac{\pi}{2} + \frac{t^2}{2!} - \frac{1}{240} t^6 + \dots$$

Example 8: Consider the following one – dimensional Burgers equation:

$$U_t = U_{xx} - UU_x; \tag{99}$$

Subject to the initial conditions:

$$U(x,0) = x. \tag{100}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned} &U_0(x,t) = x, \\ &U_{k+1}(x,t) = S^{-1}\left[uS\left[(U_k)_{xx}\right]\right] - S^{-1}[uS[A_k]], k \geq 0. \end{aligned} \tag{101}$$

Using the Adomian polynomials we obtain;

$$\begin{aligned} &U_0(x,t) = x, \\ &U_1(x,t) = S^{-1}\left[a\,uS\left[(U_0)_{xx}\right]\right] - S^{-1}[uS[A_0]] = -xt , \\ &U_2(x,t) = S^{-1}\left[a\,uS\left[(U_1)_{xx}\right]\right] - S^{-1}[uS[A_1]] = xt^2, \\ &U_3(x,t) = S^{-1}\left[a\,uS\left[(U_2)_{xx}\right]\right] - S^{-1}[uS[A_2]] = -xt^3. \end{aligned} \tag{102}$$

Summing these iterates gives the series solution;

$$U(x,t) = x\left(1 - t + t^2 - t^3 + \ \dots\right); \tag{103}$$

Consequently, the exact solution is given by;

$$U(x,t) = \frac{x}{1+t}.$$

Example 9: Consider the following nonlinear Schrodinger equation:

$$i\,U_t + U_{xx} - 2|U|^2U = 0; \tag{104}$$

Subject to the initial condition:

$$U(x,0) = e^{ix}. \tag{105}$$

Following the discussion presented above, we find;

$$\begin{aligned}
&U_0(x,t) = e^{ix},\\
&U_1(x,t) = S^{-1}\left[iuS\left[(U_0)_{xx}\right]\right] - S^{-1}[2iuS[A_0]] = -3ite^{ix},\\
&U_2(x,t) = S^{-1}\left[iuS\left[(U_1)_{xx}\right]\right] - S^{-1}[2iuS[A_1]] = \frac{1}{2!}\,(3it)^2e^{ix},\\
&U_3(x,t) = S^{-1}\left[iuS\left[(U_2)_{xx}\right]\right] - S^{-1}[2iuS[A_2]] = -\frac{1}{3!}\,(3it)^3e^{ix}.
\end{aligned} \tag{106}$$

In a few of (106), the series solution is given by;

$$U(x,t) = e^{ix}\left(1 - (3it) + \frac{1}{2!}\,(3it)^2 - \frac{1}{3!}\,(3it)^3 + \ \dots\right); \tag{107}$$

The exact solution is;

$$U(x,t) = e^{i\,(x-3t)}$$

Example 9: Consider the following homogeneous nonlinear KdV equation:

$$U_t - 6UU_x + \ U_{xxx} = 0; \tag{108}$$

Subject to the initial condition;

$$U(x,0) = 6x. \tag{109}$$

Following the discussion presented above, we find a recursive relation;

$$\begin{aligned}
&U_0(x,t) = 6x,\\
&U_{k+1}(x,t) = -S^{-1}\left[uS\left[(U_k)_{xxx}\right]\right] + S^{-1}[6uS[A_k]], k \geq 0.
\end{aligned} \tag{110}$$

That gives the first few the components by;

$$\begin{aligned}
&U_0(x,t) = 6x,\\
&U_1(x,t) = -S^{-1}\left[uS\left[(U_0)_{xxx}\right]\right] + S^{-1}[6uS[A_0]] = 6^3xt,\\
&U_2(x,t) = -S^{-1}\left[b\,uS\left[(U_1)_{xxx}\right]\right] + S^{-1}[auS[A_1]] = 6^5xt^2,\\
&U_3(x,t) = -S^{-1}\left[buS\left[(U_2)_{xxx}\right]\right] + S^{-1}[6uS[A_2]] = 6^7xt^3.
\end{aligned} \tag{111}$$

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In a few of (111), the series solution is given by;

$$U(x,t) = 6x\left(1 + (36t) + \ (36t)^2 + (36t)^3 + \ \ \ldots\right); \tag{112}$$

The exact solution is;

$$U(x,t) = \frac{6x}{1-36t}\ ,|36t| < 1.$$

9. Conclusion

In this chapter, we have combined the Adomian decomposition method and Sumudu transform to solve some of the nonlinear partial differential equations. This method has advantages of converting to the exact or approximate solutions and can easily handle a wide class of nonlinear differential equations.

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Chapter 13

Positive Periodic Solutions for First-Order Difference Equations with Impulses

Mesliza Mohamed, Gafurjan Ibragimov and Seripah Awang Kechil

Abstract

This paper investigates the first-order impulsive difference equations with periodic boundary conditions

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \in J, \ \ n \neq n_k, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad k = 1, 2, \ldots, m, \\ x(0) &= x(T), \end{aligned}$$

where $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$. By using a cone theoretic fixed point theorem, new existence theorems on positive periodic solutions are established. Our main results enrich and complement those available in the literature.

Keywords: Impulse, difference equation, boundary value problem, Green's function, fixed point theorem

1. Introduction

Let $\mathbb{R}$ denote the real numbers and $\mathbb{R}^+$ the positive real numbers. Let $J = [0, T] = \{0, 1, \cdots, T\}$. In this paper we investigate the existence of positive periodic solutions for nonlinear impulsive difference equations

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \in J, \ \ n \neq n_k, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad k = 1, 2, \ldots, m, \\ x(0) &= x(T), \end{aligned} \tag{1}$$

where Δ denotes the forward difference operators, i.e., $\Delta x_n = x_{n+1} - x_n$, $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$.

Boundary value problems for difference equations and impulsive differential equation have been widely received attentions from many authors (see [1–12]) and reference therein. However, as far as we know, the theory of difference equation for boundary value problems (BVPs) with impulses is rather less, there are still lots of work and research that should be done. In [3], He and Zhang obtained the criteria on the existence of minimal and maximal solutions of (1) by using the method of upper and lower solutions and monotone iterative technique. The similar techniques were applied in [11] for the problem

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \in J, \quad n \neq n_k, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad k = 1, 2, \cdots, p, \\ Mx(0) &- Nx(T) = C, \end{aligned} \tag{2}$$

where $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$. Motivated by the work of [3, 13] we establish the existence of positive periodic solutions for (1) by using the fixed point-theorem in cones following the ideas of [13]. The results herein improve some of the results in [3, 11].

Throughout the paper, we make the following assumptions:

1. $0 < M < 1,\ 0 < L_k < M, f(n, x(n)) + Mx(n) \geq 0$.

2. $(M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k) \geq 0$.

This paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, by applying the fixed point theorem in cones, we obtain some new existence theorems for the impulsive difference equations with periodic boundary conditions.

2. Preliminaries

Let

$$E = \{x : J \to \mathbb{R} : x(0) = x(T)\}$$

with the norm $\|x\| = \max_{n \in J} |x(n)|$. Then E is a Banach space.

Consider the following linear impulsive difference equations with periodic boundary condition

$$\begin{aligned} \Delta x(n) + Mx(n) &= \sigma(n), \quad n \in J, n \neq n_k, \\ \Delta x(n_k) = -L_k x(n_k) &+ I_k(x(n_k)) + L_k x(n_k), \quad k = 1, 2, \ldots, m, \\ x(0 &= x(T), \end{aligned} \tag{3}$$

where M is a constant, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\sigma \in C(J, \mathbb{R}^+)$.

The following lemma transforms the analysis of PBVP (3) to the analysis of summation equation. We denote $G(n,j)$, the Green's function of the problem (3).

Lemma 1. *Let $(A1)$ and $(A2)$ hold, $\sigma \in C(J, \mathbb{R}^+)$. Then $x \in E$ is a solution of PBVP (3) if and only if x is a solution of the following impulsive summation equation*

$$\begin{aligned} x(n) = &\sum_{j=0, j \neq n_k}^{T-1} G(n,j)\sigma(j) \\ &+ \sum_{0 < n_k \leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + +L_k x(n_k)), \end{aligned} \tag{4}$$

where

$$G(n,j) = \frac{1}{1 - (1 - M)^T} \begin{cases} \dfrac{(1 - M)^n}{(1 - M)^{j+1}}, & 0 \leq j \leq n - 1, \\ \dfrac{(1 - M)^{T+n}}{(1 - M)^{j+1}}, & n \leq j \leq T - 1. \end{cases} \tag{5}$$

Proof For convenience, we give the proof for the corresponding linear case (3). Consider first the homogeneous equation

$$\Delta u(n) + Mu(n) = 0, \qquad n \in J, \ n \neq n_k,$$

which is easily solved by iteration. We have

$$u(n) = (1-M)^n,$$

with $u(0) = 1$. Now the first Eq. (3) can be solved by substituting $x(n) = u(n)y(n)$ into Eq. (3), where y is to be determined:

$$u(n+1)y(n+1) - u(n)y(n)(1-M) = \sigma(n)$$

or

$$\Delta y(n) = \frac{\sigma(n)}{Eu(n)}, n \neq n_k.$$

So from (3), we see that $y(n)$ satisfies

$$\begin{gathered} \Delta y(n) = \frac{\sigma(n)}{(1-M)^{n+1}}, n \neq n_k, \\ \Delta y(n_k) = \frac{M-L_k}{1-M} y(n_k) + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}}, \ \ k = 1, 2, \dots, m, \\ y(0) = y(T)(1-M)^T. \end{gathered} \tag{6}$$

From (6), we have

$$\begin{aligned} y(n) = y(0) + & \sum_{j=0, j \neq n_k}^{n-1} \frac{\sigma(j)}{(1-M)^{j+1}} \\ & + \sum_{0 < n_k \le n-1} \left(\frac{M-L_k}{1-M} y(n_k) + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}} \right). \end{aligned} \tag{7}$$

Letting $n = T$ in (7), we have

$$\begin{aligned} y(T) = y(0) + & \sum_{j=0, j \neq n_k}^{T-1} \frac{\sigma(j)}{(1-M)^{j+1}} \\ & + \sum_{0 < n_k \le T-1} \left(\frac{M-L_k}{1-M} y(n_k) + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}} \right). \end{aligned} \tag{8}$$

Applying (8) and the boundary condition $y(0) = y(T)(1-M)^T$, we get

$$\begin{aligned} y(0) = \frac{(1-M)^T}{1-(1-M)^T} & \left(\sum_{j=0, j \neq n_k}^{T-1} \frac{\sigma(j)}{(1-M)^{j+1}} + \sum_{0 < n_k \le T-1} \left(\frac{M-L_k}{1-M} y(n_k) \right.\right. \\ & \left.\left. + \frac{I_k(x(n_k)) + L_k x(n_k)}{(1-M)^{n_k+1}} \right) \right). \end{aligned} \tag{9}$$

Substituting (9) into (7) and using $y(n) = \frac{x(n)}{(1-M)^n}, n \in J$, we get the unique solution of (3)

$$x(n) = \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) + \sum_{0<n_k\leq T-1} G(n,n_k)((M-L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))),$$

where $G(n,j)$ is given in (5). The proof is complete. □.
Consider the PBVP (1). To define a cone, we observe that

$$\frac{(1-M)^n}{1-(1-M)^T} \leq |G(n,n_j)| \leq \frac{1}{1-(1-M)^T}.$$

Define

$$A := \frac{(1-M)^n}{1-(1-M)^T},$$
$$B := \frac{1}{1-(1-M)^T}.$$

We denote the cone in E by

$$K = \{x \in E \ , \ n \in [0,T] \text{ and } x(n) \geq \sigma\|x\|\},$$

where $\sigma = \frac{A}{B} \in (0,1)$. Define a mapping $T: E \to E$ by

$$Tx(n) = \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) + \sum_{0<n_k\leq T-1} G(n,n_k)((M-L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))). \tag{10}$$

Since f is continuous, T is also a continuous map. Before starting the main results, we shall give some important lemmas. The next Lemma is essential in obtaining our results.
Lemma 2. *The mapping T maps K into K, i.e $TK \subset K$.*
Proof For any $x \in K$, it is easy to see that $Tx \in E$. From (10) we have

$$\begin{aligned} Tx(n) &= \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\ &\quad + \sum_{0<n_k\leq T-1} G(n,n_k)((M-L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\ &\leq \sum_{j=0, j\neq n_k}^{T-1} B\sigma(j) + \sum_{0<n_k\leq T-1} B((M-L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))). \end{aligned}$$

Noting that

$$G(n,j)\sigma(j) \geq 0, \quad G(n,n_k)((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k)) \geq 0.$$

We can also obtain

$$\begin{aligned} Tx(n) = & \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\ & + \sum_{0<n_k\leq T-1} G(n,n_k)((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k))) \\ & \geq \sum_{j=0,j\neq n_k}^{T-1} A\sigma(j) + \sum_{0<n_k\leq T-1} A((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k))) \\ & \geq \frac{A}{B}|Tx(n)| \\ & \geq \sigma\|Tx\|. \end{aligned}$$

Hence $TK \subset K$. The proof is complete. □.

The following lemma is crucial to prove our main results.

Lemma 3. *(Guo–Krasnoselskii's fixed point theorem [14])* Let E be a Banach space and let $K \subset E$ be a cone E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let

$$T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either.

i. $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$; or

ii. $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

We are now in a position to apply the preceding results to obtain the existence of positive periodic solutions to (1).

3. Main results

In this section, we state and prove our main findings.

Theorem 1. *Suppose that the following conditions hold*

$$lim_{x\to 0^+} \frac{f(n,x)}{x} = -M, \qquad lim_{x\to 0^+} \sum_{k=1}^{m} \frac{I_k(x)}{x} = 0, \tag{11}$$

$$lim_{x\to +\infty} \frac{f(n,x)}{x} = \infty, \qquad lim_{x\to +\infty} \sum_{k=1}^{m} \frac{I_k(x)}{x} = \infty. \tag{12}$$

Then the problem (1) has at least one positive periodic solution.

Proof $0 < r < R < \infty$, setting

$$\Omega_1 = \{x \in E : \|x\| < r\}, \quad \Omega_2 = \{x \in E : \|x\| < R\}.$$

We have $0 \in \Omega_1$, $\Omega_1 \subseteq \Omega_2$. It follows from (11) that there exists $r > 0$ so that for any $0 < x \leq r$,

$$f(n, x(n)) \leq c_1 x - Mx, \quad \sum_{k=1}^{m} I_k \leq c_2 x,$$

where c_1, c_2 are positive constants satisfying

$$\sigma B(Tc_1 + (T - m)(M + c_2)) < 1.$$

Therefore for $x \in K$, with $\|x\| = r$,

$$f(n, x(n)) + Mx \leq c_1 x, \quad \sum_{k=1}^{m} I_k \leq c_2 x.$$

Moreover $0 < \sigma\|x\| \leq x(n) \leq \|x\| = r$. Thus

$$\begin{aligned} Tx(n) = & \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\ & + \sum_{0<n_k\leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\ & \leq \sum_{n=0, n\neq n_k}^{T-1} B(f(n, x(n)) + Mx(n)) \\ & + \sum_{0<n_k\leq T-1} B((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\ & \leq TBc_1\sigma\|x\| + (T - m)B\sigma(M - L_k + c_2 + L_k)\|x\| \\ & \leq \sigma B(Tc_1 + (T - m)(M + c_2))\|x\| \\ & \leq \|x\|, \end{aligned}$$

which implies $\|Tx\| \leq \|x\|$ for $\forall x \in K \cap \partial\Omega_1$.

On the other hand (12) yields the existence of $\hat{R} > 0$ such that for any $x \geq \hat{R}$

$$f(n, x(n)) \geq \eta_1 x, \quad \sum_{k=1}^{m} I_k \geq \eta_2 x$$

where $\eta_1, \eta_2 > 0$ are constants large enough such that

$$A\sigma(T(\eta_1 + M) + (T - m)(M + \eta_2)) > 1.$$

Fixing $R \geq \max\left\{r, \frac{\hat{R}}{\sigma}\right\}$. and letting $x \in K$ with $\|x\| = R$, we get $x(n) \geq \sigma\|x\| = \sigma R > \hat{R}$ and $f(n, x(n)) + Mx \geq \eta_1 x + Mx \geq \sigma(\eta_1 + M)\|x\|$.

Thus

$$\begin{aligned} Tx(n) = & \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\ & + \sum_{0<n_k\leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\ & \geq \sum_{n=0, n\neq n_k}^{T-1} A(f(n, x(n)) + Mx(n)) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0<n_k\le T-1} A((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k))) \\
&\ge TA\sigma(\eta_1+M)\|x\| + A(T-m)\sigma(M-L_k+\eta_2+L_k)\|x\| \\
&\ge A\sigma(T(\eta_1+M)+(T-m)(M+\eta_2))\|x\| \\
&\ge \|x\|.
\end{aligned}
$$

In particular $\|Tx\|\ge\|x\|$, $\forall x\in K\cap\partial\Omega_2$.
Consequently by Lemma 3(i), T has a fixed point in

$$K\cap(\bar{\Omega}_2\backslash\{\Omega_1\}),$$

which is a positive periodic solution of (1). The proof is complete. $\square$.
Theorem 2. *Suppose that the following conditions hold*

$$lim_{x\to0^+}\frac{f(n,x)}{x}=\infty,\qquad lim_{x\to0^+}\sum_{k=1}^{m}\frac{I_k(x)}{x}=\infty, \tag{13}$$

$$lim_{x\to+\infty}\frac{f(n,x)}{x}=-M,\qquad lim_{x\to+\infty}\sum_{k=1}^{m}\frac{I_k(x)}{x}=0. \tag{14}$$

Then the problem (1) has at least one positive periodic solution.
Proof We follow the same strategy and notations as in the proof of Theorem 1. Firstly, we show that for $r>0$ sufficiently large

$$\|Tx\|\ge\ \|x\|,\ \ \forall x\in K\cap\Omega_1.$$

From (13) it follows that there exists $0<x<\hat{r}$, where β_1,β_2 are constants large enough such that $\sigma A(T(\beta_1+M)+(T-m)(M+\beta_2))>1$. Therefore, for $0<x<\hat{r}$, if $x\in K$ and $\|x\|=r$, then from (12),

$$f(n,x(n))+Mx\ge\beta_1x+Mx\ge\sigma(\beta_1+M)\|x\|,$$

$$\sum_{k=1}^{m}I_k\ge\beta_2x\ge\sigma\beta_2\|x\|.$$

Furthermore, we obtain

$$
\begin{aligned}
Tx(n) = &\sum_{j=0,j\ne n_k}^{T-1} G(n,j)\sigma(j) \\
&+ \sum_{0<n_k\le T-1} G(n,n_k)((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k))) \\
&\ge \sum_{n=0,n\ne n_k}^{T-1} A(f(n,x(n))+Mx(n)) \\
&+ \sum_{0<n_k\le T-1} A((M-L_k)x(n_k)+I_k(x(n_k))+L_kx(n_k))) \\
&\ge TA\sigma(\beta_1+M)\|x\| + (T-m)A\sigma(M-L_k+\beta_2+L_k)\|x\| \\
&\ge \sigma A(T(\beta_1+M)+(T-m)(M+\beta_2))\|x\| \\
&\ge \|x\|,
\end{aligned}
$$

which implies $\|Tx\| \geq \|x\|$, for each $x \in K \cap \partial\Omega_1$.

Next we show that for $R > 0$ sufficiently large, $\|Tx\| \leq \|x\|$, $\forall x \in K \cap \partial\Omega_2$. On the other hand (14) yields the existence of $\hat{R} > 0$ such that for any $x \geq \hat{R}$

$$f(n, x(n)) \leq \eta_1 x - Mx, \quad \sum_{k=1}^{m} I_k \leq \eta_2 x,$$

where $\eta_1, \eta_2 > 0$ are constants such that

$$B\sigma(T\eta_1 + (T - m)(M + \eta_2)) < 1.$$

Fixing $R \geq \max\left\{r, \frac{\hat{R}}{\sigma}\right\}$. and letting $x \in K$ with $\|x\| = R$, we get $x(n) \geq \sigma\|x\| = \sigma R > \hat{R}$ and $f(n, x(n)) + Mx \leq \eta_1 x \leq \eta_1 \sigma \|x\|$ and

$$\begin{aligned}
Tx(n) = & \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\
& + \sum_{0<n_k\leq T-1} G(n, n_k)((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\
& \leq \sum_{n=0, n\neq n_k}^{T-1} B(f(n, x(n)) + Mx(n)) \\
& + \sum_{0<n_k\leq T-1} B((M - L_k)x(n_k) + I_k(x(n_k)) + L_k x(n_k))) \\
& \leq TB\sigma\eta_1\|x\| + B(T - m)\sigma((M - L_k)\|x\| + \eta_2\|x\| + L_k\|x\|) \\
& \leq B\sigma(T\eta_1 + (T - m)(M + \eta_2))\|x\| \\
& \leq \|x\|.
\end{aligned}$$

which implies $\|Tx\| \leq \|x\|$, $\forall x \in K \cap \partial\Omega_2$.

Finally, it follows from Lemma 3(ii) that T has a fixed point in

$$K \cap (\bar{\Omega}_2 \backslash \Omega_1),$$

which is a positive periodic solution of (1). The proof is complete. □

4. Conclusion

By applying the fixed point theorem in cones, we establish new existence theorems on positive periodic solutions for impulsive difference equations with periodic boundary conditions. Our main findings enrich and complement those available in the literature.

Additional classifications

AMS subject classification: 39A10, 34B37

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Chapter 14

Peculiarities of the Fundamental Solution of Parabolic Systems with a Negative Genus

Vladyslav Antonovich Litovchenko

Abstract

For the parabolic Shilov-type systems with a negative genus, a method of studying the properties of a fundamental solution of the Cauchy problem is proposed. This method allows to improve the known estimates of Zhitomirskii fundamental solution for systems with dissipative parabolicity and describe the features of this solution more accurately. It opens wide possibilities for constructing a classical theory of the Cauchy problem for parabolic systems with negative genus and variable coefficients.

Keywords: parabolic Shilov systems, negative genus, fundamental solution, Cauchy problem, matriciant, dissipative parabolicity

1. Introduction

The theory of parabolic equations dates back to the time of the classical equation of thermal conductivity [1]. However, it acquired its most distinct features from the fundamental work by I.G. Petrovskii [2] published in 1938. There he describes and investigates a fairly wide class of systems of linear equations with partial derivatives, the fundamental solution of which has typical properties of the fundamental solution of the thermal conductivity equation:

$$G_{\circ}(t-\tau;x)=\left(\sqrt{4\pi a(t-\tau)}\right)^{-n}e^{-\frac{\|x\|^2}{4a(t-\tau)}},\quad t>\tau\geq 0, x\in\mathbb{R}^n \tag{1}$$

(here a – is the coefficient of thermal conductivity, and $\|\cdot\|$ – is the Euclidean norm in $\mathbb{R}^n$). These systems were later called "parabolic by Petrovskii" or "$2b$-parabolic" systems. Due to the efforts of many researchers, the theory of $2b$-parabolic systems developed rapidly throughout the second half of the 20th century. At that, there were considered the systems with both fixed and variable coefficients having different properties. Comprehensive results were obtained on the structure and properties of solutions, as well as on the correct solvability of boundary value problems, in particular, the Cauchy problem, in different functional spaces [3–13].

In 1955, G.Ye. Shilov formulates a new definition of parabolicity, which generalizes the concept of "$2b$ -parabolicity" and significantly expands the class of Petrovskii's systems with constant coefficients by those systems, in which the order p is no longer necessarily even, and may not coincide with the parabolicity index h [14]. The parabolic Shilov-type systems, mostly with constant coefficients, were studied in [15–24].

The presence of a gap between p and h in such systems produces a peculiar "dissipation" effect, the measure of which may be a special characteristic of the system – its genus μ: $1 - (p - h) \le \mu \le 1$. The parabolic systems, in which $p = h$, – the classical equation of thermal conductivity, in particular, as well as all $2b$-parabolic systems, – have the genus $\mu = 1$, while for the systems with $p \neq h$, generally speaking, the genus is $\mu < 1$. Besides, the more the parabolicity index h deviates from the order of the system p, the more its genus μ, decreasing, gets further away from 1. In systems with such a dissipation, even with constant coefficients, deviations from the standards set by the classical thermal equation are observed. First of all, for their fundamental solution $G(t, \tau; \cdot)$, the analytic properties in the complex space $\mathbb{C}^n$ [15] are getting worse, and the order of exponential behavior on the real hyperplane $\mathbb{R}^n$ changes [16]:

$$|\partial_x^k G(t, \tau; x)| \le A_k (t - \tau)^{-\frac{n+\gamma+|k|_+}{h}} \begin{cases} e^{-\delta_0 \left(\frac{\|x\|}{(t-\tau)^{\mu/p}}\right)^{\frac{p}{p-\mu}}}, & 0 < \mu \le 1, \\ e^{-\delta_0 \left(\frac{\|x\|}{(t-\tau)^{\mu/h}}\right)^{\frac{h}{h-\mu}}}, & \mu \le 0, \end{cases} \quad \gamma \ge 0. \qquad (2)$$

Another anomalous phenomenon of the systems with "dissipative parabolicity" is their parabolic instability with respect to changes in the coefficients, even of those found at zero derivative. This fact was first pointed out by U Hou-Sin in 1960, who gave the example of a parabolically unstable system [17]. In this regard, the question of the study of parabolic Shilov-type systems with variable coefficients is problematic and still remains open.

Zhitomirskii's estimates (2) show that the fundamental solution of $G(t, \tau; x)$ parabolic systems with the positive genus μ on the set $(\tau; +\infty) \times \mathbb{R}^n$ shows the behavior typical for $G_\circ(t - \tau; x)$: it decreases exponentially and has a peculiarity at only one point $(t; x) = (\tau; 0)$. This fact allowed us to successfully develop the classical theory of the Cauchy problem for parabolic systems with variable coefficients and non-negative genus μ in [25–28]. However, according to these estimates, in the case of $\mu < 0$ the function $G(t, \tau; x)$ may have a peculiarity on the entire hyperplane $t = \tau$, $x \in \mathbb{R}^n$. This point significantly complicates the substantiation of the convergence of the process of successive approximations, in particular, while making the fundamental solution of the Cauchy problem for systems with variable coefficients using the Levy method. In this regard, a natural question arises: How accurate are the estimates (2) for systems of the genus $\mu < 0$?

The answer to this question is given in this paper. A method for studying the function $G(t, \tau; x)$ for parabolic Shilov-type systems of genus $\mu < 0$, which allows us to more accurately describe the behavior of this function in the vicinity of the point $(t; x) = (\tau; 0)$ is also suggested in this research paper. In addition, one class of systems with dissipative parabolicity is also defined here. These systems are parabolically stable to changes in their lower coefficients.

The main content of the work is as follows. Section 2 contains the necessary information on the concept of parabolicity by Shilov. One class of systems with dissipative parabolicity and variable coefficients is described in Section 3. The study of the properties of the fundamental solution of the Cauchy problem for parabolic Shilov-type systems with a negative genus is carried out in Section 4. The final Section 5 is the conclusions.

2. Preliminary information

Let $\mathbb{N}$ – be the set of all natural numbers; $\mathbb{N}_m = \{1, \ldots; m\}$; $\mathbb{R}^n$ and $\mathbb{C}^n$ – real and complex space of $n \ge 1$ dimension respectively; $\mathbb{Z}_+^n$ – the set of all n-dimensional multi-indices; $\mathbb{R} := \mathbb{R}^1, \mathbb{C} := \mathbb{C}^1$, $\mathbb{Z}_+ := \mathbb{Z}_+^1$; i – imaginary unit; $(\cdot, \cdot)$ – scalar product in the space $\mathbb{R}^n$; $|x + iy| := (x^2 + y^2)^{\frac{1}{2}}$, if $\{x, y\} \subset \mathbb{R}$; $z^l := z_1^{l_1} \dots z_n^{l_n}$, $|z|^l := |z_1|^{l_1} \dots |z_n|^{l_n}$, $|z|_+^h := |z_1|^h + \dots + |z_n|^h$, $|z|_+ := |z|_+^1$, if $z := (z_1; \dots; z_n) \in \mathbb{C}^n$, $l := (l_1; \dots; l_n) \in \mathbb{Z}_+^n$, $h \in \mathbb{R}$; $\partial_\xi \cdot$ – is the partial derivative with the variable ξ.

Let us fix $\{m, p\} \subset \mathbb{N}$, $T \in (0; +\infty)$ arbitrarily and consider the system of partial differential equations of p order

$$\partial_t u(t; x) = A(t; i\partial_x) u(t; x), \ \ (t; x) \in \Pi_{(0; T]}, \tag{3}$$

in which $\Pi_{(0; T]} := (0; T] \times \mathbb{R}^n$, $u(t; x) := \mathrm{col}(u_1(t; x); \dots; u_m(t; x))$ – is an unknown vector-function and

$$A(t; i\partial_x) := \left(\sum_{|k|_+ \le p} a_k^{jl}(t) i^{|k|_+} \partial_x^k \right)_{j,l=1}^m \tag{4}$$

matrix differential expression with coefficients $a_k^{jl}(\cdot)$.

Let us denote by $\mathcal{A}$ the matrix symbol of the differential expression $A(t; i\partial_x)$:

$$\mathcal{A}(t; s) = \left(\sum_{|k|_+ \le p} a_k^{jl}(t) s^k \right)_{j,l=1}^m, t \in (0; T], s \in \mathbb{C}^n. \tag{5}$$

The Shilov-type parabolicity of the system (3) depending on the constancy or variability of its coefficients, is defined differently.

In the case when the coefficients a_k^{jl} are constant, i.e., when

$$A(t; i\partial_x) \equiv A(i\partial_x), \quad \mathcal{A}(t; \cdot) \equiv \mathcal{A}(\cdot), \tag{6}$$

the system (3) on the set $\Pi_{[0; T]}$ is referred to as *Shilov-type parabolic* system with the parabolicity index h, $0 < h \le p$, if [15]

$$\exists \delta_0 > 0 \exists \delta \ge 0 \forall \xi \in \mathbb{R}^n : \qquad \max_{j \in \mathbb{N}_m} \mathrm{Re}\, \lambda_j(\xi) \le -\delta_0 \|\xi\|^h + \delta, \tag{7}$$

where $\lambda_j(s)$ - characteristic numbers of the matrix symbol $\mathcal{A}(s)$, $s \in \mathbb{C}^n$.

If the coefficients of the system (3) depend on t (continuously), then the Shilov-type parabolicity of this system is defined somewhat differently, using the concept of the matriciant of the linear differential equations system.

For the system (3) we shall write the corresponding dual by Fourier system

$$\partial_t v(t; \xi) = \mathcal{A}(t; \xi) v(t; \xi), \ \ 0 \le \tau < t \le T, \xi \in \mathbb{R}^n. \tag{8}$$

The *matriciant of the system* (8) is such a matrix solution of the system $\Theta_\tau^t(\cdot)$, $0 \le \tau < t \le T$, that

$$\Theta_\tau^t(\cdot)\big|_{t=\tau} = E \quad (\forall \tau \in [0; T]) \tag{9}$$

(here E – a single matrix of m order).

Under the condition of continuity of the coefficients of the system (3), the matriciant $\Theta_\tau^t(\cdot)$ has the structure [29]

$$\Theta_\tau^t(\cdot) = E + \sum_{r=1}^{\infty} \int_\tau^t \int_\tau^{t_1} \dots \int_\tau^{t_{r-1}} \left(\prod_{j=1}^{r} \mathcal{A}(t_j;\cdot) \right) dt_r \dots dt_2 dt_1. \tag{10}$$

The system (3) with continuous coefficients on $[0;T]$ is called a *Shilov-type parabolic* system on the set $\Pi_{[0;T]}$ with parabolicity index h, $0 < h \le p$, if for the matriciant $\Theta_\tau^t(\cdot)$, $0 \le \tau < t \le T$, of the corresponding dual by Fourier system (8) the following estimation is performed [15]

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma) e^{-\delta(t-\tau)\|\xi\|^h}, \quad (t;\xi) \in \Pi_{(\tau;T]}, \tag{11}$$

with some positive constants c and δ. Here

$$\gamma := (p-h)(m-1), \quad |(a_{jl})_{j=1,l=1}^{k,m}| := \max_{jl} |a_{jl}|. \tag{12}$$

It should be noted that for Shilov-type parabolic systems with constant coefficients, the condition (11) is a direct consequence of the corresponding condition of parabolicity (7) [15]. For parabolic systems (3) with t-dependent coefficients at $p \ne h$, this fact generally cannot be confirmed by classical means of the theory of parabolic systems due to the parabolic instability of such systems to changing their coefficients.

The Eq. (10) allows us to extend the matriciant $\Theta_\tau^t(\cdot)$ into the complex space $\mathbb{C}^n$ to the complete analytical function. Taking into account the estimation

$$|\mathcal{A}(t;s)| \le c(1 + \|s\|^p), \ \ 0 \le t \le T, s \in \mathbb{C}^n, \tag{13}$$

we find that

$$|\Theta_\tau^t(s)| \le c_0 e^{\delta_0 (t-\tau)\|s\|^p}, \ \ 0 \le \tau < t \le T, s \in \mathbb{C}^n \tag{14}$$

(here, a c_0 and δ_0 are positive constants independent of τ, t and s).

The smoothness of the matriciant $\Theta_\tau^t(\cdot)$ together with the estimates (11), (14), according to the statement of the theorem of the Phragmén-Lindelöf type [30, p. 247], ensure the existence of the area

$$\mathbb{K}_\nu = \{\xi + i\eta \in \mathbb{C}^n : \|\eta\| \le K(1 + \|\xi\|)^\nu\} \tag{15}$$

from ν with $[1-(p-h);1]$, in which the following estimate is performed

$$|\Theta_\tau^t(\xi + i\eta)| \le c_1(1 + \|\xi\|^\gamma) e^{-\delta_1(t-\tau)\|\xi\|^h}, \quad 0 \le \tau < t \le T. \tag{16}$$

The *genus* μ of the Shilov-type parabolic system (3) is the exact upper boundary of the indices ν, with which in the domain $\mathbb{K}_\nu$ for the matriciant $\Theta_\tau^t(\cdot)$ the estimate (16) is performed [15]

Similarly to $2b$-parabolicity, it is convenient to call the Shilov-type parabolicity a *$\{p,h\}$-parabolicity*.

It should be noted that the fundamental solution of the Cauchy problem for $\{p,h\}$-parabolic system (3) is represented by the function [15]

$$G(t,\tau;x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(x,\xi)} \Theta_\tau^t(\xi) d\xi. \tag{17}$$

The following section gives an example of a $\{p, h\}$-parabolic system and defines a class of systems with dissipative parabolicity, each of which is a $\{p, h\}$-parabolic system with variable coefficients.

3. One class of parabolically resistant systems

Due to the difficulty of establishing the fundamental condition (11), for the system (3) with variable coefficients, the definition of parabolability according to Shilov is somewhat specific. It is known [4] that the corresponding condition (11) is satisfied for $2b$-parabolic systems (3) with continuous coefficients. However, it is impossible to confirm the fulfillment of this condition in a similar way for systems (3) with $p \neq h$ based on the condition (7). Therefore, it is important to be aware of the richness of the class of the Shilov-type systems with variable coefficients, in particular, of the examples of such systems that are not parabolic by Petrovskii.

Let us consider a system of Eq. (3), in which the differential expression $A(t; i\partial_x)$ allows an image

$$A(t; i\partial_x) = A_0(i\partial_x) + A_1(t; i\partial_x), \tag{18}$$

where

$$A_0(i\partial_x) := \left(\sum_{|k|_+ \leq p} a_k^{lj} i^{|k|_+} \partial_x^k \right)_{l,j=1}^{m}, \quad A_1(t; i\partial_x) := \left(\sum_{|k|_+ \leq p_1} a_k^{lj}(t) i^{|k|_+} \partial_x^k \right)_{l,j=1}^{m}. \tag{19}$$

Let us assume that the corresponding system

$$\partial_t u(t; x) = A_0(i\partial_x) u(t; x), \quad (t; x) \in \Pi_{(0; T]}, \tag{20}$$

is $\{p, h\}$-parabolic on the set $\Pi_{(\tau; T]}$, and the coefficients of the differential expression $A_1(t; i\partial_x)$ are continuous complex-valued functions defined on $[0; T]$, while the values p, p_1 and h satisfy the condition

(A): $\quad 0 \leq p_1 + (p - h)(m - 1) < h$.

Example of system (3) with condition (A). Let $n = 1$, $m = 2$, $a > 0$ and $c_j(\cdot)$, $j \in \mathbb{N}_5$, are some continuous on $[0; T]$ complex-valued functions. Then the system

$$\begin{cases} \partial_t u_1 = \{-a\partial_x^4 + c_1(t)\partial_x^2\} u_1 + \{\partial_x^5 - \partial_x^3 + c_2(t)\partial_x\} u_2, \\ \partial_t u_2 = \{c_3(t)\partial_x^2 - \partial_x^3\} u_1 - \{a\partial_x^4 - c_4(t)\partial_x^2 - c_5(t)\} u_2, \end{cases} \tag{21}$$

is the system of kind (3) with condition (A). Indeed, putting

$$A_0(i\partial_x) = \begin{pmatrix} -a\partial_x^4 & \partial_x^5 - \partial_x^3 \\ -\partial_x^3 & -a\partial_x^4 \end{pmatrix}, \tag{22}$$

$$A_1(t; i\partial_x) = \begin{pmatrix} c_1(t)\partial_x^2 & c_2(t)\partial_x \\ c_3(t)\partial_x^2 & c_4(t)\partial_x^2 + c_5(t) \end{pmatrix} \tag{23}$$

and solving the appropriate equation

$$\det(\mathcal{A}_0(s) - \lambda E) = 0, \quad s \in \mathbb{C}^n, \tag{24}$$

we obtain that $\lambda_{1,2}(s) = -as^4 \pm i\sqrt{s^8 + s^6}$, $p = 5$, $p_1 = 2$ and $h = 4$. For these values p, p_1 and h, obviously the condition (A) holds.

Theorem 1 *Let (3) be a system with continuous coefficients, for which the conditions formulated in this clause are satisfied. Then it is an $\{p,h\}$-parabolic system with variable coefficients.*

Proof. According to the definition of $\{p,h\}$-parabolicity for the system (3) with variable coefficients, it is enough to show that for the matrix $\Theta_\tau^t(\cdot)$ of the corresponding dual by Fourier system (8) on the set $\Pi_{[\tau;T]}$, $\tau \in [0;T)$, the estimate (11) is performed.

On condition of continuity of the coefficients, the matriciant $\Theta_\tau^t(\cdot)$ is the only solution of the Cauchy problem for the system (8) with the initial condition

$$v(t;\cdot)|_{t=\tau} = E. \tag{25}$$

Thus, the correct equality

$$\partial_t \Theta_\tau^t(\xi) = A_0(\xi)\Theta_\tau^t(\xi) + Q(\tau,t;\xi), \tag{26}$$

in which

$$Q(\tau,t;\xi) := A_1(t;\xi)\Theta_\tau^t(\xi). \tag{27}$$

Having solved the Cauchy problem (26), (25), we obtain the image

$$\Theta_\tau^t(\xi) = e^{(t-\tau)P_0(\xi)} + \int_\tau^t e^{(t-\beta)P_0(\xi)} Q(\tau,\beta;\xi)d\beta, \quad (t;\xi) \in \Pi_{(\tau;T]}, \quad \tau \in [0;T). \tag{28}$$

It should be noted that $e^{(t-\tau)P_0(\cdot)}$ is the matriciant of the dual by Fourier system to $\{p,h\}$-parabolic system (20), therefore, the estimate (11) is performed for it. Hence, considering the inequality

$$|Q(\tau,t;\xi)| \le c_0(1 + \|\xi\|^{p_1})|\Theta_\tau^t(\xi)|, \quad (t;\xi) \in \Pi_{(\tau;T]}, \quad \tau \in [0;T) \tag{29}$$

(here the positive constant c_0 in independent of τ, t and ξ), the next estimate is obtained

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma)e^{-\delta(t-\tau)\|\xi\|^h} + c_1(1 + \|\xi\|^\gamma)(1 + \|\xi\|^{p_1}) \int_\tau^t e^{-\delta(t-\beta)\|\xi\|^h} |\Theta_\tau^\beta(\xi)| d\beta, \tag{30}$$

from which we come to the ratio

$$\frac{|\Theta_\tau^t(\xi)| e^{\delta(t-\tau)\|\xi\|^h}}{(1 + \|\xi\|^\gamma)} \le c + c_1(1 + \|\xi\|^\gamma)(1 + \|\xi\|^{p_1}) \int_\tau^t \frac{|\Theta_\tau^\beta(\xi)| e^{\delta(\beta-\tau)\|\xi\|^h}}{(1 + \|\xi\|^\gamma)} d\beta. \tag{31}$$

Using now the classic Grönwall's lemma [4], we get

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma)e^{-(t-\tau)\left(\delta\|\xi\|^h - c_1(1+\|\xi\|^\gamma)(1+\|\xi\|^{p_1})\right)}, \quad (t;\xi) \in \Pi_{(\tau;T]}, \quad \tau \in [0;T). \tag{32}$$

This inequality, in combination with condition (A), ensures the existence of positive constants c and δ, with which for all $(t;\xi) \in \Pi_{(\tau;T]}$, $\tau \in [0;T)$, the estimate (11) is performed.

The theorem is proved.

Remark 1 *The proof of Theorem 1 is based on the classical idea of establishing an estimate (11) for 2b-parabolic systems with the coefficients continuously depending on t. Therefore, analyzing this proof, especially its last part, we can understand why, in contrast to the 2b-parabolicity, in the case of* $p \neq h$ *the difficulties in establishing the condition (11).*

The study of the properties of the matriciant $\Theta_\tau^t(\cdot)$ for systems with a negative genus μ will be continued in the next section.

4. Properties of fundamental solution

Let us move on to the search for an answer to the question posed in Section 1 concerning the accuracy of Zhitomirskii's estimates (2) in the case of a system (3) of genus $\mu < 0$.

Theorem 2 *Let the system (3)* $\{p, h\}$ *be parabolic with the negative genus* μ*, and let* $l \geq 0$ *and* $\alpha \geq 0$ *be such arbitrarily fixed numbers that* $l \leq 1 + \alpha h$ *and* $(\alpha h - l)\mu \geq \alpha h$*. Then*

$$\exists\{c, \delta, A, B\} \subset (0; +\infty)\forall k \in \mathbb{Z}_+^n \forall q \in \mathbb{Z}_+ \forall x \in \mathbb{R}^n \setminus \{0\} \forall \tau \in [0; T) \forall t \in (\tau; T]:$$

$$|\partial_x^k G(t, \tau; x)| \leq \frac{cA^q B^{|k|_+}}{\|x\|^q} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}} (t-\tau)^{\frac{(l+\mu)q-n-|k|_+-l_0\gamma}{h}} e^{-\delta\left(\frac{|x|_+}{(t-\tau)^{(l+\mu)/h}}\right)^{\frac{1}{1-\mu/h}}}, \tag{33}$$

where $l_0 := \max\{1; l\}$.

Proof. To simplify the calculations, we put $\tau = 0$. The general case $\tau > 0$ is realized similarly.

Let us consider the functional matrix

$$\mathfrak{I}_l(t; \xi) := \Theta_0^t\left(t^{-l/h}\xi\right), \quad l \geq 0, t \in (0; T], \xi \in \mathbb{R}^n, \tag{34}$$

for which, according to the definition of the genus μ of the system (3), on the set

$$\mathbb{K}_\mu = \left\{\xi + i\eta \in \mathbb{C}^n : t^{-l/h}\|\eta\| \leq K_0\left(1 + t^{-l/h}\|\xi\|\right)^\mu\right\} \tag{35}$$

the estimate is performed

$$|\mathfrak{I}_l(t; \xi + i\eta)| \leq c\left(1 + t^{-l/h}\|\xi\|\right)^\gamma e^{-\delta t^{1-l}|\xi|_+^h}, \quad t \in (0; T], \tag{36}$$

with positive values c and δ, independent of t, ξ and η.

To estimate the derivatives $\partial_\xi^q \mathfrak{I}_l$ we use the Cauchy integral formula

$$\partial_\xi^q \mathfrak{I}_l(t; \xi) = \prod_{j=1}^n \frac{q_j!}{2\pi i} \int_{\Gamma_{R_j}} \frac{\mathfrak{I}_l(t; \sigma) d\sigma_j}{\left(\sigma_j - \xi_j\right)^{q_j+1}}, \quad q \in \mathbb{Z}_+^n, \xi \in \mathbb{R}^n, \ t \in (0; T], \tag{37}$$

in which Γ_{R_j} – is a circle with the center in the point $\xi_j + i0$ of the radius

$$R_j = K_0\left(1 + t^{-l/h}|\xi_j|\right)^\mu, \quad 0 < K_0 << 1. \tag{38}$$

Let us put $\Gamma_R := \Gamma_{R_1} \times \ldots \times \Gamma_{R_n}$ and fix a fairly small positive K_0 so that $\Gamma_R \subset \mathbb{K}_\mu$ (the existence of such K_0 is substantiated in ([30], p. 287) when proving the theorem 4 of the Phragmén-Lindelöf type in the case of n independent variables). Then, according to the estimate (36), we have

$$|\partial_\xi^q \mathfrak{J}_l(t;\xi)| \le c\left(1 + t^{-l/h}\|\hat{\xi}\|\right)^\gamma e^{-\delta t^{1-l}|\check{\xi}|_+^h} \prod_{j=1}^{n} \frac{q_j!}{R_j^{q_j}}, \tag{39}$$

where $\{\hat{\xi};\check{\xi}\} \subset \mathbb{R}^n$ – fixed points with such coordinates

$$\left\{\hat{\xi}_j;\check{\xi}_j\right\} \subset \left[\xi_j - R_j;\xi_j + R_j\right], \quad j \in \mathbb{N}_n, \tag{40}$$

that

$$\hat{\xi}_j^{\,2} = \max_{\left[\xi_j - R_j;\xi_j + R_j\right]} x_j^2, \quad |\check{\xi}_j| = \min_{\left[\xi_j - R_j;\xi_j + R_j\right]} |x_j|, \tag{41}$$

that is

$$\hat{\xi}_j = \xi_j + \chi_j R_j, \quad \check{\xi}_j = \xi_j + \zeta_j R_j, \tag{42}$$

at some $\left\{\chi_j, \zeta_j\right\} \subset [-1;1]$.
First of all it should be noted that

$$R_j = \frac{K_0}{\left(1 + t^{-l/h}|\xi_j|\right)^{|\mu|}} \le K_0, \quad \xi_j \in \mathbb{R}, t \in (0;T]. \tag{43}$$

Since

$$\|\xi\| \le \sqrt{n}|\xi|_+, \quad \xi \in \mathbb{R}^n, \tag{44}$$

then

$$\begin{aligned} \|\hat{\xi}\| &\le \sqrt{n}\sum_{j=1}^{n} |\xi_j + \chi_j R_j| \le \sqrt{n}\sum_{j=1}^{n}\left(|\xi_j| + R_j\right) \le \sqrt{n}\sum_{j=1}^{n}\left(|\xi_j| + K_0\right) \le \\ &\le \sqrt{n}\left(1 + |\xi|_+\right), \quad K_0 \le 1/n, \xi \in \mathbb{R}^n, t \in (0;T]. \end{aligned} \tag{45}$$

Now let us estimate the value $e^{-\delta t^{1-l}|\check{\xi}|_+^h}$.
Let us start with the simpler case when $t \in [1;T]$.
We assume that $|\xi_j| \ge 2K_0$, then

$$\left|\check{\xi}_j\right|^h = \left(|\xi_j| - R_j\right)^h \ge \left(|\xi_j| - K_0\right)^h \ge 2^{-h}\left|\xi_j\right|^h. \tag{46}$$

If $|\xi_j| < 2K_0$, then

$$e^{-\delta t^{1-l}\left|\check{\xi}_j\right|^h} \le 1 = e^{\delta_0 t^{1-l}\left|\xi_j\right|^h} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h} \le e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h + a} \quad (\forall \delta_0 > 0), \tag{47}$$

where $a = \delta_0 (2K_0)^h \max\limits_{t \in [1;T]} t^{1-l}$.

Therefore, for each $\delta > 0$ there are such positive constants c_0 and δ_0 that for all $\xi_j \in \mathbb{R}$ and $t \in [1; T]$ the estimate is performed

$$e^{-\delta t^{1-l}\left|\check{\xi}_j\right|^h} \le c_0 e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h}. \tag{48}$$

We show that the statement (48) is also true in the case of $t \in (0; 1)$.

We shall fix arbitrarily $\alpha \ge 0$ and further consider that $l \le 1 + \alpha h$. Then for $|\xi_j| < t^\alpha$, we have:

$$e^{-\delta t^{1-l}\left|\check{\xi}_j\right|^h} \le e^{\delta_0 t^{1-l}\left|\xi_j\right|^h - \delta_0 t^{1-l}\left|\xi_j\right|^h} \le e^{-\delta_0\left(t^{1-l}\left|\xi_j\right|^h - t^{1+\alpha h - l}\right)} \le e^{-\delta_0\left(t^{1-l}\left|\xi_j\right|^h - 1\right)} \quad (\forall \delta_0 > 0). \tag{49}$$

Now let $t^\alpha \le |\xi_j|$, and α be such that the condition: $(l - \alpha h)|\mu| \ge \alpha h$ is satisfied. Taking into consideration that

$$R_j \le \frac{K_0}{\left(1 + t^{\alpha - l/h}\right)^{|\mu|}} \le K_0 t^{(l/h - \alpha)|\mu|} \le K_0 t^\alpha, \tag{50}$$

we obtain:

$$\begin{aligned} \left|\check{\xi}_j\right|^h &= \left(|\xi_j| - R_j\right)^h \ge \left(|\xi_j| - K_0 t^\alpha\right)^h = \left|\xi_j\right|^h \left(1 - K_0 t^\alpha / |\xi_j|\right)^h \ge \\ &\ge \left|\xi_j\right|^h (1 - K_0)^h \ge 2^{-h}\left|\xi_j\right|^h. \end{aligned} \tag{51}$$

Hence we arrive at performing (48) at $t \in (0; 1)$.

According to the estimates (45), (48) and equality

$$\sup_{y \ge 0}\left\{y^\beta e^{-\delta y}\right\} = \left(\frac{\beta}{e\delta}\right)^\beta, \quad \beta > 0, \delta > 0, \tag{52}$$

we find:

$$\begin{aligned} &c_0^{-1}\left(1 + t^{-l/h}\|\hat{\xi}\|\right)^\gamma e^{-\frac{\delta}{3}t^{1-l}\left|\check{\xi}\right|_+^h} \le \left(2\sqrt{n}\right)^\gamma t^{-l\gamma/h}\left(1 + |\xi|_+\right)^\gamma e^{-\delta_0 t^{1-l}|\xi|_+^h} \le \\ &\le \left(2\sqrt{n}\right)^\gamma t^{-l\gamma/h}\left(1 + |\xi|_+ e^{-\frac{\delta_0}{\gamma}t^{1-l}|\xi|_+^h}\right)^\gamma \le \left(2\sqrt{n}\right)^\gamma t^{-l\gamma/h}\left(1 + n\left(\frac{\gamma t^{l-1}}{he\delta_0}\right)^{1/h}\right)^\gamma; \\ &c_0^{-1}R_j^{-q_j} e^{-\frac{\delta}{3n}t^{1-l}\left|\check{\xi}\right|_+^h} \le R_j^{-q_j} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h} = K_0^{-q_j}\left(1 + t^{-l/h}|\xi_j|\right)^{|\mu|q_j} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h} \le \\ &\le \left(2^\mu K_0\right)^{-q_j}\left(1 + t^{\mu l q_j/h}\left|\xi_j\right|^{|\mu|q_j} e^{-\delta_0 t^{1-l}\left|\xi_j\right|^h}\right) \le \left(2^\mu K_0\right)^{-q_j}\left(1 + \left(\frac{|\mu|q_j}{he\delta_0 t}\right)^{|\mu|q_j/h}\right). \end{aligned} \tag{53}$$

Together with (39), these estimates ensure the existence of such positive constants c, A and δ that for all $\xi \in \mathbb{R}^n$, $t \in (0; T]$ and $q \in \mathbb{Z}_+^n$ the following inequality is true

$$\left|\partial_\xi^q \mathfrak{I}_l(t; \xi)\right| \le c A^{|q|_+} q^{\left(1 - \frac{\mu}{h}\right)q} t^{\frac{\mu|q|_+ - l_0\gamma}{h}} e^{-\delta t^{1-l}|\xi|_+^h}, \tag{54}$$

in which $l_0 = \max\{1; l\}$.

Next, we shall use the image

$$G(t,0;x) = (2\pi)^{-n} t^{-nl/h} \int\limits_{\mathbb{R}^n} e^{-i\left(x, t^{-l/h}\xi\right)} \mathfrak{I}(t;\xi)\, d\xi, \quad (t;x) \in \Pi_{(0;T]}. \tag{55}$$

Identity

$$t^{l/h} L_{\xi;x}\left[e^{-i\left(x, t^{-l/h}\xi\right)}\right] = e^{-i\left(x, t^{-l/h}\xi\right)}, \tag{56}$$

in which

$$L_{\xi;x} = i\|x\|^{-2} \sum_{j=1}^{n} x_j \partial_{\xi_j}, \tag{57}$$

at $x \neq 0$ allows to write the previous equality in the form

$$G(t,0;x) = (2\pi)^{-n} t^{l(q-n)/h} \int\limits_{\mathbb{R}^n} L^q_{\xi;x}\left[e^{-i\left(x, t^{-l/h}\xi\right)}\right] \mathfrak{I}(t;\xi)\, d\xi \quad (\forall q \in \mathbb{Z}_+). \tag{58}$$

Hence, after integrating by parts q times, we arrive at the relation

$$G(t,0;x) = (-1)^q (2\pi)^{-n} t^{l(q-n)/h} \int\limits_{\mathbb{R}^n} e^{-i\left(x, t^{-l/h}\xi\right)} L^q_{\xi;x}[\mathfrak{I}(t;\xi)]\, d\xi \quad (\forall q \in \mathbb{Z}_+), \tag{59}$$

from which we obtain that

$$|x^r \partial_x^k G(t,0;x)| \le (2\pi)^{-n} t^{\frac{l(q-n-|k-r|_+)}{h}} \int\limits_{\mathbb{R}^n} |\xi|^k |\partial_\xi^r \left(L^q_{\xi;x}[\mathfrak{I}(t;\xi)]\right)| d\xi, \tag{60}$$

for all $\{r,k\} \subset \mathbb{Z}^n_+$ and $q \in \mathbb{Z}_+$.
Having considered the estimate (54), for $(t;\xi) \in \Pi_{(0;T]}$ and $x \neq 0$ we find:

$$|\partial_\xi^r \left(L^q_{\xi;x}[\mathfrak{I}(t;\xi)]\right)| \le c A^{q+|r|_+} \|x\|^{-q} t^{\frac{\mu(q+|r|_+)-l_0\gamma}{h}} r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} e^{-\delta t^{1-l}|\xi|^h_+}. \tag{61}$$

Then

$$\begin{aligned} |x^r \partial_x^k G(t,0;x)| &\le c_1 A^{q+|r|_+} \|x\|^{-q} t^{\frac{(l+\mu)(q+|r|_+)-l(n+|k|_+)-l_0\gamma}{h}} r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} \times \\ &\times \int\limits_{\mathbb{R}^n} |\xi|^k e^{-\delta l^{1-l}|\xi|^h_+} d\xi \le c_1 A^{q+|r|_+} \|x\|^{-q} t^{\frac{(l+\mu)(q+|r|_+)-n-|k|_+-l_0\gamma}{h}} r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} \times \\ &\times \left(\prod_{j=1}^{n} \sup_{y \ge 0}\left\{y^{\frac{k_j}{h}} e^{-\frac{\delta}{2}y}\right\}\right) \int\limits_{\mathbb{R}^n} e^{-\frac{\delta}{2}|\zeta^h|_+} d\zeta \le c_2 A^{q+|r|_+} B^{|k|_+} \|x\|^{-q} t^{\frac{(l+\mu)(q+|r|_+)-n-|k|_+-l_0\gamma}{h}} \times \\ &\times r^{\left(1-\frac{\mu}{h}\right)r} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}}, \quad t \in (0;T], x \neq 0, q \in \mathbb{Z}_+, \{k,r\} \subset \mathbb{Z}^n_+ \end{aligned} \tag{62}$$

(here positive values c_2, A and B do not depend on t, x, q, k and r).

Thus, for all $t \in (0;T]$, $x \in \mathbb{R}^n \backslash \{0\}$, $q \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_+^n$ the correct estimates are

$$|\partial_x^k G(t,0;x)| \le c_2 A^q B^{|k|_+} \|x\|^{-q} t^{\frac{(l+\mu)(q+|r|_+)-n-|k|_+-l_0\gamma}{h}} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}} \times$$

$$\times \left(\prod_{j=1}^{n} \inf_{r_j} \left\{ \left(t^{\frac{l+\mu}{h}} A\right)^{r_j} r_j^{r_j\left(1-\frac{\mu}{h}\right)} |x_j|^{-r_j} \right\} \right) \le \tag{63}$$

$$\le c A^q B^{|k|_+} \|x\|^{-q} t^{\frac{(l+\mu)q-n-|k|_+-l_0\gamma}{h}} q^{\left(1-\frac{\mu}{h}\right)q} k^{\frac{k}{h}} e^{-\delta\left(\frac{|x|_+}{t^{(l+\mu)/h}}\right)^{\frac{1}{1-\mu/h}}},$$

in which the values $c > 0$, $A > 0$, $B > 0$ and $\delta > 0$ do not depend on k, q, t and x.

The theorem is proved.

Remark 2 *Zhitomirskii's estimates (2) are obtained from (33) for* $q = 0$, $l = 0$ *and* $\alpha = 0$.

Given that $l = 1 + \alpha h$, $(\alpha h - l)\mu = \alpha h$ and $q = 0$, from the theorem 2 we arrive at the following statement.

Corollary 1 *For* $\{p, h\}$*-parabolic system (3) with genus* $\mu < 0$ *there are such positive constants c, B and* δ *that for all* $k \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$, $\tau \in [0;T)$ *and* $t \in (\tau;T]$ *the next estimate is performed*

$$|\partial_x^k G(t,\tau;x)| \le c B^{|k|_+} k^{\frac{k}{h}} (t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} e^{-\delta\left(\frac{|x|_+}{(t-\tau)^{1/h}}\right)^{\frac{1}{1-\mu/h}}}. \tag{64}$$

Therefore, according to the corollary 1, the fundamental solution G in the case of negative genus μ also has a singularity only at the point $(t;x) = (\tau;0)$.

Corollary 2 *Let (3)* $\{p, h\}$ *be a parabolic system with negative genus* μ*, then for all* $t \in (\tau;T]$, $\tau \in [0;T)$, $x \in \mathbb{R}^n \backslash \{0\}$ *and* $k \in \mathbb{Z}_+^n$ *estimate is performed*

$$|\partial_x^k G(t,\tau;x)| \le \frac{c B^{|k|_+} k^{\frac{k}{h}} |k|_+^{\left(1-\frac{\mu}{h}\right)|k|_+}}{\|x\|^{n+[\gamma]+1+|k|_+}} (t-\tau)^{\frac{1-\{\gamma\}}{h}} e^{-\delta\left(\frac{|x|_+}{(t-\tau)^{1/h}}\right)^{\frac{1}{1-\mu/h}}}, \tag{65}$$

in which the positive values c, δ and B do not depend on t, τ, x and k; $[\cdot]$ and $\{\cdot\}$ are integer and fractional parts of the number respectively.

Proof. Estimates (65) are obtained directly from (33) at $l = 1 + \alpha h$, $(\alpha h - l)\mu = \alpha h$ and $q = n + [\gamma] + 1 + |k|_+$.

The established estimates (65) provide exponential decrease when changing $t \to \tau + 0$ on the set $\mathbb{R}^n \backslash \{0\}$ derivatives of the function $G(t, \tau; \cdot)$ in case $\mu < 0$. Similarly to the case $\mu \ge 0$ considered in [25–28], this will allow us to successfully study the Cauchy problem for wide classes of $\{p, h\}$-parabolic systems (3) with negative genus μ and variable coefficients $a_k^{jl}(t;x)$. Moreover, this will also allow us to describe in a similar way the sets of classical solutions of such systems with generalized limit values f on the initial hyperplane and to study the local behavior of these solutions when changing $t \to \tau + 0$ on that part of $\mathbb{R}^n$ where the functional f has good properties etc.

5. Conclusions

The class of systems with dissipative parabolicity and variable coefficients defined in Section 3 proves that the class of parabolic Shilov-type systems with coefficients $a_k^{jl}(t)$ is quite broad and cannot be confined to the class of $2b$-parabolic systems (3) with continuous coefficients only.

Analyzing the obtained estimates (33) of the fundamental solution of the systems (3) with dissipative parabolicity, we conclude that in the case of the negative genus μ the function $G(t, \tau; x)$ on the set $(\tau; T] \times \mathbb{R}^n$ has only one singular point $(t; x) = (\tau; 0)$. Similarly to the case $\mu \geq 0$, these estimates allow to perform the expansion of the Shilov class $\{p, h\}$-parabolic systems by supplementing it with the systems with negative genus μ and coefficients depending on space variable, and to successfully develop the theory of the Cauchy problem for it using the classical means. Moreover, the estimates (33) open wide possibilities for studying the properties of solutions of parabolic systems of the genus $\mu < 0$ at the approximation of the initial hyperplane.

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Chapter 15

Boundary Element Modeling and Simulation Algorithm for Fractional Bio-Thermomechanical Problems of Anisotropic Soft Tissues

Mohamed Abdelsabour Fahmy

Abstract

The main purpose of this chapter is to propose a novel boundary element modeling and simulation algorithm for solving fractional bio-thermomechanical problems in anisotropic soft tissues. The governing equations are studied on the basis of the thermal wave model of bio-heat transfer (TWMBT) and Biot's theory. These governing equations are solved using the boundary element method (BEM), which is a flexible and effective approach since it deals with more complex shapes of soft tissues and does not need the internal domain to be discretized, also, it has low RAM and CPU usage. The transpose-free quasi-minimal residual (TFQMR) solver are implemented with a dual-threshold incomplete LU factorization technique (ILUT) preconditioner to solve the linear systems arising from BEM. Numerical findings are depicted graphically to illustrate the influence of fractional order parameter on the problem variables and confirm the validity, efficiency and accuracy of the proposed BEM technique.

Keywords: boundary element method, modeling and simulation algorithm, bio-heat transfer, fractional bio-thermomechanical problems, anisotropic soft tissues

1. Introduction

Human body is a complex thermal system, Arsene d'Arsonval and Claude Bernard have shown that the temperature difference between arterial blood and venous blood is due to oxygenation of blood [1]. A large number of research papers in bio-heat transfer over the past few decades have focused on an understanding of the impact of blood flow on the temperature distribution within living tissues. Pennes [2] was the first attempt to explain the temperature distribution in human tissue with blood flow effect. The improvement of numerical models for determination of temperature distribution in living tissues has been a topic of interest for numerous researchers. Askarizadeh and Ahmadikia [3] introduced analytical solutions for the transient Fourier and non-Fourier bio-heat transfer equations. Li et al. [4] studied the bio-thermomechanical interactions within the human skin in the context of generalized thermoelasticity.

Analytical solutions for the current problem [5, 6] are very difficult to obtain, so numerical methods have become the main way for solving these problems [7–10]. The boundary element method (BEM) [11–21] is one of the numerical methods used to solve the current general problem [22–31]. Generally, Laplace-domain fundamental solutions are easier to obtain than time-domain fundamental solutions for engineering and scientific problems [32, 33]. consequently, the BEM is very helpful for problems that did not have time-domain fundamental solutions, because it requires the Laplace-domain fundamental solutions of the problem's governing equations. So, BEM expands the range of engineering problems that can be solved with the classical time-domain BEM.

The main aim of this chapter is to propose a new boundary element fractional model for describing the bio-thermomechanical properties of anisotropic soft tissues. The dual reciprocity boundary element method has been used to solve the TWMBT for obtaining the temperature distribution, and then the BEM has been used to obtain the displacement and stress at each time step. The linear systems from BEM were solved by the TFQMR solver with the ILUT preconditioner which reduces the number of iterations and the total CPU time.

A brief summary of the chapter is as follows: Section 1 introduces the background and provides the readers with the necessary information to books and articles for a better understanding of bio-thermomechanical problems in anisotropic soft tissues Section 2 describes the BEM modeling of the bio-thermomechanical interactions and introduces the partial differential equations that govern its related problems. Section 3 outlines the dual reciprocity boundary element method (DRBEM) for temperature field. Section 4 discusses the convolution quadrature boundary element method (CQBEM) for poro-elastic field. Section 5 presents the new numerical results that describe the bio-thermomechanical problems in anisotropic soft tissues.

2. Formulation of the problem

Consider an anisotropic soft tissue in the Cartesian coordinate system $Ox_1x_2x_3$ as shown in **Figure 1**. It occupies the region $\Omega = \left\{(x_1, x_2, x_3) : 0 < x_1 < \underline{\alpha}, 0 < x_2 < \underline{\beta}, 0 < x_3 < \underline{\gamma}\right\}$ with boundary Γ that is subdivided into two non-intersective parts Γ_D and Γ_N.

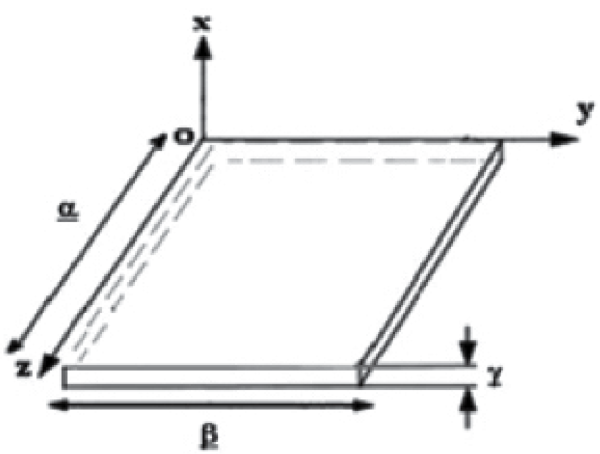

Figure 1.
Geometry of the current problem.

The governing equations which model the fractional bio-thermomechanical problems in anisotropic soft tissues can be written as follows [34, 35].

$$\left(\nabla^T\sigma\right)^T + F = \rho\ddot{u} + \phi\rho_f\left(\ddot{u}_f - \ddot{u}\right) \tag{1}$$

$$\dot{\zeta} + \nabla^T q = 0 \tag{2}$$

$$\sigma = \left(C_{ailg}\,\mathrm{tr}\in - Ap\right)I - \mathfrak{B}\theta, \tag{3}$$

$$\in = \frac{1}{2}\left(\nabla u^T + \left(\nabla u^T\right)^T\right) \tag{4}$$

$$\zeta = A\ \mathrm{tr}\in + \frac{\phi^2}{R}P \tag{5}$$

where the fluid was modeled by the following Darcy's law [36].

$$q = -K\left(\nabla p + \rho_f\ddot{u} + \frac{\rho_a + \phi\rho_f}{\phi}\left(\ddot{u}_f - \ddot{u}\right)\right) \tag{6}$$

The fractional order equation which describes the TWMBT can be expressed as [37].

$$\begin{aligned}&\nabla\mathcal{K}\nabla\mathcal{T}(r,\tau) + W_bC_b(\mathcal{T}_b - \mathcal{T}) + Q_{met} + Q_{ext} + \frac{\overline{\tau}}{\alpha!}\left(-W_bC_bD_\tau^\alpha\mathcal{T} + D_\tau^\alpha Q_{met} + D_\tau^\alpha Q_{ext}\right)\\ &= \rho C\left[\frac{\overline{\tau}}{\alpha!}D_\tau^{\alpha+1}\mathcal{T} + \frac{\partial\mathcal{T}}{\partial\tau}\right], 0<\alpha\leq 1\end{aligned} \tag{7}$$

where σ, $\in$, C_{ajlg}, $\rho = \rho_s(1-\phi) + \phi\rho_f$, ρ_s, ρ_f, u, u_f, FF and qq are total stress tensor, linear strain tensor, constant elastic moduli, bulk density, solid density, fluid density, solid displacement, fluid displacement, bulk body forces and specific flux of the fluid, respectively, $\mathfrak{B}$ are stress-temperature coefficients, tr denotes the trace, $A = \phi(1 + Q/R)$ is Biot's coefficient, Q and R are the solid–fluid coupling parameters, p is the fluid pressure in the vasculature, ζ is the fluid volume variation measured in unit reference volume, $\phi = \frac{V^f}{V}$ is the porosity, V^f is the fluid volume, $V = V^f + V^s$ is the bulk volume, V^s is the solid volume, τ is the time, K is the permeability, $\rho_a = \mathbb{C}\phi\rho_f$ where $\mathbb{C} = 0.66$ at low frequency [38], $\boldsymbol{K}$ is the soft tissue thermal conductivity, W_b is the blood perfusion rate, C_b is the blood specific heat, T_b is the arterial blood temperature, T is the soft tissue temperature, $\overline{\tau}$ is the thermal relaxation time ρ is the soft tissue density, C is the soft tissue specific heat, Q_{met} is the metabolic heat generation and Q_{ext} is the external heat generation.

According to Bonnet [39], our problem can be expressed as a matrix system as [40].

$$\left.\begin{aligned}&\hat{B}_{\tilde{x}}\hat{u}^g(\tilde{x}) = 0 \text{ for } \tilde{x}\in\Omega\\ &\hat{u}^g(x) = \hat{g}_D \text{ for } x\in\Gamma_D\\ &\hat{t}^g(x) = \hat{g}_N \text{ for } x\in\Gamma_N\end{aligned}\right\} \tag{8}$$

where

$$\hat{B}_{\tilde{x}} = \begin{bmatrix} B_{\tilde{x}}^e + s^2\left(\rho - \beta\rho_f\right)I & (\alpha-\beta)\nabla_{\tilde{x}} & -\mathfrak{B}\nabla_{\tilde{x}}\\ s(\alpha-\beta)\nabla_{\tilde{x}}^T & -\frac{\beta}{s\rho_f}\Delta_{\tilde{x}} + \frac{s\phi^2}{R} & 0\end{bmatrix} \tag{9}$$

$$\hat{t}^{g}(x) = \begin{bmatrix} T_x^e & -\alpha n_x & 0 \\ s\beta n_x^T & \dfrac{\beta}{s\rho_f} n_x^T \nabla_x & 0 \end{bmatrix} \begin{bmatrix} \hat{u}(x) \\ \hat{p}(x) \\ \theta(x) \end{bmatrix}, \quad \beta = \frac{\phi^2 s K \rho_f}{\phi^2 + sK\left(\rho_a + \phi\rho_f\right)} \tag{10}$$

3. Boundary element implementation for bioheat transfer field

Through this chapter, we supposed that Q_{met} and $\mathcal{T}_b$ are constants and $\theta(r,\tau) = \mathcal{T}(r,\tau) - \mathcal{T}(r,0)$. Thus, Eq. (7) can be written as

$$\rho C \frac{\overline{\tau}}{\alpha!} D_\tau^{\alpha+1}\theta + \rho C \frac{\partial \theta}{\partial \tau} + \frac{\overline{\tau}}{\alpha!} W_b C_b D_\tau^{\alpha}\theta + W_b C_b \theta = \mathcal{K}\frac{\partial^2 \theta}{\partial x^2} + q, \quad 0 < \alpha \leq 1 \tag{11}$$

According to finite difference scheme of Caputo [22] and using the fundamental solution of difference equation resulting from fractional bio-heat Eq. (11) [41], we can write the following dual reciprocity boundary integral equation

$$C_i\theta_i + \int_\Gamma q^*\theta d\Gamma - \int_\Gamma \theta^* q d\Gamma = \sum_{j=1}^{N+L} \alpha_j \left(C_j \hat{\theta}_{ij} + \int_\Gamma q^* \hat{\theta}_j d\Gamma - \int_\Gamma \theta^* \hat{q}_j d\Gamma \right) \tag{12}$$

in which

$$C_i = \frac{\gamma}{2\pi}, \quad q = \frac{\partial \theta}{\partial n}, \quad q^* = \frac{\partial \theta^*}{\partial n}, \quad \theta^* = \ln\left(\frac{1}{r}\right) \tag{13}$$

where n is the outward unit normal vector to boundary Γ, r is the distance between source point $\boldsymbol{i}$ and considered point j, N is the number of boundary nodes and L is the number of internal nodes.

where

$$\alpha = \mathbb{R}^{-1}\tilde{f} = \mathbb{R}^{-1}\left(\tilde{a}\frac{\partial^2 \theta}{\partial \tau^2} + \tilde{b}\frac{\partial \theta}{\partial \tau} + \tilde{c}\theta + \tilde{d} \right) \tag{14}$$

The discretization process for Eq. (12) leads to

$$\begin{aligned} &C_i\theta_i + \sum_{k=1}^{N} \int_{\Gamma_k} q^*\theta d\Gamma - \sum_{k=1}^{N} \int_{\Gamma_k} \theta^* q d\Gamma \\ &= \sum_{j=1}^{N+L} \alpha_j \left(C_i \hat{\theta}_{ij} + \sum_{k=1}^{N} \int_{\Gamma_k} Z_{ik}\hat{\theta}_{kj} d\Gamma - \sum_{k=1}^{N} \int_{\Gamma_k} G_{ik}\hat{q}_{kj} d\Gamma \right) \end{aligned} \tag{15}$$

After interpolation and integration processes over boundary elements, Eq. (15) can be expressed as

$$C_i\theta_i + \sum_{k=1}^{N} Z_{ik}\theta_k - \sum_{k=1}^{N} G_{ik} q_k = \sum_{j=1}^{N+L} \alpha_j \left(C_i\hat{\theta}_{ij} + \sum_{k=1}^{N} Z_{ik}\hat{\theta}_{kj} - \sum_{k=1}^{N} G_{ik}\hat{q}_{kj} \right) \tag{16}$$

The matrix form of Eq. (16) can be written using (14) as

$$Z\theta - Gq = \left(Z\hat{\Theta} - G\hat{Q} \right)\mathbb{R}^{-1}\left(\tilde{a}\frac{\partial^2 \theta}{\partial \tau^2} + \tilde{b}\frac{\partial \theta}{\partial \tau} + \tilde{c}\theta + \tilde{d} \right) \tag{17}$$

which also can be written

$$X\left(\tilde{a}\frac{\partial^2\theta}{\partial\tau^2}+\tilde{b}\frac{\partial\theta}{\partial\tau}+\tilde{c}\theta+\tilde{d}\right)+Z\theta=Gq \tag{18}$$

where

$$X=\left(Z\hat{\Theta}-G\hat{Q}\right)\mathbb{R}^{-1}$$

The boundary and initial conditions

$$\theta(x,y;\tau)=0 \tag{19}$$

$$\frac{\partial\theta(x,y;0)}{\partial\tau}=\vartheta(x,y;0)=0 \tag{20}$$

$$\theta(x,y;0)=\begin{cases}1^{\circ}\mathrm{C} & -0.02\le x,y\le 0.02\\ 0 & \text{other } x,y\end{cases} \tag{21}$$

The time discretization has been performed as follows

$$q=\left(1-\theta_q\right)q^m+\theta_q q^{m+1} \tag{22}$$

$$\theta=(1-\theta_u)\theta^m+\theta_u\theta^{m+1} \tag{23}$$

$$\frac{\partial\theta}{\partial\tau}=\frac{1}{\Delta\tau}\left(\theta^{m+1}-\theta^m\right) \tag{24}$$

$$\frac{\partial^2\theta}{\partial\tau^2}=\frac{1}{\Delta\tau^2}\left(\theta^{m+1}+\theta^{m-1}-2\theta^m\right) \tag{25}$$

Substituting from Eqs. (22)–(25) into (20), we obtain

$$\begin{gathered}\left(\frac{X\tilde{a}}{\Delta\tau^2}+\frac{X\tilde{b}}{\Delta\tau}+X\tilde{c}\theta_u+\theta_u Z\right)\theta^{m+1}-\theta_q G q^{m+1}+X\tilde{d}\\ =\left(\frac{2X\tilde{a}}{\Delta\tau^2}+\frac{X\tilde{b}}{\Delta\tau}-X\tilde{c}(1-\theta_u)-Z(1-\theta_u)\right)\theta^m-\frac{X\tilde{a}}{\Delta\tau^2}\theta^{m-1}+\left(1-\theta_q\right)Gq^m\end{gathered} \tag{26}$$

Thus, with the temperature θ determined, the remaining task is to solve the problem (8).

4. Boundary element implementation for the poro-elastic fields

The representation formula of (8) that describes the unknown field $\hat{u}^g$ can be written as

$$\hat{u}^g(\tilde{x})=\left(\hat{V}\hat{t}^g\right)_{\Omega}(\tilde{x})-\left(\hat{K}\hat{u}^g\right)_{\Omega}(\tilde{x})\ \text{for}\ \tilde{x}\in\boldsymbol{\Omega} \tag{27}$$

where

$$\left(\hat{V}\hat{t}^g\right)_{\Omega}(\tilde{x})=\int_{\Gamma}\hat{U}^T(y-\tilde{x})\hat{t}^g(y)ds_y \tag{28}$$

$$\left(\hat{K}\hat{u}^{g}\right)_{\Omega}(\tilde{x}) = \int_{\Gamma} \left(\hat{T}_{y}\hat{U}\right)^{T}(y-\tilde{x})\hat{u}^{g}(y)ds_{y} \tag{29}$$

For anisotropic case, the Laplace domain fundamental solution $\hat{U}(r)$ and the corresponding traction $\hat{T}_{v}$ can be expressed as [40].

$$\hat{U}(r) = \begin{bmatrix} \hat{U}^{s}(r) & \hat{U}^{f}(r) & 0 \\ \left(\hat{P}^{s}\right)^{T}(r) & \hat{P}^{f}(r) & 0 \end{bmatrix}, \quad \hat{T}_{y} = \begin{bmatrix} T_{y}^{e} & san_{y} & 0 \\ -\beta n_{y}^{T} & \dfrac{\beta}{s\rho^{f}} n_{y}^{T}\nabla & 0 \end{bmatrix} \quad \text{with} \quad r := |y-x| \tag{30}$$

where the solid displacement fundamental solution $\hat{U}^{s}(r)$ may be expressed as

$$\hat{U}^{s}(r) = \frac{1}{4\pi r\left(\rho-\beta\rho^{f}\right)}\left[\mathbb{R}_{1}\frac{\left(k_{4}^{2}-k_{2}^{2}\right)}{\left(k_{1}^{2}-k_{2}^{2}\right)}e^{-k_{1}r} - \mathbb{R}_{2}\frac{\left(k_{4}^{2}-k_{1}^{2}\right)}{\left(k_{1}^{2}-k_{2}^{2}\right)}e^{-k_{2}r} + \left(Ik_{3}^{2}-\mathbb{R}_{3}\right)e^{-k_{3}r}\right] \tag{31}$$

with

$$\mathbb{R}_{j} = \frac{3\nabla_{y}r\nabla_{y}^{T}r - I}{r^{2}} + k_{j}\frac{3\nabla_{y}r\nabla_{y}^{T}r - I}{r} + k_{j}^{2}\nabla_{y}r\nabla_{y}^{T}r \tag{32}$$

which can be expressed as [36].

$$\hat{U}^{s}(r) = \frac{1}{4\pi\mu r(\lambda+2\mu)}\left[(\lambda+\mu)\nabla_{y}r\nabla_{y}^{T}r + I(\lambda+3\mu)\right] + O\left(r^{0}\right) \tag{33}$$

The fundamental solution of solid displacement $\hat{U}^{s}(r)$ can be dismantled into singular $\hat{U}_{s}^{s}(r)$ and regular $\hat{\mathrm{U}}_{\mathrm{r}}^{\mathrm{s}}(\mathrm{r})$ parts as

$$\begin{aligned} \hat{U}^{s}(r) &= \hat{U}_{s}^{s}(r) + \hat{U}_{r}^{s}(r) \text{ with } r := |y-x| \\ &= \frac{1}{\mu}\left[I\Delta_{y} - \frac{\lambda+\mu}{\lambda+2\mu}\nabla_{y}\nabla_{y}^{T}\right]\Delta_{y}\hat{x}(r) \\ &-\frac{1}{\mu}\left[\left(\left(k_{1}^{2}+k_{2}^{2}\right)\Delta_{y} - k_{1}^{2}k_{2}^{2}\right)I - \left(k_{1}^{2}+k_{2}^{2}-k_{4}^{2}-\frac{k_{1}^{2}k_{2}^{2}}{k_{3}^{2}}\right)\nabla_{y}\nabla_{y}^{T}\right]\hat{x}(r) \end{aligned} \tag{34}$$

in which

$$\begin{aligned} \hat{x}(r) &= \frac{1}{4\pi r}\left[\frac{e^{-k_{1}r}}{\left(k_{2}^{2}-k_{1}^{2}\right)\left(k_{3}^{2}-k_{1}^{2}\right)} + \frac{e^{-k_{2}r}}{\left(k_{2}^{2}-k_{1}^{2}\right)\left(k_{2}^{2}-k_{3}^{2}\right)} + \frac{e^{-k_{3}r}}{\left(k_{1}^{2}-k_{3}^{2}\right)\left(k_{2}^{2}-k_{3}^{2}\right)}\right] \\ &= -\frac{1}{\left(k_{1}^{2}-k_{2}^{2}\right)\left(k_{1}^{2}-k_{3}^{2}\right)\left(k_{3}^{2}-k_{2}^{2}\right)} + O\left(r^{2}\right) \end{aligned} \tag{35}$$

The remaining parts of $\hat{U}(r)$ as in (30) can be described as [36].

$$\hat{U}^f(r) = \frac{\rho^f(\alpha-\beta)\nabla_y r}{4\pi r\beta(\lambda+2\mu)\left(k_1^2-k_2^2\right)}\left[\left(k_1+\frac{1}{r}\right)e^{-k_1 r}-\left(k_2+\frac{1}{r}\right)e^{-k_2 r}\right] = O\left(r^0\right) \quad (36)$$

$$\hat{P}^s(r) = \frac{\hat{U}^f(r)}{s} = O\left(r^0\right) \quad (37)$$

$$\hat{P}^f(r) = \frac{s\rho^f}{4\pi r\beta\left(k_1^2-k_2^2\right)}\left[\left(k_1^2-k_4^2\right)e^{-k_1 r}-\left(k_2^2-k_4^2\right)e^{-k_2 r}\right] = \frac{s\rho^f}{4\pi r\beta}+O\left(r^0\right) \quad (38)$$

On the basis of limiting process $\tilde{x}\in\Omega\to x\in\Gamma$ on (28) we get

$$\lim_{\tilde{x}\in\Omega\to x\in\Gamma}\left(\hat{V}\hat{t}^g\right)_\Omega(\tilde{x}) = \left(\hat{V}\hat{x}^g\right)(x) := \int_\Gamma \hat{U}^T(y-x)\hat{t}^g(y)ds_y \quad (39)$$

According to limiting process $\tilde{x}\in\Omega\to x\in\Gamma$ on (28) we obtain [42].

$$\lim_{\tilde{x}\in\Omega\to x\in\Gamma}\left(\hat{K}\hat{u}^g\right)_\Omega(\tilde{x}) = [-I(x)+C(x)]\hat{u}^g(x)+\left(\hat{K}\hat{u}^g\right)(x) \quad (40)$$

where

$$C(x) = \lim_{\varepsilon\to 0}\int_{y\in\Omega:|y-x|=\varepsilon}\left(\hat{T}_y\hat{U}\right)^T(y-x)ds_y \quad (41)$$

and

$$\left(\hat{K}\hat{u}^g\right)(x) = \lim_{\varepsilon\to 0}\int_{|y-x|\geq\varepsilon}\left(\hat{T}_y\hat{U}\right)^T(y-x)\hat{u}^g(y)ds_y \quad (42)$$

By using (39)-(42), we can write

$$C(x)\hat{u}^g(x) = \left(\hat{V}\hat{t}^g\right)(x)-\left(\hat{K}\hat{u}^g\right)(x) \quad (43)$$

By applying the inverse Laplace transform, we obtain

$$C(x)u^g(x,t) = (V*t^g)(x,t)-(Ku^g)(x,t) \quad (44)$$

where $*$ is the time convolution.
According to [40], the fundamental solution is

$$\left(\hat{T}_y\hat{U}\right)^T = \left[\begin{bmatrix}\hat{T}_y^e & san_y\\ -\beta n_y^T & \dfrac{\beta}{s\rho_0^f}n_y^T\nabla_y\end{bmatrix}\begin{bmatrix}\hat{U}^s & \hat{U}^f\\ \left(\hat{P}^s\right)^T & \hat{P}^f\end{bmatrix}\right]^T = \begin{bmatrix}\hat{T}^s & \hat{T}^f\\ \left(\hat{Q}^s\right)^T & \hat{Q}^f\end{bmatrix} \quad (45)$$

On the basis of Stokes theorem, we obtain

$$\int_\Gamma(\nabla_y\times \mathrm{a},n_y)ds_y = -\int_{\partial\Gamma}(\mathrm{a},v)d\gamma_y = -\int_\phi(\mathrm{a},v)d\gamma_y = 0 \quad (46)$$

which can be expressed as

$$\int_{\Gamma}^{\cdot} (n_y \times \nabla_y, \mathrm{a}) ds_y = 0 \tag{47}$$

On the basis of [40], we get

$$\int_{\Gamma}^{\cdot} (M_y \ \mathrm{a}) ds_y = 0 \tag{48}$$

in which $M_y = \left(\nabla_y \nabla_y^T\right)^T - \nabla_y \nabla_y^T$.
By applying (48) to a formula $\mathrm{a} = vu$ we obtain [43].

$$\int_{\Gamma}^{\cdot} (M_y \ v) u ds_y = -\int_{\Gamma}^{,} v (M_y \ u) ds_y \tag{49}$$

$$\int_{\Gamma}^{\cdot} (M_y \ v)^T u ds_y = -\int_{\Gamma}^{\cdot} v^T (M_y \ u) ds_y \tag{50}$$

Making use of (34) and (45), we can express $\hat{T}^s$ as

$$\left(\hat{T}^s\right)^T = \left(T_y^e \left(\hat{U}_{sing}^s + \hat{U}_{reg}^s\right)\right)^T + s\alpha \hat{P}^s n_y^T = \left(T_y^e \hat{U}_{sing}^s\right)^T + O\left(r^0\right) \tag{51}$$

On the basis of [40], we obtain

$$\left(\hat{T}^s\right)^T = (\lambda + 2\mu) n_y \nabla_y^T \hat{U}_{sing}^s - \mu\left(n_y \times \left(\nabla_y \times \hat{U}_{sing}^s\right)\right) + 2\mu M_y \hat{u}_{sing}^s + O\left(r^0\right) \tag{52}$$

which may be expressed using (34) as

$$\left(\hat{T}^s\right)^T = M_y \Delta_y^2 \hat{X} + I\left(\mathrm{n}^T \nabla_y\right) \Delta_y^2 \hat{X} + 2\mu\left(M_y \hat{U}_{sing}^s\right)^T + o\left(r^0\right) \tag{53}$$

By applying (29) (53), we obtain

$$\left(\hat{k}\hat{u}\right)_{\Omega}^s (\tilde{x}) = \int_{\Gamma}^{\cdot} \left[\left(M_y \Delta_y^2 \hat{X}\right)\hat{u} + \left(I\left(\mathrm{n}^T \nabla_y\right) \Delta_y^2 \hat{X}\right)\hat{u} + 2\mu\left(M_y \hat{U}_{sing}^s\right)^T \hat{u} + o\left(r^0\right)\hat{u}\right] ds_y \tag{54}$$

Based on [42], we have

$$\left(\hat{K}\hat{u}\right)_{\Omega}^s (\tilde{x}) = \int_{\Gamma}^{\cdot} \left[-\Delta_y^2 \hat{X}\left(M_y \hat{u}\right) + \left(I\left(n^T \nabla_y\right) \Delta_y^2 \hat{X}\right)\hat{u} + 2\mu \hat{U}_s^s\left(M_y \hat{u}\right) + o\left(r^0\right)\hat{u}\right] ds_y \tag{55}$$

In the in right-side of (55), we can write second term as follows

$$\left(n^T \nabla_y\right) \Delta_y^2 \hat{x}(r) = \frac{n^T \nabla_y r}{4\pi r^2} + O\left(r^0\right) \tag{56}$$

in which

$$C^s(x) = I(x)\,c(x) \text{ with } c(x) = \frac{\phi(x)}{4\pi} \tag{57}$$

According to [40], we can write

$$\lim_{\Omega \in \tilde{x} \to x \in \Gamma} \left(\hat{K}\hat{u}\right)^s_\Omega(\tilde{x}) = -I(x)[-1 + c(x)]\hat{u}(x) + \left(\hat{K}\hat{u}\right)^s(x) \tag{58}$$

By augmenting $\hat{U}^s_s$ to $\hat{U}^s$, we obtain

$$\left(\hat{k}\hat{u}\right)^s_\Omega(\tilde{x}) = \int_\Gamma -\Delta_y^2\hat{x}\left(M_y\hat{u}\right) + \left(I\left(n^T\nabla_y\right)\Delta_y^2\hat{x}\right)\hat{u} + 2\mu\hat{U}^s\left(M_y\hat{u}\right) + O\left(r^0\right)\hat{u}\,ds_y \tag{59}$$

According to [41], we get

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau \text{ for } t \in [0, T] \tag{60}$$

where

$$(f * g)(t_n) \approx \sum_{k=0}^{n} \omega^{\Delta t}_{n-k}\left(\hat{f}\right)g(t_k) \tag{61}$$

On the basis of Lubich [44, 45], the integration weights ω_n are calculated using Cauchy's integral formula as

$$\omega^{\Delta t}_{n-k}\left(\hat{f}\right) := \frac{1}{2\pi i}\int_{|z|=R} \hat{f}\left(\frac{\gamma(z)}{\Delta t}\right)z^{-(n+1)}dz \tag{62}$$

Polar coordinate transformation $z = Re^{-i\varphi}$ is used with the trapezoidal rule to approximate the integral (62) as

$$\omega^{\Delta t}_n\left(\hat{f}\right) \approx \frac{R^{-1}}{L+1}\sum_{\ell=o}^{L}\hat{f}(s_\ell)\zeta^{\ell n} \quad \text{with } \zeta = e^{\frac{2\pi i}{L+1}} \; and \; s_\ell = \frac{\gamma\left(R\zeta^{-\ell}\right)}{\Delta t} \tag{63}$$

Substitution of Eq. (63) into Eq. (61), we get

$$\begin{aligned}(f * g)(t_n) &\approx \sum_{k=0}^{N}\frac{R^{-(n-k)}}{N+1}\sum_{\ell=0}^{N}\hat{f}(s_\ell)\zeta^{\ell(n-k)}g(t_k) \\ &\approx \frac{R^{-n}}{N+1}\sum_{\ell=0}^{N}\hat{f}(s_\ell)\hat{g}(s_\ell)\zeta^{\ell n}\end{aligned} \tag{64}$$

with

$$\hat{g}(s_\ell) = \sum_{k=0}^{N} R^k g(t_k)\zeta^{-\ell k}. \tag{65}$$

According to the procedure [43], the convolution operator (44) can be expressed as

$$C(x)u^g(x,t) = (v * t^g)(x,t) - (k * u^g)(x,t) \tag{66}$$

which may be written as

$$C(x)\hat{u}^g(x,s_\ell) = \left(\hat{v}\hat{t}^g\right)(x,s_\ell) - \left(\hat{k}\hat{u}^g\right)(x,s_\ell), \quad \ell = 0\ldots\ldots..N \tag{67}$$

Let the boundary Γ is discretized into N_e boundary elements τ_e as follows

$$\Gamma \approx \Gamma_h = \bigcup_{e=1}^{N_e} \tau_e \tag{68}$$

Now, we assume that we have

$$S_h[k](\Gamma_{N,h}) := span\{\varphi_i^\alpha[k]\}_{i=1}^{\mathbb{i}}, \quad \alpha \geq 1 \tag{69}$$

$$S_h[k](\Gamma_{D,h}) := span\left\{\psi_i^\beta[k]\right\}_{j=1}^{\mathbb{j}}, \quad \beta \geq 0 \tag{70}$$

where

$$\hat{u}^g[k](x) \approx \hat{u}_h^g[k](x) = \sum_{i=1}^{\mathbb{i}} \hat{u}_{h,i}^g[k]\varphi_i^\alpha[k](x) \in S_h[k](\Gamma_{N,h}) \tag{71}$$

$$\hat{t}^g[k](x) \approx \hat{t}_h^g[k](x) = \sum_{j=1}^{\mathbb{j}} \hat{t}_{h,j}^g[k]\psi_j^\beta[k](x) \in S_h[k](\Gamma_{D,h}) \tag{72}$$

where $k = 1, 2, 3, 4$ are the poro-elastic degrees of freedom, $\varphi_i^\alpha[k]$ are $\mathbb{i}$ continuous polynomial shape functions and $\psi_i^\beta[k]$ are $\mathbb{j}$ piecewise discontinuous polynomial shape functions.

Thus, from (67), we can write the following $N + 1$ algebraic systems of equations

$$\begin{bmatrix} \hat{V}_{DD} - \hat{K}_{DN} \\ \hat{V}_{ND} - \left(C + \hat{K}_{NN}\right) \end{bmatrix}_\ell \begin{bmatrix} \hat{t}_{D,h}^g \\ \hat{u}_{N,h}^g \end{bmatrix}_\ell = \begin{bmatrix} -\hat{V}_{DN} & \left(C + \hat{K}_{DD}\right) \\ -\hat{V}_{NN} & \hat{K}_{ND} \end{bmatrix}_\ell \begin{bmatrix} \hat{g}_{N,h}^g \\ \hat{g}_{D,h}^g \end{bmatrix}_\ell \quad \ell = 0\ldots N \tag{73}$$

5. Numerical results and discussion

In the current study, a Krylov subspace iterative method is used for solving the resulting linear systems. In order to reduce the number of iterations, a dual threshold incomplete LU factorization technique (ILUT) which is one of the well-known preconditioning techniques is implemented as a robust preconditioner for TFQMR (Transpose-free quasi minimal residual) [46] to accelerate the convergence of the solver TFQMR.

To illustrate the numerical calculations computed by the proposed technique, the physical parameters for transversely isotropic soft tissue are given as follows [47]:

The elasticity tensor

$$C_{ablg} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{74}$$

in which

$$\begin{aligned} C_{11} &= \frac{E^2 v_0^2 - EE_0}{(1+v)\left(2Ev_0^2 + E_0(v-1)\right)}, C_{12} = -\frac{E^2 v_0^2 + EE_0 v}{(1+v)\left(2Ev_0^2 + E_0(v-1)\right)} \\ C_{13} &= -\frac{EE_0 v}{2Ev_0^2 + E_0(v-1)}, C_{33} = -\frac{E_0^2(v-1)}{2Ev_0^2 + E_0(v-1)} \\ C_{44} &= \mu_0, \;\; C_{66} = \frac{1}{2}(C_{11} - C_{12}) \end{aligned} \tag{75}$$

where

$$v = 0.196, v_0 = 0.163, \mu_0 = 20.98 \text{ GPa}, E = 68.34 \text{ GPa}, E_0 = 51.35 \text{ GPa} \tag{76}$$

and therefore

$$k_1 = 108.39 \text{ GPa}, k_2 = -21.70 \text{ GPa} \tag{77}$$

where E and E_0 are the respectively, v and v_0 are Poisson's ratio in the isotropy plane and in the fiber direction respectively, and μ_0 is the shear moduli in any direction within a plane perpendicular to isotropy plane.

Since for strongly anisotropic soft tissue both bulk moduli are positive, we used the following physical parameters for anisotropic soft tissue [48].

$$v = 0.95, v_0 = 0.49, \mu_0 = 20.98 \text{ GPa}, E = 22 \text{ kPa}, E_0 = 447 \text{ kPa} \tag{78}$$

and therefore

$$k_1 = 1243 \text{ kPa}, k_2 = 442 \text{ kPa} \tag{79}$$

and other constants considered in the calculations are as follows.

$$\begin{gathered} \rho_s = 1600\, kg/m^3,\; \rho_{\mathcal{F}} = 1113\, kg/m^3, p = 25\, MPap = 25MPa, \\ \phi = 0.15 \text{ and } Q/R = 0.65. \end{gathered} \tag{80}$$

The domain boundary of the current problem has been discretized into 21 boundary elements and 42 internal points as depicted in **Figure 2**. The computation was done using Matlab R2018a on a MacBook Pro with 2.9GHz quad-core Intel Core i7 processor and 16GB RAM.

Figure 3 shows the variation of the temperature T along x-axis for different values of fractional order parameter. It can be seen from this figure that the fractional order parameter has a significant influence on the temperature.

Figure 4 illustrates the variation of the displacement u_1 along x-axis for different values of fractional order parameter. It can be seen from this figure that the fractional order parameter has a significant influence on the displacement u_1.

Figure 5 shows the variation of the displacement u_2 along x-axis for different values of fractional order parameter. It can be seen from this figure that the fractional order parameter has a significant influence on the displacement u_2.

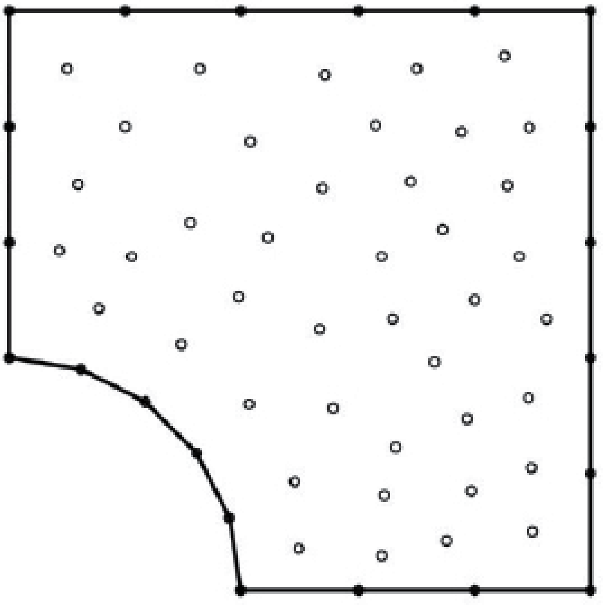

Figure 2.
Boundary element model of the current problem.

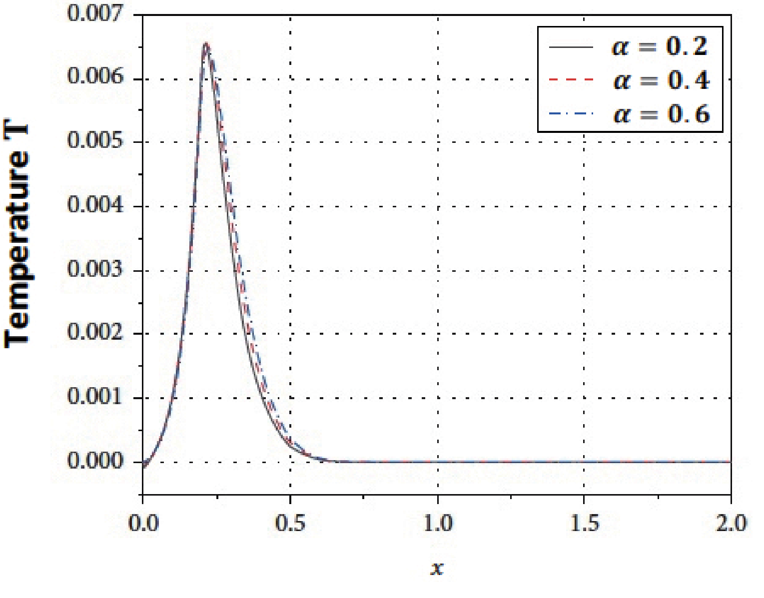

Figure 3.
*Variation of the temperature **T** along **x-axis**.*

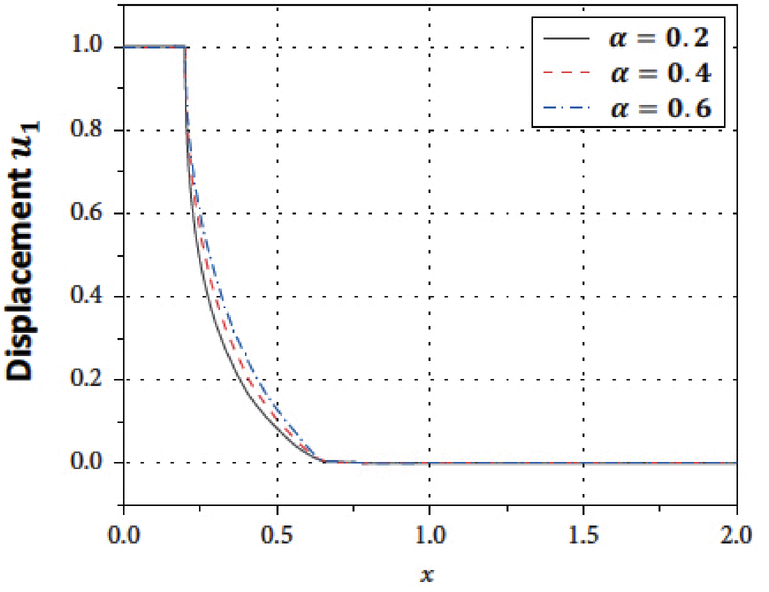

Figure 4.
Variation of the displacement u_1 along x-axis.

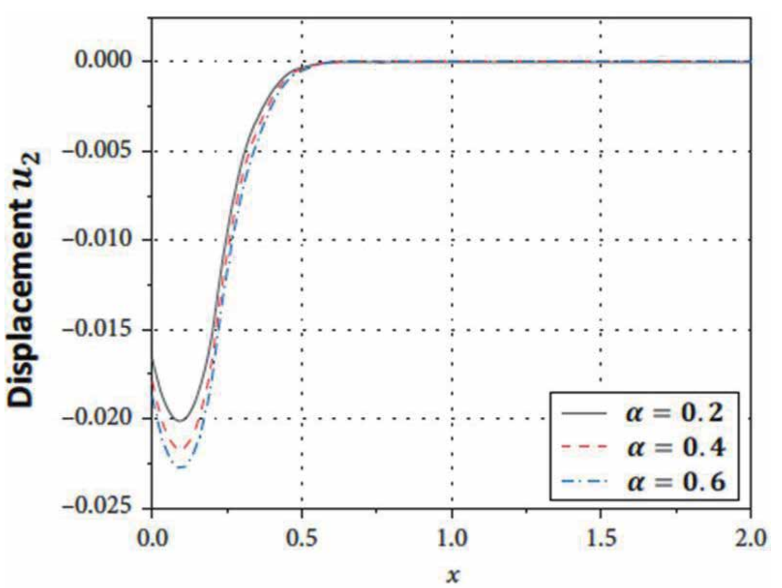

Figure 5.
Variation of the displacement u_2 along x-axis.

Figure 6 shows the variation of the fluid pressure p along x-axis for different values of fractional order parameter. It can be seen from this figure that the fractional order parameter has a significant influence on the fluid pressure p.

Figure 7 shows the variation of the bio-thermal stress σ_{11} along x-axis for different values of fractional order parameter. It can be seen from this figure that the fractional order parameter has an important influence on the bio-thermal stress σ_{11}.

Since there are no findings available for the problem under consideration. Therefore, some literatures may be regarded as special cases from our general problem. In the special case under consideration, the results of the bio-thermal

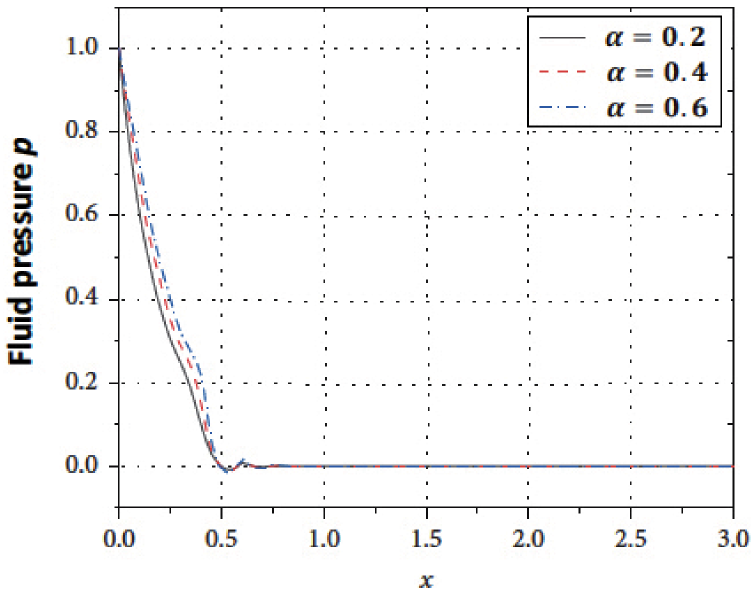

Figure 6.
Variation of the fluid pressure $\mathbf{p}$ *along* $\boldsymbol{x}$*-axis.*

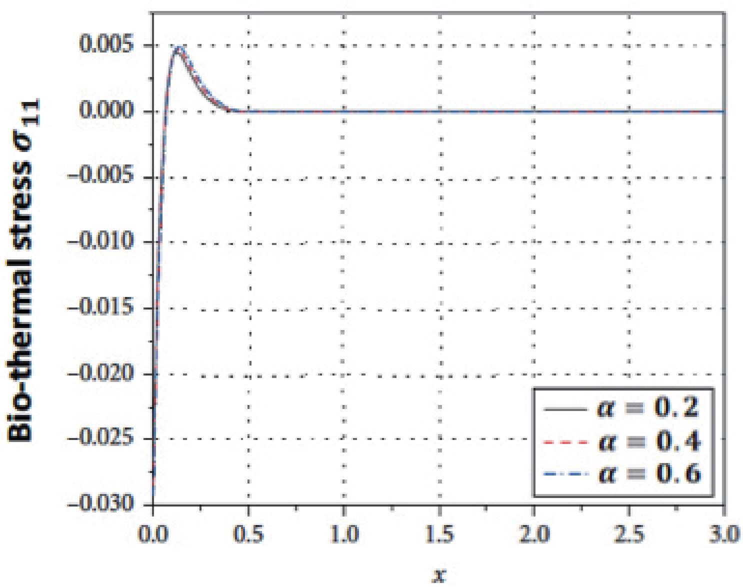

Figure 7.
Variation of the bio-thermal stress $\boldsymbol{\sigma_{11}}$ *along* $\boldsymbol{x}$*-axis.*

stress caused by coupling between the temperature and displacement fields are plotted in **Figure 8** to illustrate the variation of the bio-thermal stress σ_{11} along x-axis for BEM, FDM and FEM, where the boundary of the special case problem has been discretized into 21, 42 and 84 boundary elements (bes). The validity, accuracy and efficiency of our proposed technique have been confirmed by a graphical comparison of the three different boundary elements (21, 42 and 84) with those obtained using the FDM results of Shen and Zhang [49] and FEM results of Torvi and Dale [50] for the special case under consideration, the increase of BEM boundary elements leads to improve the accuracy and efficiency of the BEM, also, it can

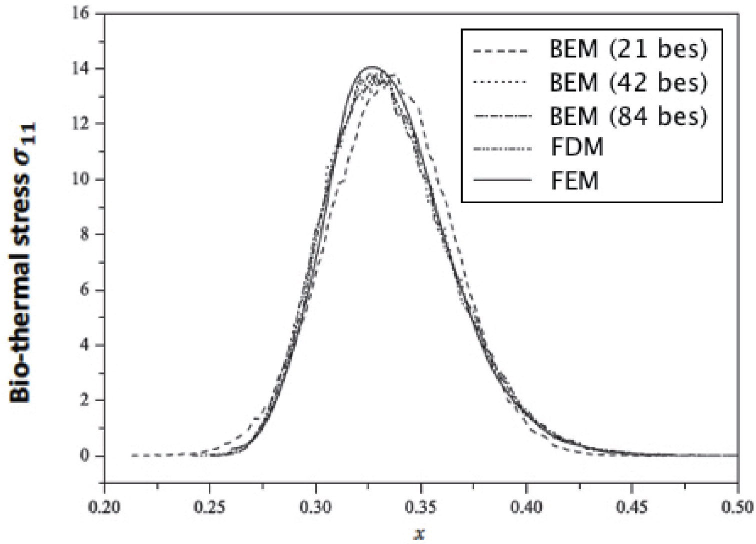

Figure 8.
*Variation of the bio-thermal stress $\boldsymbol{\sigma_{11}}$ along **x-axis** for BEM, FDM and FEM.*

be noted that the BEM findings are in excellent agreement with the FDM and FEM results, we refer the interested reader to recent work [51–55] for understanding the BEM methodology.

6. Conclusion

1. A novel boundary element model based on the TWMBT and Biot's theory was established for describing the bio-thermomechanical interactions in anisotropic soft tissues.

2. The bio-heat transfer equation has been solved using the dual reciprocity boundary element method (DRBEM) to obtain the temperature distribution.

3. The mechanical equation has been solved using the convolution quadrature boundary element method (CQBEM) to obtain the displacement and fluid pressure for different temperature distributions at each time step.

4. Due to the advantages of DRBEM and CQBEM such as dealing with more complex shapes of soft tissues and not needing the discretization of the internal domain, also, they have low RAM and CPU usage. Therefore, they are a versatile and powerful methods for modeling of fractional bio-thermomechanical problems in anisotropic soft tissues.

5. The linear systems resulting from BEM have been solved by TFQMR solver with the ILUT preconditioner which reduces the number of iterations and the total CPU time.

6. Numerical findings are presented graphically to show the effect of fractional order parameter on the problem variables temperature, displacements and fluid pressure.

7. Numerical findings confirm the validity, efficiency and accuracy of the proposed BEM technique.

8. The proposed technique can be applied to a wide variety of fractional bio-thermomechanical problems in anisotropic soft tissues.

9. For open boundary problems of soft tissues, such as the considered problem, the BEM users need only to deal with real geometry boundaries. But for these problems, FDM and FEM use artificial boundaries, which are far away from the real soft tissues. Also, these artificial boundaries are also becoming a big challenge for FDM users and FEM users. So, BEM becomes the best method for the considered problem.

10. The presence of fractional order parameter in the current study plays a significant role in all the physical quantities during modeling and simulation in medicine and healthcare.

11. From the research that has been performed, it is possible to conclude that the proposed BEM is an easier, effective, predictable, and stable technique in the treatment of the bio-thermomechanical soft tissue models.

12. It can be concluded from this chapter that Biot's equations for the dynamic response of poroelastic media can be combined with the bio-heat transfer models to describe the fractional bio-thermomechanical interactions of anisotropic soft tissues.

13. Current numerical results for our complex and general problem may provide interesting information for researchers and scientists in bioengineering, heat transfer, mechanics, neurophysiology, biology and clinicians.

Nomenclature

$A = \phi(1 + Q/R)$	Biot's coefficient
$B_{\tilde{x}}^{e}$	linear elastostatics operator
Γ	considered boundary
Γ_D	Dirichlet boundary
Γ_N	Neumann boundary
C	specific heat of soft tissue
$\mathbb{C}$	shape factor
C_b	specific heat of the blood
$C_{p\,jkl}$	specific heat of the blood
F	bulk body forces
$\hat{g}_D$	Dirichlet datum
$\hat{g}_N$	Neumann datum
K	dynamic permeability
$\mathcal{K}$	thermal conductivity of soft tissue
m	iterative parameter
p	pore pressure
$P_0(\tau)$	heating power
Q, R	solid–fluid coupling parameters
Q_{met}	metabolic heat source

Q_{ext}	external heat source
S_0	scattering coefficient
T	soft tissue temperature
$\breve{T}_b$	arterial blood temperature
T_x^e	traction derivative
u	solid displacement
u_f	fluid displacement
$V = V^f + V^s$	bulk volume
V^f	fluid volume
V^s	solid volume
W_b	blood perfusion rate
$\mathfrak{B}$	stress-temperature coefficients
$\in$	linear strain tensor
ζ	fluid volume variation
$\rho = \rho_s(1-\phi) + \phi\rho_f$	bulk density
$\rho_s = \mathbb{C}\phi\rho_f$	mass density of soft tissue
ρ_f	blood density
σ	total stress tensor
τ	time
τ_q	phase lag for heat flux
τ_T	phase lag for temperature gradient
$\varphi_i^\alpha[k]$	continuous polynomial shape functions
$\phi = \frac{V^f}{V}$	porosity
$\psi_j^\beta[k]$	discontinuous polynomial shape functions
Ω	considered region

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Chapter 16

Solving Second-Order Differential Equations by Decomposition

Fritz Schwarz

Abstract

The subject of this article are linear and quasilinear differential equations of second order that may be decomposed into a first-order component with guaranteed solution procedure for obtaining closed-form solutions. These are homogeneous or inhomogeneous linear components, special Riccati components, Bernoulli, Clairaut or d'Alembert components. Procedures are described how they may be determined and how solutions of the originally given second order equation may be obtained from them. This makes it possible to solve new classes of differential equations and opens up a new area of research. Applying decomposition to linear *inhomogeneous* equations a simple procedure for determining a special solution follows. It is not based on the method of *variation of constants* of Lagrange, and consequently does not require the knowledge of a fundamental system. Algorithms based on these results are implemented in the computer algebra system ALLTYPES which is available on the website www.alltypes.de.

Keywords: ordinary differential equations, decomposition, exact solutions, computer algebra

1. Introduction

The history of differential equations begins shortly after the establishment of the analysis by Newton and Leibniz in the 17th century. A brief overview of its first hundred years can be found in Appendix A of Ince's book [1]. These early investigations were mainly limited to first-order equations, associated with the names Riccati, Bernoulli and Euler. Starting in the early 18th century special linear equations of higher order were also investigated.

A more systematic search for solution methods was initiated by the results of Galois for solving algebraic equations in the early 19th century. Inspired by these results, Picard and Pessiot in Paris founded a solution theory for *linear* differential equations, known as Picard-Vessiot theory or differential Galois theory. A good introduction into their work and its extensions by Loewy may be found in the books [2, 3]. Completely independent of these activities Sophus Lie in Leipzig founded the so-called symmetry analysis for solving *nonlinear* differential Equations [4, 5]. Its main weaknesses are that most differential equations have no symmetries and therefore it cannot be applied. Furthermore, there are many differential equations with fairly simple closed form solutions that have no symmetries. That was essentially the status in the early twentieth century, which did not fundamentally change until its end.

In this situation, a new solution method based on *decompositions* was proposed [6]. Essentially a decomposition means to find a *component* of lower order such that the original equation may be represented as a differential polynomial in terms of this component. Its existence is based on the following observation. Let $F(x, y, y', y'') = 0$ be a second-order differential equation for a function y depending on a variable x, and $\omega(x, y, C_1, C_2) = 0$ its general solution depending on two undetermined constants. It describes a two-parameter family of curves in the $x - y$-plane. If C_1 and C_2 are constrained by a relation $\varphi(C_1, C_2) = 0$ the resulting expression for ω contains effectively a single parameter C. It describes a family of curves that may obey a first-order differential equation called a component. Its solutions are also solutions of the originally given second-order equation.

Every second-order equation has an infinite number of first-order components corresponding to the choice of $\varphi(C_1, C_2)$. Any such component has the form

$$F(x, y, y', y'') = f(x, y, z, z', C)(z \equiv g(x, y, y', C)). \tag{1}$$

Its meaning may be described as follows. If $z \equiv g(x, y, y', C)$ is substituted into $f(x, y, z, z', C)$ the second-order equation on the left-hand side is obtained. The constant C does not necessarily occur in f and g, the same is true for y and its occurence in f.

Solving a second-order equation by decomposition involves two steps. First a decomposition of a certain type has to be found. Then the first order equation has to be solved in order to get the solutions of the original second-order equation. Of particular interest are those components the solution of which can always be determined. These are linear homogeneous and inhomogeneous components, special Riccati components, Bernoulli, Clairaut or d'Alembert components.

In this article equations of second order for an unknown function y depending on x with leading term y'' or $y'y''$ are considered. They are assumed to be linear in y'', polynomial in the derivatives y', and rational in y and x. Equations of this kind are fairly common in applications, therefore many special examples of them are given in the collections by Kamke [7], Murphy [8], Polyanin [9], Sachdev [10] and Zwillinger [11]. Many interesting applications of such differential equations can be found in the textbooks by MacCluer et al. [12] and Swift and Wirkus [13].

In the following Section 2 equations with leading term y'' are considered, and possible linear or Bernoulli components are determined. For linear inhomogeneous equations it is shown how decomposition leads to a new procedure for determining a special solution without first having to know a fundamental system. Equations with leading term $y'y''$ and possible components of Clairaut or d'Alembert type are the subject of Section 3. Most of the examples do not have Lie symmetries, so decomposition is the only way to solve them. The last Section 4 discusses various possible generalizations of the decomposition method, on the one hand more general equations to be solved, on the other hand more general first-order components.

2. Equations with leading term y''

Equations that are linear in the highest derivative y'', but may contain powers of y' with coefficients that are rational in y and x are considered in this section. Moreover it is assumed that they are primitive, i.e. the leading coefficient is unity. Their general form is

$$y'' + \sum_{k=0}^{K} c_k(x, y) y'^k = 0 \text{ with } c_k(x, y) \in \mathbb{Q}(x, y), \quad K \in \mathbb{N}. \tag{2}$$

Equations of this form appear in numerous applications, as can be seen in the collections of solved examples quoted above. The following proposition has been proved in [6], it is the basis for generating quasilinear first-order components; as usual $y' \equiv \frac{dy}{dx}$ and $D \equiv \frac{d}{dx}$.

Proposition 1 Let a second-order quasilinear Eq. (2) be given. A first-order component $z \equiv y' + r(x, y)$ exists if $r(x, y)$ satisfies

$$r_x - rr_y - \sum_{k=0}^{K} (-1)^k c_k(x, y) r^k = 0. \tag{3}$$

Then the original second-order equation can be decomposed as

$$\left(z' - r_y z + \sum_{k=1}^{K} c_k(x, y) \left((z - r)^k - (-1)^k r^k \right) \right) (z \equiv y' + r) = 0. \tag{4}$$

The proof may be found in Section 2 of [6]. As a first application linear first-order components of the form $z \equiv y' + a(x)y + b(x)$ are searched for, i.e. with the above notation $r(x, y) = a(x)y + b(x)$; its coefficients a and b are solutions of the so-called *determining system*, they may be in any field extension of $\mathbb{Q}(x)$. The following proposition describes how they may be obtained.

Proposition 2. Let a second-order quasilinear Eq. (2) be given. In order that it has a linear first-order component $z \equiv y' + a(x)y + b(x)$ the coefficients $a(x)$ and $b(x)$ have to satisfy

$$\left(a' - a^2\right)y + b' - ab - \sum_{k=0}^{K} (-1)^k c_k(x, y)(ay + b)^k = 0. \tag{5}$$

Then (2) may be written as follows

$$\left(z' - az + \sum_{k=1}^{K} c_k(x, y) \left((z - ay - b)^k - (-1)^k (ay + b)^k \right) \right) (z \equiv y' + ay + b) = 0. \tag{6}$$

The coefficients a and b are solutions of a first-order algebro-differential system. Its general form is

$$a' - a^2 + p(a, b, x) = 0, \quad b' - ab + q(a, b, x) = 0, \quad r_i(a, b, x) = 0 \tag{7}$$

for $i = 1, 2 \ldots$; $p(a, b, x)$, $q(a, b, x)$ and $r_i(a, b, x)$ are polynomials in a and b, and rational in x; their maximal degree in a and b is K. The $r_i(a, b, x)$ generate an ideal $I_{ab} \in \mathbb{Q}(x)[a, b]$.

Proof. Substituting $r = a(x)y + b(x)$ into (3) yields (5). At this point y is considered as an undetermined function. Therefore the left-hand side of (5) is represented as a partial fraction in y. Equating its coefficients to zero yields sufficient conditions in order that (5) vanishes and z is a component of (2). The first order ode's for a and b in the determining system (7) originate from the coefficients of first and zeroth degree in y of (5). The polynomials in a and b, i.e. $p(a, b, x)$, $q(a, b, x)$ and $r_i(a, b, x)$ in (7) originate from the powers of $ay + b$ and the rational coefficients $c_k(x, y)$ in (5), i.e. exclusively from the nonlinearities of (2). Substitution of $y' = z - ay - b$ into (4) yields (6). As a result, the sums at the left-hand side of (6) are a polynomial in z the coefficients of which may depend explicitly on y.

It is important to represent the left side of (5) as a partial fraction in y, only in this way the structure of the system (7) is assured.

2.1 Linear equations

If $K = 1$, $c_1(x, y) = c_1(x)$ and $c_0(x, y) = c_0(x)y + c_r(x)$ the above proposition contains the decomposition of linear equations as a special case as shown next.

Corollary 1 Let $K = 1$, $c_1(x, y) = c_1(x)$, $c_0(x, y) = c_0(x)y + c_r(x)$ and the linear inhomogeneous second-order equation

$$y'' + c_1(x)y' + c_0(x)y + c_r(x) = 0 \tag{8}$$

be given. A first-order component $z \equiv y' + a(x)y + b(x)$ exists if a and b are solutions of the determining system

$$a' - a^2 + c_1(x)a - c_0(x) = 0 \text{ and } b' + (c_1(x) - a)b - c_r(x) = 0. \tag{9}$$

If it is satisfied Eq. (8) may be written as

$$(z' + (c_1 - a)z)(z \equiv y' + ay + b) = 0. \tag{10}$$

Proof. The system (9) follows from (5) for the given special values of K and the coefficients c_k. Then reduction of (8) w.r.t. z yields (10). □

It is remarkable that in the case of linear equations the algebraic conditions $r_i(a, b, x)$ are missing, i.e. they are the most significant contributions originating from possible nonlinearities in (2).

For linear *homogeneous* ode's, i.e. for $c_r = 0$ and $b = 0$, Loewy decompositions have been shown to be an effective method for determining a fundamental system [3]. It is based on a factorization of the linear differential operator corresponding to the given equation over its base field, i.e. restricting the coefficients of the factors to the field of the coefficients of the given second-order equation. This restriction does not apply in the above corollary, the coefficients may be in any field extension.

For linear *inhomogeneous* equations in addition to a fundamental system a special solution has to be found. The above corollary avoids the usual method of variation of constants that somehow appears like an *ad hoc* method. The method described in the above corollary requires only a special solution of a Riccati equation and subsequently solving a linear first-order equation in order to obtain the general solution of the second-order Eq. (8). The following example applies this procedure.

Example 1 The equation

$$y'' - y' - \frac{1}{x}y = (x + 1)e^x \tag{11}$$

is Equation 2.109 in Kamke's collection [7]. Here $c_1 = -1$, $c_0 = -\frac{1}{x}$ and $c_r = -(x + 1)e^x$. The Riccati equation $a' - a^2 - a + \frac{1}{x} = 0$ has the special solution $a = -1 - \frac{1}{x}$. From $b' + \frac{1}{x}b = (x + 1)e^x$ follows $b = \frac{C}{x} + \frac{1}{x}(x^2 - x + 1)e^x$ and leads to the component

$$z = y' - \left(1 + \frac{1}{x}\right)y + \frac{1}{x}\left(C + \left(x^2 - x + 1\right)e^x\right).$$

Integration yields the general solution

$$y = C_1 x e^x + C_2 x e^x \int \exp(-x) \frac{dx}{x^2} + \left(x^2 - x\log(x) - 1\right)e^x.$$

This is also the general solution of Eq. (11).

It may occur that a fundamental system of a second-order equation is rather complicated. Usually this is the case when the Riccati equation for a in (9) does not have a special rational solution and the usual algorithms for solving it do not apply, but one of the special cases of Section 4.9 (a), ... (e) in [7]. Then it may be advantageous to assume that all integration constants in (9) are zero and only a special solution is determined as shown next.

Example 2 Consider the equation

$$y'' - \frac{1}{2x} y' + xy + 1 = 0. \tag{12}$$

Here $c_1 = -\frac{1}{2x}$, $c_0 = x$ and $c_r = 1$. The Riccati equation $a' - a^2 - \frac{1}{2x}a - x = 0$ has the special solution $a = \sqrt{x}\tan\left(-\frac{2}{3}x\sqrt{x}\right)$, it yields

$$b' - \left(\frac{1}{2x} + \sqrt{x}\tan\left(-\frac{2}{3}\sqrt{x}\right)\right)b - 1 = 0.$$

Its special solution leads to the component

$$z \equiv y' + \sqrt{x}\tan\left(-\frac{2}{3}x\sqrt{x}\right)y + \frac{\sqrt{x}}{\cos\left(-\frac{2}{3}x\sqrt{x}\right)} \int \cos\left(-\frac{2}{3}x\sqrt{x}\right)\frac{dx}{\sqrt{x}}.$$

One more integration yields a special solution of (12).

$$y_0 = -\cos\left(-\frac{2}{3}x\sqrt{x}\right) \int \frac{\sqrt{x}}{\cos\left(-\frac{2}{3}x\sqrt{x}\right)^2} \int \cos\left(-\frac{2}{3}x\sqrt{x}\right)\frac{dx}{\sqrt{x}} dx. \qquad \square$$

The application of Corollary 1 is particularly convenient if the coefficients c_1 and c_0 are constant and the solutions of the algebraic equation $a^2 - c_1 a + c_0 = 0$ are also solutions of the Riccati equation for a. The following example is of this type.

Example 3 The equation $y'' + 4y' + 4y = \cosh(x)$ has coefficients $c_1 = c_0 = 4$ and $c_r = -\cosh(x)$. The solution of $(a-2)^2 = 0$ is $a = 2$. It leads to $b' + 2b = \cosh(x)$ and the component

$$z = y' + 2y + C\exp(-2x) + \frac{1}{3}\sinh(x) - \frac{2}{3}\sinh(x).$$

Its general solution

$$y = C_1 \exp(-2x) + C_2 x \exp(-2x) + \frac{5}{9}\cosh(x) - \frac{4}{9}\sinh(x)$$

is also the general solution of the given second-order equation. $\square$

2.2 Quasilinear equations

The most interesting applications of Proposition 2 relate to nonlinear equations, of course. They differ from the linear case mainly by the occurence of the ideal I_{ab}

in (7), which defines algebraic conditions $r_i(a, b, x) = 0$ for the coefficients of a possible component. Furthermore, the first-order ode's for a and b are modified due to the nonlinearity by additional terms. The structure of the determining system (7) suggests the following solution procedure.

At first the algebraic system $r_i(a, b, x) = 0$ is established and a Gröbner basis for the ideal I_{ab} is generated. Usually it may be determined rather efficiently.

If it is inconsistent a linear component does not exist in any field extension. This applies to a generic nonlinear equation of the form (2).

If the ideal I_{ab} is finite-dimensional each solution that satisfies the two first-order ode's yields a component that may be integrated and leads to a one-parameter family of solutions of the given second-order equation.

Finally, the algebraic equations may generate a relation between a and b; substitution into the first-order differential equations may lead to one of the above cases, or to a solution depending on a parameter. In the latter case a one-parameter family of linear components exists, integrating the corresponding equation yields the general solution of the given second-order equation containing two undetermined constants.

Subsequently this proceeding will be illustrated by several examples. They show that all of the alternatives mentioned actually exist.

Example 4 Consider the equation

$$y'' + xy'^2 + (x-1)yy' + \frac{x}{x+1}y' - y^2 - \frac{1}{x+1}y = 0.$$

Its coefficients $c_2 = x$, $c_1 = (x-1)y + \frac{x}{x+1}$ and $c_0 = -y^2 - \frac{y}{x+1}$ result in the system

$$a' - a^2 - 2abx + \frac{ax}{x+1} + b(x-1) = 0, \;\; b' - ab - b^2x + \frac{bx}{x+1} + \frac{1}{x+1} = 0,$$
$$a^2 - \frac{x-1}{x}a - \frac{1}{x} = 0.$$

The single algebraic equation has the solutions $a = -\frac{1}{x}$ and $a = 1$ and the decompositions

$$\left(z' + xz^2 + \frac{x^3y + 2x^2y + x^2 + xy + x + 1}{x^2 + x}z\right)\left(z \equiv y' - \frac{1}{x}y\right) = 0,$$
$$\left(z' + xz^2 - \frac{x^2y + 2xy + y + 1}{x+1}z\right)(z \equiv y' + y) = 0$$

follow. Integration of the two components yields the two one-parameter families $y = C\exp(-x)$ and $y = Cx$ of solution curves. It is not obvious how the general solution of the second-order equation involving two constants is supported by them. □

The most interesting, of course, are equations that allow a one-parameter family of linear components and whose integration gives the general solution. The next example is of this type.

Example 5 Consider the equation

$$yy'' - y'^2 + \frac{2}{3}y' - \frac{1}{x}y^2 = 0 \qquad (13)$$

with $K = 2$ and the coefficients $c_2 = -\frac{1}{y}$, $c_1 = \frac{2}{3y}$ and $c_0 = -\frac{y}{x}$; they generate the determining system

$$a' + \frac{1}{x} = 0, \;\; b' + ab + \frac{2}{3}a = 0, \;\; b^2 + \frac{2}{3}b = 0$$

with solution $a = -\log(x) + C$, $b = -\frac{2}{3}$ from which the decomposition

$$\left(z' - \frac{1}{y}z^2 - \frac{1}{y}\left(\log(x)y - Cy + \frac{2}{3}\right)z\right)\left(z \equiv y' - (\log(x) - C)y - \frac{2}{3}\right) = 0$$

is obtained. Integration of the first-order component leads to the general solution

$$y = \frac{2}{3}x^x \exp(C_1 x)\left(\int \exp(-C_1 x)\frac{dx}{x^x} + C_2\right)$$

of Eq. (13), it does not have a Lie symmetry.

Here the question arises how exceptional are the equations that have a one-parameter family of linear components of the first order and thus have a general solution in closed form. The following example is a generalization of the previous one. A family of second order equations is constructed whose general solution can be given explicitly.

Example 6 The equation

$$yy'' - y'^2 + p(x)y' + q(x)y^2 = 0 \tag{14}$$

with undetermined coefficients $p(x)$ and $q(x)$ generalizes the preceding example. Here $c_2 = \frac{1}{y}$, $c_1 = \frac{p(x)}{y}$ and $c_0 = q(x)y$. A first-order linear component $z \equiv y' + ay + b$ exists if a and b are solutions of the system

$$a' - q(x) = 0, \;\; b' + ab + p(x)a = 0, \;\; b(b + p(x)) = 0.$$

The result may be described as follows. If $p(x) = k$ is a constant, and $q(x)$ is an undetermined function then $a = \int q(x)dx + C$, $b = -k$ and the decomposition

$$\begin{aligned} &y'' - \frac{1}{y}y'^2 + \frac{k}{y}y' + q(x)y \\ &\quad = \left(z' - \frac{1}{y}z^2 + \left(\int q(x)dx + C\right)z - \frac{k}{y}z\right)\left(z \equiv y' + \left(\int q(x)dx + C\right)y - k\right) = 0 \end{aligned}$$

exists. Defining $Q(x, C_1) \equiv \int q(x)dx + C_1$ integration of the first-order component yields

$$y = \exp\left(-\int Q(x, C_1)dx\right)\left(k\int \exp\left(\int Q(x, C_1)dx\right) + C_2\right).$$

This is the general solution of Eq. (14). □

It turns out that a behavior similar to that in the previous example often applies, i.e. first-order linear components often exist not only for isolated equations, but for entire families, which are parameterized by indefinite functions. This explains the existence of families of solvable equations as those given in the collections mentioned above.

Bernoulli equations are another class of first-order ode's with guaranteed closed form general solutions. In addition to a term linear in y they contain a nonlinearity y^n where n is an integer; $n = 1$ or $n = 0$ correspond to linear homogeneous or linear inhomogeneous equations, respectively. Similar as for linear components, a special Bernoulli component guarantees a one-parameter set of solution curves of a given second-order equation, and a one-parameter family of such components guarantees the general solution of the latter. The main result of this section is the following proposition.

Proposition 3 Let a second-order quasilinear Eq. (2) be given. In order that it has a first-order Bernoulli component $z \equiv y' + a(x)y^n + b(x)y$, $n \in \mathbb{N}$, the coefficients a and b have to satisfy

$$(a' - (n+1)ab)y^n + \left(b' - b^2\right)y - na^2y^{2n-1} - \sum_{k=0}^{K} (-1)^k c_k(x,y)(ay^n + by)^k = 0. \tag{15}$$

Then (2) may be written as follows

$$\left(z' - \left(nay^{n-1} + b\right)z + \sum_{k=1}^{K}\left((z - ay^n - by)^k + (-1)^{k+1}(ay^n + by)^k\right)\right) \tag{16}$$
$$(z \equiv y' + ay^n + by) = 0.$$

The coefficients a and b may be obtained from a first-order algebro-differential system; its general form is

$$a' - (n+1)ab + p(a,b,x) = 0, \quad b' - b^2 + q(a,b,x) = 0, \quad r_i(a,b,x) = 0; \tag{17}$$

$p(a,b,x)$, $q(a,b,x)$ and $r_i(a,b,x)$ are polynomials in a and b, and rational in x; the maximal degree in a and b is K, they generate an ideal I_{ab} in the ring $\mathbb{Q}(x)[a,b]$.

Proof. Substituting $r = a(x)y^n + b(x)y$ into (3) yields condition (15). Representing its left-hand side as partial fraction in the variable y, the coefficients of the various terms yield sufficient conditions for its vanishing. They form the algebro-differential system (17). The first order ode's for a and b originate from the coefficients of nth and first degree in y, respectively; $p(a,b,x)$, $q(a,b,x)$ and $r_i(a,b,x)$ originate from the coefficients $c_k(x,y)$ and the powers of $ay^n + by$.

Substitution of $y' = z - ay^n - by$ into (4) yields (16). As a result, the sums at the left-hand side of (16) are a polynomial in z and z' the coefficients of which may depend explicitly on y. □

The structure of the system (17) is similar as for linear components considered above, and consequently also the proceeding for its solution. The following examples applies the above proposition.

Example 7 The equation

$$yy'' - y'^2 + 2y^3y' + xy^2 = 0 \tag{18}$$

with $K = 2$ has coefficients $c_2 = -\frac{1}{y}$, $c_1 = 2y^2$ and $c_0 = xy$. For generic n the condition

$$(a' - (n-1)ab)y^n + \left(b' - x\right)y - (n-1)a^2y^{2n-1} + 2ay^{n+2} + 2by^3 = 0 \tag{19}$$

follows. The two equations $b' - x = 0$ and $b = 0$ originating from the coefficients of y and y^3, respectively, are inconsistent. In order for a Bernoulli component to exist, this inconsistency must be compensated by other coefficients for a suitable choice of n. To this end either $n = 1$ or $n = 3$ is required. The former leads to the inconsistency $x = 0$, whereas the latter yields

$$a' - 2ab + 2b = 0, \quad b' - x = 0, \quad a^2 - a = 0.$$

This system has the solution $a = 1$, $b = -\frac{1}{2}x^2 + C$ from which the decomposition

$$\left(z' - \frac{1}{y}z^2 + \left(y^2 + \frac{1}{2}x^2 + C\right)z\right)\left(z \equiv y' + y^3 + \left(\frac{1}{2}x^2 + C\right)y\right) = 0$$

follows. Integrating the right component yields the general solution

$$y = \frac{1}{\sqrt{(2)}\exp\left(C_1 x + \frac{1}{6}x^3\right)\left(\int \exp\left(-2C_1 x - \frac{1}{3}x^3\right)dx + \frac{1}{2}C_2\right)^{1/2}}$$

of Eq. (18). It does not have a Lie symmetry. □

The next example deals with a problem in hydrodynamics. The boundary layer at a circular cylinder immersed in the uniform flow of liquid is considered [14], see also Eq. 6.210 of [7].

Example 8 The equation

$$y^3 y'' + yy'' - 3y^2 y'^2 + y'^2 = 0 \tag{20}$$

has the only nonvanishing coefficient $c_2 = -\frac{3y^2-1}{y(y^2+1)}$. Substitution into (15) yields

$$\begin{aligned}&((a' - (n+1)ab)y^n + \left(b' - b^2\right)y - na^2 y^{2n-1}\\ &\qquad -\left(\frac{1}{y} - \frac{4y}{y^2+1}\right)\left(a^2 y^{2n} + 2aby^{n+1} + b^2 y^2\right) = 0.\end{aligned} \tag{21}$$

It turns out that for $n = 3$ this condition specializes to

$$\left(\left(a' - 4a^2 + 2ab\right)y^3 + \left(b' + 4a^2 - 8ab + 2b^2\right)y - 4(a-b)^2\frac{y}{y^2+1} = 0.$$

After some simplifications the resulting system for a and b is $a' - 2a^2 = 0$ and $b = a$; Its solution $a = b = -\frac{\frac{1}{2}}{x+C}$ leads to the Bernoulli equation $y' - -\frac{\frac{1}{2}}{x+C}y^3 - \frac{\frac{1}{2}}{x+C} = 0$ with general solution

$$y = \frac{\sqrt{1 - C_1 x - C_2}}{\sqrt{C_1 x + C_2}}.$$

This is also the general solution of Eq. (20) □

In general it is *a priori* not known whether there exists a Bernoulli component of any order. If a component for small values of n cannot be found it is desirable to determine bounds for its possible existence. The next example shows that this is possible in special cases.

Example 9 Consider the equation

$$y'' + \frac{x}{y-1}y'^2 + y' + xy = 0 \tag{22}$$

with $K = 2$ and non-vanishing coefficients $c_2 = \frac{x}{y-1}$, $c_1 = 1$ and $c_0 = xy$.
Substitution into (15) yields

$$(a' - (n+1)ab)y^n + (b' - b^2)y - na^2y^{2n-1} - xy + ay^n + by - \frac{x}{y-1}(ay^n + by)^2.$$

Expanding the last term into unique partial fractions by using the general formula

$$\frac{y^n}{y-k} = \sum_{\nu=0}^{n-1} k^{n-\nu-1}y^\nu + \frac{k^n}{y-k} \tag{23}$$

leads to

$$\begin{aligned}&(a' - (n+1)ab)y^n + (b' - b^2)y - na^2y^{2n-1} - xy + ay^n + by\\ &-a^2x\sum_{\nu=0}^{2n-1} y^\nu - 2abx\sum_{\nu=0}^{n} y^\nu - \frac{(a+b)^2x}{y-1} = 0.\end{aligned}$$

The coefficients of the various terms yield a system for the unknowns a, b and n. There is always the subsystem $a' + a^2 + a + x = 0$, $a + b = 0$ independent of n, it originates from the coefficient of y and the term independent of y. Furthermore, the leading term of the first sum in the above equation requires $a = 0$ for for any $n \geq 2$. These equations combined are inconsistent, i.e. the above Eq. (22) does not allow a Bernoulli component for any nonnegative natural number n. A similar reasoning exists for negative values of n. □

At the moment an algorithm for determining bounds for n is not known, it is not even clear whether the existence of bounds is decidable in general.

3. Equations with leading term $y'y''$

Another important class of differential equations are those with leading term $y'y''$, they are considered in this section. Their general form is

$$y'y'' + c(x,y)y'' + \sum_{k=0}^{K} c_k(x,y)y'^k = 0 \text{ with } c(x,y), c_k(x,y) \in \mathbb{Q}(x)[y], \quad K \in \mathbb{N}. \tag{24}$$

Components of Clairaut or d'Alembert type $z \equiv y - xf(y') - g(y')$ may lead to partial or even general solutions in closed form, mostly in a parameter representation. The main result of this section is given in the following proposition.

Proposition 4 Let a second-order differential Eq. (24) be given. A first-order component $z \equiv y - xf(y') - g(y')$ exists if $f(y')$ and $g(y')$ satisfy

$$\begin{aligned}&(p - f(p))(c(x, xf(p) + g(p)) + p)\\ &\quad + (xf'(p) + g'(p))\sum_{k=0}^{K} c_k(x, xf(p) + g(p))p^k = 0\end{aligned} \tag{25}$$

where $p \equiv y'$ has been defined. Representing the left hand side of (25) as a partial fraction w.r.t. x and equating the coefficients of the various terms to zero, a system of first-order quasilinear ode's for $f(p)$ and $g(p)$ is obtained; its degree in $f(p)$ and $g(p)$ is not higher than the degree in y of the coefficients $c(x,y)$ and $c_k(x,y)$.

Proof. Reduction of (24) w.r.t. $z \equiv y - xf(y') - g(y')$ leads to Eq. (25). Their properties follow directly from the assumptions about the coefficients $c(x,y)$ and $c_k(x,y)$ in (24), and representing the left hand side of (25) as a partial fraction in x.

The determining system for the two functions $f(p)$ and $g(p)$ may be obtained explicitly from (25) if the coefficients $c(x,y)$ and $c_k(x,y)$ are known. Without restrictions on the coefficients $c_k(x,y)$ the derivatives $f'(p)$ and $g'(p)$ may occur linearly in any equation obtained after separation w.r.t. x, and an algebraic system in $p, f(p), g(p)$, $f'(p)$ and $g'(p)$ follows. It turns out that an algebraic Gröbner basis algorithm including factorization is a suitable tool for solving them in many cases. If a solution has been obtained the corresponding component may be applied for generating the decomposition of the given equation explicitly. The following example uses this proceeding.

Example 10 Consider the equation

$$y'y'' + \frac{1}{2}yy'' + \frac{x-1}{2x}y'^2 - \frac{1}{2x}yy' + \frac{1}{2x} = 0. \tag{26}$$

Here $J = 1$ and $K = 2$, its nonvanishing coefficients are $c(x,y) = \frac{1}{2}y$, $c_2 = \frac{x-1}{2x}$, $c_1 = -\frac{y}{2x}$ and $c_0 = \frac{1}{2x}$. A linear or Bernoulli component does not exist. Proposition 4 leads to the system

$$g'gp + g'p^2 - g' = 0, \quad f'fp - f'p^2 + f^2 - fp = 0,$$

$$f'gp + f'p^2 - f' + fg'p + fg + 2fp - g'p^2 - gp - 2p^2 = 0.$$

Transforming the left-hand sides into algebraic Gröbner bases in the term order $f'(p) > g'(p) > f(p) > g(p) > p$, the following two systems and their solutions are obtained.

$$f - p = 0, gp + p^2 - 1 = 0, \ \rightarrow f = p, g = \frac{1}{p} - p,$$

$$f'p + f = 0, gp + p^2 - 1 = 0, \ \rightarrow f = C\frac{1}{p}, g = \frac{1}{p} - p.$$

They lead to the decompositions

$$\left(zz' + \frac{xy'^2 + y'^2 + 1}{y'}z' + \frac{xy'^2 - y'^2 - 1}{xy'}z\right)\left(z \equiv y - xy' + y' - \frac{1}{y'}\right) = 0,$$

$$\left(zz' + \frac{y'^2 + Cx + 1}{y'}z' - \frac{xy'^2 + y'^2 + 1}{xy'}z\right)\left(z \equiv y - \frac{C}{y'}x + y' - \frac{1}{y'}\right) = 0,$$

respectively. The former decomposition generates a Clairaut component. It yields the solution $y = Cx + \frac{1}{C} - C$ of (26), C is an undetermined constant. Its parameter solution $x = \frac{1}{p^2} + 1, y = \frac{2}{p}$ does not solve it, it annihilates a lower-order factor of the expression in the left-hand bracket of this decomposition and has to be discarded.

Integration of the d'Alembert component $z \equiv y'^2 + yy' - Cx - 1$ leads to the general solution of (26) in a parameter representation

$$x = \frac{1}{C_1\sqrt{C_1 + p^2}}\left(\sqrt{C_1 + p^2} - C_1 p \log\frac{\sqrt{C_1 + p^2} + p}{\sqrt{C_1}} + (C_1C_2 + 1)p\right),$$

$$y = -\frac{1}{\sqrt{C_1 + p^2}}\left(p\sqrt{C_1 + p^2} - C_1 \log\frac{\sqrt{C_1 + p^2} + p}{\sqrt{C_1}} + C_1C_2 + 1\right).$$

Eq. (26) does not have a Lie symmetry. □

This example shows that solutions of a component must be tested to see if they meet the second order equation, otherwise they have to be discarded; this phenomenon seems to be quite common.

4. Conclusions

The structure of the determining systems for linear or Bernoulli components of a nonlinear Eq. (2) given in Propositions 2 or 3, respectively, show clearly its relation to the corresponding system for the decomposition of a linear equation. For a generic equation of the second order this appears to be the best possible result. The same applies to the verious solution steps given on page 6. The corresponding result for determining Clairaut and d'Alembert components given in Proposition 4 is less specific. However, it should be possible, to obtain more detailed results if special classes of second-order equations are considered. In general, this area is only at an early stage and a better understanding of the underlying mechanisms generating the solutions and also its limitations would be highly desirable.

There are numerous possible generalizations fairly obvious. On the one hand, this concerns the equations to be solved. More general function fields for its coefficients like e.g. algebraic or elementary functions may be allowed. Equations of order three or four would be interesting in many applications. The greatest challenge however is certainly to develop similar procedures for partial differential equations as it has been indicated in Section 5 of [6].

On the other hand, the component type offers space for extensions too. In principle all equations of first order, as described for example in Kamke's book [7], Part A, Section 4, are possible components. Components that guarantee at least a partial solution are of course particularly useful, the most important of them have been discussed in this article.

In order to apply decompositions to concrete problems the implementation of the procedures described in this article are available on the website www.alltypes.de [15].

Beyond that there are a number of general problems related to decompositions. For instance the question how rare are equations that allow a particular decomposition, Example 6 provides a partial answer. If two or more one-parameter families of solution curves are known as in Example 4, does it faciliate generating the general solution? The exact relation between Lie's symmetry analysis and solution by decompositions is another subject of interest.

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Chapter 17

The Uniformly Parabolic Equations of Higher Order with Discontinuous Data in Generalized Morrey Spaces and Elliptic Equations in Unbounded Domains

Tair Gadjiev and Konul Suleymanova

Abstract

We study the regularity of the solutions of the Cauchy-Dirichlet problem for linear uniformly parabolic equations of higher order with vanishing mean oscillation (VMO) coefficients. We prove continuity in generalized parabolic Morrey spaces $M_{p,\varphi}$ of sublinear operators generated by the parabolic Calderon-Zygmund operator and by the commutator of this operator with bounded mean oscillation (BMO) functions. We obtain strong solution belongs to the generalized Sobolev-Morrey space $\overset{\circ}{W}{}^{m,1}_{p,\varphi}(Q)$. Also we consider elliptic equation in unbounded domains.

Keywords: higher order parabolic equations, generalized Morrey spaces, sublinear operators, Calderon-Zygmund integrals, VMO, Cauchy-Dirichlet problem, elliptic equations, unbounded domain

1. Introduction

We consider the higher order linear Cauchy-Dirichlet problem in $Q = \Omega \times (0, T)$, being a cylinder in $\mathbb{R}^{n+1}$, $\Omega \subset R^n$ be a bounded domain $0 < T < \infty$

$$u_t - \sum_{\substack{|\alpha| \le m, \\ |\beta| \le m}} a_{\alpha\beta}(x,t) D^{\alpha\beta} u(x,t) = f(x,t), \ a.e. \text{ in } Q \tag{1}$$

$$u(x,t) = 0 \text{ on } \partial_p Q, \tag{2}$$

where $\partial_p Q = (\partial\Omega \times [0,T]) \cup (\Omega \times \{t = 0\})$ stands for the parabolic boundary of Q and $D^{\alpha\beta} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}, \cdots, \partial x_n^{\alpha_n}} \cdots \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1}, \cdots, \partial y_n^{\beta_n}}$, $|\alpha| = \sum_{k=1}^n \alpha_k$, $\beta = \sum_{k=1}^n \beta_k$.

The unique strong solvability of this type problem was proved in [1]. In [2] the regularity of the solution in the Morrey spaces $L_{p,\lambda}(\mathbb{R}^{n+1})$ with $p \in (1, \infty)$,

$\lambda \in (0, n+2)$ and also its Hölder regularity was studied. In [3] Nakai extend these studies on generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^{n+1})$ with a weight φ satisfying the integral condition

$$\int_r^\infty \frac{\varphi(a,s)}{s} ds \leq c\varphi(a,r), \forall a \in \mathbb{R}^{n+1}, r > 0.$$

The generalized Morrey space is then defined to be the set of all $f \in L_{p,loc}(\mathbb{R}^{n+1})$ such that

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} = \sup_{\mathcal{E}} \frac{1}{\varphi(\mathcal{E})} \left(\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |f(x)|^p dx \right)^{\frac{1}{p}},$$

where the supremum is taken over all parabolic balls $\mathcal{E}$ with respect to the parabolic distance.

The main results connected with these spaces is the following celebrated lemma: let $|Df| \in L_{p,n-\lambda}$ even locally, with $n - \lambda < p$, then u is Holder continuous of exponent $\alpha = 1 - \frac{n-\lambda}{p}$. This result has found many applications in theory elliptic and parabolic equations. In [2] showed boundedness of the maximal operator in $L_{p,\lambda}(\mathbb{R}^{n+1})$ that allows them to prove continuity in these spaces of some classical integral operators. So was put the beginning of the study of the generalized Morrey spaces $M_{p,\varphi}, p > 1$ with φ belonging to various classes of weight functions. In [3] proved boundedness of maximal and Calderon-Zygmund operators in $M_{p,\varphi}$ imposing suitable integral and doubling conditions on φ. These results allow to study the regularity of the solutions of various linear elliptic and parabolic value problems in $M_{p,\varphi}$ (see [4–6]). Here we consider a supremum condition on the weight which is optimal and ensure the boundedness of the maximal operator in $M_{p,\varphi}$. We use maximal inequality to obtain the Calderon-Zygmund type estimate for the gradient of the solution of the problem (1) and (2) in the $M_{p,\varphi}$.

The results presented here are a natural extension of the previous paper [7] to parabolic equations. Here we study the boundedness of the sublinear operators, generated by Calderon-Zygmund operators in generalized Morrey spaces and the regularity of the solutions of higher order uniformly elliptic boundary value problem in local generalized Morrey spaces where domain is bounded. Also hear we study higher order uniformly elliptic boundary value problem where domain is unbounded.

In paper [8] Byun, Palagachev and Wang is study the regularity problem for parabolic equation in classical Lebesgue classes and of Byun, Palagchev and Softova [9, 10] where the problem studied in weighted Lebesgue and Orlicz spaces with a Muckenhoupt weight and the classical Morrey spaces $L_{p,\lambda}(Q)$ with $\lambda \in (0, n+2)$.

In papers [11, 12] the authors studied second order linear elliptic and parabolic equations with VMO coefficients.

Denote by a the coefficient $a(x,t) = \{a_{\alpha\beta}(x,t)\} : Q \to M^{n \times n}$ and by $f(x,t)$ nonhomogeneous term. Suppose that the operator is uniformly parabolic.

The paper is organized as follows. In section 2 we introduce some notations and give the definition of the generalized Morrey spaces $M_{p,\varphi}(Q)$. In section 3 we study sublinear operators generated by parabolic singular integrals in generalized Morrey spaces. In section 4 we is consider sublinear operators generated by non-singular integrals, in section 5 singular and non-singular integrals in generalized Morrey spaces. In section 6 we consider uniformly parabolic equations of higher order with VMO coefficients and proved regularity of solutions. In section 7 we study uniformly elliptic equations in unbounded domains.

2. Some notation and definition

The following notations are used in this paper:

$$x = (x', t), y = (y', \tau) \in \mathbb{R}^{n+1} = R^n \times R^n, \mathbb{R}_+^{n+1} = R^n \times R_+;$$

$$x = (x'', x_n, t) \in D_+^{n+1} = R^{n-1} \times R_+ \times R_+, D_-^{n+1} = R^{n-1} \times R_- \times R_+;$$

$|\cdot|$ is the Euclidean metric, $|x| = \left(\sum_{i=1}^n x_i^2 + t^2\right)^{\frac{1}{2}}$; $B_r(x') = \{y' \in R^n : |x' - y'| < r\}$, $|B_r| = c \cdot r^n$; $\mathcal{I}_r(x', t) = \left\{y \in \mathbb{R}^{n+1} : |x' - y'| < r, |t - \tau| < r^2\right\}, |\mathcal{I}_r(x', t)| = c \cdot r^{n+2}$; $Q_r = \mathcal{I}_r(x, \tau) \cap Q$ for each $(x, \tau) \in Q$, $2\mathcal{I}_r(x, \tau) = \mathcal{I}_{2r}(x, \tau)$.

S^n is the unit sphere in $\mathbb{R}^{n+1}$;

$$D_i u = \frac{\partial u}{\partial x_i}, Du = (D_1 u, \dots, D_n u), u_t = \frac{\partial u}{\partial t};$$

$$D^{\alpha\beta} u = \frac{\partial^{|\alpha|} \partial^{|\beta|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} \cdot \partial y_1^{\beta_1} \cdot \partial y_n^{\beta_n}}$$

the letter C is used for various positive constants.

In the following, besides the standard parabolic metric $\rho(x, t) = \max\left(|x'|, |t|^{\frac{1}{2}}\right)$. We use the equivalent one

$$\rho(x, t) = \left(\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}\right)^{\frac{1}{2}}$$

considered by Fabes and Riviere in [13]. The topology induced by $\rho(x, t)$ consists of the ellipsoids

$$\mathcal{E}_r(x) = \left\{y \in \mathbb{R}^{n+1} : \frac{|x' - y'|^2}{r^2} + \frac{|t - \tau|^2}{r^4} < 1\right\}, |\mathcal{E}_r| = C \cdot r^{n+2},$$

$$\mathcal{E}_1(x) \equiv B_1(x).$$

It is easy to see that the this metrics ore equivalent. In fact, for each $\mathcal{E}_r$ there exist parabolic cylinders $\underline{\mathcal{I}}$ and $\overline{\mathcal{I}}$ with measure comparable to r^{n+2} such that $\underline{\mathcal{I}} \subset \mathcal{E}_r \subset \overline{\mathcal{I}}$.

Let $Q = \Omega \times (0, T), T > 0,$ be a cylinder in R_+^{n+1}. We give the definitions of the functional spaces that we are going to use. Let $a \in L_{1,loc}\left(\mathbb{R}^{n+1}\right.$ and let $a_{\mathcal{E}_r} = |\mathcal{E}_r|^{-1} \int_{\mathcal{E}_r} a(y) dy$ be the mean value of the integral of a. Denote

$$\eta_a(R) = \sup_{r \le R} \frac{1}{|\mathcal{E}_r|} \int_{\mathcal{E}_r} \left| f(y) - f_{\mathcal{E}_r} \right| dy \text{ for every } R > 0,$$

where $\mathcal{E}_r$ ranges over all ellipsoids in $\mathbb{R}^{n+1}$. We say $a \in BMO$ (bounded mean oscillation [14]) if

$$\|a\|_* = \sup_{R > 0} \eta_\alpha(R)$$

is finite. $\|\cdot\|_*$ is a norm in a BMO constant functions.

We say $a \in VMO$ (vanishing mean oscillation) [14] if $a \in BMO$ and

$$\lim_{R \to 0} \eta_a(R) = 0$$

$\eta_a(R)$ is called the VMO-modulus of a. For any bounded cylinder Q we define $BMO(Q)$ and $VMO(Q)$ taking $a \in L_1(Q)$ and $Q_r = Q \cap \mathcal{E}_r(x),\ x \in Q,$ instead of $\mathcal{E}_r$ in the definition above. If a function $a \in BMO$ or VMO, it is possible to extend the function in the whole of $\mathbb{R}^{n+1}$ preserving its BMO-norm or VMO-modulus, respectively (see [15]). Any bounded uniformly continuous (BUC) function f with modulus of continuity $\omega_f(R)$ belongs to VMO with $\eta_f(R) = \omega_f(R)$. Besides, BMO and VMO also contain discontinuous functions, and the following example shows the inclusion $W^1_{n+2}(\mathbb{R}^{n+1}) \subset VMO \subset BMO$.

Example 2.1. We have that $f(x) = |\log \rho(x,t)| \in BMO \backslash VMO$; $\sin f(x) \in BMO \cap L_\infty(\mathbb{R}^{n+1})$; $f_\alpha(x) = |\log \rho(x,t)|^\alpha \in VMO$ for any $\alpha \in (0,1)$; $f_\alpha \in W^1_{n+2}(\mathbb{R}^{n+1})$ for $\alpha \in \left(0, 1 - \frac{1}{n+2}\right)$; $f_\alpha \notin W^1_{n+2}(\mathbb{R}^{n+1})$ for $\alpha \in \left[1 - \frac{1}{n+2}, 1\right)$.

Let $\varphi : \mathbb{R}^{n+1} \times R_+ \to R_+$ be a measurable function and $p \in [1, \infty)$. The generalized parabolic Morrey space $M_{p,\varphi}(\mathbb{R}^{n+1})$ consists of all $f \in L_{p,loc}(\mathbb{R}^{n+1})$ such that

$$\|f\|_{p,\varphi;\mathbb{R}^{n+1}} = \sup_{(x,r) \in \mathbb{R}^{n+1} \times R_+} \varphi^{-1}(x,r) \left(r^{-n-2} \int_{\mathcal{E}_r(x)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

The space $M_{p,\varphi}(Q)$ consists of $L_p(Q)$ functions provided the following norm is finite

$$\|f\|_{p,\varphi;Q} = \sup_{(x,r) \in \mathbb{R}^{n+1} \times R_+} \varphi^{-1}(x,r) \left(r^{-n-2} \int_{Q_r(x)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

The generalized weak parabolic Morrey space $WM_{1,\alpha}(R_{n+1})$ consists of all measurable functions such that

$$\|f\|_{WM_{1,\alpha}(\mathbb{R}^{n+1})} = \sup_{(x,r) \in \mathbb{R}^{n+1} \times R_+} \varphi^{-1}(x,r) r^{-n-2} \|f\|_{WL_1(\mathcal{E}_r(x))},$$

where WL_1 denotes the weak L_1 space. The generalized Sobolev-Morrey space $W^{2m,1}_{p,\varphi}(Q),\ p \in [1,\infty),$ consists of all Sobolev functions $U \in W^{2m,1}_p(Q)$ with distributional derivatives $D^l_t D^s_x u \in M_{p,\varphi}(Q),\ 0 \le 2l + |s| \le 2m,$ endowed by the norm

$$\|u\|_{W^{2m,1}_{p,\varphi}(Q)} = \|u_t\|_{p,\varphi;Q} + \sum_{|\delta| \le 2m} \|D^s u\|_{p,\varphi;Q}.$$

We also define the space

$$\overset{0}{W}{}^{2m,1}_{p,\varphi}(Q) = \left\{ u \in W^{2m,1}_{p,\varphi}(Q) : u(x) = 0,\ x \in \partial Q \right\},$$
$$\|u\|_{\overset{0}{W}{}^{2m,1}_{p,\varphi}(Q)} = \left\{ \|u\|_{W^{m,1}_{p,\varphi}(Q)} \right\},$$

where ∂Q means the parabolic boundary $\Omega \cup (\partial\Omega \times (0,T))$. In problem (1) and (2) the coefficient matrix $a(x,t) = \left(a_{\alpha,\beta}(x,t)\right)^n_{i,j=1}$, $|\alpha|, |\beta| = m$ satisfies

$$\exists\gamma > 0 \ \gamma \sum_{|\alpha|=m} \xi_\alpha^2 \le \sum_{\substack{|\alpha|=m \\ |\beta|=m}} a_{\alpha,\beta}(x,t)\xi_\alpha\xi_\beta, \tag{3}$$

for a.e. $(x,t) \in Q, \forall \xi \in R^n, \xi = \{\xi_\alpha, |\alpha = m| \in R^N\}$, N–number different multiindeks with length equal to m, $a_{\alpha,\beta}(x,t) = a_{\beta,\alpha}(x,t)$, which implies $a_{\alpha,\beta}(x,t) \in L_\infty(Q)$.

Theorem 2.1. *(Main results) Let $a(x,t) \in VMO(Q)$ with $\eta_{\alpha,\beta} = \sum_{i,j=1}^n \eta_{\alpha\beta ij}$ satisfy* (3), *and, for each $p \in (1,\infty)$, let $u(x,t) \in \overset{0}{W}{}_p^{2m,1}(Q)$ be a strong solution* (1) and (2). *If $f \in M_{p,\varphi}(Q)$ with $\varphi(x,r)$ being a measurable positive function satisfying*

$$\int_r^\infty \left(1 + \ln \frac{s}{r}\right) \frac{\operatorname*{ess\ inf}_{s<\xi<\infty} \varphi(x,\xi)\xi^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}} + 1} ds \le C\varphi(x,r) \tag{4}$$

$(x,r) \in Q \times R_+$, *then $u(x,t) \in \overset{0}{W}{}_{p,\varphi}^{2m,1}(Q)$ and*

$$\|u\|_{\overset{0}{W}{}_{p,\varphi}^{2m,1}(Q)} \le C\|f\|_{p,\varphi;Q} \tag{5}$$

with $C = C\big(n,p,\gamma,\partial\Omega,T,\eta_\alpha,\|a\|_{\infty;Q}\big)$.

3. Sublinear operators generated by parabolic singular integrals in generalized Morrey spaces

Let $f \in L_1(\mathbb{R}^{n+1})$ be a function with a compact support and $a \in BMO$. For any $x \notin suppf$ define the sublinear operators T and T_a such that

$$|Tf(x)| \le c\int_{\mathbb{R}^{n+1}} \frac{|f(y)|}{\rho^{n+2}(x-y)} dy, \tag{6}$$

$$|T_af(x)| \le c\int_{\mathbb{R}^{n+1}} |a(x) - a(y)| \frac{|f(y)|}{\rho^{n+2}(x-y)} dy, \tag{7}$$

This operators are bounded in $L_p(\mathbb{R}^{n+1})$ satisfy the estimates

$$\|Tf\|_{L_p} \le C\|f\|_{L_p}, \|T_af\|_{L_p} \le C\|a\|_* \|f\|_{L_p}, \tag{8}$$

where constants independent of a and f. Let we have the Hardy operator $Hg(r) = \frac{1}{r}\int_0^r g(s)ds, r > 0$.

Theorem 3.1. *(see [12]) The inequality*

$$\operatorname*{ess\ sup}_{r>0} \omega(r)Hg(r) \le A \operatorname*{ess\ sup}_{r>0} \vartheta(r)g(r) \tag{9}$$

holds for all non-increasing functions $g : R_+ \to R_+$ if and only if

$$A = C\sup_{r>0} \frac{\omega(r)}{r} \int_0^r \frac{ds}{\underset{0<\xi<s}{\text{ess sup}}\,\vartheta(\xi)} < \infty \tag{10}$$

Lemma 3.1. *(see [12]) Let $f \in L_{p,loc}(\mathbb{R}^{n+1}), p \in [1, \infty)$, be such that*

$$\int_r^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p(\mathcal{E}_s(\gamma_0))} ds < \infty \ \ \forall (x_0, r) \in \mathbb{R}^{n+1} \times R_+ \tag{11}$$

and let T be a sublinear operator satisfying (6).
i. *If $p > 1$ and T is bounded on $L_p(\mathbb{R}^{n+1})$, then*

$$\|Tf\|_{L_p(\mathcal{E}_r)(x_0)} \le cr^{\frac{n+2}{p}} \int_{2r}^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p(\mathcal{E}_s(\gamma_0))} ds \tag{12}$$

ii. *If $p = 1$ and T is bounded from $L_1(\mathbb{R}^{n+1})$ on $WL_1(\mathbb{R}^{n+1})$, then*

$$\|Tf\|_{WL_1(\mathcal{E}_r)(x_0)} \le cr^{n+2} \int_{2r}^\infty s^{-(n+3)} \|f\|_{L_1(\mathcal{E}_s(x_0))} ds, \tag{13}$$

where the constants are independent of r, x_0 and f.
Theorem 3.2. *(see [12]) Let $p \in [1, \infty)$ and $\varphi(x, r)$ be a measurable positive function satisfying*

$$\int_r^\infty \frac{\underset{s<\xi<\infty}{\text{ess inf}}\,\varphi(x,\xi)\xi^{\frac{n+2}{p}}}{s^{\frac{n+2}{p}+1}} ds \le C\varphi(x, r), \ \ \forall (x, r) \in \mathbb{R}^{n+1} \times R_+ \tag{14}$$

and let T be a sublinear operator satisfying (6).

i. *If $p > 1$ and T is bounded on $L_p(\mathbb{R}^{n+1})$, then T is bounded on $M_{p,\varphi}(\mathbb{R}^{n+1})$, and*

$$\|Tf\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} \le C\|f\|_{M_{p,\varphi}(\mathbb{R}^{n+1})} \tag{15}$$

ii. *If $p = 1$ and T is bounded from $L_1(\mathbb{R}^{n+1})$ to $WL_1(\mathbb{R}^{n+1})$, then it is bounded from $M_{1,\varphi}(\mathbb{R}^{n+1})$ to $WM_{1,\varphi}(\mathbb{R}^{n+1})$, and*

$$\|Tf\|_{WM_{1,\varphi}(\mathbb{R}^{n+1})} \le C\|f\|_{M_{1,\varphi}(\mathbb{R}^{n+1})} \tag{16}$$

with constants independent of f.
Our next step is to show boundedness of T_a in $M_{p,\varphi}(\mathbb{R}^{n+1})$. For this we recall some properties of the *BMO* functions.
Lemma 3.2. *John-Nirenberg lemma [[12], Lemma 2.8]. Let $a \in BMO$ and $p \in [1, \infty)$. Then, for any $\mathcal{E}_r$,*

$$\left(\frac{1}{|\mathcal{E}_r|} \int_{\mathcal{E}_r} |a(y) - a_{\mathcal{E}_r}|^p dy \right)^{\frac{1}{p}} \le c(p)\|a\|_*.$$

As an immediate consequence of (7) we get the following property.
Corollary 3.1. *Let $a \in BMO$. Then, for all $0 < 2r < s$,*

$$|a_{\mathcal{E}_r} - a_{\mathcal{E}_s}| \le C(n)\left(1 + \ln\frac{s}{r}\right) \cdot \|a\|_* \tag{17}$$

Now we estimate the norm of T_a.

Lemma 3.3. *(see [12]) Let $a \in BMO$. and T_a be a bounded operator in $L_p\left(\mathbb{R}^{n+1}\right), p \in (1,\infty)$, satisfying (7) and (8). Suppose that, for any $f \in L_{p,loc}\left(\mathbb{R}^{n+1}\right)$,*

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right) \|f\|_{L_p(\mathcal{E}_s(x_0))} \frac{ds}{s^{\frac{n+2}{p}+1}} < \infty \ \forall (x_0, r) \in \mathbb{R}^{n+1} \times R^+ \tag{18}$$

Then,

$$\|T_a f\|_{L_p(\mathcal{E}_s(x_0))} \le c \cdot \|a\|_* r^{\frac{n+2}{p}} \int_{2r}^\infty \left(1 + \ln\frac{s}{r}\right) \|f\|_{L_p(\mathcal{E}_s(x_0))} \frac{ds}{s^{\frac{n+2}{p}+1}} \tag{19}$$

where C is independent of a, f, x_0 and r.

Theorem 3.3. *Let $p \in (1,\infty)$ and $\varphi(x,r)$ be measurable positive functions such that*

$$\int_r^\infty \left(1 + \ln\frac{s}{r}\right) \frac{\operatorname{ess}\inf_{s<\xi<\infty} \varphi(x,\xi)\xi^{\frac{n+2}{p}}}{s^{\frac{(n+2+p)}{p}}} ds \le C\varphi(x,r) \tag{20}$$

for $\forall (x,r) \in \mathbb{R}^{n+1} \times R_+$, where C is independent of x and r. Suppose that $a \in BMO$ and let T_a be a sublinear operator satisfying (7). If T_a is bounded in $L_p\left(\mathbb{R}^{n+1}\right)$, then bounded in $M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)$, and

$$\|T_a f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \le C\|a\|_* \cdot \|f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \tag{21}$$

constant C independent of a and f.

Then basic results of the theorem follows by Lemma 3.3 and Theorem 3.1 in the same manner as for Theorem 3.2. For example the functions $\varphi(x,r) = r^{\beta - \frac{n+2}{p}}$, $\varphi(x,r) = r^{\beta - \frac{n+2}{p}} \cdot \log^m(l + r)$ with $0 < \beta < \frac{n+2}{p}$ and $m \ge 1$, are weight functions satisfying the condition (20).

4. Non-singular integrals in generalized Morrey spaces

Let $x \in D_+^{n+1}$, define $\overline{x} = (x'', -x_n, t) \in D_-^{n+1}$ and $x^0 = (x'', 0, 0) \in R^{n-1}$. Consider the semi-ellipsoids $\mathcal{E}_r^+(x^0) = \mathcal{E}_r^+(x^0) \cap D_-^{n+1}$. Let $f \in L_1\left(D_+^{n+1}\right)$, $a \in BMO\left(D_+^{n+1}\right)$, and $\overline{T}, \overline{T}_a$ be sublinear operators such that

$$|\overline{T}f(x)| \le C \int_{D_+^{n+1}} \frac{|f(y)|}{\rho(\overline{x} - y)^{n+2}} dy \tag{22}$$

$$|\overline{T}_a f(x)| \le C \int_{D_+^{n+1}} |a(x) - a(y)| \frac{|f(y)|}{\rho(\overline{x} - y)^{n+2}} dy \tag{23}$$

Let both the operators be bounded in $L_p\left(D_+^{n+1}\right)$, satisfy the estimates

$$\|\overline{T}f\|_{L_p\left(D_+^{n+1}\right)} \le C\|f\|_{L_p\left(D_+^{n+1}\right)}, \ \|\overline{T}_a f\|_{L_p\left(D_+^{n+1}\right)} \le C\|a\|_* \|f\|_{L_p\left(D_+^{n+1}\right)} \tag{24}$$

constants C independent of a and f.

The following results hold, which can be proved in the some manner as in Section 3 (see [12]).

Lemma 4.1. *Let* $f \in L_{p,loc}\left(D_+^{n+1}\right)$, $p \in (1,\infty)$ *and for all* $(x^0, r) \in R^{n-1} \times R_+$

$$\int_r^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} ds < \infty. \tag{25}$$

If $\overline{T}$ *is bounded on* $L_p\left(D_+^{n+1}\right)$, *then*

$$\|\overline{T}f\|_{L_p\left(\mathcal{E}_r^+(x^0)\right)} \le c r^{\frac{n+2}{p}} \int_{2r}^\infty s^{-\frac{n+2}{p}-1} \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} ds, \tag{26}$$

where the constant c is independent of r, x^0 and f.

Theorem 4.1. *Suppose* φ *be a weight function satisfying* (14), *and let* $\overline{T}$ *be a sublinear operator satisfying* (22) *and* (24). *Then* $\overline{T}$ *is bounded in* $M_{p,\varphi}\left(D_+^{n+1}\right)$, $p \in (1,\infty)$ *and*

$$\|\overline{T}f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} \le C \|f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} ds, \tag{27}$$

with a constant c independent of f.

Lemma 4.2. *Let* $p \in (1,\infty)$, $a \in BMO\left(D_+^{n+1}\right)$ *and* $\overline{T}_a$ *satisfy* (23) *and* (24). *Suppose that, for all* $f \in L_{p,loc}\left(D_+^{n+1}\right)$,

$$\int_r^\infty \left(1 + \ln \frac{s}{r}\right) \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} s^{-\frac{n+2}{p}-1} ds < \infty, \ \forall \left(x^0, r\right) \in \mathbb{R}^{n+1} \times R_+. \tag{28}$$

Then

$$\|\overline{T}_a f\|_{L_p\left(\mathcal{E}_r^+(x^0)\right)} \le C \|a\|_* r^{\frac{n+2}{p}} \int_{2r}^\infty \left(1 + \ln \frac{s}{r}\right) \|f\|_{L_p\left(\mathcal{E}_s^+(x^0)\right)} \frac{ds}{s^{\frac{n+2}{p}+1}}$$

with a constant c independent of a, f, x^0 and r.

Theorem 4.2. *Let* $p \in (1,\infty)$, $a \in BMO\left(D_+^{n+1}\right)$, *let* $\varphi(x^0, r)$ *be a weight function satisfying* (20) *and* $\overline{T}_a$ *be a sublinear operator satisfying* (7), (8). *Then sublinear operator* $\overline{T}_a$ *is bounded in* $M_{p,\varphi}\left(D_+^{n+1}\right)$ *and*

$$\|\overline{T}_a f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} \le C \|a\|_* \|f\|_{M_{p,\varphi}\left(D_+^{n+1}\right)} \tag{29}$$

constant c independent of a and f.

5. Singular and non-singular integrals in generalized Morrey spaces

We apply the above results to Calderon-Zygmund-type operators with parabolic kernel. Since these operators are sublinear and bounded in $L_p\left(\mathbb{R}^{n+1}\right)$, their continuity in $M_{p,\varphi}$ follows immediately. We are called a parabolic Calderon-Zygmund kernel if the following a measurable function $K(x,\xi) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \backslash \{0\} \to R$.

1.$K(x,\cdot)$ is a parabolic Calderon-Zygmund kernel for a.e. $x \in \mathbb{R}^{n+1}$:

1_a. $K(x,\cdot) \in C^\infty\left(\mathbb{R}^{n+1}\right) \backslash \{0\}$,

1_b. $K(x, (\mu\xi', \mu^2 s)) = \mu^{-n-2} K(x,\xi)$ for all $\mu > 0$, $\xi = (\xi', s)$,

1_c. $\int_{S^n} K(x,\xi) d\sigma_\xi = 0$, $\int_{S^n} |K(x,\xi)| d\sigma_\xi < +\infty$.

2. $\|D_\xi^\beta K\|_{L_\infty\left(\mathbb{R}^{n+1}\times S^n\right)} \le M(\beta) < \infty$ for every multi-index β.

Moreover,

$$|K(x,x-y)| \le \rho(x-y)^{-n-2}|K\left(x,\left(\frac{x'-y'}{\rho(x-y)},\frac{t-\tau}{\rho^2(x-y)}\right)\right)| \le \frac{M}{\rho(x-y)^{n+2}},$$

which means the singular integrals

$$\mathcal{B}f(x) = PV\int_{\mathbb{R}^{n+1}} K(x,x-y)f(y)dy, \tag{30}$$

$$\mathcal{C}[a,f](x) = PV\int_{\mathbb{R}^{n+1}} K(x,x-y)[a(y)-a(x)]f(y)dy$$

are sublinear and bounded in $L_p\left(\mathbb{R}^{n+1}\right)$ according to the results in [1, 13].

Theorem 5.1. *Let $f \in M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)m$ then there exist constants c depending on n,p and the kernel such that*

$$\|\mathcal{B}f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \le C\|f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)}, \tag{31}$$

$$\|\mathcal{C}[a,f]\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \le C\|a\|_*\|f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)}.$$

Corollary 5.1. *For any cylinder Q in $\mathbb{R}_+^{n+1}$, $f \in M_{p,\varphi}(Q)$, $a \in BMO(Q)$ and $K(x,\xi) : Q \times \mathbb{R}_+^{n+1}\backslash\{0\} \to R$. Then the operators (30) are bounded in $M_{p,\varphi}(Q)$ and*

$$\|\mathcal{B}f\|_{M_{p,\varphi}(Q)} \le C\|f\|_{M_{p,\varphi}(Q)}, \ \|\mathcal{C}[a,f]\|_{M_{p,\varphi}(Q)} \le C\|a\|_*\|f\|_{M_{p,\varphi}(Q)}. \tag{32}$$

constant c independent of a and f.
We define the extensions

$$\overline{K}(x,\xi) = \begin{cases} K(x,\xi), & (x,\xi) \in Q \times R_+^{n+1}\backslash\{0\} \\ 0, & \text{elsewhere} \end{cases}, \ \overline{f}(x) = \begin{cases} f(x), & x \in Q \\ 0, & x \notin Q \end{cases}$$

and then the singular integral satisfying inequalities

$$|\mathcal{B}f(x)| \le |\overline{\mathcal{B}}f(x)| \le C\int_{\mathbb{R}^{n+1}} \frac{|f(y)|}{\rho(x-y)^{n+2}}dy$$

and

$$\|\mathcal{B}f\|_{M_{p,\varphi}(Q)} \le \|\overline{\mathcal{B}}f\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} \le C\|\overline{f}\|_{M_{p,\varphi}\left(\mathbb{R}^{n+1}\right)} = C\|f\|_{M_{p,\varphi}(Q)}.$$

Corollary 5.2. *Let $a \in VMO$. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon,\eta_a)$ such that for any $\mathcal{E}_r(x_0)$ with a radius $r \in (0,r_0)$ and all $f \in M_{p,\varphi}(\mathcal{E}_r(x_0))$*

$$\|\mathcal{C}[a,f]\|_{M_{p,\varphi}(\mathcal{E}_r(x_0))} \le C\varepsilon\|f\|_{M_{p,\varphi}(\mathcal{E}_r(x_0))}, \tag{33}$$

where c is independent of $\mathcal{E},f,r,$ and x_0.
For the proof of corollary see [12].
For any $x' \in R_+^n$ and any fixed $t > 0$, define the generalized reflexion

$$\tau(x) = (\tau'(x), t),\ \tau'(x) = x' - 2x_n \frac{a^n_{\alpha\beta(x',t)}}{a^{nn}_{\alpha\beta(x',t)}}, \tag{34}$$

where $|\alpha| \leq m,\ |\beta| \leq m,\ a^n_{\alpha\beta}(x)$ is the last row of the coefficients matrix $a(x) = \left(a_{\alpha\beta}(x)\right)$ of (1). The function $\tau'(x)$ maps R^n_+ into R^n_-, and the kernel $K(x, \tau(x) - y) = K(x, \tau'(x) - y', t - \tau)$ is non-singular for any $x, y \in D^{n+1}_+$. Taking $\overline{x} \in D^{n+1}_+$, there exists positive constants K_1 and K_2 such that

$$K_1\rho(\overline{x} - y) \leq \rho(\tau(x) - y) \leq K_2\rho(\overline{x} - y). \tag{35}$$

Let $f \in M_{p,\varphi}\left(D^{n+1}_+\right)$, $a \in BMO\left(D^{n+1}_+\right)$ define the non-singular integral operators

$$\begin{aligned} \overline{\mathcal{B}}f(x) &= \int_{D^{n+1}_+} K(x, \tau(x) - y)f(y)dy, \\ \overline{\mathcal{C}}[a,f](x) &= \int_{D^{n+1}_+} K(x, \tau(x) - y)[a(y) - a(x)]f(y)dy. \end{aligned} \tag{36}$$

Since $K(x, \tau(x) - y)$ is still homogeneous and satisfies 1_b, we have

$$|K(x, \tau(x) - y)| \leq \frac{M}{\rho(\tau(x) - y)^{n+2}} \leq \frac{C}{\rho(\overline{x} - y)^{n+2}}.$$

Hence, the operators (36) are sublinear and bounded in $L_p\left(D^{n+1}_+\right)$, $p \in (1, \infty)$. From section 4 the following results are obtained.

Theorem 5.2. *Let $a \in BMO\left(D^{n+1}_+\right)$ and $f \in M_{p,\varphi}\left(D^{n+1}_+\right)$ with (p, φ) as in (8) Then the non-singular operators are continuous in $M_{p,\varphi}\left(D^{n+1}_+\right)$ and*

$$\begin{aligned} \|\overline{\mathcal{B}}f(x)\|_{M_{p,\varphi}(D^{n+1}} &\leq C\|f\|_{D^{n+1}_+}, \\ \|\overline{\mathcal{C}}[a,f](x)\|_{M_{p,\varphi}(D^{n+1}} &\leq C\|a\|_*\,\|f\|_{D^{n+1}_+} \end{aligned} \tag{37}$$

constant C independent of a and f.

Corollary 5.3. *For any $a \in VMO$. Then there exists a positive number $r_0 = r_0(\varepsilon, \varphi_a)$ such that for any $\mathcal{E}_r(x_0)$ with a radius $r \in (0, r_0)$ and all $\|f\|_{M_{p,\varphi}\left(\mathcal{E}^+_r(x^0)\right)}$*

$$\|\mathcal{C}[a,f]\|_{M_{p,\varphi}\left(\mathcal{E}^+_r(x^0)\right)} \leq C\varepsilon\|f\|_{M_{p,\varphi}\left(\mathcal{E}^+_r(x^0)\right)} \tag{38}$$

where C is independent of $\mathcal{E}, f, r$ and x^0, $\varepsilon > 0$.

6. Proof of the first main result

Now using boundedness of singular integral of Calderon-Zygmund operators in generalized Morrey spaces we will get interval estimates for solutions of problem (1), (2) with coefficients from *VMO* spaces.

Let Ω to be open bounded domain in $R^n, n \geq 3$ and we suppose that its boundary is sufficiently smoothness.

Let coefficients $a_{\alpha\beta}(x)$, $|\alpha|, |\beta| \leq m$ are symmetric and satisfying to the condition uniform ellipticity, essential boundedness of the coefficient $a_{\alpha\beta}(x) \in L_\infty(Q)$ and regularity $a_{\alpha\beta}(x) \in VMO(Q)$. Let $f \in M_{p,\varphi}(Q)$, (p, φ) as in (8) Since $M_{p,\varphi}(Q)$ is a proper subset of $L_p(Q)$, (1) and (2) is uniquely solvable and the solution $u(x)$ belongs at least to $W_p^{2m,1}(Q)$. Our aim is to show that this solution also belong to $\overset{0}{W}{}_{p,\varphi}^{2m,1}(Q)$. For this we need an a priori estimate of u, which we prove in two steps. Before we give interior estimate. For any $x_0 \in R_+^{n+1}$ define the parabolic semi-cylinders $C_r(x_0) = B_r(x_0') \times (t_0 - r^2, t_0)$. Let $\vartheta \in C_0^\infty(C_r)$ and suppose that $\vartheta(x,t) = 0$, for $t \le 0$. According to [1, 7, 16], for any $x \in supp\,\vartheta$ the following representation formula for the higher derivatives of ϑ holds true if $u \in \overset{0}{W}{}_p^{2m}(Q)$

$$\begin{aligned} D^{|\alpha|}u(x) &= P.V.\int_{\mathbb{R}^{n+1}} D^{|\alpha|}\Gamma(x, x-y)\left[\sum_{|\alpha|,\,|\beta| \le 2m} \left(a_{\alpha\beta}(x) - a_{\alpha\beta}(y)\right) D^{\alpha,\beta}\vartheta(y)\right] dy \\ &+ P.V.\int_{\mathbb{R}^{n+1}} D^{|\alpha|}\Gamma(x, x-y) L\vartheta(y) dy + L\vartheta(x)\int_{S^n} D^{|\beta|}\Gamma(x,y)\nu_i d\sigma_y \end{aligned} \tag{39}$$

where $\nu = (\nu_1, \dots, \nu_{n+1})$ is the outward normal to S^n. Here, $\Gamma(x,\xi)$ is the fundamental solution of the operator L. $\Gamma(x,t)$ can be represented in form

$$\Gamma(x,\xi) = \frac{1}{(n-2)\omega_n\left(det a_{\alpha\beta}\right)^{\frac{1}{2}}}\left(\sum_{i,j=1}^{n} A_{\alpha\beta}(x)\xi_i\xi_j\right)^{\frac{2-n}{2}}$$

for a.e. $x \in \mathbb{R}^{n+1}$ and $\forall \xi \in R^n \backslash \{0\}$, where $\left(A_{\alpha\beta}\right)_{n\times n}$ is inverse matrix for $\left(a_{\alpha\beta}\right)_{n\times n}$. Since any function $\vartheta \in W_p^{2m,1}(Q)$ can be approximated by C_0^∞ functions, the representation formula (39) still holds for any $\vartheta \in W_p^{2m,1}(C_r(x_0))$. The properties of the fundamental solution (see [7, 17]) imply that $D^{|\alpha|}\Gamma(x,y)$ are variable Calderon-Zygmund kernels in the sense of our definition above. By notation above, we can write

$$\begin{gathered} D^{\alpha,\beta}\vartheta(x) = D^{\alpha,\beta}C\left[a_{\alpha,\beta},\vartheta\right](x) + D^{\alpha,\beta}B(L\vartheta)(x) + L\vartheta(x)\int_{S^n} D^{\alpha}\Gamma(x,y)\nu_i d\sigma y. \\ |\alpha|, |\beta| \le m. \end{gathered} \tag{40}$$

The operators $D^{\alpha,\beta}B$ and $D^{\alpha,\beta}C$ are defined by (30) with $K(x, x-y) = D^{\alpha,\beta}\Gamma(x, x-y)$. Due to (30) and (31) and the equivalence of the metrics we obtain for $\mathcal{E} > 0$ there exists $r_0(\mathcal{E})$ such that for any $r < r_0(\mathcal{E})$

$$\|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_r(x_0))} \le C\left(\|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_r(x_0))} + \|L\vartheta\|_{M_{p,\varphi}(C_r(x_0))}\right) \tag{41}$$

for some r small enough. From (41) we get that

$$\|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_r(x_0))} \le C(n,p,\varphi_\alpha)\cdot\|D^{\alpha,\beta}\Gamma\|_{L_\infty(Q)})\|L\vartheta\|_{M_{p,\varphi}(C_r(x_0))}.$$

Define a cut-off function $\psi(x) = \psi_1(x')\psi_2(t)$, with $\psi_1 \in C_0^\infty\left(B_r(x_0')\right)$, $\psi_2 \in C_0^\infty(R)$ such that

$$\psi_1(x') = \begin{matrix} 1, & x' \in B_{\theta_r}(x_0') \\ 0, & x' \notin B_{\theta_r'}(x_0'), \end{matrix}$$

$$\psi_2(t) = \begin{matrix} 1, & t \in \left(t_0 - (\theta r)^2, t_0\right] \\ 0, & t < (t_0 - (\theta/r)^2 \end{matrix}$$

with $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > 0$ and $|D^\alpha \psi| \le C[\theta(1-\theta)r]^{-\alpha}$, $|\alpha| \le 2m$, $|\psi_t| \sim |D^\alpha \psi|$. For any solution $u \in W_p^{2m,1}(Q)$ of (1) and (2) define $\vartheta(x) = \varphi(x)u(x) \in W_p^{2m,1}(C_r)$. Hence,

$$\begin{aligned}
&\|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \|D^{\alpha,\beta}\vartheta\|_{M_{p,\varphi}(C_{\theta' r}(x_0))} \\
&\le C\|L\vartheta\|_{M_{p,\varphi}(C_{\theta' r}(x_0))} \le C\|f\|_{M_{p,\varphi}(C_{\theta' r}(x_0))} + \frac{\|D^\alpha u\|_{M_{p,\varphi}(C_{\theta' r}(x_0))}}{\theta(1-\theta)r} + \frac{\|u\|_{M_{p,\varphi}(C_{\theta' r}(x_0))}}{[\theta(1-\theta)r]^2}.
\end{aligned}$$

As so,

$$\begin{aligned}
&[\theta(1-\theta)r]^2 \|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \\
&\le C\left(r^2\|f\|_{M_{p,\varphi}(Q)}\right) + \theta'(1-\theta')r\|D^\alpha u\|_{M_{p,\varphi}(C_{\theta'\cdot r}(x_0))} + \|u\|_{M_{p,\varphi}(C_{\theta'\cdot r}(x_0))}.
\end{aligned}$$

We introduce

$$\theta_\alpha = \sup_{0<\theta<1} [\theta(1-\theta)r]^\alpha \|D^\alpha u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))}, \quad |\alpha| \le 2m,$$

the above inequality becomes

$$[\theta(1-\theta)r]^2 \cdot \|D^\alpha u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \theta_{2m} \le C\left(r^2\|f\|_{M_{p,\varphi}(Q)+\theta_m+\theta_0}\right) \tag{42}$$

Now we use following interpolation inequality (see [5])

$$\theta_m \le \varepsilon \cdot \theta_{2m} + \frac{c}{\varepsilon}\theta_0 \ \ for \ \ any \ \ \varepsilon \in (0, 2m).$$

where there exists a positive constant C independent of r. Thus (42) becomes

$$[\theta(1-\theta)r]^2 \|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))} \le \theta_{2m} \le C\left(r^2 + \theta_0\right), \ \forall \theta \in (0,1).$$

Taking $\theta = \frac{1}{2}$ we obtain the Caccioppoli-type estimate

$$\|D^{\alpha,\beta}u\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} \le C\left(\|f\|_{M_{p,\varphi}(Q)} + \frac{1}{r^2}\|u\|_{M_{p,\varphi}(C_{\theta\cdot r}(x_0))}\right)$$

We get the boundedness of the coefficients

$$\begin{aligned}
&\|u_t\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} \le \|a\|_{L_\infty(Q)} \cdot \|D^{\alpha,\beta}u\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} + \\
&+\|f\|_{M_{p,\varphi}\left(C_{r/2}(x_0)\right)} \le C\left(\|f\|_{M_{p,\varphi}(Q)+\frac{1}{r_2}}\|u\|_{M_{p,\varphi}(C_r(x_0))}\right).
\end{aligned}$$

Let $Q' = \Omega' \times (0,T)$ and $Q'' = \Omega'' \times (0,T)$ the cylinders with $\Omega' \Subset \Omega'' \Subset \Omega$. By the standard covering procedure and partition of the unity we obtain that

$$\|u\|_{W^{2m,1}_{p,\varphi}(Q')} \le C\left(\|f\|_{M_{p,\varphi}(Q)}\right) + \|u\|_{M_{p,\varphi}(Q''))} \tag{43}$$

where C depends on $n, p, \Lambda, T, \|D\Gamma\|_{L_\infty(Q)}, \eta_\alpha$, $\|a\|_{L_\infty(Q)}$ and $dist(\Omega', \partial\Omega'')$. Now we give boundary estimates. For any fixed $(x^0, r) \in \mathbb{R}^{n+1} \times R_+$ define the semi-cylinders

$$C_r^+(x^0) = B_r^+(x^{0\prime}) \times (0, r^2) = |x^0 - x'| < r, \; x_n > 0, \; 0 < t < r^2$$

with $S_r^+ = (x'', 0, t) : |x^0 - x''| < r, \; 0 < t < r^2$. For any solution $u \in W_p^{2m,1}(C_r^+(x^0))$ with $supp\, u \in C_r^+(x^0)$, the following boundary representation formula holds (see [1, 7, 16]).

$$D^{\alpha,\beta}u(x) = \mathcal{C}_{ij}\left[a_{\alpha,\beta}, D^{\alpha,\beta}u\right](x) + \mathcal{B}_{ij}(Lu)(x) + Lu(x)\int_{S^n} D^\alpha \Gamma \nu_i d\sigma_y - J_{ij}(x),$$

where

$$J_{ij}(x) = \mathcal{B}_{ij}(Lu)(x) + \tilde{\mathcal{C}}_{ij}\left[a_{\alpha,\beta,D^{\alpha,\beta}}u\right](x), \; i,j = 1, \dots, n-1,$$

$$J_{in}(x) = J_{ni}(x) = \sum_{i=1}^{n} \left(\frac{\partial \tau(x)}{\partial x_n}\right)^l \left[\overline{\mathcal{C}}_{il}\left[a_{\alpha,\beta,D^{\alpha,\beta}u}\right](x) + \overline{\mathcal{B}}_{il}(Lu)(x)\right], \; i = 1, \dots, n$$

$$J_{nn}(x) = \sum_{r,l=1}^{n} \left(\frac{\partial \tau(x)}{\partial x_n}\right)^r \left(\frac{\partial \tau(x)}{\partial x_n}\right)^l \left[\overline{\mathcal{C}}_{il}\left[c, D^{\alpha,\beta}u\right](x) + \overline{\mathcal{B}}_{il}(Lu)(x)\right],$$

$$\left(\frac{\partial \tau(x)}{\partial x_n}\right) = \left(-2\frac{a_{\alpha,\beta}^{n1}(x)}{a_{\alpha,\beta}^{nn}(x)}, \dots, -2\frac{a_{\alpha,\beta}^{nn-1}(x)}{a_{\alpha,\beta}^{nn}(x)}, -1, 0\right).$$

Here $\overline{B}_{ij}$ and $\overline{C}_{ij}$ are non-singular operators defined by (36) with a kernel $K(x, \tau(x) - y) = D^{\alpha,\beta}\Gamma(x, \tau(x) - y)$. Applying the estimates (37) and (38) and having in mind that the components of the vector $\left(\frac{\partial \tau(x)}{\partial x_n}\right)$ are bounded, we obtain that

$$\|D^{\alpha,\beta}u\|_{M_{p,\varphi}(C_r(x_0))} \le C\left(\|f\|_{M_{p,\varphi}(Q)} + r^2\|u\|_{M_{p,\varphi}(C_r(x_0))}\right)$$

Taking r small enough we can move the norm of u on the left-hand side, obtaining that

$$\|u\|_{M_{p,\varphi}(C_r(x_0))} \le C\|f\|_{M_{p,\varphi}(Q)}$$

with a constant C depending on $n, p, \Lambda, T, \eta_\alpha, \|a\|_{L_\infty(Q)}$. By covering the boundary with small cylinders, using a partition of the unit subordinated by that covering and local flattening of $\partial\Omega$ we get that

$$\|u\|_{W_{p,\varphi}^{2m,1}(Q \setminus Q')} \le C\|f\|_{M_{p,\varphi}(Q)} \tag{44}$$

Using (43) and (44), we obtain (5).

7. The higher order elliptic equations in unbounded domains

Now we are consider boundary value the Dirichlet problem for higher order nondivergence uniformly elliptic equations with coefficients in modified Morrey spaces in unbounded domains Ω

$$\begin{aligned} Lu &= \sum_{|\alpha| \le |\beta| \le m} a_{\alpha,\beta} D^{\alpha,\beta}u = f(x) \; \text{ in } \Omega \\ D^\alpha u &= g(x) \; |\alpha| \le m-1 \; \text{ on } \partial\Omega \end{aligned} \tag{45}$$

where the coefficients matrix $a(x) = \left\{a^{ij}_{\alpha,\beta}(x)\right\}^{n}_{i,j=1}$ satisfies

$$\exists \Lambda > 0 \ \Lambda \sum_{|\alpha|=m} \xi_\alpha^2 \leq \sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}\xi_\alpha\xi_\beta, \tag{46}$$

for a.e. $x \in \Omega$, $\forall \xi \in R^n$, $a_{\alpha,\beta} = a_{\beta,\alpha}, \xi = \left\{\xi_\alpha, \||\alpha| = m \in R^N\right\}$, N–number different multiindeks with length equal to m.

Under these assumptions we prove that the maximal operator M are bounded from the modified Morrey space $\tilde{L}_{p,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$ if and only if,

$$\frac{\alpha}{n} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n-\lambda}.$$

For $x \in R^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius t and ${}^{\complement}B(x,t) = R^n \backslash B(x,t)$. One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x) = \sup_{t>0} \left|B(x,t)\right|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy, \ \ 0 \leq \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball B(x,t). The fractional maximal function $M_\alpha f$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$.

Let $1 \leq p < \infty, \ 0 \leq \lambda \leq n, \ [t]_1 = \min\{1,t\}$. We denote by $\tilde{L}_{p,\lambda}(R^n)$ the modified Morrey space, as the set of locally integrable functions $f(x), \ x \in R^n$, with the finite norm

$$\|f\|_{\tilde{L}_{p,\lambda}} = \sup_{x \in R^n, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p dy\right)^{\frac{1}{p}}$$

Note that

$$\tilde{L}_{p,0}(R^n) = L_{p,0}(R^n) = L_p(R^n),$$

$$\tilde{L}_{p,\lambda}(R^n) \hookrightarrow L_{p,\lambda}(R^n) \cap L_p(R^n) \ \text{and} \ \max\left\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\right\} \leq \|f\|_{\tilde{L}_{p,\lambda}},$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(R^n) = \tilde{L}_{p,\lambda}(R^n) = \theta$, where θ is the set of all functions equivalent to 0 on R^n. $W\tilde{L}_{p,\lambda}(R^n)$-the modified weak Morrey space as the set of locally integrable functions $f(x), \ x \in R^n$ with finite norm

$$\|f\|_{W\tilde{L}_{p,\lambda}} = \sup_{r>0} r \sup_{x \in R^n, t>0} \left([t]_1^{-\lambda} |\{y \in B(x,t) : |f(y)| > r\}|\right)^{\frac{1}{p}}.$$

Note that

$$W\tilde{L}_{p,0}(R^n) = WL_{p,0}(R^n) = WL_p(R^n),$$
$$\tilde{L}_{p,\lambda}(R^n) \subset W\tilde{L}_{p,\lambda}(R^n) \ \text{and} \ \|f\|_{W\tilde{L}_{p,\lambda}} \leq \|f\|_{\tilde{L}_{p,\lambda}}.$$

We study the $\tilde{L}_{p,\lambda}$-boundedness of the maximal operator M.

The classical result by Hardy-Littlewood-Sobolev states that if $1<p<q<\infty$, then the Riesz potential I_α is bounded from $L_p(R^n)$ to $L_q(R^n)$ if and only if $\alpha = n\left(\frac{1}{p}-\frac{1}{q}\right)$ and for $p=1<q<\infty$, I_α is bounded from $L_1(R^n)$ to $WL_q(R^n)$ if and only if $\alpha = n\left(1-\frac{1}{q}\right)$. D.R. Adams studied the boundedness of the I_α in Morrey spaces and proved the follows statement.

Theorem **(Adams)** Let $0<\alpha<n$ and $0\le\lambda<n-\alpha$, $1\le p<\frac{n-\lambda}{\alpha}$.

1. If $1<p<\frac{n-\lambda}{\alpha}$, then condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{p,\lambda}(R^n)$ to $L_{q,\lambda}(R^n)$.

2. If $p=1$, then condition $1-\frac{1}{q}=\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{1,\lambda}(R^n)$ to $WL_{q,\lambda}(R^n)$.

If $\alpha=\frac{n}{p}-\frac{n}{q}$, then $\lambda=0$ and the statement of Theorem reduced to the aforementioned result by Hardy-Littlewood-Sobolev Theorem also implies the boundedness of the fractional maximal operator M_α.

In this section we study the fractional maximal integral and the Riesz potential in the modified Morrey space. In the case $p=1$ we prove that the operator I_α is bounded from $\tilde{L}_{1,\lambda}(R^n)$ to $W\tilde{L}_{q,\lambda}(R^n)$ if and only if, $\frac{\alpha}{n}\le 1-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$. In the case $1<p<\frac{n-\lambda}{\alpha}$ we prove that the operator I_α is bounded from $\tilde{L}_{p,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$ if and only if, $\frac{\alpha}{n}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$.

Theorem 7.1. *If* $f\in\tilde{L}_{p,\lambda}(R^n),\ 1<p<\infty,\ 0\le\lambda<n,$ *then* $Mf\in\tilde{L}_{p,\lambda}(R^n)$ *and*

$$\|Mf\|_{\tilde{L}_{p,\lambda}}\le C_{p,\lambda}\|f\|_{\tilde{L}_{p,\lambda}},$$

where $C_{p,\lambda}$ depends only on p, λ and n.

Proof. We use Fefferman-Stein inequality and get

$$\int_{B(x,t)}(Mf(y))^p dy\le C\int_{R^n}|f(y)|^p M_{\chi_{B(x,t)}}(y)dy.$$

Later from some estimates for $M_{\chi_{B(x,t)}}$ we have the following inequalities

$$\int_{B(x,t)}(Mf(y))^p dy\le C\left(\int_{B(x,t)}|f(y)|^p dy+\right.$$
$$\left.+\sum_{j=0}^{\infty}\int_{B\left(x,2^{j+1}t\right)\backslash B\left(x,2^j t\right)}\frac{t^n|f(y)|^p dy}{(|x-y|+t)^n}\right)\le C\,[t]_1^\lambda\cdot\|f\|_{\tilde{L}_{p,\lambda}}^p.$$

□

Theorem 7.2. *(see [18]) Let* $0<\alpha<n,\ 0\le\lambda<n-\alpha$ *and* $1\le p<\frac{n-\lambda}{\alpha}$.

1. If $1<p<\frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the Riesz potential operator I_α from $\tilde{L}_{p,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$.

2. If $p=1<\frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n}\le 1-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $\tilde{L}_{1,\lambda}(R^n)$ to $\tilde{L}_{q,\lambda}(R^n)$.

Recall that, for $0 < \alpha < n$

$$M_\alpha f(x) \le \nu_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x)$$

where ν_n is the volume of the unit ball in R^n. From [7] for unbounded domains $\Omega \subset R^n$ we have following result.

Theorem 7.3. *Let $\Omega \subset R^n$ be an unbounded domains with noncompact boundary $\partial\Omega$, and $0 < \alpha < n$, $0 \le \lambda < n - \alpha$ and $1 < p < \frac{n-\lambda}{\alpha}$. Also let satisfies conditions $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$, $f \in \tilde{L}_{q,\lambda}(\Omega)$, function $U(x)$ is a solution of problem (45). Then there is exist constant C which dependent only at n, λ, p, q, Ω such that*

$$\|U\|_{\tilde{W}^{2m}_{p,\lambda}}(\Omega) \le C\|f\|_{\tilde{L}_{q,\lambda}(\Omega)}, \tag{47}$$

where $\tilde{W}^{2m}_{p,\lambda}$-is correspondingly modified Sobolev-Morrey space.

The proved Theorem 7.3 consequence from methods of [7] and Theorem 7.1 and 7.2.

Additional classifications

Mathematics Subject Classifications (2010): 35 J25, 35B45,42B20, 47B38

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Chapter 18

Gradient Optimal Control of the Bilinear Reaction–Diffusion Equation

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Abstract

In this chapter, we study a problem of gradient optimal control for a bilinear reaction–diffusion equation evolving in a spatial domain $\Omega \subset \mathbb{R}^n$ using distributed and bounded controls. Then, we minimize a functional constituted of the deviation between the desired gradient and the reached one and the energy term. We prove the existence of an optimal control solution of the minimization problem. Then this control is characterized as solution to an optimality system. Moreover, we discuss two special cases of controls: the ones are time dependent, and the others are space dependent. A numerical approach is given and successfully illustrated by simulations.

Keywords: distributed bilinear systems, reaction–diffusion equation, controllability, optimal control

1. Introduction

The controllability of distributed bilinear systems governed by partial differential equations has been studied by many authors: in [1], the authors developed the weak controllability of the beam and rod equations in the mono-dimensional case. In [2], the author considered the controllability of semilinear parabolic and hyperbolic systemse using distributed controls. In [3], the author studied the exact controllability of the semilinear wave equations in one space dimension. The optimal control problem for a class of infinite dimensional bilinear systems have been consedered in many works. In [4], the author proved the existence and characterization of an optimal control of a bilinear convective-diffusive fluid model using bounded controls. In [5], the author developed optimal control problem of a bilinear heat equation with distributed bounded control. In [6], the authors studied optimal control for a class of bilinear systems using unbounded control. In [7], the authors considered the optimal control problem of the wave equation using bounded boundary control. In [8], the authors considered the optimal control problem of the Kirchhoff plate equation with distributed bounded controls. In [9], the author proved the optimal control of the bilinear wave equation using distributed and bounded controls. The regional optimal control problem of a class of infinite dimensional bilinear systems with unbounded controls was developed in [10], then the authors studied the existence and characterization of an optimal control.

In [11], the authors studied the constrained regional optimal control of a bilinear plate equation using distributed and bounded controls. The notion of gradient controllability is very important, since its close to real applications and there exist systems that cannot be controllable but gradient of the state is controllable. Then in [12], the authors proved the regional controllability of parabolic systems using HUM method.

In the present work, we study the gradient optimal control problem of the bilinear reaction–diffusion equation using distributed and bounded controls. Then, we examine the existence and we give characterization of an optimal control. Also, an algorithm and simulations are given. Let Ω be an open bounded domain of $\mathbb{R}^n$, $(n \geq 1)$ with a $\mathcal{C}^2$ boundary $\partial\Omega$, we denote by $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$, and we consider the bilinear reaction–diffusion equation

$$\begin{cases} y_t(x,t) - \Delta y(x,t) = u(x,t)y(x,t) & \text{in } Q \\ y(x,0) = y_0(x) & \text{in } \Omega \\ y(x,t) = 0 & \text{on } \Sigma, \end{cases} \tag{1}$$

where $u \in \mathcal{U}_\rho := \{u \in L^\infty(Q) \mid -\rho \leq u \leq \rho \ \ a.e.\ in\, Q\}$ is a scalar control function, and ρ is a positive constant.

Let us consider the following state space

$$\mathcal{H} := L^2\left(0, T; H_0^1(\Omega)\right).$$

For all $y_0 \in H_0^1(\Omega)$ and $u \in \mathcal{U}_\rho$, the system (1) has a unique weak solution $y \in \mathcal{H}$ (see for example [13, 14]).

Define the operator

$$\nabla : H_0^1(\Omega) \to \left(L^2(\Omega)\right)^n$$
$$y \to \nabla y = \left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}\right),$$

and ∇^* its adjoint.

Let us recall that the system (1) is weakly gradient controllable if for all $y^d \in \left(L^2(\Omega)\right)^n$ and $\varepsilon > 0$, there exist a control $u \in \mathcal{U}_\rho$ such that

$$\|\nabla y(.,T) - y^d(.)\|_{\left(L^2(\Omega)\right)^n} \leq \varepsilon,$$

where $y^d = \left(y_1^d, \dots, y_n^d\right)$ is the gradient of the desired state in $\left(L^2(\Omega)\right)^n$.

Our problem consists in finding a control u that steers the gradient of state close to y^d, over the time interval $[0, T]$ with a reasonable amount of energy. This may be stated as the following minimization problem

$$\min_{u \in \mathcal{U}_\rho} J(u), \tag{2}$$

where

$$J(u) = \frac{1}{2}\int_0^T \|\nabla y(.,t) - y^d(.)\|^2_{\left(L^2(\Omega)\right)^n} dt + \frac{\beta}{2}\int_Q u^2(x,t)dQ, \tag{3}$$

with $\beta > 0$.

The rest of the paper is organized as follows: in section 2, we study the existence of an optimal control solution of (2). In section 3, we give a characterization of an optimal control solution of the problem (2), and we discuss two special cases of an optimal control solution of such problem. Finally, in section 4, we present an algorithm and simulations.

2. Existence of an optimal control

The main result of the existence of an optimal control solution of (2) is given by the following theorem.

Theorem 1. There exists an optimal control $u^* \in \mathcal{U}_\rho$, solution of (2).

Proof: Let u^n be a minimizing sequence in $\mathcal{U}_\rho$, such that

$$\lim_{n\to+\infty} \inf J(u^n) = \inf_{u\in\mathcal{U}_\rho} J(u). \tag{4}$$

Then, according to the nature of the cost function J, we can deduce that u^n is uniformly bounded in $\mathcal{U}_\rho$.

So, we can extract from u^n a subsequence also denoted by u^n such that $u^n \rightharpoonup u$ weakly in $\mathcal{U}_\rho$.

In other hand, using the weak form of system (1), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|y^n\|^2_{L^2(\Omega)} + \int_\Omega \nabla y^n \nabla y^n dx = \int_\Omega u^n |y^n|^2 dx. \tag{5}$$

By integration with respect to time and using the function u^n is uniformly bounded in $L^\infty(Q)$, we have

$$\|y^n\|^2_{L^2(\Omega)} + \int_0^t \|y^n\|_{H_0^1(\Omega)} ds \le c_1 \int_0^t \|y^n\|^2_{L^2(\Omega)} ds, \tag{6}$$

where c_1 is a positive constant.

Using Gronwall's Lemma, we deduce that y^n uniformly bounded in $L^\infty(0,T;L^2(\Omega))$, and then y^n uniformly bounded in $L^2(0,T;H_0^1(\Omega))$.

Using the previous result and system (1), we obtain that y_t^n is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$, and then y^n is uniformly bounded in $\mathcal{H}$.

Using the above bounds, we can extract a subsequence satisfying the following convergence properties

$$y^n \rightharpoonup y^* \quad \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)) \tag{7}$$

$$y^n \to y^* \quad \text{strongly in} \quad L^2(Q) \tag{8}$$

$$u^n \rightharpoonup u^* \quad \text{weakly in} \quad L^2(Q). \tag{9}$$

Since $\mathcal{U}_\rho$ is a closed and convex subset of $L^\infty(Q) \subset L^2(Q)$, $\mathcal{U}_\rho$ is weakly closed in $L^2(Q)$. Then $u^* \in \mathcal{U}_\rho \subset L^2(Q)$. On the other hand, since $-\rho \le u^n \le \rho$ for all n, $u^n \rightharpoonup u^{**}$ weakly* in $L^\infty(Q)$, and hence $u^n \rightharpoonup u^{**}$ weakly in $L^2(Q)$. By the uniqueness of the weak limit, we obtain $u^* = u^{**}$ and $u^* \in \mathcal{U}_\rho \subset L^\infty(Q)$.

Now, we show that $u^n y^n \to u^* y^*$ weakly in $L^2(Q)$.

Since $u^n y^n - u^* y^* = u^n(y^n - y^*) + (u^n - u^*)y^*$, and using (7), (8) and (9), we obtain $u^n y^n \to u^* y^*$ weakly in $L^2(Q)$.

Thus $y^* = y(u^*)$ is the solution of system (1) with control u^*.

Since the functional J is lower semi-continuous with respect to weak convergence (basically Fatou's lemma), we obtain

$$\begin{aligned} J(u^*) &\le \frac{1}{2} \liminf_{n\to+\infty} \int_0^T \|\nabla y^n(.,t) - y^d(.)\|^2_{(L^2(\Omega))^n} dt + \frac{\beta}{2} \liminf_{n\to+\infty} \int_Q (u^n)^2(x,t)dxdt \\ &\le \liminf_{n\to+\infty} J(u^n) \\ &= \inf_{u\in\mathcal{U}_\rho} J(u). \end{aligned}$$

Finally, we conclude that u^* is an optimal control.

3. Characterization of an optimal control

This section is devoted to characterization of an optimal control solution of the problem (2).

3.1 Time and space control dependent

In this part, we give characterization of an optimal control that depend on time and space.

The following result give the differentiability of the mapping $u \to y(u)$.

Lemma 1 The mapping $u \in \mathcal{U}_\rho \to y(u) \in \mathcal{H}$ is differentiable in the following sense

$$\frac{y(u+\varepsilon h) - y(u)}{\varepsilon} \rightharpoonup \phi \ weakly \ \ in \ \ \mathcal{H} \ \ as \ \varepsilon \to 0, for\, any\, u, u+\varepsilon h \in \mathcal{U}_\rho$$

Moreover, $\phi = \phi(y,h)$ satisfies the following system

$$\begin{cases} \phi_t(x,t) - \Delta\phi(x,t) = u(x,t)\phi(x,t) + h(x,t)y(x,t) & \text{on} \quad Q \\ \phi(x,0) = 0 & \text{in} \quad \Omega \\ \phi(x,t) = 0 & \text{in} \quad \Sigma. \end{cases} \tag{10}$$

Proof: Consider $y^\varepsilon = y(u+\varepsilon h)$ and $y = y(u)$. Then $\left(\frac{y^\varepsilon - y}{\varepsilon}\right)$ is a weak solution of

$$\begin{cases} \left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t - \Delta\left(\frac{y^\varepsilon - y}{\varepsilon}\right) = u\left(\frac{y^\varepsilon - y}{\varepsilon}\right) + hy^\varepsilon & \text{on} \quad Q \\ \left(\frac{y^\varepsilon - y}{\varepsilon}\right)(x,0) = 0 & \text{in} \quad \Omega \\ \left(\frac{y^\varepsilon - y}{\varepsilon}\right)(x,t) = 0 & \text{in} \quad \Sigma. \end{cases}$$

Using the result (6), it follows that

$$\left\|\frac{y^\varepsilon - y}{\varepsilon}\right\|_{\mathcal{H}} \le C,$$

where C depends on the L^∞ bound on h, but is independent of ε. Hence on a subsequence, by weak compactness, we have

$$\frac{y^\varepsilon - y}{\varepsilon} \rightharpoonup \phi \quad \text{weakly} \quad \text{in} \quad L^\infty\left([0,T]; H_0^1(\Omega)\right)$$
$$\left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t \rightharpoonup \phi_t \quad \text{weakly} \quad \text{in} \quad L^\infty\left([0,T]; H^{-1}(\Omega)\right).$$

By the definition of weak solution, we have

$$\left\langle \left(\frac{y^\varepsilon - y}{\varepsilon}\right)_t, \psi \right\rangle - \int_\Omega \nabla \left(\frac{y^\varepsilon - y}{\varepsilon}\right) \nabla \psi dx = \int_\Omega u \left(\frac{y^\varepsilon - y}{\varepsilon}\right) \psi dx + \int_\Omega h y^\varepsilon \psi dx, \tag{11}$$

for any $\psi \in H_0^1(\Omega)$, and a.e $0 \le t \le T$.
Letting $\varepsilon \to 0$ in (11), we conclude that ϕ is the weak solution of system (10).
Now, we give characterization of an optimal control that depend on time and space.
Theorem 2 An optimal control solution of problem (2) is given by the formula

$$u^*(x,t) = \max\left(-\rho, \min\left(-\frac{1}{\beta}\sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t), \rho\right)\right), \tag{12}$$

where $p \in \mathcal{C}([0,T]; \mathcal{H})$ is the weak solution of the adjoint system

$$\begin{cases} p_{i_t}(x,t) - \Delta p_i(x,t) = -u^*(x,t) p_i(x,t) & \text{on} \quad Q \\ p_i(x,T) = \left(\dfrac{\partial y(T)}{\partial x_i} - y_i^d\right) & \text{in} \quad \Omega \\ p_i(x,t) = 0 & \text{in} \quad \Sigma. \end{cases} \tag{13}$$

Proof: Let $u^* \in \mathcal{U}_\rho$ and $y = y(u^*)$ be the corresponding weak solution, and let $u^* + \varepsilon h \in \mathcal{U}_\rho$, for $\varepsilon > 0$ and $y^\varepsilon = y(u^* + \varepsilon h)$.
Since J reaches its minimum at u^*, then

$$\begin{aligned} 0 \le \lim_{\varepsilon \to 0^+} \frac{J(u^* + \varepsilon h) - J(u^*)}{\varepsilon} &= \lim_{\varepsilon \to 0^+} \sum_{i=1}^n \frac{1}{2} \int_\Omega \left(\int_0^T \frac{\partial \phi}{\partial x_i} \frac{\partial p_i}{\partial t} dt \right. \\ &+ \left. \int_0^T \left(-\Delta \frac{\partial \phi}{\partial x_i} + u \frac{\partial \phi}{\partial x_i} + h \frac{\partial y}{\partial x_i} p_i \right) dt \right) dx \\ &+ \lim_{\varepsilon \to 0^+} \frac{\beta}{2} \int_Q \left(2hu^* + \varepsilon h^2\right) dQ. \end{aligned}$$

Then

$$0 \le \int_Q \beta h u \, dQ + \sum_{i=1}^n \int_Q h \frac{\partial y}{\partial x_i} p_i dQ = \int_Q h \left(\beta u + \sum_{i=1}^n \int_Q h \frac{\partial y}{\partial x_i} p_i \right) dQ.$$

Using a standard control argument based on the choices for the variation $h(x,t)$, an optimal control is given by

$$u^*(x,t) = \max\left(-\rho, \min\left(-\frac{1}{\beta}\sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t), \rho\right)\right).$$

3.2 Time or space control dependent

In this subsection, we study two cases of controls: the first ones are time dependent $u(t)$, and the others are space dependent $u(x)$.

- **Case 1**: $u = u(t)$.

Here, we consider the admissible controls set

$$\mathcal{U}_\rho = \{u \in L^\infty(0,T) \;\;:\;\; -\rho \le u \le \rho \qquad \text{a.e in } \;(0,T)\} \tag{14}$$

and we take the functional cost

$$J(u) = \frac{1}{2}\int_0^T \|\nabla y(.,t) - y^d(.)\|^2_{\left(L^2(\Omega)\right)^n} dt + \frac{\beta}{2}\int_0^T u^2(t)dt. \tag{15}$$

Corollary 1 Under conditions (14) and (15), an optimal control is given by the formula

$$u(t) = \max\left(-\rho,\; \min\left(-\frac{1}{\beta}\int_\Omega \sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t)dx, \rho\right)\right), \tag{16}$$

where y is the weak solution of the equation

$$\begin{cases} y_t(x,t) - \Delta y(x,t) = u(t)y(x,t) & on \;\; Q \\ y(x,0) = y_0(x), & in \;\; \Omega \\ y(x,t) = 0 & in \;\; \Sigma, \end{cases}$$

and p_i is the weak solution of the adjoint equation

$$\begin{cases} p_{i_t}(x,t) - \Delta p_i(x,t) = -u^*(t)p_i(x,t) & \text{on} \quad Q \\ p_i(x,T) = \left(\dfrac{\partial y(T)}{\partial x_i} - y_i^d\right) & \text{in} \quad \Omega \\ p_i(x,t) = 0 & \text{in} \quad \Sigma. \end{cases}$$

Proof: Using the same steps as in the proof of Theorem 2, let $h = h(t)$ be an arbitrary function with $u + \varepsilon h \in \mathcal{U}_\rho$ for small ε.

We have

$$\int_0^T h(t)\left(\int_\Omega \sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t)dx + \beta u(t)\right)dt \ge 0.$$

By using a standard control argument concerning the sign of the variation h, we obtain

$$u(t) = \max\left(-\rho,\; \min\left(-\frac{1}{\beta}\int_\Omega \sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t)dx, \rho\right)\right).$$

- **Case 2:** $u = u(x)$.

We consider the admissible controls set

$$\mathcal{U}_\rho = \{u \in L^\infty(\Omega) \;\;:\;\; -\rho \le u \le \rho \qquad \text{a.e in } \;\Omega\} \tag{17}$$

and we take the functional cost

$$J(u) = \frac{1}{2}\int_0^T \|\nabla y(.,t) - y^d(.)\|^2_{\left(L^2(\Omega)\right)^n} dt + \frac{\beta}{2}\int_\Omega u^2(x)dx. \tag{18}$$

Corollary 2 Under conditions (17) and (18), an optimal control satisfies

$$u(x) = \max\left(-\rho, \min\left(-\frac{1}{\beta}\int_0^T \sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t)dt, \rho\right)\right), \tag{19}$$

where y is the solution of system

$$\begin{cases} y_t(x,t) - \Delta y(x,t) = u(x)y(x,t) & \text{on } Q \\ y(x,0) = y_0(x), & \text{in } \Omega \\ y(x,t) = 0 & \text{in } \Sigma, \end{cases}$$

and p_i is the solution of system

$$\begin{cases} p_{i_t}(x,t) - \Delta p_i(x,t) = -u^*(x,t)p_i(x,t) & \text{on } \quad Q \\ p_i(x,T) = \left(\dfrac{\partial y(T)}{\partial x_i} - y_i^d\right) & \text{in } \quad \Omega \\ p_i(x,t) = 0 & \text{in } \quad \Sigma. \end{cases}$$

Proof: Using the same notations as in the proof of Theorem 2, let $h = h(x)$ be an arbitrary function with $u + \varepsilon h \in \mathcal{U}_\rho$ for small ε.

We have

$$\int_\Omega h(x)\left(\int_0^T \sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t)dt + \beta u(x)\right)dx \geq 0.$$

A standard control argument gives

$$u(x) = \max\left(-\rho, \min\left(-\frac{1}{\beta}\int_0^T \sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t)dt, \rho\right)\right).$$

4. Algorithm and simulations

We have the solution of the problem (2) is given by the formula

$$u^*(x,t) = \max\left(-\rho, \min\left(-\frac{1}{\beta}\sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t), \rho\right)\right),$$

where y^* is the weak solution of the Eq. (1) and p_i is the weak solution of the adjoint Eq. (13).

The computation of an optimal control solution the problem (2) can be realized by

$$\begin{cases} u^*_{n+1}(x,t) = \max\left(-\rho, \min\left(-\dfrac{1}{\beta}\sum_{i=1}^n \dfrac{\partial y^n(x,t)}{\partial x_i} p_i^n(x,t), \rho\right)\right), \\ u^*_0 = 0, \end{cases} \tag{20}$$

where y^n is the solution of the Eq. (1) associated to u_n^* and p^n is the solution of the adjoint Eq. (13). Then, we consider the following algorithm

Step 1 : Initialization

$\odot$Initial state y_0, u_0^* and y^d.

$\odot$Threshold accuracy ε and the final time T.

Step 2 :

$\odot$Solving the system (1) gives y^n.

$\odot$Solving the system (13) gives p^n.

$\odot$Calculate u_{n+1}^* by the formula (20).

Until $\|u_{n+1}^* - u_n^*\|_{L^\infty(Q)} \leq \varepsilon$ stop, else $n = n+1$ go to step 2.

Step 3 : The control u_n^* is optimal.

4.1 Simulations

On $\Omega =]0,1[$, we consider the following equation

$$\begin{cases} y_t(x,t) - \Delta y(x,t) = u(t)y(x,t) & \text{on} \quad Q \\ y(x,0) = x(1-x)(1+x), & \text{in} \quad \Omega \\ y(x,t) = 0 & \text{in} \quad \Sigma, \end{cases} \tag{21}$$

and consider problem (2) with the control set

$$\mathcal{U}_\rho = \{u \in L^\infty(0,T) \quad : \quad -\rho \leq u \leq \rho \quad \text{a.e in} \quad (0,T)\}.$$

An optimal control solution of problem (2) is given by the following formula

$$u^*(t) = \max\left(-\rho, \min\left(-\frac{1}{\beta}\int_0^1 \sum_{i=1}^n \frac{\partial y(x,t)}{\partial x_i} p_i(x,t)dx, \rho\right)\right),$$

where y^* is solution of the Eq. (21) associated to the control u^* and p is the solution of the following adjoint system

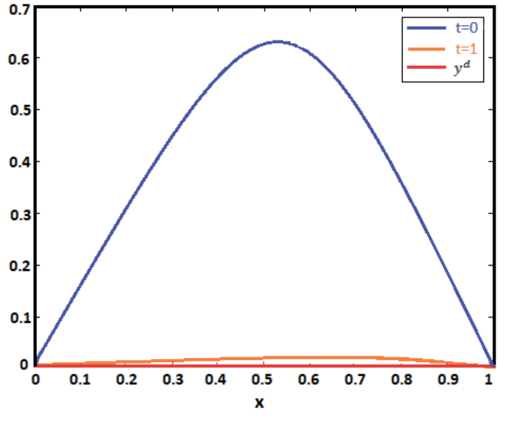

Figure 1.
The gradient of the state on]0, 1[.

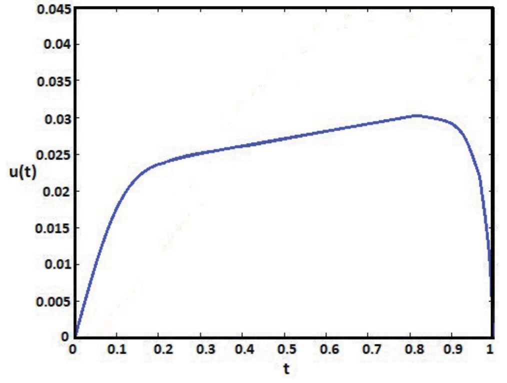

Figure 2.
Evolution of the control function.

$$\begin{cases} p_{i_t}(x,t) - \Delta p_i(x,t) = u^*(t)p_i(x,t) & \text{on} \quad Q \\ p_i(x,T) = \left(\dfrac{\partial y(T)}{\partial x_i} - y_i^d\right) & \text{in} \quad \Omega \\ p_i(x,t) = 0 & \text{in} \quad \Sigma. \end{cases}$$

We take $T = 1$, $\rho = 1$, $\beta = 0.1$, $y_0(x) = x(1-x)(1+x)$, and $y^d(x) = 0$. Applying the previous algorithm, with $\varepsilon = 10^{-4}$ we obtain.

Figure 1 shows that the gradient state is very close to the desired one with error $\|\nabla y(T)\| = 5.33 \times 10^{-5}$ and the evolution of control is given by **Figure 2**.

5. Conclusion

Gradient optimal control problem of the bilinear diffusion equation with distributed and bounded controls is considered. The existence and characterized of an optimal control are proved. The obtained results are tested by numerical examples. Questions are still open, as is the case of gradient optimal control problem of the semilinear reaction–diffusion equation.

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Chapter 19

Boundary Element Method for the Mixed BBM-KdV Equation Compared to Non Standard Boundary Conditions

Mostafa Abounouh, Hassan Al-Moatassime, Sabah Kaouri and Youssef Ouakrim

Abstract

In this chapter, we are interested in the numerical resolution of the mixed BBM-KdV equation defined in unbounded domain. Boundary Element Method (BEM) are introduced to truncate the equation into a considered bounded domain. BEM uses domain decomposition techniques to construct Boundary Condition (BC) as transmission between the bounded domain and its complementary. We then present a suitable approximation of these BC using Discrete Galerkin Method. Numerical simulations are made to show the efficiency of these BC. We also compare with another method that truncates the equation from unbounded to bounded domain, called Non Standard Boundary Conditions (NSBC) which introduces new variables to catch information at the boundary and compose a system to connect all these variables in the bounded domain. Further discussions are made to highlight the advantages of each method as well as the difficulties encountered in the numerical resolution.

Keywords: wave equations, transparent boundary condition, boundary element method, non-standard boundary conditions, finite difference method

1. Introduction

We consider a combination of two linearized typical dispersive partial differential equations that model solitary waves and all interactions between them, given as follows

$$\begin{cases} \partial_t u(t,x) + \alpha\partial^3_{xxx}u(t,x) - \beta\partial^3_{txx}u(t,x) + \gamma\partial_x u(t,x) = & 0 & (t,x)\in\mathbb{R}^*_+\times\mathbb{R} \\ u(0,x) = & u_0(x) & x\in\mathbb{R} \\ \lim\limits_{|x|\to\pm\infty} u(t,x) = & 0 & t\in\mathbb{R}^*_+ \end{cases} \tag{1}$$

such that α, β are dispersion parameters and are positive numbers, while $\gamma\in\mathbb{R}$ is the velocity number. In the case $\alpha=0$, we obtain the BBM equation [1] and when $\beta=0$, we get the KdV equation [2]. Our main purpose is to obtain numerical approximation of Eq. (1) when taken in a bounded domain $[0,T]\times[a,b]$ with suitable boundary conditions with no spurious reflections. For this regard, we use two different techniques that are BEM and NSBC.

The Boundary Element Method (BEM), also known as the Boundary Integral Equation Method (BIEM), is an alternative deterministic method that incorporates a mesh located, only, at domain boundaries and therefore attractive for free surface problems. There are two types of BEM, the direct BEM which requires a closed boundary so that the physical variables (e.g. pressure and normal velocity in acoustics) can only be considered from one side of the surface (interior or exterior), while the indirect (IBEM) can consider both sides of the surface and does not need a closed surface. In the first part of this chapter, we use this technique of BEM to derive the BC to the Eq. (1) in the domain $[0, T] \times [a, b]$. More precisely, we are going to introduce the BEM to establish BC satisfied by the Eq. (1) on two interface points a and b by solving the same equation in the complementary domain $\mathbb{R}\backslash[a, b]$. The BEM has significant advantages over the finite element or difference methods (FEM or FDM), as there is no need for discretizing the domain $\mathbb{R}\backslash[a, b]$ into elements. It only uses infinite boundary condition and transmission condition to compute the solution at a and b as integral equations. Consequently, this integral equations will be fixed as the boundary conditions of the problem (1) on the bounded domain $[0, T] \times [a, b]$. Therefore, the boundary condition are approximated as Fredholm Integral Equations of second kind.

Despite the meshing effort is limited and the system matrices are smaller, the BEM also has disadvantages over the Finite Element Method or Difference Finite Method. In fact, the BEM matrices are mostly populated with complex coefficients. Furthermore, singularities may arise in the solution. These deteriorate the efficiency of the solution and must be prevented [2].

The outline of this chapter is organized as follows. In section 2, we describe the BEM for the mixed BBM-KdV equation [3]. Next, we discuss the special case of the BBM equation and give the approximation of the resulting equation Finite Difference Method. Section 3 presents briefly another method to derive boundary conditions for BBM equation called NSBC introduced in [4]. Finally in section 4, comparison of both methods is given with numerical experiments to highlight the transparency of both BC obtained in sections 2 and 3.

2. Boundary element method for the mixed BBM-KdV equation

Being in one dimensional space, $\mathbb{R}$, the boundary of any bounded interval reduces to two points. Hence, we use the BEM to find two values that might depend on time. For this regard, we consider a bounded domain $\Omega_T =]0, T[\times\Omega$ where $\Omega =]a, b[$ and $a, b, T \in \mathbb{R}$ such that $a < b, T > 0$. Note $\Sigma = \{a, b\}$ and $\Sigma_T =]0, T[\times\Sigma$. we take the decomposition $\mathbb{R} = \Omega_g \cup \Omega \cup \Omega_d$, such that, $\Omega_g =]-\infty, a]$ and $\Omega_d = [b, +\infty[$. The corresponding equations to (1) using Dirichlet-to-Neumann domain decomposition write

$$\begin{cases} \partial_t u(t,x) + \alpha\partial^3_{xxx}u(t,x) - \beta\partial^3_{txx}u(t,x) + \gamma\partial_x u(t,x) & = 0 \quad \text{in}\,\Omega_T \\ u(0,x) & = u_0(x) \quad \text{at}\,\Omega \\ \partial_n u & = \partial_n w \quad \text{at}\,\Sigma_T \end{cases} \tag{2}$$

$$\begin{cases} \partial_t w(t,x) + \alpha\partial^3_{xxx}w(t,x) - \beta\partial^3_{txx}w(t,x) + \gamma\partial_x w(t,x) & = 0 \quad \text{in}\,\Omega_{g_T} \cup \Omega_{d_T} \\ w(0,x) & = 0 \quad \text{in}\,\Omega_g \cup \Omega_d \\ w & = u \quad \text{in}\,\Sigma_{g_T} \cup \Sigma_{d_T} \\ \lim\limits_{|x|\to+\infty} w(t,x) & = 0 \quad \text{at}\,]0, T[\end{cases} \tag{3}$$

The main object of this section is to prove the following result.

Lemma 2.1 The solution of the evolution Eq. (3) satisfies the following integral equations

$$\begin{aligned} w(t,a) - U_2\mathcal{L}^{-1}\left(\frac{\lambda_1(s)^2}{s}\right) * w_x(t,a) - U_2\mathcal{L}^{-1}\left(\frac{\lambda_1(s)}{s}\right) * w_{xx}(t,a) = 0 \\ w(t,b) - \mathcal{L}^{-1}\left(\frac{1}{\lambda_1(s)^2}\right) * w_{xx}(t,b) = 0, \\ w_x(t,b) - \mathcal{L}^{-1}\left(\frac{1}{\lambda_1(s)}\right) * w_{xx}(t,b) = 0 \end{aligned} \tag{4}$$

where $\mathcal{L}^{-1}(f(s))$ stands for the inverse Laplace transform of f, $*$ denotes the convolution operator and λ_1 a function of the time co-variable s.

Proof. We apply the Laplace transformation with respect to the time variable t to the exterior problems (3), recall the Laplace transformation

$$\mathcal{L}(w)(s,x) := \tilde{w}(s,x) = \int_0^{+\infty} w(t,x)e^{-ts}dt, \tag{5}$$

where s stands for the co-variable of time t and verify $\Re(s) > 0$.

We obtain

$$s\tilde{w}(s,x) + \alpha\partial_{xxx}\tilde{w} - \beta s\partial_{xx}\tilde{w} + \gamma\partial_x\tilde{w} = 0, \quad x \geq b, x \leq a, \Re(s) > 0 \tag{6}$$

which is a cubic ordinary differential equation whose solutions are of the form are given explicitly by

$$\hat{w}(s,x) = c_1(s)e^{\lambda_1(s)x} + c_2(s)e^{\lambda_2(s)x} + c_3(s)e^{\lambda_3(s)x}, \quad x \in \mathbb{R}\backslash[a,b] \tag{7}$$

where $\lambda_1(s), \lambda_2(s), \lambda_3(s)$ denote the roots of the depressed cubic equation

$$\alpha\lambda^3 - \beta s\lambda^2 + \gamma\lambda + s = 0 \tag{8}$$

The three solutions are given by

$$\lambda_k(s) = \mathrm{j}^{k-1}\zeta(s) - \frac{\Theta_1(s)}{\mathrm{j}^{k-1}\zeta(s)} + \Theta_2(s), \quad k = 1, 2, 3 \tag{9}$$

where the complex j is given by $\mathrm{j} = \exp(2i\pi/3)$,

$$\begin{aligned} \zeta(s) &= \left(\frac{\sqrt{4\alpha\gamma^3 - \beta^2 s^2\gamma^2 + 18\alpha\beta s^2\gamma - 4\beta^3 s^4 + 27\alpha^2 s^2}}{2 3^{32}\alpha^2} - \frac{9\alpha\beta s\gamma - 2\beta^3 s^3 + 27\alpha^2 s}{54\alpha^3}\right)^{13}, \\ \Theta_1(s) &= \frac{3\alpha\gamma - \beta^2 s^2}{9\alpha^2}, \\ \Theta_2(s) &= \frac{\beta s}{3\alpha}. \end{aligned} \tag{10}$$

Assume that $\Re(s) > \frac{2\beta^3 + 9\gamma\alpha^2}{27\alpha^2}$, then roots of the cubic Eq. (8) possess the following separation property

$$\Re(\lambda_1) < \frac{\beta}{3\alpha}, \quad \Re(\lambda_2) > \frac{\beta}{3\alpha} \quad \text{and} \quad \Re(\lambda_3) > \frac{\beta}{3\alpha}. \tag{11}$$

In fact, we consider the change of variable $\lambda = z - \frac{\beta}{3\alpha}$. Then the cubic Eq. (8) becomes $z^3 + pz + q = 0$ such that $p = \frac{3\alpha\gamma - \beta^2}{3\alpha^2}$ and $q = \frac{27\alpha^2 s + 2\beta^3 - 9\alpha\beta\gamma}{27\alpha}$.

Hence under the condition $\Re(q) = \frac{27\alpha^2\Re(s) + 2\beta^3 - 9\alpha\beta\gamma}{27\alpha}$, it follows that the roots $r_i, i = 1, 2, 3$ of the equation $z^3 + pz + q = 0$ satisfy

$$\Re(r_1) < 0, \quad \Re(r_2) > 0, \quad \Re(r_3) > 0$$

Now back to Eq. (7), for $x \geq b$ we have from the infinite condition that the coefficients c_2 and c_3 must vanishe, hence $\tilde{w}(x,s) = c_1(s)e^{r_1(s)x}$, deriving over x and using the continuity of w in the interface yield

$$\hat{w}(s,b) - \frac{1}{\lambda_1^2(s)}\hat{w}_{xx}(s,b) = 0, \quad \hat{w}_x(s,b) - \frac{1}{\lambda_1(s)}\hat{w}_{xx}(s,b) = 0 \tag{12}$$

Idem for $x \leq a$, we have $c_1 = 0$ and hence

$$\hat{w}_{xx}(s,a) - (\lambda_2(s) + \lambda_3(s))\hat{w}_x(s,a) + \lambda_2(s)\lambda_3(s)\hat{w}(s,a) = 0. \tag{13}$$

As $\lambda_1, \lambda_2,$ and λ_3 are roots of the cubic Eq. (8) we obtain immediately

$$\lambda_1(s)\lambda_2(s)\lambda_3(s) = -\frac{s}{\alpha} \text{ and } \lambda_2(s) + \lambda_3(s) + \lambda_1(s) = -\frac{\beta}{\alpha} \tag{14}$$

Then the Eq. (13) becomes in terms of $\lambda_1(s)$

$$\hat{w}_{xx}(s,a) + \left(\lambda_1(s) + \frac{\beta}{s}\right)\hat{w}_x(s,a) - \frac{s}{\alpha\lambda_1(s)}\hat{w}(s,a) = 0 \tag{15}$$

Now applying the inverse Laplace transform to Eqs. (8) and (10), we infer

$$\begin{gathered} w(t,a) - \alpha\mathcal{L}^{-1}\left(\frac{\lambda_1^2(s)s + \lambda_1(s)\beta}{s}\right) * w_x(t,a) - \alpha\mathcal{L}^{-1}\left(\frac{\lambda_1(s)}{s}\right) * w_{xx}(t,a) = 0 \\ w(t,b) - \mathcal{L}^{-1}\left(\frac{1}{\lambda_1^2(s)}\right) * w_{xx}(t,b) = 0, \quad w_x(t,b) - \mathcal{L}^{-1}\left(\frac{1}{\lambda_1(s)}\right) * w_{xx}(t,b) = 0 \end{gathered} \tag{16}$$

Therefore, we get the following result describing the problem in the bounded domain satisfied by the restriction on Ω_T of the original problem (1).

Theorem 1.1 Let α, β be non negative numbers and $\gamma \in \mathbb{R}$. The restriction of (1) to Ω is described by the following Initial Boundary Value Problem (IBVP)

$$\begin{cases} \partial_t u + \alpha\partial_{xxx}u - \beta\partial_{txx}u + \gamma\partial_x u = 0 & \text{in}\,\Omega_T \\ u(0,x) = u_0(x) & \text{at}\,\Omega \\ \partial_n u(t,x) = Bu(t,x) & \text{at}\,\Sigma_T \end{cases} \tag{17}$$

where B is derived on Σ_T from equations

$$u(t,a) - \alpha\mathcal{L}^{-1}\left(\frac{\lambda_1^2(s)s + \beta\lambda_1(s)}{s}\right) * u_x(t,a) - \alpha\mathcal{L}^{-1}\left(\frac{\lambda_1(s)}{s}\right) * u_{xx}(t,a) = 0$$

$$u(t,b) - \mathcal{L}^{-1}\left(\frac{1}{\lambda_1^2(s)}\right) * u_{xx}(t,b) = 0, \quad u_x(t,b) - \mathcal{L}^{-1}\left(\frac{1}{\lambda_1(s)}\right) * u_{xx}(t,b) = 0$$

We emphasize that those boundary conditions strongly depend on α and β through the root $\lambda_1(s)$. Some simplifications can be obtained for particular cases allowing direct evaluation of the inverse Laplace transform. Taking for example the BBM equation (for $\alpha = 0$), we can get after applying Laplace transformation to (3),

$$\begin{aligned} \partial_x \tilde{w}(s,b) &= \frac{\gamma}{2\beta s}\tilde{w}(s,b) - \frac{\sqrt{\gamma^2 + 4\beta s^2}}{2\beta s}\tilde{w}(s,b) \\ &= \frac{\gamma}{2\beta}\frac{\tilde{w}(s,b)}{s} - \frac{\gamma^2}{2\beta}\frac{1}{\sqrt{\gamma^2+4\beta s^2}}\frac{\tilde{w}(s,b)}{s} - 2\frac{s}{\sqrt{\gamma^2+4\beta s^2}}\tilde{w}(s,b). \end{aligned} \tag{18}$$

$$\begin{aligned} \partial_x \tilde{w}(s,a) &= \frac{\gamma}{2\beta s}\tilde{w}(s,a) + \frac{\sqrt{\gamma^2 + 4\beta s^2}}{2\beta s}\tilde{w}(s,a) \\ &= \frac{\gamma}{2\beta}\frac{\tilde{w}(s,a)}{s} + \frac{\gamma^2}{2\beta}\frac{1}{\sqrt{\gamma^2+4\beta s^2}}\frac{\tilde{w}(s,a)}{s} + 2\frac{s}{\sqrt{\gamma^2+4\beta s^2}}\tilde{w}(s,a). \end{aligned} \tag{19}$$

In this case, we obtain convolution products with Bessel functions after the Laplace inverse transformation as follows

$$\begin{aligned} \partial_x w(t,b) &= C_1 I_t(w)(t) - C_2\left(J_0^c * I_t(w)\right)(t) + C_3\left(J_1^c * w\right)(t), \\ \partial_x w(t,a) &= C_1 I_t(w)(t) + C_2\left(J_0^c * I_t(w)\right)(t) - C_3\left(J_1^c * w\right)(t). \end{aligned}$$

where we have used the expressions

$$\sqrt{\frac{c}{c^2+s^2}} = \mathcal{L}(H(t)J_0(ct))(s) \coloneqq \mathcal{L}\left(H(t)J_0^c(t)\right)(s) \ \text{, and } \left(J_0^c\right)' = -J_1^c,$$

and the notations $c = \frac{\gamma}{2\sqrt{\beta}}, \ C_1 = \frac{\gamma}{2\beta}, \ C_2 = \left(\frac{\gamma}{2}\right)^{\frac{3}{2}}\beta^{-\frac{5}{4}}, \ C_3 = \sqrt{\frac{2}{\gamma\sqrt{\beta}}}.$.
Recall that the Bessel functions can be defined by the following integrals

$$J_0^c(t) = \frac{1}{\pi}\int_0^\pi cos\,(ctsin\tau)d\tau, \quad J_1^c(t) = \frac{1}{\pi}\int_0^\pi cos\,(ctsin\tau - \tau)d\tau.$$

From this, we may compute

$$\begin{aligned} \partial_n w(t,b) &= -C_1 I_t(w)(t) + C_2\left(J_0^c * I_t(w)\right)(t) - C_3\left(J_1^c * w\right)(t), \\ \partial_n w(t,a) &= C_1 I_t(w)(t) + C_2\left(J_0^c * I_t(w)\right)(t) - C_3\left(J_1^c * w\right)(t). \end{aligned}$$

Thus the boundary operator B in (2) writes, in the case $\alpha = 0$,

$$Bu(t,x) \coloneqq \begin{cases} C_1 I_t u(t) + C_2\left(J_0^c * I_t u\right)(t) - C_3\left(J_1^c * u\right)(t) & x = a \\ -C_1 I_t u(t) + C_2\left(J_0^c * I_t u\right)(t) - C_3\left(J_1^c * u\right)(t) & x = b \end{cases} \tag{20}$$

Next, we propose an approximation, always for the case $\alpha = 0$, of the BBM equation in Ω_T supplemented with constructed boundary conditions.

2.1 Numerical approximation

This subsection is devoted to the numerical approximation of the obtained IBVP (17) for $\alpha = 0$ and B given in (20). Our strategy is to seek numerical simulations that permits to avoid any boundary reflections and in some way renders the fully discrete scheme unconditionally stable.

Let N, M be integers, we define time step $\Delta t = \frac{T}{M}$ and spatial step $h = \frac{b-a}{N}$. The grids $t_n = n\Delta t, 0 \le n \le M$ and $x_i = a + ih, 0 \le i \le N$ are used to discretize Ω_T. Throughout this paper, we denote u_i^n the considered approximation of $u(t_n, x_i)$ and the set $\mathbb{N}_k^l = \{m \in \mathbb{N},\ k \le m \le l\}$.

2.1.1 Approximation of the governing equation

We describe a discretization for the BBM equation by the Crank-Nicholson time scheme as follows

$$\frac{u(t_{n+1},x)-u(t_n,x)}{\Delta t} - \beta\frac{\partial_{xx}u(t_{n+1},x)-\partial_{xx}u(t_n,x)}{\Delta t} + \gamma\frac{\partial_x u(t_{n+1},x)+\partial_x u(t_n,x)}{2} = 0, \tag{21}$$

$$u(t_0,x) = u_0(x), \qquad n \in \mathbb{N}_0^{M-1}.$$

For the space finite difference scheme, we use the approximations

$$\partial_x u(t,x) \approx \frac{1}{2h}(u(t,x+h) - u(t,x-h)),$$
$$\partial_{xx} u(t,x) \approx \frac{1}{h^2}(u(t,x+h) - 2u(t,x) + u(t,x-h)).$$

The fully discretization then writes,

$$\begin{aligned} &\frac{u_i^{n+1}-u_i^n}{\Delta t} - \beta\frac{\left(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}\right)-\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)}{h^2\Delta t} \\ &\qquad +\gamma\frac{\left(u_{i+1}^{n+1}-u_{i-1}^{n+1}\right)+\left(u_{i+1}^n-u_{i-1}^n\right)}{4h} = 0, \quad (i,n)\in\mathbb{N}_1^{N-1}\times\mathbb{N}_0^{M-1} \\ &u_i^0 = u_0(x_i), \qquad\qquad i\in\mathbb{N}_0^N. \end{aligned} \tag{22}$$

2.1.2 Approximation of the boundary condition

The constructed boundary conditions (BC) contains time convolutions that are non-local and introduces many difficulties, for example, using a direct implementation leads to long and low accuracy. Several techniques have been used to overcome these problems by trying to localize the BC, see [5–8] for more details. The resulting localized BC are easy to implement and more efficient but tends to depend sensitively on the initial data. In our case, we utilize the Discrete Galerkin Method. The BC are formulated as Fredholm integral equations of second kind. The basic idea is to write the boundary condition on (20) in the form

$$\partial_n u(t,a) - \int_0^t \overline{K}_1(t,s)u(s,a)ds = 0, \tag{23}$$

$$\partial_n u(t,b) - \int_0^t \overline{K}_2(t,s)u(s,b)ds = 0. \tag{24}$$

where, the introduced Kernels $\overline{K}_1, \overline{K}_2$ represent a linear combination of the two Bessel functions of order 0 and 1 at the time $(t-s)$. After a space discretization we obtain

$$u(t,x_0) - \int_0^t K_1(t,s)u(s,x_0)ds = u(t,x_1), \tag{25}$$

$$u(t,x_N) - \int_0^t K_2(t,s)u(s,x_N)ds = u(t,x_{N-1}), \tag{26}$$

where $K_1 = -h\overline{K}_1$ and $K_2 = h\overline{K}_2$. Both resulting Eqs. (25) and (26) can be identified to the linear integral equation

$$y(t) - \int_D K(t,s)y(s)d\sigma(s) = z(t), \quad t \in D. \tag{27}$$

The Eq. (27) is a Fredholm integral equation of second kind, where D is a closed bounded set in $\mathbb{R}^m$, with $m \geq 1$. The approximation of such integral equation could be made by a discrete Galerkin method using the quadrature rule of Gauss-Legendre as presented in [9]. Based on this, the BC can be similarly discretized while considering the domain D as the time interval $[0,t]$. Precisely, we use the Gauss Legendre Quadrature of order q, labeled GLQ_q with zeros ξ_j and weights w_j being in the interval $[-1,1]$ for $j \in \mathbb{N}_0^q$. Let $i \in \mathbb{N}_0^{N-1}$, we introduce the following transformation

$$\begin{array}{llll} F_i: & [-1,1] & \rightarrow & [t_i, t_{i+1}] \\ & \xi & \mapsto & t_i \dfrac{1-\xi}{2} + t_{i+1}\dfrac{1+\xi}{2} \end{array} \tag{28}$$

The approximation of the BC is now given by

$$u(t_{n+1},x_0) - \int_0^{t_{n+1}} K_1(t_{n+1},s)u(s,x_0)ds = u(t_{n+1},x_1) \tag{29}$$

$$u(t_{n+1},x_N) - \int_0^{t_{n+1}} K_2(t_{n+1},s)u(s,x_N)ds = u(t_{n+1},x_{N-1}) \tag{30}$$

that is

$$u_0^{n+1} - \int_0^{t_{n+1}} K_1(t_{n+1},s)u(s,x_0)ds = u_1^{n+1} \tag{31}$$

$$u_N^{n+1} - \int_0^{t_{n+1}} K_2(t_{n+1},s)u(s,x_N)ds = u_{N-1}^{n+1} \tag{32}$$

For the seek of simplicity, we rewrite the integral terms of (31) and (32) in the form

$$\begin{aligned} \int_0^{t_{n+1}} K_i(t_{n+1},s)u(s,x_k)ds &= A_i\int_0^{t_{n+1}} u(s,x_k)ds + B_i\int_0^{t_{n+1}} J_0^c(t_{n+1}-s)\left(\int_0^s u(r,x_k)dr\right)ds \\ &\quad + D_i\int_0^{t_{n+1}} J_0^c(t_{n+1}-s)u(s,x_k)ds \\ &:= A_iI_1(t_{n+1},x_k) + B_iI_2(t_{n+1},x_k) + D_iI_3(t_{n+1},x_k), \end{aligned}$$

such that $(i,k) \in \{(1,0),(2,N)\}$, $A_1 = A_2 = -hC_1, B_2 = -B_1 = hC_2$ and $D_1 = -D_2 = hC_3$, all constants C_i are defined in (??). Thus, basing on the approach presented in [9], the GLQ_q applied to the integrals previously defined is described by the following, for $n \in \mathbb{N}_0^{M-1}$ and $k \in \{0, N\}$,

$$I_1(t_{n+1}, x_k) = \frac{\Delta t}{2} \sum_{i=0}^{n} \sum_{j=0}^{q} w_j u\left(F_i\left(\xi_j\right), x_k\right). \tag{33}$$

The second integral is more complicated since it involves two composing integrals, using Gauss-Legendre quadrature twice yields

$$I_2(t_{n+1}, x_k) = \frac{\Delta t^2}{4} \sum_{i=0}^{n} \sum_{j=0}^{q} w_j J_0\left(t_{n+1} - F_i\left(\xi_j\right)\right) \sum_{l=0}^{j} \sum_{m=0}^{q} w_m u(F_l(\xi_m), x_k) \tag{34}$$

and the remained integral is approximated by

$$I_3(t_{n+1}, x_k) = \frac{\Delta t}{2} \sum_{i=0}^{n} \sum_{j=0}^{q} w_j J_1\left(t_{n+1} - F_i\left(\xi_j\right)\right) u\left(F_i\left(\xi_j\right), x_k\right). \tag{35}$$

From approximations (33)–(35), the numerical solution on the interface of (17) can be given by

$$u_0^{n+1} = u_1^{n+1} + f_1^{n+1}, \tag{36}$$

$$u_N^{n+1} = u_{N-1}^{n+1} + f_2^{n+1}. \tag{37}$$

We accomplish this by simply adding (36) and (37) to the discretization of the interior governing Eq. (22). We obtain an implicit scheme that we illustrate by the following system in matrix form

$$\begin{cases} \left(I + \tilde{A} + \tilde{B}\right) U^{n+1} = (I + A - B) U^n + F^n, \quad n \in \mathbb{N}_0^{M-1}, \\ u_0^{n+1} = u_1^{n+1} + f_1^{n+1}, \\ u_N^{n+1} = u_{N-1}^{n+1} + f_2^{n+1}. \end{cases} \tag{38}$$

with

$$U^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_{N-1}^n \end{pmatrix} \quad I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad F^n = \begin{pmatrix} (C_{d_1} + C_{d_2})\left(f_1^n - u_1^n\right) \\ 0 \\ \vdots \\ 0 \\ (C_{d_1} - C_{d_2})\left(f_2^n - u_{N-1}^n\right) \end{pmatrix},$$

$$A = C_{d_1} \begin{pmatrix} 2 & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = C_{d_2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & \ddots & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and

$$\tilde{A} = C_{d_1} \begin{pmatrix} 2 - C_{d_1} & -1 & 0 \\ -1 & \ddots & -1 \\ 0 & -1 & 2 - C_{d_1} \end{pmatrix}, \quad \tilde{B} = C_{d_2} \begin{pmatrix} -C_{d_2} & 1 & 0 \\ -1 & \ddots & 1 \\ 0 & -1 & C_{d_2} \end{pmatrix}.$$

where the discretization constants are $C_{d_1} = \frac{\beta}{h^2}, C_{d_2} = \frac{\gamma \Delta t}{4h}$.

3. Non standard boundary conditions for the BBM equation

In [4], we have presented a new method to derive transparent boundary conditions for the BBM equation. These boundary conditions have the advantage of being local in time but needs an additional function construct the BC which means bigger system to be solved. We recall that the problem designed to be the restriction in Ω_T of the BBM initial Eq. (1) with $\alpha = 0$ is given by

$$\begin{cases} \partial_t u - \beta \partial_{txx} u + \gamma \partial_x u = f & \text{in }]0, T] \times]a, b[\\ \partial_t v - \beta \partial_{txx} v + \gamma \partial_x v = g & \text{in }]0, T] \times]a, b[\\ \partial_x u = v & \text{on }]0, T] \times \{a, b\} \\ \beta \partial_{tx} v - \gamma v = \partial_t u - f & \text{on }]0, T] \times \{a, b\} \\ u(0, x) = u_0(x) & \text{on } [a, b] \\ v(0, x) = v_0(x) & \text{on } [a, b] \end{cases} \tag{39}$$

4. Numerical examples

We take an initial condition as solitary wave like function locally supported in Ω. The evolution of the solutions are plotted in different time steps before, under and after traveling the right boundary of the considered bounded domain. We save a reference solution that is numerically calculated in a broaden domain of Ω with Dirichlet boundary condition. We compute infinite error between numerical solutions using both formulations presented in this paper and the reference solution. We denote GLQ_i for approached solution with BEM and Gauss Legendre Quadrature in (2) for $i \in \{0, 1, 2\}$, while NSBC refers to numerical solution with non standard boundary conditions given in (3). We define the following errors

$$\|(u - u_{ref})(t, a)\|_\infty = \sup_{t \in [0, T]} |u(t, a) - u_{ref}(t, a)|, \tag{40}$$

$$\|(u - u_{ref})(t, b)\|_\infty = \sup_{t \in [0, T]} |u(t, b) - u_{ref}(t, b)|, \tag{41}$$

$$\|(u - u_{ref})(t, x)\|_\infty = \sup_{(t, x) \in [0, T] \times]a, b\left[|u(t, x) - u_{ref}(t, x)|\right.}. \tag{42}$$

Let $\beta = \gamma = 1$, $T = 10$ and $u_0(x) = sech^2(x)$. The considered initial data is locally supported in the interval $\Omega =]-10, 10[$, since $u_0(10) = u_0(-10) \approx 8, 25.10^{-9}$. We fix $h = 10^{-2}$ and we vary the time step Δt. For a better comparison of these methods, we compute CPU time, in seconds, needed for each one to obtain numerical solution.

BC	dt	$\|(u-u_{\text{ref}})(t,a)\|_\infty$	$\|(u-u_{\text{ref}})(t,b)\|_\infty$	$\|(u-u_{\text{ref}})(t,x)\|_\infty$	CPU time(s)
GLQ_0	10^{-2}	1.06×10^{-4}	6.33×10^{-2}	6.03×10^{-2}	4
GLQ_1		1.06×10^{-4}	6.3×10^{-2}	5.9304×10^{-2}	9
GLQ_2		1.06×10^{-4}	5.895×10^{-2}	5.9187×10^{-2}	15
NSBC		7.39×10^{-5}	5.8×10^{-3}	6.8×10^{-3}	5
GLQ_0	10^{-3}	1.06×10^{-4}	5.92×10^{-2}	5.86×10^{-2}	39
GLQ_1		1.06×10^{-4}	5.86×10^{-2}	5.857×10^{-2}	62
GLQ_2		1.06×10^{-4}	5.852×10^{-2}	5.85×10^{-2}	80
NSBC		6.7×10^{-5}	1.41×10^{-3}	1.4×10^{-3}	50
GLQ_0	10^{-4}	1.06×10^{-4}	5.855×10^{-2}	5.858×10^{-2}	436
GLQ_1		1.06×10^{-4}	5.853×10^{-2}	5.852×10^{-2}	1040
GLQ_2		1.06×10^{-4}	5.85×10^{-2}	5.85×10^{-2}	3205
NSBC		5.9×10^{-6}	7.35×10^{-4}	7.3×10^{-4}	1015

Table 1.
Infinite errors using different boundary conditions.

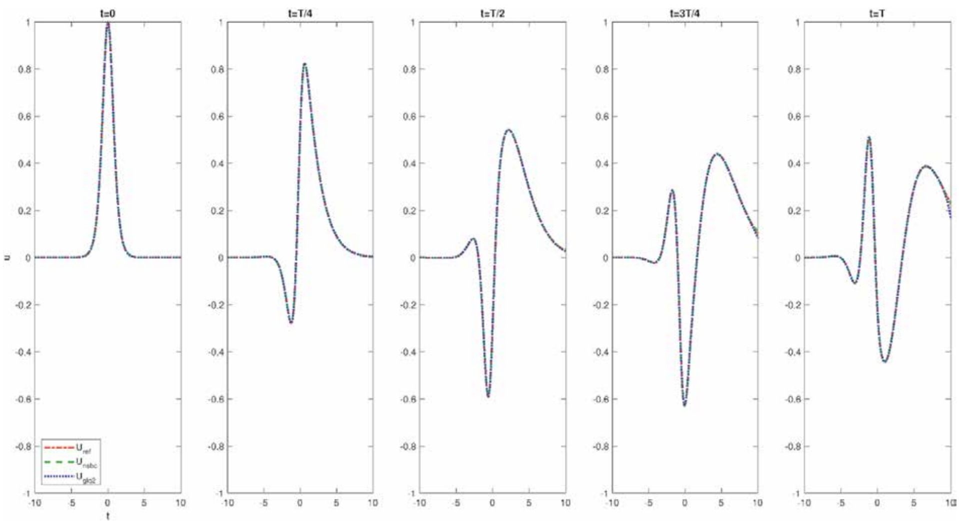

Figure 1.
Reference solution and approximated solutions NSBC and GLQ_i for $i = 2$ at different times for $\Delta t = 10^{-3}$.

Table 1 shows that both methods give a good approximation of the restriction to Ω_T of the reference solution. *NSBC* give better approximation than *GLQ*. We can remark a slow convergence of GLQ_i with respect to i and also time step. However, NSBC gives a good approximation in as much as Δt goes to zero. Furthermore, GLQ_i is more expensive in CPU time when i increases than NSBC due to the presence of non local convolutions in time in the boundary condition.

We also plot in **Figure 1**, captions at different times of either reference solution and approximated solutions using NSBC and GLQ_i for $i = 2$. We can see that NSBC follows the refrence solution better than GLQ especially at last times in the right figure. One remarks that no reflections turn back to the bounded domain when the wave is going out from the right boundary using both methods.

5. Conclusion

We have compared two methods of deriving and approaching boundary conditions for the BBM equation. We presented the BEM for a general equation that is the mixed BBM-KdV equation and that shows the hardness to put easy implemented BC. Furthermore, being non local in time, BC seems to be low accurate and slowly convergent as presented in numerical example. However, this point opens many possibilities trying to improve the accuracy of such BC whether by improving the approximation of convolution product, that comes from Inverse Laplace transformation, via quadrature or exploring a numerical equivalent to such operation such as $\mathcal{Z}$ transformation. We have proposed an other manner to derive local BC that gives better approximation than non local BC. All these conclusions have been made in one space dimension but nothing can be said about the comparison in higher dimension to decide which method is more adapted, this matter will be our interest in future works.

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Chapter 20

On the Generalized Simplest Equations: Toward the Solution of Nonlinear Differential Equations with Variable Coefficients

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Abstract

The simplest equations with variable coefficients are considered in this research. The purpose of this study is to extend the procedure for solving the nonlinear differential equation with variable coefficients. In this case, the generalized Riccati equation is solved and becomes a basis to tackle the nonlinear differential equations with variable coefficients. The method shows that Jacobi and Weierstrass equations can be rearranged to become Riccati equation. It is also important to highlight that the solving procedure also involves the reduction of higher order polynomials with examples of Korteweg de Vries and elliptic-like equations. The generalization of the method is also explained for the case of first order polynomial differential equation.

Keywords: the simplest equation, Riccati equation, nonlinear differential equations, reduction of polynomial

1. Introduction

Despite the advent of supercomputers in numerical methods, increasing activities are devoted to solving nonlinear differential equations by analytical method in recent years [1–3]. Analytical solutions have their own importance concerning the physical phenomena as they are often pave the way to the construction of right theory [4]. Many methods have been proposed concerning this important problem and generally, for the problems with constant coefficients. One of the useful methods is the method of simplest Equation [5] or for some authors, the auxiliary Equations [6]. The method is built by the utilization of the first integral of simplest nonlinear differential equations, such as Bernoulli and Riccati Equations [7]. The method had produced many new solutions of the considered nonlinear differential equations, generally with constant coefficients [8, 9].

For the more general cases, we have found that the method can be extended such as involving the solution of the nonlinear differential equations with variable coefficients. The nature of variable coefficients often arises in the equation describing the heterogenous media and composites [10] or in other cases are produced by the coordinate transformation of the partial differential Equations [11]. Those two categories are developed rapidly in recent years with the capacity of high-speed super computers which sufficient for computing nonlinear problem with complex geometries [12, 13], as sometimes desired by engineering design activities. The role of analytical solutions is as a benchmark to validate the computer algorithm with simpler geometries as it is usually performed [14].

In this chapter, the solution method of the simplest equations is different from the cases of constant coefficients except, on Bernoulli equation. Hence, we will start from Riccati equation instead of Bernoulli equation as the simplest equation to highlight the novelty of the procedure. The method is then followed by examples and conclusion.

2. The first integral of the simplest equations with variable coefficients

2.1 Riccati equation

Consider the Riccati equation with variable coefficients as follows,

$$A_\xi = a_1(\xi)A^2 + a_2(\xi)A + a_3(\xi) \tag{1}$$

Let $A = \beta_1\beta_2$ and the above equation can be rearranged as,

$$\beta_2\beta_{1\xi} + \beta_1\beta_{2\xi} = a_1\beta_1^2\beta_2^2 + a_2\beta_1\beta_2 + a_3 \text{ or}$$

$$\beta_2\beta_{1\xi} - a_1\beta_1^2\beta_2^2 - a_2\beta_1\beta_2 = -\beta_1\beta_{2\xi} + a_3 = \gamma\beta_1\beta_2.$$

and is separated as

$$\beta_{1\xi} - a_1\beta_1^2\beta_2 - (a_2+\gamma)\beta_1 = 0 \text{ and } \beta_{2\xi} + \gamma\beta_2 - \frac{a_3}{\beta_1} = 0 \tag{2}$$

The solutions for β_1 and β_2 are

$$\beta_2 = e^{\int_\xi (a_2+\gamma)d\xi}\left[\int_\xi e^{\int_\xi (a_2+\gamma)d\xi} a_1\beta_2 d\xi + C_1\right]^{-1}$$

and

$$\beta_3 = e^{-\int_\xi \gamma d\xi}\left(\int_\xi e^{\int_\xi \gamma d\xi}\frac{a_3}{\beta_1}d\xi + C_2\right) \tag{3}$$

The relation for $A = \beta_1\beta_2$ is thus,

$$A = \beta_1\beta_2 = e^{\int_\xi a_2 d\xi}\left[\int_\xi e^{\int_\xi (a_2+\gamma)d\xi} a_1\beta_2 d\xi + C_1\right]^{-1}\left(\int_\xi e^{\int_\xi \gamma d\xi}\frac{a_3}{\beta_1}d\xi + C_2\right) \tag{4}$$

Without loss of generality, suppose that $\beta_1 = e^{\int_\xi \gamma d\xi}$ and thus the above relation is performed as,

$$\beta_1\beta_2 = e^{\int_\xi a_2 d\xi}\left[\int_\xi e^{\int_\xi a_2 d\xi} a_1\beta_1\beta_2 d\xi + C_1\right]^{-1}\left(\int_\xi a_3 d\xi + C_2\right) \tag{5}$$

Rearrange Eq. (5) and integrate once,

$$a_1 e^{\int_\xi a_2 d\xi} \beta_1 \beta_2 \left[\int_\xi e^{\int_\xi a_2 d\xi} a_1 \beta_1 \beta_2 d\xi + C_1 \right] = a_1 e^{2\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right)$$

or

$$\left[\int_\xi e^{\int_\xi a_2 d\xi} a_1 \beta_1 \beta_2 d\xi + C_1 \right]^2 = 2\int_\xi a_1 e^{2\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right) d\xi + C_3$$

The solution for A is then,

$$A = \beta_1 \beta_2 = \frac{\sqrt{2}}{2} e^{\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right) \left[\int_\xi a_1 e^{2\int_\xi a_2 d\xi} \left(\int_\xi a_3 d\xi + C_2 \right) d\xi + C_3 \right]^{-\frac{1}{2}} \tag{6}$$

where the coefficients a_i will be determined later from the substitution into the considered nonlinear differential equations.

2.2 The Jacobi and Weierstrass equations

It is interesting to know that other simplest equations can also be rearranged into the Riccati-type equations. The famous examples are Jacobi [15] and Weierstrass Equations [16], which can solve a large class of nonlinear differential equations. Let us consider Jacobi type equation with variable coefficients,

$$\phi_\xi^2 = b_1(\xi)\phi^4 + b_2(\xi)\phi^3 + b_3(\xi)\phi^2 + b_4(\xi)\phi + b_5(\xi) \tag{7}$$

and the Weierstrass equation as follows,

$$\phi_\xi^2 = b_1(\xi)\phi^3 + b_2(\xi)\phi^2 + b_3(\xi)\phi + b_4(\xi) \tag{8}$$

Here, the reader should not be confused by the coefficients which represent different functions with the same index. Take $\phi = \frac{1}{\nu} + a(\xi)$ and the Weierstrass equation becomes Jacobi equation which admits the similar method of solution.

Concerning the search for obtaining solution of (7) and (8), the balancing principle suggests the substitution of the first order series $\phi = b_6 + b_7 A$ as in the following,

$$\begin{aligned} (b_{6\xi} + b_{7\xi}A + b_7 A_\xi)^2 &= b_1 b_7^4 A^4 + \left(4b_1 b_6 b_7^3 + b_2 b_7^3\right)A^3 + \left(6b_1 b_6^2 b_7^2 + 3b_2 b_6 b_7^2 + b_3 b_7^2\right)A^2 \\ &+ \left(4b_1 b_6^3 b_7 + 3b_2 b_6^2 b_7 + 2b_3 b_6 b_7 + b_4 b_7\right)A + b_1 b_6^4 + b_2 b_6^3 + b_3 b_6^2 + b_4 b_6 + b_5 \end{aligned} \tag{9}$$

Performing the Riccati equation $A_\xi = a_1 b_7 A^2 + a_2 A + a_3$ into (9) and we generate the following expression,

$$\begin{aligned} (b_{6\xi} + b_{7\xi}A + b_7 A_\xi)^2 &= \left[a_1 b_7^2 A^2 + (a_2 b_7 + b_{7\xi})A + a_3 b_7 + b_{6\xi}\right]^2 = a_1^2 b_7^4 A^4 + 2a_1 b_7 (a_2 b_7 + b_{7\xi})A^3 \\ &+ \left[2a_1 b_7 (a_3 b_7 + b_{6\xi}) + (a_2 b_7 + b_{7\xi})^2\right]A^2 + 2(a_2 b_7 + b_{7\xi})(a_3 b_7 + b_{6\xi})A + (a_3 b_7 + b_{6\xi})^2 \end{aligned}$$

The coefficients of polynomial are then related with the coefficients in (9) in order to determine a_1, a_2, a_3, b_6, b_7 as functions of the known b_1, b_2, b_3, b_4 and b_5 as follows,

$$
\begin{aligned}
&b_1 b_7^4 = a_1^2 b_7^4 \\
&4b_1 b_6 b_7^3 + b_2 b_7^3 = 2a_1 b_7 (a_2 b_7 + b_{7\xi}) \\
&6b_1 b_6^2 b_7^2 + 3b_2 b_6 b_7^2 + b_3 b_7^2 = 2a_1 b_7 (a_3 b_7 + b_{6\xi}) + (a_2 b_7 + b_{7\xi})^2 \\
&4b_1 b_6^3 b_7 + 3b_2 b_6^2 b_7 + 2b_3 b_6 b_7 + b_4 b_7 = 2(a_2 b_7 + b_{7\xi})(a_3 b_7 + b_{6\xi}) \\
&b_1 b_6^4 + b_2 b_6^3 + b_3 b_6^2 + b_4 b_6 + b_5 = (a_3 b_7 + b_{6\xi})^2
\end{aligned}
\tag{10}
$$

Hence, the first equation gives,

$$a_1 = f_0(b_1) \tag{11}$$

and the second equation is then,

$$a_2 b_7 + b_{7\xi} = \frac{4b_1 b_6 b_7^2 + b_2 b_7^2}{2f_0} \tag{12}$$

The next relation produces,

$$a_3 b_7 + b_{6\xi} = \frac{6b_1 b_6^2 b_7 + 3b_2 b_6 b_7 + b_3 b_7}{2f_0} - \frac{1}{8f_0^3}(4b_1 b_6 + b_2)^2 b_7^3 \tag{13}$$

Eqs. (12) and (13) are thus substituted into the fourth relation of (12) to form the third order polynomial equation in term of b_6 as follows,

$$
\begin{aligned}
&(32f_0^4 b_1 + 64b_1^3 b_7^4 - 192f_0^2 b_1^2 b_7^2)b_6^3 + (24f_0^4 b_2 + 32b_1^2 b_2 b_7^4 - 16b_1^2 b_2 b_7^4 - 96f_0^2 b_1 b_2 b_7^2 - 48f_0^2 b_1 b_2 b_7^2)b_6^2 \\
&+(16f_0^4 b_3 + 4b_1 b_2^2 b_7^4 + 8b_1 b_2^2 b_7^2 - 32f_0^2 b_1 b_3 b_7^2 - 24f_0^2 b_2^2 b_7^2)b_6 + 8f_0^4 b_4 + b_2^3 b_7^4 - 8f_0^2 b_2 b_3 b_7^2 = 0
\end{aligned}
\tag{14}
$$

which the roots will determine the solution for b_6 as functions of b_1, b_2, b_3, b_4, b_7, or $b_6 = f_1(b_7)$ in simple unknown variable. The step now is to find the polynomial expression for b_7 from the last relation of (10) as,

$$
\begin{aligned}
&b_1 f_1^4(b_7) + b_2 f_1^3(b_7) + b_3 f_1^2(b_7) + b_4 f_1(b_7) + b_5 = \\
&\left[\frac{6b_1 f_1^2(b_7) b_7 + 3b_2 f_1(b_7) b_7 + b_3 b_7}{2f_0} - \frac{1}{8f_0^3}\left(4b_1 f_1(b_7) + b_2\right)^2 b_7^3\right]^2
\end{aligned}
\tag{15}
$$

Therefore, the last equation gives the expression for b_7 as polynomial equation of higher order, and the generated polynomial is,

$$a_n b_7^n + a_{n-1} b_7^{n-1} + a_{n-2} b_7^{n-2} + a_{n-3} b_7^{n-3} + \ldots\ldots\ldots\ldots + a_2 b_7^2 + a_1 b_7 + a_0 = 0 \tag{16}$$

In this case the higher order polynomial will be solved by reducing the order.

3. Reduction of higher order polynomial

Consider the sixth order polynomial equation as in the following,

$$b_7^6 + a_4 b_7^5 + a_5 b_7^4 + a_6 b_7^3 + a_7 b_7^2 + a_8 b_7 + a_9 = 0$$

First, multiply the above equation with the functionα and rearranged as,

$$B^6 + a_4\alpha B^5 + a_5\alpha^2 B^4 + a_6\alpha^3 B^3 + a_7\alpha^4 B^2 + a_8\alpha^5 B + a_9\alpha^6 + \varphi = \varphi \tag{17}$$

where $B = \alpha b_7$. The polynomial equation is cut as in the following,

$$\left(B^4 + b_1B^3 + b_2B^2 + b_3B + b_4\right)B^2 + b_5B^2 - \left(B^4 + b_1B^3 + b_2B^2 + b_3B + b_4\right)b_6 = \varphi$$

Note that, the coefficients b_i in this section is different from the previous section. Expanding for the new coefficients,

$$B^6 + b_1B^5 + (b_2 - b_5)B^4 + (b_3 - b_1b_5)B^3 + (b_4 + b_5 - b_2b_5)B^2 - b_3b_5B - b_4b_6 = \varphi \tag{18}$$

Hence, the relation for coefficients is,

$$\begin{aligned}
&b_1 = a_1\alpha \\
&b_2 - b_6 = a_2\alpha^2 \\
&b_3 - b_1b_6 = a_3\alpha^3 \\
&b_4 + b_5 - b_2b_6 = a_4\alpha^4 \qquad\qquad \text{or} \\
&-b_3b_6 = a_5\alpha^5 \\
&-b_4b_6 = a_6\alpha^6 + \varphi \\
&b_6 = \frac{\varphi}{b_5}
\end{aligned}$$

$$\begin{aligned}
&b_1 = a_1\alpha \\
&b_2 - b_6 = a_2\alpha^2 \\
&b_3 - a_1\alpha\left(b_2 - a_2\alpha^2\right) = a_3\alpha^3 \\
&b_4 + b_5 - \left(b_2 - a_2\alpha^2\right)^2 = a_4\alpha^4 + a_2\alpha^2\left(b_2 - a_2\alpha^2\right) \\
&-\left[a_3\alpha^3 + a_1\alpha\left(b_2 - a_2\alpha^2\right)\right]\left(b_2 - a_2\alpha^2\right) = a_5\alpha^5 \\
&-\left[a_4\alpha^4 + a_2\alpha^2\left(b_2 - a_2\alpha^2\right) + \left(b_2 - a_2\alpha^2\right)^2 - b_5\right]\left(b_2 - a_2\alpha^2\right) = a_6\alpha^6 + \varphi \\
&\left(b_2 - a_2\alpha^2\right) = -\frac{\left[a_4\alpha^4 + a_2\alpha^2(b_2 - a_2\alpha^2) + (b_2 - a_2\alpha^2)^2 - b_5\right](b_2 - a_2\alpha^2) + a_6\alpha^6}{b_5}
\end{aligned}.$$

The fifth coefficient relation is rearranged as,

$$\left(b_2 - a_2\alpha^2\right)^2 + \frac{a_3}{a_1}\alpha^2\left(b_2 - a_2\alpha^2\right) + \frac{a_5}{a_1}\alpha^4 = 0 \tag{19}$$

and the roots are,

$$\left(b_2 - a_2\alpha^2\right) = \frac{1}{2}\alpha^2\left[-\frac{a_3}{a_1} \pm \left(\frac{a_3^2}{a_1^2} - 4\frac{a_5}{a_1}\right)^{\frac{1}{2}}\right] = f_0\alpha^2 \tag{20}$$

Also, the last relation is rewritten as,

$$\left(b_2 - a_2\alpha^2\right)^3 + a_2\alpha^2\left(b_2 - a_2\alpha^2\right)^2 + a_4\alpha^4\left(b_2 - a_2\alpha^2\right) + a_6\alpha^6 = 0 \tag{21}$$

Note that performing (19) into (21) will remove b_5 and α. Thus, it is necessary to take other relation, i.e. $a_6\alpha^6 = \alpha^{12} + b_5$, which will produce the cubic equation as follows,

$$\alpha^{12} + \left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)\alpha^6 + b_5 = 0 \tag{22}$$

which has the roots as,

$$\alpha^6 = -\frac{1}{2}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}} \tag{23}$$

Substituting back into $a_6\alpha^6 = \alpha^{12} + b_5$ to get,

$$\left\{-\frac{1}{2}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}}\right\}^2$$
$$+b_5 = -\frac{1}{2}a_6\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}a_6\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}} \quad \text{or}$$

$$\frac{1}{4}\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right] - \frac{1}{2}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}}$$
$$+\frac{1}{4}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 + b_5 = -\frac{1}{2}a_6\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right) \pm \frac{1}{2}a_6\left[\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - 4b_5\right]^{\frac{1}{2}}$$
or

$$b_5 = \frac{1}{4}\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 - \frac{1}{4}\left[\frac{\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)^2 + a_6\left(f_0^3 + a_2 f_0^2 + a_4 f_0\right)}{\left(f_0^3 + a_2 f_0^2 + a_4 f_0 + a_6\right)}\right]^2 \tag{24}$$

Therefore,α is also determined by (24) and so all the coefficients, $b_1, b_2, b_3, b_4, b_6, \varphi$. The polynomial equation of sixth order is then re-expressed as,

$$\left(B^4 + b_1B^3 + b_2B^2 + b_3B + b_4\right)\left(B^2 - b_6\right) = -b_5\left(B^2 - \frac{\varphi}{b_5}\right) \tag{25}$$

as reduced into the quartic equation the roots can be obtained by radical solution.

The procedure described by (17–25) can be applied and iterated into (16) until the polynomial equation of b_7 is reduced into quartic equation. Hence, all the coefficients for Riccati equation the first order series, i.e. a_1, a_2, a_3, b_6, b_7 are determined and produce the solution as,

$$\phi = b_6 + b_7A \text{ or}$$
$$\phi = b_6 + \frac{\sqrt{2}}{2}b_7 e^{\int_\xi a_2 d\xi}\left(\int_\xi a_3 d\xi + C_2\right)\left[\int_\xi a_1 b_7 e^{2\int_\xi a_2 d\xi}\left(\int_\xi a_3 d\xi + C_2\right)d\xi + C_3\right]^{-\frac{1}{2}} \tag{26}$$

Thus, following the method explained by (2–6) and (17–25), we have arrived at the solution of Jacobi and Weierstrass equations with variable coefficients.

4. Solution examples

4.1 The elliptic-like equation

As an application, consider the elliptic-like equation with forcing function,

$$\phi_{\xi\xi} + b_1(\xi)\phi^3 + b_2(\xi)\phi = b_3(\xi) \tag{27}$$

The balance principle suggests that the solution should be in the form,

$$\phi = b_4 + b_5 A \tag{28}$$

Substituting into (27) will reproduce the following expression,

$$b_5 A_{\xi\xi} + 2b_{5\xi}A_\xi + b_1 b_5^3 A^3 + 3b_1 b_4 b_5^2 A^2 + \left(b_{5\xi\xi} + 3b_1 b_4^2 b_5 + b_2 b_5\right)A + b_{4\xi\xi} + b_1 b_4^3 + b_2 b_4 = b_3 \tag{29}$$

The next step is to differentiate the Riccati equation once,

$$A_{\xi\xi} = 2a_1^2 A^3 + (3a_1 a_2 + a_{1\xi})A^2 + \left(2a_1 a_3 + a_2^2 + a_{2\xi}\right)A + a_2 a_3 + a_{3\xi}$$

Substituting into Eq. (29) and it will produce the polynomial equation as in the following,

$$\left(2a_1^2 b_5 + b_1 b_5^3\right)A^3 + \left(3a_1 a_2 b_5 + a_{1\xi} b_5 + 2a_1 b_{5\xi} + 3b_1 b_4 b_5^2\right)A^2$$

$$+\left(2a_1 a_3 b_5 + a_2^2 b_5 + a_{2\xi} b_5 + 2a_2 b_{5\xi} + b_{5\xi\xi} + 3b_1 b_4^2 b_5 + b_2 b_5\right)A$$

$$+a_2 a_3 b_5 + a_{3\xi} b_5 + 2a_3 b_{5\xi} + b_{4\xi\xi} + b_1 b_4^3 + b_2 b_4 = b_3$$

or the next step is to relate the coefficients as,

$$2a_1^2 b_5 + b_1 b_5^3 = 0$$

$$3a_1 a_2 b_5 + a_{1\xi} b_5 + 2a_1 b_{5\xi} + 3b_1 b_4 b_5^2 = 0$$

$$2a_1 a_3 b_5 + a_2^2 b_5 + a_{2\xi} b_5 + 2a_2 b_{5\xi} + b_{5\xi\xi} + 3b_1 b_4^2 b_5 + b_2 b_5 = 0$$

$$a_2 a_3 b_5 + a_{3\xi} b_5 + 2a_3 b_{5\xi} + b_{4\xi\xi} + b_1 b_4^3 + b_2 b_4 = b_3$$

In this case, the first equation gives,

$$a_1 = f(b_1) b_5 \tag{30}$$

For the second equation,

$$3fa_2 b_5 + f_\xi b_5 + 3fb_{5\xi} + 3b_1 b_4 b_5 = 0$$

Thus, provide the expression for b_5 as,

$$b_5 = C_4 f^{-\frac{1}{3}} e^{-\int_\xi \left(a_2 + \frac{b_1 b_4}{f}\right) d\xi} \tag{31}$$

The third and fourth equations produce,

$$a_3 b_5 = b_5^{-1} e^{-\int_\xi a_2 d\xi} \left[\int_\xi b_5 e^{\int_\xi a_2 d\xi} \left(b_3 - b_{4\xi\xi} - b_1 b_4^3 - b_2 b_4 \right) d\xi + C_4 \right] \tag{32}$$

Substituting (32) into (31),

$$-a_2^2 - a_{2\xi} - 2a_2 \frac{b_{5\xi}}{b_5} - \frac{b_{5\xi\xi}}{b_5} - 3b_1 b_4^2 - b_2 = 2f b_5^{-1} e^{-\int_\xi a_2 d\xi} \left[\int_\xi b_5 e^{\int_\xi a_2 d\xi} \left(b_3 - b_{4\xi\xi} - b_1 b_4^3 - b_2 b_4 \right) d\xi + C_4 \right]$$

Replace b_5 with (10),

$$\begin{aligned} &\frac{2}{3}\frac{f_\xi}{f} a_2 + \frac{1}{3}\frac{f_{\xi\xi}}{f} - \frac{4}{9}\left(\frac{f_\xi}{f}\right)^2 - \left(\frac{b_1 b_4}{f}\right)^2 + \left(\frac{b_1 b_4}{f}\right)_\xi - 3b_1 b_4^2 - b_2 = \\ &2f e^{\int_\xi \frac{b_1 b_4}{f} d\xi} \left[\int_\xi e^{-\int_\xi \frac{b_1 b_4}{f} d\xi} \left(b_3 - b_{4\xi\xi} - b_1 b_4^3 - b_2 b_4 \right) d\xi + C_4 \right] \end{aligned} \tag{33}$$

which then solves a_2 regardless of b_4. In this case, we take b_4 as the chosen fundamental variable and the resulted coefficients, b_5, a_1, a_2, a_3 depend on b_4 and with the known coefficients b_1, b_2, b_3. Therefore, the solution of (27) is generated as,

$$\begin{aligned} \phi(\xi) = b_5(b_4) &\left\{ \frac{\sqrt{2}}{2} e^{\int_\xi a_2(b_4) d\xi} \left(\int_\xi a_3(b_4) d\xi + C_2 \right) \left[\int_\xi a_1(b_4) e^{2\int_\xi a_2(b_4) d\xi} \left(\int_\xi a_3(b_4) d\xi + C_2 \right) d\xi + C_3 \right]^{-\frac{1}{2}} \right\} \\ &+ b_4 \end{aligned} \tag{34}$$

4.2 Korteweg de Vries equation

The next example is for the Korteweg de Vries type equation,

$$\phi_{\xi\xi\xi} + b_1(\xi)\phi\phi_\xi + b_2(\xi)\phi_\xi + b_3(\xi)\phi + b_4(\xi) = 0 \tag{35}$$

The balancing principle with application of Riccati equation will determined the ansatz,

$$\phi = b_5 + b_6 A + b_7 A^2 \tag{36}$$

Performing into (35) will produce,

$$\begin{aligned} &2b_7 A A_{\xi\xi\xi} + b_6 A_{\xi\xi\xi} + 6b_7 A_\xi A_{\xi\xi} + 6b_{7\xi} A_\xi^2 + 6b_{7\xi} A A_{\xi\xi} + 2b_1 b_7^2 A^3 A_\xi + 3b_1 b_6 b_7 A^2 A_\xi \\ &+ \left(6b_{7\xi\xi} + b_1 b_6^2 + 2b_1 b_5 b_7 + 2b_2 b_7\right) A A_\xi + 3b_{6\xi} A_{\xi\xi} + (3b_{6\xi\xi} + b_1 b_5 b_6 + b_2 b_6) A_\xi + b_1 b_7 b_{7\xi} A^4 \\ &+ (b_1 b_{6\xi} b_7 + b_1 b_6 b_{7\xi}) A^3 + (b_{7\xi\xi\xi} + b_1 b_5 b_{7\xi} + b_1 b_6 b_{6\xi} + b_1 b_{5\xi} b_7 + b_2 b_{7\xi} + b_3 b_7) A^2 \\ &+ (b_{6\xi\xi\xi} + b_1 b_5 b_{6\xi} + b_1 b_{5\xi} b_6 + b_2 b_{6\xi} + b_3 b_6) A + b_{5\xi\xi\xi} + b_1 b_5 b_{5\xi} + b_2 b_{5\xi} + b_3 b_5 + b_4 = 0 \end{aligned} \tag{37}$$

Performing the Riccati equation into (37) will produce the following polynomial,

$$
\begin{aligned}
&\left(12b_7a_1^3+2b_1b_7^2a_1+12a_1^3b_7\right)A^5+\begin{pmatrix}18a_1a_{1\xi}b_7+54a_1^2a_2b_7+6a_1^3b_6+b_1b_7b_{7\xi}+2a_2b_1b_7^2\\+18a_1^2b_{7\xi}+3a_1b_1b_6b_7\end{pmatrix}A^4\\
&+\begin{pmatrix}40a_1^2a_3b_7+32a_1a_2^2b_7+12a_{1\xi}a_2b_7+18a_1a_{2\xi}b_7+2a_{1\xi\xi}b_7+6a_1a_{1\xi}b_6+12a_1^2a_2b_6+b_1b_{6\xi}b_7\\+b_1b_6b_{7\xi}+2a_3b_1b_7^2+6a_2^3b_7+30a_1a_2b_{7\xi}+6a_{1\xi}b_{7\xi}+6a_1^2b_{6\xi}+3a_2b_1b_6b_7+6a_1b_{7\xi\xi}\\+a_1b_1b_6^2+2a_1b_1b_5b_7+2a_1b_2b_7\end{pmatrix}A^3\\
&+\begin{pmatrix}52a_1a_2a_3b_7+14a_{1\xi}a_3b_7+10a_1a_{3\xi}b_7+12a_2a_{2\xi}b_7+2a_{2\xi\xi}b_7+8a_2^3b_7+8a_1^2a_3b_6+7a_1a_2^2b_6\\+3a_{1\xi}a_2b_6+6a_1a_{2\xi}b_6+a_{1\xi\xi}b_6+24a_1a_3b_{7\xi}+12a_2^2b_{7\xi}+6a_{2\xi}b_{7\xi}+9a_1a_2b_{6\xi}+3a_{1\xi}b_{6\xi}+b_{7\xi\xi\xi}\\+b_1b_5b_{7\xi}+b_1b_6b_{6\xi}+b_1b_{5\xi}b_7+b_2b_{7\xi}+b_3b_7+3a_3b_1b_6b_7+6a_2b_{7\xi\xi}+a_2b_1b_6^2+2a_2b_1b_5b_7\\+2a_2b_2b_7+3a_1b_{6\xi\xi}+a_1b_1b_5b_6+a_1b_2b_6\end{pmatrix}A^2\\
&+\begin{pmatrix}16a_1a_3^2b_7+14a_2^2a_3b_7+10a_{2\xi}a_3b_7+8a_2a_{3\xi}b_7+2a_{3\xi\xi}b_7+8a_1a_2a_3b_6+4a_{1\xi}a_3b_6+2a_1a_{3\xi}b_6\\+3a_2a_{2\xi}b_6+a_{2\xi\xi}b_6+a_2^3b_6+18a_2a_3b_{7\xi}+6a_{3\xi}b_{7\xi}+6a_1a_3b_{6\xi}+3a_2^2b_{6\xi}+3a_{2\xi}b_{6\xi}+b_{6\xi\xi\xi}\\+b_1b_5b_{6\xi}+b_1b_{5\xi}b_6+b_2b_{6\xi}+b_3b_6+6a_3b_{7\xi\xi}+a_3b_1b_6^2+2a_3b_1b_5b_7+2a_3b_2b_7+3a_2b_{6\xi\xi}\\+a_2b_1b_5b_6+a_2b_2b_6\end{pmatrix}A\\
&+\begin{pmatrix}2a_1a_3^2b_6+a_2^2a_3b_6+2a_{2\xi}a_3b_6+a_2a_{3\xi}b_6+a_{3\xi\xi}b_6+6a_2a_3^2b_7+6a_3a_{3\xi}b_7+3a_2a_3b_{6\xi}\\+3a_{3\xi}b_{6\xi}+b_{5\xi\xi\xi}+b_1b_5b_{5\xi}+b_2b_{5\xi}+6a_3^2b_{7\xi}+3a_3b_{6\xi\xi}+a_3b_1b_5b_6+a_3b_2b_6+b_3b_5+b_4\end{pmatrix}=0
\end{aligned}
\tag{38}
$$

From this step on, there is a little hope to solve all the coefficients as they are equal to zero. As it has also to be reduced, it is important to note that the problem of reduction here is different from the case of Jacobi equation since all the coefficients are in principle solvable in algebraic form. In this case, it is not practical to reduce the fifth order polynomial as the even highest power, i.e., as a tenth order polynomial equation. The calculation will become too tedious as the detail expression is needed in the reduced polynomial equation. The next sub section will illustrate the reduction of an odd highest power polynomial equation.

4.3 Reduction of fifth order polynomial

Consider Eq. (38) as follows,

$$d_1A^5+d_2A^4+d_3A^3+d_4A^2+d_5A+d_6=0$$

Multiply by the function, β and rearrange,

$$d_1B^5+d_2\beta B^4+d_3\beta^2B^3+d_4\beta^3B^2+d_5\beta^4B+d_6\beta^5+\varphi=\varphi \tag{39}$$

where, $B=\beta A$. Rearranged Eq. (39) as given by,

$$\left(b_1B^3+b_2B^2+b_3B+b_4\right)B^2+b_5B^2-\left(b_1B^3+b_2B^2+b_3B+b_4\right)b_6=\varphi \tag{40}$$

Expanding the all the coefficients as,

$$b_1B^5+b_2B^4+(b_3-b_1b_6)B^3+(b_4+b_5-b_2b_6)B^2-b_3b_6B-b_4b_6=\varphi$$

Relate the coefficients as in the following,

$$
\begin{aligned}
& b_1 = d_1 \\
& b_2 = d_2\beta \\
& b_3 - b_1 b_6 = d_3\beta^2 \\
& b_4 + b_5 - d_2\beta \frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = d_4\beta^3 \\
& -\frac{1}{d_1^2}\left(b_3 - d_3\beta^2\right)^2 = d_5\beta^4 + d_3\beta^2 \frac{1}{d_1}\left(b_3 - d_3\beta^2\right) \\
& -\left[d_4\beta^3 + d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) - b_5\right]\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = d_6\beta^5 + \varphi \\
& \frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = -\left\{\frac{\left[d_4\beta^3 + d_2\beta\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) - b_5\right]\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) + d_5\beta^5}{b_5}\right\}
\end{aligned}
\tag{41}
$$

The fifth equation of (41) gives the roots as,

$$
\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) = \frac{1}{2}\beta^2\left[d_3 \pm \left(d_3^2 - 4d_5\right)^{\frac{1}{2}}\right] = \beta^2 f_0 \tag{42}
$$

Moving to the last equation, the functions b_5 and β disappear from the operation. In this case we will consider the test function, $b_5 + \beta^{10} = d_6\beta^5$, and will perform as,

$$
\beta^{10} + d_4\beta^3\frac{1}{d_1}\left(b_3 - d_3\beta^2\right) + d_2\beta\frac{1}{d_1^2}\left(b_3 - d_3\beta^2\right)^2 + b_5 = 0 \tag{43}
$$

Substituting for b_3, the expression for β is,

$$
\begin{aligned}
& \beta^{10} + \left(d_4 f_0 + d_2 f_0^2\right) d_4\beta^5 + b_5 = 0 \\
& \beta^5 = -\frac{1}{2}\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}}
\end{aligned}
\tag{44}
$$

Substitute back to, $b_5 + \beta^{10} = d_6\beta^5$ as follows

$$
\begin{aligned}
& \left\{-\frac{1}{2}\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}}\right\}^2 + b_5 \\
& \quad = -\frac{1}{2}d_6\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}d_6\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}} \text{ or}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{4}\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right] - \frac{1}{2}\left(d_4 f_0 + d_2 f_0^2\right)\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}} \\
& + \frac{1}{4}\left(d_4 f_0 + d_2 f_0^2\right)^2 + b_5 = -\frac{1}{2}d_6\left(d_4 f_0 + d_2 f_0^2\right) \pm \frac{1}{2}d_6\left[\left(d_4 f_0 + d_2 f_0^2\right)^2 - 4b_5\right]^{\frac{1}{2}}
\end{aligned}
$$

or

$$
b_5 = \frac{1}{4}\left(d_4 f_0 + d_2 f_0^2\right)^2 - \frac{1}{4}\left[\frac{\left(d_4 f_0 + d_2 f_0^2\right)^2 + d_6\left(d_4 f_0 + d_2 f_0^2\right)}{\left(d_4 f_0 + d_2 f_0^2 + d_6\right)}\right]^2 \tag{45}
$$

which then solves b_5, β, φ and thus generates all the coefficients of b_i. The polynomial is then rewritten as,

$$\left(b_1B^3+b_2B^2+b_3B+b_4\right)\left(B^2-b_6\right)=-b_5\left(B^2-\frac{\varphi}{b_5}\right)$$

which is reduced as,

$$d_1A^3+d_2A^2+\left\{\frac{1}{2}d_1\left[d_3\pm\left(d_3^2-4d_5\right)^{\frac{1}{2}}\right]+d_3\right\}A+d_4+\frac{1}{2}d_2\left[d_3\pm\left(d_3^2-4d_5\right)^{\frac{1}{2}}\right]=0 \tag{46}$$

Eq. (46) dictates that the relations, $d_1=d_2=d_3=d_4=0$ will satisfy for the solution. Hence, the coefficients are then,

$$\begin{aligned}
&12b_7a_1^3+2b_1b_7^2a_1+12a_1^3b_7=0\\
&18a_1a_{1\xi}b_7+54a_1^2a_2b_7+6a_1^3b_6+b_1b_7b_{7\xi}+2a_2b_1b_7^2+18a_1^2b_{7\xi}+3a_1b_1b_6b_7=0\\
&40a_1^2a_3b_7+32a_1a_2^2b_7+12a_{1\xi}a_2b_7+18a_1a_{2\xi}b_7+2a_{1\xi\xi}b_7+6a_1a_{1\xi}b_6+12a_1^2a_2b_6\\
&+b_1b_{6\xi}b_7+b_1b_6b_{7\xi}+2a_3b_1b_7^2+6a_2^3b_7+30a_1a_2b_{7\xi}+6a_{1\xi}b_{7\xi}+6a_1^2b_{6\xi}+3a_2b_1b_6b_7\\
&+6a_1b_{7\xi\xi}+a_1b_1b_6^2+2a_1b_1b_5b_7+2a_1b_2b_7=0\\
&52a_1a_2a_3b_7+14a_{1\xi}a_3b_7+10a_1a_{3\xi}b_7+12a_2a_{2\xi}b_7+2a_{2\xi\xi}b_7+8a_2^3b_7+8a_1^2a_3b_6\\
&+7a_1a_2^2b_6+3a_{1\xi}a_2b_6+6a_1a_{2\xi}b_6+a_{1\xi\xi}b_6+24a_1a_3b_{7\xi}+12a_2^2b_{7\xi}+6a_{2\xi}b_{7\xi}+9a_1a_2b_{6\xi}\\
&+3a_{1\xi}b_{6\xi}+b_{7\xi\xi\xi}+b_1b_5b_{7\xi}+b_1b_6b_{6\xi}+b_1b_{5\xi}b_7+b_2b_{7\xi}+b_3b_7+3a_3b_1b_6b_7+6a_2b_{7\xi\xi}\\
&+a_2b_1b_6^2+2a_2b_1b_5b_7+2a_2b_2b_7+3a_1b_{6\xi\xi}+a_1b_1b_5b_6+a_1b_2b_6=0
\end{aligned} \tag{47}$$

The first equation gives,

$$b_7=f(b_1)a_1^2 \tag{48}$$

The second equation is rewritten as,

$$18a_{1\xi}fa_1^3+54fa_1^3a_2+6a_1^3b_6+b_1ff_{\xi}a_1^4+2b_1f^2a_1^3a_{1\xi}+2a_2b_1f^2a_1^4+18f_{\xi}a_1^4+18fa_1^3a_{1\xi}+3b_1b_6fa_1^2=0 \text{ or}$$

$$18a_{1\xi}f+54fa_2+6b_6+b_1ff_{\xi}a_1+2b_1f^2a_{1\xi}+2a_2b_1f^2a_1+18f_{\xi}a_1+18fa_{1\xi}+3b_1b_6f=0 \text{ or}$$

$$\left(36f+2b_1f^2\right)a_{1\xi}+54fa_2+6b_6+\left(b_1ff_{\xi}+2a_2b_1f^2+18f_{\xi}\right)a_1+3b_1b_6f=0$$

The solution for b_6 is then,

$$\left(36f+2b_1f^2\right)a_{1\xi}=-\left(b_1ff_{\xi}+2a_2b_1f^2+18f_{\xi}\right)a_1-54fa_2-\left(6+3fb_1\right)b_6$$

$$b_6=-\frac{1}{\left(6+3fb_1\right)}\left[\left(b_1ff_{\xi}+2a_2b_1f^2+18f_{\xi}\right)a_1+\left(36f+2b_1f^2\right)a_{1\xi}+54fa_2\right]=h_1(a_1,a_2) \tag{49}$$

The third equation will produce,

$$40a_1^2a_3f + 32a_1^2a_2^2f + 12a_{1\xi}a_2fa_1 + 18a_{2\xi}fa_1^2 + 2a_{1\xi\xi}fa_1 + 6a_{1\xi}b_6 + 12a_1a_2b_6 + b_1b_{6\xi}fa_1 + b_1b_6f_{\xi}a_1$$
$$+2fa_{1\xi}b_1b_6 + 2a_3b_1f^2a_1^3 + 6a_2^3fa_1 + 30f_{\xi}a_1^2a_2 + 30ff_{\xi}a_1^2a_2 + 60f^2a_{1\xi}a_1a_2 + 6a_{1\xi}f_{\xi}a_1 + 12fa_{1\xi}^2$$
$$+6a_1b_{6\xi} + 3a_2b_1b_6fa_1 + 6f_{\xi\xi}a_1^2 + 24f_{\xi}a_{1\xi}a_1 + 12fa_{1\xi\xi}a_1 + 12fa_{1\xi}^2 + b_1b_6^2 + 2b_1b_5fa_1^2 + 2b_2fa_1^2 = 0$$

Take the expression for b_5 as,

$$b_5 = -\frac{1}{2b_1fa_1^2}\left[h_2(a_1,a_2) + \left(40fa_1^2 + 2b_1f^2a_1^3\right)a_3\right]$$

with,

$$h_2(a_1,a_2) = 32a_1^2a_2^2f + 12a_{1\xi}a_2fa_1 + 18a_{2\xi}fa_1^2 + 2a_{1\xi\xi}fa_1 + 6a_{1\xi}h_1 + 12a_1a_2h_1$$
$$+b_1h_{1\xi}fa_1 + b_1h_1f_{\xi}a_1 + 24f_{\xi}a_{1\xi}a_1 + 6a_2^3fa_1 + 30f_{\xi}a_1^2a_2 + 30ff_{\xi}a_1^2a_2 + 60f^2a_{1\xi}a_1a_2$$
$$+6a_{1\xi}f_{\xi}a_1 + 12fa_{1\xi}^2 + 6a_1h_{1\xi} + 3a_2b_1h_1fa_1 + 6f_{\xi\xi}a_1^2 + 2fa_{1\xi}b_1h_1 + 12fa_{1\xi\xi}a_1 + 12fa_{1\xi}^2$$
$$+b_1h_1^2 + 2b_2fa_1^2 \tag{50}$$

The fourth relation of (47) will generate,

$$h_3(a_1,a_2) = \left(24f_{\xi}a_1^3 + 48fa_1^2a_{1\xi} + 3b_1b_6fa_1^2\right)a_3 + 6fa_{1\xi}a_{1\xi\xi}12a_2a_{2\xi}fa_1^2 + 2a_{2\xi\xi}fa_1^2$$
$$+8a_2^3fa_1^2 + 7a_1a_2^2h_1 + 3a_{1\xi}a_2h_1 + 6a_1a_{2\xi}h_1 + a_{1\xi\xi}h_1 + 12f_{\xi}a_1^2a_2^2 + 24fa_1a_{1\xi}a_2^2 + 6f_{\xi}a_1^2a_{2\xi}$$
$$+12fa_1a_{1\xi}a_{2\xi} + 9a_1a_2h_{1\xi} + 3a_{1\xi}h_{1\xi} + f_{\xi\xi\xi}a_1^2 + 2f_{\xi\xi}a_1a_{1\xi} + 4f_{\xi\xi}a_1a_{1\xi} + 6f_{\xi}a_{1\xi}^2 + 6f_{\xi}a_1a_{1\xi\xi}$$
$$+2fa_1a_{1\xi\xi\xi} + b_1h_1h_{1\xi} + f_{\xi}a_1^2b_2 + 2fa_1a_{1\xi}b_2 + b_3fa_1^2 + 2a_2b_2fa_1^2 + 3a_1h_{1\xi\xi} + a_1b_2h_1$$
$$+6f_{\xi\xi}a_1^2a_2 + 24f_{\xi}a_1a_{1\xi}a_2 + 12fa_{1\xi}^2a_2 + 12fa_1a_{1\xi\xi}a_2 + a_2b_1h_1^2 + \frac{h_2}{2b_1fa_1^2} + b_1fa_1^2\left(\frac{h_2}{2b_1fa_1^2}\right)_{\xi} \tag{51}$$

which then produce the solution of a_3.

Note that a_1 and a_2 are the chosen fundamental variables and according to (48–51) and with the known coefficients b_1, b_2, b_3, b_4, they will define, $b_5, b_6, b_7, a_3, \beta$. Therefore, the solution of Korteweg de Vries equation is generated as,

$$\phi(\xi) = b_5(a_1,a_2) + b_6(a_1,a_2)\left\{\frac{\sqrt{2}}{2}\frac{e^{\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1,a_2)d\xi + C_2\right)}{\left[\int_{\xi}a_1e^{2\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1,a_2)d\xi + C_2\right)d\xi + C_3\right]^{\frac{1}{2}}}\right\}$$
$$+b_7(a_1,a_2)\left\{\frac{1}{2}\frac{e^{2\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1,a_2)d\xi + C_2\right)^2}{\left[\int_{\xi}a_1e^{2\int_{\xi}a_2d\xi}\left(\int_{\xi}a_3(a_1,a_2)d\xi + C_2\right)d\xi + C_3\right]}\right\} \tag{52}$$

Since only a few of the considered equation has a special polynomial to be solved by equating all the variable coefficients to zero, it is important to note that the reduction of polynomial order would be an important step. Solving all coefficients to zero often be an obstacle because the difficulty would be the same or even more than the original nonlinear ODEs. In this case, the reduction of polynomial manipulates and reduces the need for solving all coefficients.

However, it is possible not to search for the expression of variable coefficients, i.e., b_5, b_6, b_7, a_1, a_2 and a_3. First the roots of Eq. (37) are determined first as ϕ, and then Eq. (1) is decomposed as,

$$D_\xi = a_1 B D^2 + \left(a_2 - \frac{B_\xi}{B}\right) D + \frac{a_3}{B} \tag{53}$$

with $A = BD$. The solution of (53) is then,

$$D = \frac{\sqrt{2}}{2B} e^{\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) d\xi + C_3 \right]^{-\frac{1}{2}} \text{ or}$$

$$A = BD = \frac{\sqrt{2}}{2} e^{\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) d\xi + C_3 \right]^{-\frac{1}{2}} \tag{54}$$

The definition for B is determined by substituting the polynomial solution, ϕ into (54) as in the following,

$$A = \phi = \frac{\sqrt{2}}{2} e^{\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) d\xi + C_3 \right]^{-\frac{1}{2}}$$

Rearranging the above equation as,

$$\phi^2 \left[\int_\xi \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) d\xi + C_3 \right] = \frac{1}{2} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right)^2$$

Differentiating once,

$$\begin{aligned} \frac{a_1}{B} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) &= \left(\frac{a_2}{\phi^2} e^{2\int_\xi a_2 d\xi} - \frac{\phi_\xi}{\phi^3} e^{2\int_\xi a_2 d\xi} \right) \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right)^2 \\ &\quad + \frac{a_3}{B} \frac{1}{\phi^2} e^{2\int_\xi a_2 d\xi} \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) \text{ or} \end{aligned}$$

$$\frac{a_1}{B} = \left(\frac{a_2}{\phi^2} - \frac{\phi_\xi}{\phi^3} \right) \left(\int_\xi \frac{a_3}{B} d\xi + C_2 \right) + \frac{a_3}{B} \frac{1}{\phi^2} \tag{55}$$

Eq. (55) is a first order ODE in B and can be easily solved, which then prove that $A = \phi$ without establishing the explicit expression for variable coefficients.

5. Generalized method

In this section, the method of solution to the Riccati equation is extended for the class of the first order polynomial differential equation as,

$$A_\xi = a_n A^n + a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \,\ldots\ldots\ldots + a_3 A^3 + a_2 A^2 + a_1 A + a_0 \tag{56}$$

The above equation can be always re-expressed as,

$$A_\xi = \left(b_nA^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \,\dots\dots\dots\, + b_3A + b_2\right)A^2 + b_1A \\ + \left(b_nA^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \,\dots\dots\dots\, + b_3A + b_2\right)b_0$$

or

$$A_\xi = b_nA^n + b_{n-1}A^{n-1} + (b_{n-2} + b_nb_0)A^{n-2} + \,\dots\dots\, + (b_3 + b_{n-1}b_0)A^3 \\ + (b_2 + b_{n-2}b_0)A^2 + (b_1 + b_3b_0)A + b_2b_0 \tag{57}$$

which the coefficients will be reformulated as,

$$\begin{aligned} &b_n = a_n, b_{n-1} = a_{n-1}, b_2b_0 = a_0 \\ &b_{n-2} + b_nb_0 = a_{n-2} \\ &b_3 + b_{n-1}b_0 = a_3 \\ &b_2 + b_{n-2}b_0 = a_2 \\ &b_1 + b_3b_0 = a_1 \end{aligned} \quad \text{or} \quad \begin{aligned} &b_n = a_n, b_{n-1} = a_{n-1}, b_2b_0 = a_0 \\ &a_2b_2^2 - b_2^3 + a_na_0 = a_{n-2}a_0b_2 \\ &b_3 + a_{n-1}\frac{a_0}{b_2} = a_3 \\ &b_2^2 + b_{n-2}a_0 = a_2b_2 \\ &b_1 + b_3\frac{a_0}{b_2} = a_1 \end{aligned} \tag{58}$$

In this case, we will always obtain the new coefficients b_i. Proceeding into the other equations andmultiply the equation by the function α, to get,

$$\alpha A_\xi = \left(b_nA^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \,\dots\dots\dots\, + b_3A + b_2\right)\alpha A^2 + b_1\alpha A \\ + \left(b_nA^{n-2} + b_{n-1}A^{n-3} + b_{n-2}A^{n-4} + \,\dots\dots\dots\, + b_3A + b_2\right)\alpha b_0 \quad \text{or}$$

$$B_\xi = \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \,\dots\dots\dots\, + b_3\alpha^{-1}B + b_2\right)\alpha^{-1}B^2 + \left(b_1 + \frac{\alpha_t}{\alpha}\right)B \\ + \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \,\dots\dots\dots\, + b_3\alpha^{-1}B + b_2\right)\alpha b_0 \tag{59}$$

where $B = \alpha A$. Then, all the new coefficients in b_i will be determined. The step is now to solve the Riccati equation. Let $B = \beta_2\beta_3$, the equation can be rearranged as,

$$\beta_3\beta_{2\xi} + \beta_2\beta_{3\xi} = \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \,\dots\dots\dots\, + b_3\alpha^{-1}B + b_2\right)\alpha^{-1}\beta_2^2\beta_3^2 \\ + \left(b_1 + \frac{\alpha_\xi}{\alpha}\right)\beta_2\beta_3 + \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \,\dots\dots\dots\, + b_3\alpha^{-1}B + b_2\right)\alpha b_0 \quad \text{or}$$

$$\beta_3\beta_{2\xi} - \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \,\dots\dots\dots\, + b_3\alpha^{-1}B + b_2\right)\alpha^{-1}\beta_2^2\beta_3^2 \\ - \left(b_1 + \frac{\alpha_\xi}{\alpha}\right)\beta_2\beta_3 = -\beta_2\beta_{3\xi} + \begin{pmatrix} b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} \\ + \,\dots\dots\dots\, + b_3\alpha^{-1}B + b_2 \end{pmatrix}\alpha b_0 = \gamma\beta_2\beta_3 \tag{60}$$

and is separated as,

$$\beta_{2\xi} - \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \,\dots\dots\dots\, + b_3\alpha^{-1}B + b_2\right)\alpha^{-2}\beta_2^2\beta_3 \\ - \left(b_1 + \frac{\alpha_\xi}{\alpha} + \gamma\right)\beta_2 = 0$$

and

$$\beta_{3\xi}+\gamma\beta_3-\frac{1}{\beta_2}\left(b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B+b_2\right)\alpha b_0=0 \tag{61}$$

The solutions for β_2 and β_3 are,

$$\beta_2=-e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}+\gamma\right)d\xi}x$$

$$\left[\int_\xi e^{\int_t\left(b_1+\frac{\alpha_\xi}{\alpha}+\gamma\right)d\xi}\begin{pmatrix}b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}\\+\dots\dots\dots+b_3\alpha^{-1}B+b_2\end{pmatrix}\alpha^{-1}\beta_3d\xi+C_1\right]^{-1} \text{ and}$$

$$\beta_3=e^{-\int_\xi\gamma d\xi}\left[\int_\xi e^{\int_\xi\gamma d\xi}\begin{pmatrix}b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}\\+\dots\dots\dots+b_3\alpha^{-1}B+b_2\end{pmatrix}\alpha\frac{b_0}{\beta_2}d\xi+C_2\right] \tag{62}$$

The relation for $B=\beta_2\beta_3$ is thus,

$$B=\beta_2\beta_3=-e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\begin{bmatrix}\int_\xi e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}+\gamma\right)d\xi}\left(b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B+b_2\right)\alpha^{-1}\beta_3d\xi\\+C_1\end{bmatrix}^{-1}x$$

$$\left[\int_\xi e^{\int_\xi\gamma d\xi}\left(b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B+b_2\right)\alpha b_0d\xi+C_2\right]$$

Without loss of generality, suppose that $\beta_2=\varphi e^{\int_\xi\gamma d\xi}$ and the above relation is performed as,

$$\beta_2\beta_3=-e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\left[\int_\xi e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\begin{pmatrix}b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}\\+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B\\+b_2\end{pmatrix}\alpha^{-1}\varphi\beta_2\beta_3d\xi+C_1\right]^{-1} \text{ or}$$

$$\left[\int_\xi\left(b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B+b_2\right)\alpha\varphi^{-1}b_0d\xi+C_2\right]$$

$$\beta_2\beta_3\left[\int_\xi e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\begin{pmatrix}b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}\\+\dots\dots\dots+b_3\alpha^{-1}B+b_2\end{pmatrix}\alpha^{-1}\varphi\beta_2\beta_3d\xi+C_1\right]=$$

$$-e^{\int_\xi\left(b_1+\frac{\alpha_\xi}{\alpha}\right)d\xi}\left[\int_t\begin{pmatrix}b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}\\+\dots\dots\dots+b_3\alpha^{-1}B+b_2\end{pmatrix}\alpha\varphi^{-1}b_0d\xi+C_2\right]$$

Rearrange the above equation as,

$$e^{\int_\xi b_1d\xi}\left(b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B+b_2\right)\varphi\beta_2\beta_3$$

$$\left[\int_\xi e^{\int_\xi b_1d\xi}\left(b_ny^{n-2}+b_{n-1}y^{n-3}+b_{n-2}y^{n-4}+\ \dots\dots\dots+b_3y+b_2\right)\varphi\beta_2\beta_3d\xi+C_1\right]=$$

$$-e^{2\int_\xi b_1d\xi}\alpha\varphi\left(b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B+b_2\right)$$

$$\left[\int_\xi\left(b_n\alpha^{2-n}B^{n-2}+b_{n-1}\alpha^{3-n}B^{n-3}+b_{n-2}\alpha^{4-n}B^{n-4}+\ \dots\dots\dots+b_3\alpha^{-1}B+b_2\right)\alpha\varphi^{-1}b_0d\xi+C_2\right]$$

Let $e^{2\int_\xi b_1 d\xi}\alpha\varphi = \alpha\varphi^{-1}b_0$ and integrate the above equation to get,

$$\left[\int_\xi e^{\int_\xi b_1 d\xi}\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B + b_2\right)\varphi\beta_2\beta_3 dt + C_1\right]^2 =$$
$$-\left[\int_\xi \left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B + b_2\right)\alpha\varphi^{-1}b_0 d\xi + C_2\right]^2$$

or

$$e^{\int_\xi b_1 d\xi}\left(b_n\alpha^{2-n}B^{n-1} + b_{n-1}\alpha^{3-n}B^{n-2} + b_{n-2}\alpha^{4-n}B^{n-3} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B^2 + b_2 B\right)\varphi^2 =$$
$$-\left(b_n\alpha^{2-n}B^{n-2} + b_{n-1}\alpha^{3-n}B^{n-3} + b_{n-2}\alpha^{4-n}B^{n-4} + \ \ldots\ldots\ldots\ + b_3\alpha^{-1}B + b_2\right)\alpha b_0$$

The solution for B is then reduced into the solution of the polynomial equation. Thus, let $A = \alpha^{-1}B = \phi$, whereϕ is the expression from the solution of the resulting polynomial equation which is similar to (38). The expression for α can be determined by the inverse method as in (53–55) for the first order polynomial differential Eq. (56).

6. Conclusion

In this chapter, we propose the method of the simplest or the auxiliary equation to solve the nonlinear differential equation with variable coefficients. The method is based on the solution of the generalized Riccati equation as the simplest equation. It is found that the other known simplest equations, i.e., Jacobi and Weierstrass equation, are also solved by the Riccati equation. The applications with the variable coefficients elliptic-like and Korteweg de Vries equations show that the problem of solving nonlinear differential equations with variable coefficients are simplified, especially by the reduction of the resulting polynomial equation in solving the Korteweg de Vries equation. The generalization of the method is also derived in detail.

Conflict of interest

Authors declare that there is no conflict of interest.

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Chapter 21

Thermodynamic Stability Conditions as an Eigenvalues Fundamental Problem

Francisco Nogueira Lima

Abstract

Quadratic forms diagonalization methods can be used in addressing the stability of physical systems. Thermodynamic stability conditions appears as an eigenvalues fundamental problem, in particular when postulational approaches is taken. The second-order derivatives or appropriate relations between such derivatives of the energy, entropy or any considered thermodynamic potential, as Helmholtz, enthalpy and Gibbs, have interesting mathematical features that directly imply in the physical stability, obtained by use and as consequence of analytical techniques. Formal aspects on the thermal and mechanical stability become simple consequences, but no less formal, of the superposition of rigorously established physical laws, and appropriate applications of mathematical techniques.

Keywords: quadratic forms, Taylor's series, themodynamic stability, eigenvalues, thermodynamic potentials

1. Introduction

In physics, there is a time-independent theory, namely, thermodynamics that is used to determine the macroscopic equilibrium of physical systems. In practice, to compute the equilibrium conditions and the physical properties of a system, a physicist must find a function that completely describes the system, being capable of capturing all involved properties. The existence of such a function arises as a postulate of the themodynamics, having an extremum to the equilibrium states [1]. The function is called entropy and has a maximum at final equilibrium state. On the other hand, the same understanding about the physical properties of the system can be extracted through another relevant physical function, namely, energy. This treatment of using energy function instead of entropy to investigate the physical properties is completely equivalent but now the energy has a minimum and its existence also occurs by postulational reason, as for entropy function. A broad discussion on themodynamic's postulates can be found in Ref. [1].

In practical problems, it woud be impossible to computing the total energy of a system taking all time-dependent freedom degrees, such as atomic coordinates of the components of the system each with its translational, rotation energies, etc., among others time-dependent properties. The thermodynamics theory emerges from the fact that a great number of those freedom degrees are eliminated by considering statistical averages, and not macroscopically manifesting. Thus, as the physical principle of energy conservation keeps unaltered over decades, having been already rigorously tried and confirmed, a well-defined thermodynamic energy function appears somewhat intuitive. Indeed, the energy must be interpreted as a function capable of providing the macroscopic properties of the system. Besides, due to the complexity in measuring the energy of a system, it is relevant to assume some state whereby the energy is arbitrary defined as zero and measuring the energy in connection that state because only energy differences have any physical meaning [1–3].

There are equivalent approaches to investigate the thermodynamics properties of a system in terms of thermodynamic functions (or thermodynamic potentials) of Helmholtz, enthalpy and Gibbs instead of the energy or entropy. Such thermodynamic potentials are obtained by using Legendre transformations in order to change the original extensive variables, or part of them, in the function thermodynamic energy by the intensive variables. Besides, other thermodynamic functions (in addition to those already mentioned) can appear when making Legendre transformations in specific extensive parameters of the energy or in the extensive parameters of the entropy, such as grand canonical potential, and Massieu, Planck and Krammers functions. The function to be used must be defined by the practical characteristics of the problem and these last mentioned functions are less common in more elementary approaches of postulational thermodynamics [1, 4].

A solid understanding of postulational thermodynamic theory is necessary in order to investigate the thermal or mechanic stability of the most diverse systems. The increase in the thermal stability of DNA against thermal denaturation can be experimentally investigated using a methodology in which the differences or changes in the standard values of negativity and positivity of enthalpy and entropy, or even between them, are decisive for the study's conclusions [5]. The formalism of free energy (or Helmholtz potential) can be used for practical determination of the level of stored energy accumulated in material during plastic processing applied as well as the stored energy for the simple stretching of austenitic steel [6]. There are an infinity of applications of thermodynamic theory in wich the stability of a system is intimately related to some physical feature of thermodynamic functions, and whose the convenience of the choice is determined by practical situation.

Interesting formalisms or analytical techniques that combine the superposition of the thermodynamic theory and mathematical methods appear as support for problems of applied physics aimed to investigate the stability conditions of a system, either through experimental or computational studies. In order to show of a physical point of view, as arises the thermal and mechanical stability of a system, let us invoke the known physical origin of the energy U, i. e., its existence is determined by a postulate and the same way we know that U is a function of the extensive parameters, entropy S, volume V and the mole numbers of the chemical components $N_1, N_2, \dots, N_r$. This physical consideration can be mathematically written as $U = U(S, V, N_1, N_2, \dots, N_r)$. Similarly, entropy S is a function of the extensive parameters, energy U, volume V and the mole numbers of the chemical components $N_1, N_2, \dots, N_r$, and so $S = S(U, V, N_1, N_2, \dots, N_r)$ [1].

In this chapter, we discuss in details the postulate of maximum entropy or minimum energy through which it is possible to see that the thermodynamic functions S or U, or any potential/function derived them by Legendre transformations, have mathematical features that can be obtained of an eingenvalues fundamental problem, that is, the diagonalization of the hypersurfaces defined by $U = U(S, V, N_1, N_2, \dots, N_r)$ or $S = S(U, V, N_1, N_2, \dots, N_r)$ that conveniently expanded in Taylor's series provides the signs its second-order derivatives in an

$(r+2)$-dimensional thermodynamic space. Besides, some relations between these derivatives by diagonalization of the quadratic form of U, S or other thermodynamic function, naturally appear and as consequence relevant conclusions about the system stability. Quadratic forms appear in several physical problems, especially in quantum mechanics [7], and in thermodynamic theory this is not different. In particular, we precisely investigate the mathematical caracteristics of the hypersurface of energy and other thermodynamic functions for a system of single chemical component. In this case, it is possible to reduce the hypersurface $U(S, V, N_1, N_2, \dots, N_r)$, in an $(r+2)$-dimensional thermodynamic space, to a three dimensional hypersurface where $U = U(S, V, N)$ (see that $r = 1$). Analytical calculations of quadratic forms diagonalization are used to reveal the signs of the second-order derivatives of the three-dimensional thermodynamic functions. Accordingly, the stability conditions are obtained.

This chapter is organized as follows. In Section 2, we discuss the general procedures to diagonalize the thermodynamic energy as well as obtain Talyor's series in an $(r+2)$-dimensional thermodynamic space. It is also presented the same way to entropy function. In Section 3, we diagonalize thermodynamic energy in a three-dimensional space, and derived Helmholtz, enthalpy, and Gibbs potentials as well as grand canonical potential. In addition, the signs of second-order derivatives of such thermodynamic functions are calculated. In Section 4, stability conditions are presented as consequences of the obtained signs in previous section. As it turns, we summarize our main findings and draw some perspectives in Section 5.

2. The quadratic form of the energy hypersurface in an $(r+2)$-dimensional thermodynamic space

We already addressed in the introduction about the postulational existence of the thermodynamic energy function $U = U(S, V, N_1, N_2, \dots, N_r)$ that is a function on extensive parameters entropy S, volume V and the mole numbers of the chemical components $N_1, N_2, \dots, N_r$, where r represents the amount of chemical components in the system. Besides, U is capable of describing all thermodynamic macroscopic properties of treated system. A formal discussion on extensive parameters can be found in Ref. [1]. However, understand them as those are dependent on the amount of matter or mass of the system.

Remembering the most general form of Taylor's series for a function $f = f(x_1, x_2, \dots, x_n)$ of n variables expanded around $(x_{10}, x_{20}, \dots, x_{n0})$ [8]:

$$\begin{aligned} f(x_1, x_2, \dots, x_n) = f(x_{10}, x_{20}, \dots, x_{n0}) + \sum_i \frac{\partial f}{\partial x_i} \Delta x_i \\ + \frac{1}{2!} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + \dots, \end{aligned} \tag{1}$$

where $\Delta x_i = x_i - x_{i0}$, and all partial derivatives are evaluated at $(x_{10}, x_{20}, \dots, x_{n0})$. Here x_{i0} denotes the coordinates of some arbitrary stationary point around which the function is expanded, with zero index to differentiate it from all other points in the n-dimensional space.

Let us carefully expanding the energy $U(S, V, N_1, N_2, \dots, N_r)$ using Taylor's series given by Eq. (1) around $(S_0, V_0, N_{10}, N_{20}, \dots, N_{r0})$ point in $(r+2)$-dimensional space.

$$
\begin{aligned}
U(S,V,N_1,N_2,\ldots,N_r) &= U(S_0,V_0,N_{10},N_{20},\ldots,N_{r0}) + \frac{\partial U}{\partial S}(S-S_0) \\
+\frac{\partial U}{\partial V}(V-V_0) &+ \sum_{k=1}^{r}\frac{\partial U}{\partial N_k}(N_k-N_{k0}) + \frac{1}{2!}\left[\frac{\partial^2 U}{\partial S^2}(S-S_0)^2 + \frac{\partial^2 U}{\partial V^2}(V-V_0)^2\right. \\
&\left. + \sum_{k=1}^{r}\frac{\partial^2 U}{\partial N_k^2}(N_k-N_{k0})^2 + \sum_i\sum_{j(i\neq j)}\frac{\partial^2 U}{\partial X_i \partial X_j}\Delta X_i \Delta X_j\right] + \ldots,
\end{aligned}
\tag{2}
$$

where $\Delta X_i \equiv X_i - X_{i0}$, with $X_i = S, V, N_1, N_2, \ldots, N_r$ and $X_{i0} = S_0, V_0, N_{10}, N_{20}, \ldots, N_{r0}$. Notice that last term that explicitly appears in Eq. (2) in wich the simplified notation X_i is introduced represents all possible combinations of double partial derivatives obtained from the extensive variables of the energy. Besides, see that $i \neq j$ in the same term due to already computed previous terms to $i = j$.

By analogy with the one-variable differential calculus and due to the postulate of minimum energy ($d^2U > 0$, see Refs. [1–3]), taking a stationary point $(S_0, V_0, N_{10}, N_{20}, \ldots, N_{r0})$, we know that all first-order derivatives in Eq. (2) are null at this point

$$
\frac{\partial U}{\partial S} = 0,\ \frac{\partial U}{\partial V} = 0 \,\text{and}\, \frac{\partial U}{\partial N_k} = 0 \,\text{with}\, \mathrm{k} = (1,..,\mathrm{r}),
\tag{3}
$$

and therefore

$$
\begin{aligned}
U(S,V,N_1,N_2,\ldots,N_r) &= U(S_0,V_0,N_{10},N_{20},\ldots,N_{r0}) + \frac{1}{2!}\left[\frac{\partial^2 U}{\partial S^2}(S-S_0)^2\right. \\
&\left.+\frac{\partial^2 U}{\partial V^2}(V-V_0)^2 + \sum_{k=1}^{r}\frac{\partial^2 U}{\partial N_k^2}(N_k-N_{k0})^2 + \sum_i\sum_{j(i\neq j)}\frac{\partial^2 U}{\partial X_i \partial X_j}\Delta X_i \Delta X_j\right] + \ldots
\end{aligned}
\tag{4}
$$

Let us define in Eq. (4) $\Delta S \equiv S - S_0$, $\Delta V \equiv V - V_0$, $\Delta N_k \equiv N_k - N_{k0}$, $\Delta U \equiv U(S, V, N_1, N_2, \ldots, N_k) - U(S_0, V_0, N_{10}, N_{20}, \ldots, N_{r0})$, and also $\tilde{U} \equiv 2!(\Delta U)$. Thus, it is possible rewriting Eq. (4) as follows.

$$
\begin{aligned}
\tilde{U}(S,V,N_1,N_2,\ldots,N_r) = \frac{\partial^2 U}{\partial S^2}(\Delta S)^2 &+ \frac{\partial^2 U}{\partial V^2}(\Delta V)^2 + \sum_{k=1}^{r}\frac{\partial^2 U}{\partial N_k^2}(\Delta N_k)^2 \\
&+\sum_i\sum_{j(i\neq j)}\frac{\partial^2 U}{\partial X_i \partial X_j}\Delta X_i \Delta X_j + \ldots
\end{aligned}
\tag{5}
$$

Notice that $\tilde{U}$ in above expression must be interpreted the same way as the U, being only mathematically multiplied and suppressed by the constants 2! and $U(S_0, V_0, N_{10}, N_{20}, \ldots, N_{r0})$, respectively. Physically, $\tilde{U}$ also obeys minimum energy postulate and keep the dependence with the extensive parameters, $\tilde{U} = \tilde{U}(S, V, N_1, N_2, \ldots, N_r)$. On the other words, $\tilde{U}$ is the original energy function U, at less than a multiplicative constant, and additive. We should not forget that the expression given by Eq. (5) has more terms than those explicitly listed, with third-order, fourth-order derivatives and so on. However, if we take only terms until the second-order derivatives, it is possible to see that hypersurface defined by $\tilde{U}$ is a complete quadratic form, in an $(r+2)$-dimensional thermodynamic space (see quadratic forms in Refs. [8, 9]). Then, some mathematical generalities can be extracted of the thermodynamic energy written as Eq. (6) below:

$$\tilde{U} = \frac{\partial^2 U}{\partial S^2}(\Delta S)^2 + \frac{\partial^2 U}{\partial V^2}(\Delta V)^2 + \sum_{k=1}^{r} \frac{\partial^2 U}{\partial N_k^2}(\Delta N_k)^2 + \sum_i \sum_{j(i \neq j)} \frac{\partial^2 U}{\partial X_i \partial X_j} \Delta X_i \Delta X_j. \tag{6}$$

The matricial form of the quadratic expression in Eq. (6) is given by

$$\tilde{U} = (\Delta S \;\; \Delta V \;\; \Delta N_1 \;\; \Delta N_2 \;\; \dots \;\; \Delta N_r) \begin{pmatrix} \frac{\partial^2 U}{\partial S^2} & \cdots & & \cdots & . \\ \vdots & \frac{\partial^2 U}{\partial V^2} & & \frac{\partial^2 U}{\partial X_i \partial X_j} & \vdots \\ & & \frac{\partial^2 U}{\partial N_1^2} & & \\ \vdots & \frac{\partial^2 U}{\partial X_j \partial X_i} & & \ddots & \vdots \\ . & \cdots & & \cdots & \frac{\partial^2 U}{\partial N_r^2} \end{pmatrix} \begin{pmatrix} \Delta S \\ \Delta V \\ \Delta N_1 \\ \Delta N_2 \\ \vdots \\ \Delta N_r \end{pmatrix}, \tag{7}$$

where the second-order derivatives above and below of main diagonal represent all combinations of double partial derivatives in relation to the extensive variables of the energy. Explicitly showing the terms of mixed partial derivatives in the matricial equation given by Eq. (7), we have

$$\tilde{U} = (\Delta S \;\; \Delta V \;\; \Delta N_1 \;\; \Delta N_2 \;\; \dots \;\; \Delta N_r) \begin{pmatrix} \frac{\partial^2 U}{\partial S^2} & \frac{\partial^2 U}{\partial S \partial V} & \frac{\partial^2 U}{\partial S \partial N_1} & \cdots & \frac{\partial^2 U}{\partial S \partial N_r} \\ \frac{\partial^2 U}{\partial V \partial S} & \frac{\partial^2 U}{\partial V^2} & \frac{\partial^2 U}{\partial V \partial N_1} & \cdots & \frac{\partial^2 U}{\partial V \partial N_r} \\ \frac{\partial^2 U}{\partial N_1 \partial S} & \frac{\partial^2 U}{\partial N_1 \partial V} & \frac{\partial^2 U}{\partial N_1^2} & \cdots & \frac{\partial^2 U}{\partial N_1 \partial N_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 U}{\partial N_r \partial S} & \frac{\partial^2 U}{\partial N_r \partial V} & \frac{\partial^2 U}{\partial N_r \partial N_1} & \cdots & \frac{\partial^2 U}{\partial N_r^2} \end{pmatrix} \begin{pmatrix} \Delta S \\ \Delta V \\ \Delta N_1 \\ \Delta N_2 \\ \vdots \\ \Delta N_r \end{pmatrix}. \tag{8}$$

Resuming the previous discussion in which the extensive variables are compactly defined as X_i, we can also express the energy in Eq. (8) of a compact way

$$\tilde{U} = (\Delta X_i)^T M (\Delta X_i), \tag{9}$$

where

$$M \equiv \begin{pmatrix} \frac{\partial^2 U}{\partial S^2} & \frac{\partial^2 U}{\partial S \partial V} & \frac{\partial^2 U}{\partial S \partial N_1} & \cdots & \frac{\partial^2 U}{\partial S \partial N_r} \\ \frac{\partial^2 U}{\partial V \partial S} & \frac{\partial^2 U}{\partial V^2} & \frac{\partial^2 U}{\partial V \partial N_1} & \cdots & \frac{\partial^2 U}{\partial V \partial N_r} \\ \frac{\partial^2 U}{\partial N_1 \partial S} & \frac{\partial^2 U}{\partial N_1 \partial V} & \frac{\partial^2 U}{\partial N_1^2} & \cdots & \frac{\partial^2 U}{\partial N_1 \partial N_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 U}{\partial N_r \partial S} & \frac{\partial^2 U}{\partial N_r \partial V} & \frac{\partial^2 U}{\partial N_r \partial N_1} & \cdots & \frac{\partial^2 U}{\partial N_r^2} \end{pmatrix}, \tag{10}$$

ΔX_i is a column vector with ΔS, ΔV, ΔN_1, …, ΔN_r components, and ΔX_i^T is the transpose of ΔX_i. As M is a symmetric matrix, a diagonalization procedure can be applied to simplify the investigation of mathematical features of $\tilde{U}$ and its physical consequences. At first, the choice to expanding the thermodynamic energy in Taylor's series up to the second-order is due to the appearance of a complete quadratic form with a known mathematics of many-variable calculus. Accordingly, the canonical form $\tilde{U} = \Delta X_i'^T D \Delta X_i'$ obtained by diagonalization allows visualizing interesting physical features more easily. Notice that D is the eigenvalues matrix of M with $(r+2)$-components, and the $\Delta X_i'$ is the column eigenvector (with $\Delta S'$, $\Delta V'$, $\Delta N_1'$, …, $\Delta N_r'$ components) of the diagonal matrix D as well as $\Delta X'^T$ is the transpose. A review on quadratic forms diagonalization can be found in Ref. [9]. The canonical form to $\tilde{U}$ can be expressed by Eq. (11)

$$\begin{aligned}\tilde{U} = \left(\Delta X_i'\right)^T D\left(\Delta X_i'\right) = \\ \begin{pmatrix}\Delta S' & \Delta V' & \Delta N_1' & \Delta N_2' & \dots & \Delta N_r'\end{pmatrix} \begin{pmatrix} \lambda_S & 0 & 0 & 0 & 0 \\ 0 & \lambda_V & 0 & 0 & 0 \\ 0 & 0 & \lambda_{N_1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_{N_r} \end{pmatrix} \begin{pmatrix} \Delta S' \\ \Delta V' \\ \Delta N_1' \\ \Delta N'2 \\ \vdots \\ \Delta N_r' \end{pmatrix} \\ = \lambda_S(\Delta S')^2 + \lambda_V(\Delta V')^2 + \lambda_{N_1}\left(\Delta N_1'\right)^2 + \dots + \lambda_{N_r}\left(\Delta N_r'\right)^2.\end{aligned} \tag{11}$$

See that in the canonical form of $\tilde{U}$ given by Eq. (11) are eliminated the mixed partial derivatives of Eq. (6). Besides, the minimum energy postulate imposes to the function $\tilde{U}(S, V, N_1, N_2, .., N_r)$ in Eq. (11) the following mathematical condition:

$$\tilde{U} = \lambda_S(\Delta S')^2 + \lambda_V(\Delta V')^2 + \lambda_{N_1}\left(\Delta N_1'\right)^2 + \dots + \lambda_{N_r}\left(\Delta N_r'\right)^2 > 0. \tag{12}$$

It is possible to see that this conditon occurs only when $\lambda_S > 0$, $\lambda_V > 0$, $\lambda_{N_1} > 0$, …, $\lambda_{N_r} > 0$ for any sets of values of $\Delta S'$, $\Delta V'$, $\Delta N_1'$, …, $\Delta N_r'$. To obtain the λ_i $(i = S, V, N_1, \dots, N_r)$ eigenvalues, it is necessary diagonalize M (see Eq. (8)) by solving the equation $(\lambda_i I - M)X_i = 0$, where I is an indentity matrix (see Ref. [9]) that provides the determinant below

$$\begin{vmatrix} \left(\dfrac{\partial^2 U}{\partial S^2} - \lambda\right) & \dfrac{\partial^2 U}{\partial S \partial V} & \dfrac{\partial^2 U}{\partial S \partial N_1} & \dots & \dfrac{\partial^2 U}{\partial S \partial N_r} \\ \dfrac{\partial^2 U}{\partial V \partial S} & \left(\dfrac{\partial^2 U}{\partial V^2} - \lambda\right) & \dfrac{\partial^2 U}{\partial V \partial N_1} & \dots & \dfrac{\partial^2 U}{\partial V \partial N_r} \\ \dfrac{\partial^2 U}{\partial N_1 \partial S} & \dfrac{\partial^2 U}{\partial N_1 \partial V} & \left(\dfrac{\partial^2 U}{\partial N_1^2} - \lambda\right) & \dots & \dfrac{\partial^2 U}{\partial N_1 \partial N_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 U}{\partial N_r \partial S} & \dfrac{\partial^2 U}{\partial N_r \partial V} & \dfrac{\partial^2 U}{\partial N_r \partial N_1} & \dots & \left(\dfrac{\partial^2 U}{\partial N_r^2} - \lambda\right) \end{vmatrix} = 0. \tag{13}$$

Observe that Eq. (13) implies an equation in λ of $(r+2)$-degree. Besides, all λ_i are necessarily positive due to the minimum energy postulate.

So far, we have show some generalities about thermodynamic energy in an $(r+2)$-dimensional space. Notice that diagonalizing M by solving Eq. (13) is not an easy task. For a system with great number of chemical components analytical solutions of Eq. (13) can become increasingly hard.

If we take the entropy of the system instead of energy, all above formalism remains valid by simple exchanging U and S variables in the equations. In addition, due to the maximum entropy postulate, all eigenvalues of second-order derivatives matrix (similar to M by exchanging U and S) must be negatives ($\lambda_U < 0, \lambda_V < 0, \lambda_{N_1} < 0, \dots, \lambda_{N_r} < 0$). Then, in this case we have Eq. (14) instead Eq. (12).

$$\tilde{S} = \lambda_S (\Delta U')^2 + \lambda_V (\Delta V')^2 + \lambda_{N_1} \left(\Delta N_1'\right)^2 + \dots + \lambda_{N_r} \left(\Delta N_r'\right)^2 < 0. \qquad (14)$$

In a two-dimensional thermodynamic space, a discussion on the eigenvalues of M and the physical consequences of its positivity is presented in Ref. [10]. In this case, the conditions of thermal and mechanical stability are naturally demonstrated through the signs of the second-order derivatives of some thermodynamic function of two-variables. The two-dimensional problem arises when is considered a one-component system and, in particular, we can take the thermodynamic energy per mol, reducing the dependence of such energy function for only the variables entropy (s) and volume (v) per mol ($u = u(s, v)$).

The stability conditions of a thermodynamic system are intrinsically related to the signs of the second-order derivatives of the energy, being the exact calculating of the eigenvalues of Eq. (13) (of previously known signs) an important factor in order to understand the physical origin of the stability of the system. In next section, we present a discussion of eigenvalues of the energy in a three-dimensional thermodynamic space.

3. Diagonalization of the energy in a three-dimensional thermodynamic space

Let us define the energy in a three-dimensional thermodynamic space. To do this, we consider a system with one chemical component and explicitly write the energy $U = U(S, V, N)$ in terms of the involved extensive parameters S, V and N. Similarly of Eq. (3) and by analogy with one-variable calculus, we have $\frac{\partial U}{\partial S} = \frac{\partial U}{\partial V} = \frac{\partial U}{\partial N} = 0$ (at a stationary point (S_0, V_0, N_0)) due to the minimum energy principle. Besides, in order to investigate the second-order derivatives of U (or $\tilde{U}$, there are no physical difference), a simple matricial quadratic form can be obtained by application of Eqs. (6), (7) and (8), as follows:

$$\tilde{U} = (\Delta S \ \ \Delta V \ \ \Delta N) \begin{pmatrix} \frac{\partial^2 U}{\partial S^2} & \frac{\partial^2 U}{\partial S \partial V} & \frac{\partial^2 U}{\partial S \partial N} \\ \frac{\partial^2 U}{\partial V \partial S} & \frac{\partial^2 U}{\partial V^2} & \frac{\partial^2 U}{\partial V \partial N} \\ \frac{\partial^2 U}{\partial N \partial S} & \frac{\partial^2 U}{\partial N \partial V} & \frac{\partial^2 U}{\partial N^2} \end{pmatrix} \begin{pmatrix} \Delta S \\ \Delta V \\ \Delta N \end{pmatrix} = (\Delta X_i)^T M_{(3x3)} (\Delta X_i), \qquad (15)$$

where

$$M_{(3x3)} = \begin{pmatrix} \frac{\partial^2 U}{\partial S^2} & \frac{\partial^2 U}{\partial S \partial V} & \frac{\partial^2 U}{\partial S \partial N} \\ \frac{\partial^2 U}{\partial V \partial S} & \frac{\partial^2 U}{\partial V^2} & \frac{\partial^2 U}{\partial V \partial N} \\ \frac{\partial^2 U}{\partial N \partial S} & \frac{\partial^2 U}{\partial N \partial V} & \frac{\partial^2 U}{\partial N^2} \end{pmatrix} \tag{16}$$

ΔX_i is a column vector with ΔS, ΔV, ΔN components, ΔX_i^T is the transpose of ΔX_i, and $M_{(3x3)}$ is a symmetric matrix that provides three eigenvalues for $\tilde{U}$ by diagonalization of $M_{(3x3)}$. Thus, by using the canonical form of $\tilde{U}$ combined with minimum energy principle, we know that all signs of the eigenvalues λ_1, λ_2, and λ_3 of $M_{(3x3)}$ are positive

$$\begin{aligned} \tilde{U} &= \left(\Delta X_i'\right)^T D_{(3x3)} \left(\Delta X_i'\right) = \left(\Delta S' \ \ \Delta V' \ \ \Delta N'\right) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \Delta S' \\ \Delta V' \\ \Delta N' \end{pmatrix} \\ &= \lambda_1 (\Delta S')^2 + \lambda_2 (\Delta V')^2 + \lambda_3 (\Delta N')^2 > 0. \end{aligned} \tag{17}$$

Note that $D_{(3x3)}$ in Eq. (17) is the eigenvalues matrix of $M_{(3x3)}$ given by Eq. (16). As in Eq. (13), here we need solve the eigenvalues equation $\left(\lambda I_{(3x3)} - M_{(3x3)}\right) X_i = 0$ that provides the following determinant

$$\begin{vmatrix} \left(\frac{\partial^2 U}{\partial S^2} - \lambda\right) & \frac{\partial^2 U}{\partial S \partial V} & \frac{\partial^2 U}{\partial S \partial N} \\ \frac{\partial^2 U}{\partial V \partial S} & \left(\frac{\partial^2 U}{\partial V^2} - \lambda\right) & \frac{\partial^2 U}{\partial V \partial N} \\ \frac{\partial^2 U}{\partial N \partial S} & \frac{\partial^2 U}{\partial N \partial V} & \left(\frac{\partial^2 U}{\partial N^2} - \lambda\right) \end{vmatrix} = 0. \tag{18}$$

The determinant given by Eq. (18) provides a third-degree equation in λ.

$$\begin{aligned} &-\lambda^3 + \left[\frac{\partial^2 U}{\partial S^2} + \frac{\partial^2 U}{\partial V^2} + \frac{\partial^2 U}{\partial N^2}\right]\lambda^2 + \left[\frac{\partial^2 U}{\partial S^2}\frac{\partial^2 U}{\partial V^2} + \frac{\partial^2 U}{\partial S^2}\frac{\partial^2 U}{\partial N^2} + \frac{\partial^2 U}{\partial V^2}\frac{\partial^2 U}{\partial N^2} + \frac{\partial^2 U}{\partial S \partial N}\frac{\partial^2 U}{\partial N \partial S} \right. \\ &\left. + \frac{\partial^2 U}{\partial V \partial N}\frac{\partial^2 U}{\partial N \partial V} + \frac{\partial^2 U}{\partial S \partial V}\frac{\partial^2 U}{\partial V \partial S}\right]\lambda + \left[\frac{\partial^2 U}{\partial S^2}\frac{\partial^2 U}{\partial V^2}\frac{\partial^2 U}{\partial N^2} + \frac{\partial^2 U}{\partial S \partial V}\frac{\partial^2 U}{\partial V \partial N}\frac{\partial^2 U}{\partial N \partial S} \right. \\ &+ \frac{\partial^2 U}{\partial V \partial S}\frac{\partial^2 U}{\partial N \partial V}\frac{\partial^2 U}{\partial S \partial N} - \frac{\partial^2 U}{\partial N \partial V}\frac{\partial^2 U}{\partial V \partial N}\frac{\partial^2 U}{\partial S^2} - \frac{\partial^2 U}{\partial N \partial S}\frac{\partial^2 U}{\partial S \partial N}\frac{\partial^2 U}{\partial V^2} \\ &\left. - \frac{\partial^2 U}{\partial S \partial V}\frac{\partial^2 U}{\partial V \partial S}\frac{\partial^2 U}{\partial N^2}\right] = 0. \end{aligned} \tag{19}$$

The above equation is commonly known as characteristic equation, and its solution necessarily imply in three positive roots due to the minimum energy postulate. After some algebraic manipulations [8, 11, 12] in order to solve Eq. (19) and considering $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$ (three positive roots), we find the following relations

$$\frac{\partial^2 U}{\partial S^2} > 0, \frac{\partial^2 U}{\partial V^2} > 0, \frac{\partial^2 U}{\partial N^2} > 0(\text{as expected from one} - \text{variable calculus}), \tag{20}$$

$$\frac{\partial^2 U}{\partial S^2}\frac{\partial^2 U}{\partial V^2} - \frac{\partial^2 U}{\partial S \partial V}\frac{\partial^2 U}{\partial V \partial S} > 0, \tag{21}$$

$$\begin{aligned}&\left[\frac{\partial^2 U}{\partial S^2}\frac{\partial^2 U}{\partial V^2}\frac{\partial^2 U}{\partial N^2} + \frac{\partial^2 U}{\partial S \partial V}\frac{\partial^2 U}{\partial V \partial N}\frac{\partial^2 U}{\partial N \partial S} + \frac{\partial^2 U}{\partial V \partial S}\frac{\partial^2 U}{\partial N \partial V}\frac{\partial^2 U}{\partial S \partial N}\right.\\ &\left.-\frac{\partial^2 U}{\partial N \partial V}\frac{\partial^2 U}{\partial V \partial N}\frac{\partial^2 U}{\partial S^2} - \frac{\partial^2 U}{\partial N \partial S}\frac{\partial^2 U}{\partial S \partial N}\frac{\partial^2 U}{\partial V^2} - \frac{\partial^2 U}{\partial S \partial V}\frac{\partial^2 U}{\partial V \partial S}\frac{\partial^2 U}{\partial N^2}\right] > 0.\end{aligned} \tag{22}$$

Observe that Eq. (22) is equivalent to the determinant of $M_{(3x3)}$ (see Eq. (16)), being positive to energy representaion, and so $|M_{(3x3)}| > 0$. Besides, considering that the product of the three roots $x_1x_2x_3 = -d/a$ in a general third-degree equation $ax^3 + bx^2 + cx + d = 0$ is a known expression of more elementary courses, Eq. (22) can be easily obtained due to the positivity of all eigenvalues of $\tilde{U}$ (see that d is the last bracket term in Eq. (19), and $a = -1$) in the condition of minimum introduced by the thermodynamic postulate. In addition, notice that first relation in Eq. (19) is the determinant of the upper left $1x1$ submatrix of $M_{(3x3)}$, while Eq. (20) is the determinant of the upper left $2x2$ submatrix of $M_{(3x3)}$.

In short, to obtaining in which conditions at equilibrium point $(S_0, V_0, N0)$ $\tilde{U} = \tilde{U}(S, V, N)$ has a minimum in this three-dimensional thermodynamic space, the set of relations given by Eqs. (20)-(22) must occur, where the relations $\frac{\partial^2 U}{\partial V^2} > 0$ and $\frac{\partial^2 U}{\partial N^2} > 0$ in Eq. (20) were introduced for a more physical than mathematical reason during analytical solution of Eq. (19). A general approach about mathematical second derivative test for many variable functions can be found in Ref. [8].

We must solve Eq. (19) permuting U and S in an equivalent entropy representation. Besides, by imposing all negative values due to maximum entropy postulate, it is possible to obtain a set of relations as in Eqs. (20)-(22). Solving eigenvalues equation below

$$\begin{aligned}&-\lambda^3 + \left[\frac{\partial^2 S}{\partial U^2} + \frac{\partial^2 S}{\partial V^2} + \frac{\partial^2 S}{\partial N^2}\right]\lambda^2 + \left[\frac{\partial^2 S}{\partial U^2}\frac{\partial^2 S}{\partial V^2} + \frac{\partial^2 S}{\partial U^2}\frac{\partial^2 S}{\partial N^2} + \frac{\partial^2 S}{\partial V^2}\frac{\partial^2 S}{\partial N^2} + \frac{\partial^2 S}{\partial U \partial N}\frac{\partial^2 S}{\partial N \partial U}\right.\\ &\left.+\frac{\partial^2 S}{\partial V \partial N}\frac{\partial^2 S}{\partial N \partial V} + \frac{\partial^2 S}{\partial U \partial V}\frac{\partial^2 S}{\partial V \partial U}\right]\lambda + \left[\frac{\partial^2 S}{\partial U^2}\frac{\partial^2 S}{\partial V^2}\frac{\partial^2 S}{\partial N^2} + \frac{\partial^2 S}{\partial S \partial V}\frac{\partial^2 S}{\partial V \partial N}\frac{\partial^2 S}{\partial N \partial U}\right.\\ &+\frac{\partial^2 S}{\partial V \partial U}\frac{\partial^2 S}{\partial N \partial V}\frac{\partial^2 S}{\partial U \partial N} - \frac{\partial^2 S}{\partial N \partial V}\frac{\partial^2 S}{\partial V \partial N}\frac{\partial^2 S}{\partial U^2} - \frac{\partial^2 S}{\partial N \partial U}\frac{\partial^2 S}{\partial U \partial N}\frac{\partial^2 S}{\partial V^2}\\ &\left.-\frac{\partial^2 S}{\partial U \partial V}\frac{\partial^2 S}{\partial V \partial U}\frac{\partial^2 S}{\partial N^2}\right] = 0,\end{aligned} \tag{23}$$

and imposing $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$ (all negative eigenvalues due to maximum entropy postulate), we obtain

$$\frac{\partial^2 S}{\partial U^2} < 0, \frac{\partial^2 S}{\partial V^2} < 0, \frac{\partial^2 S}{\partial N^2} < 0(\text{as expected from one} - \text{variable calculus}) \tag{24}$$

$$\frac{\partial^2 S}{\partial U^2}\frac{\partial^2 S}{\partial V^2} - \frac{\partial^2 S}{\partial U \partial V}\frac{\partial^2 S}{\partial V \partial U} > 0 \tag{25}$$

$$\begin{aligned}&\left[\frac{\partial^2 S}{\partial U^2}\frac{\partial^2 S}{\partial V^2}\frac{\partial^2 S}{\partial N^2} + \frac{\partial^2 S}{\partial U \partial V}\frac{\partial^2 S}{\partial V \partial N}\frac{\partial^2 S}{\partial N \partial U} + \frac{\partial^2 S}{\partial V \partial U}\frac{\partial^2 S}{\partial N \partial V}\frac{\partial^2 S}{\partial U \partial N}\right.\\ &\left.-\frac{\partial^2 S}{\partial N \partial V}\frac{\partial^2 S}{\partial V \partial N}\frac{\partial^2 S}{\partial U^2} - \frac{\partial^2 S}{\partial N \partial U}\frac{\partial^2 S}{\partial U \partial N}\frac{\partial^2 S}{\partial V^2} - \frac{\partial^2 S}{\partial U \partial V}\frac{\partial^2 S}{\partial V \partial U}\frac{\partial^2 S}{\partial N^2}\right] < 0.\end{aligned} \tag{26}$$

As it happened for energy, here Eq. (24) is expected from one-variable calculus and its last two relations were introduced for a more physical than mathematical reason during analytical solution of Eq. (23). It is important to emphasize that although Eq. (25) keeps the same format and sign of Eq. (21), the sign in Eq. (26) for the entropy formalism is now negative. This should not cause any surprise and can be concluded even without explicitly calculate the three eigenvalues of characteristic equation due to the known expression to the product between the three roots, $x_1x_2x_3 = -d/a$ in a general third-degree equation $ax^3 + bx^2 + cx + d = 0$. Then, as all eigenvalues are now negative, Eq. (26) is easy verified from characteristic equation (see Eq. (23) where d is the last bracket term, and $a = -1$). The set of Eqs. (24)-(26) provides the mathematical conditions of maximum for entropy thermodynamic function $\tilde{S} = \tilde{S}(U, V, N)$ at (U_0, V_0, N_0).

Some physical problems require the use of thermodynamic potentials of Helmholtz, enthalpy and Gibbs as well as the grand canonical potential instead of thermodynamic energy to be more easy solved. These thermodynamic functions are introduced in the next topic.

3.1 Second-order derivatives of other thermodynamic functions

By using Legendre transformations, it is possible to change the extensive variables, or part of them, in the thermodynamic energy function. In this subsection, we are considering the same energy of three extensive variables defined by $U = U(S, V, N)$ in which making appropriate Legendre transformations the intensive variables are introduced. A discussion on extensive and intensive thermodynamic variables can be found in Ref. [1]. Legendre's transformation is, in short, a process of change of variables.

3.1.1 Helmholtz potential

In order to introduce Helmholtz potential that is an energy function that instead of being a function of S, V and N it is written in terms of T, V and N, we need to make Legendre transformation (change S by T) in extensive parameter S. This process of introducing intensive parameter T is described below. Before let us write $U(S, V, N)$ as

$$dU(S, V, N) = \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV + \frac{\partial U}{\partial N}dN, \tag{27}$$

where the temperature can be defined by $T \equiv \frac{\partial U}{\partial S}$ with V and N constant, the pressure is defined by $P \equiv -\frac{\partial U}{\partial V}$ with S and N constant, and the chemical potential is defined by $\mu \equiv \frac{\partial U}{\partial N}$ with S and V constant. With these definitions, we have to Eq. (27)

$$dU = TdS - PdV + \mu dN. \tag{28}$$

Taking

$$\begin{aligned} d(TS) &= TdS + SdT \\ TdS &= d(TS) - SdT, \end{aligned} \tag{29}$$

and substituting Eq. (29) into Eq. (28)

$$\begin{aligned} dU &= d(TS) - SdT - PdV + \mu dN \\ d(U - TS) &= -SdT - PdV + \mu dN \\ dF &= -SdT - PdV + \mu dN, \end{aligned} \tag{30}$$

where

$$F \equiv U - TS (\text{being} F \text{known as Helmholtz potential}). \tag{31}$$

See of the Eq. (30) that F is a function of T, V and N. Then $F = F(T, V, N)$, and the energy F defined as function of T, V and N has modified its concavite in relation to the new introduced parameter by Legendre transformation in S, i. e., the second-order derivatives of F on T is negative now, keeping positive the signs of F on V and N as in original energy (see Eq. (32) below).

$$\frac{\partial^2 F}{\partial T^2} < 0, \frac{\partial^2 F}{\partial V^2} > 0, \frac{\partial^2 F}{\partial N^2} > 0. \tag{32}$$

It is a general fact that Legendre transformation change the sign of the second-order derivatives of the new introduced function in relation that intensive parameter. A demonstration of this consideration to molar Helmholtz potential$f = f(s, v)$ is shown in Re. [10], and a treatment on Legendre transformations can be found in Ref. [13]. Recently, the thermodynamic stability of chignolin protein was theoretically investigated by using of a computational methodology of decomposition of the Helmholtz energy profile that indicates that intramolecular interactions predominantly stabilized certain conformations of the protein [14]. Besides, in the same study the direct Helmholtz energy decomposition provides the predominant factor in the thermodynamic stability of proteins.

Following the same procedure used to derive the stability conditions of the energy and entropy functions, it is possible to obtain a complete set of relations that Helmholtz potential must obey. Mathematically F is known as a saddle surface. This feature of F stems from the imposition that some eigenvalue of the canonical form of F (similarly to the Eq. (17)) have opposite sign to the others. The saddle surface of Helmholtz of three variables has a maximum in relation to the temperature but a minimum in relation to the volume and mole number. The relations given by Eq. (32) are sufficient to conclude on the physical stability of a system, as demonstrated in Section 4, and the other expressions to the second-order derivatives of F are not shown here. However, the curious reader can be computing all signs of the second-order derivatives to Helmoltz and to other thermodynamic functions that follow below, as already discussed to energy and entropy functions.

3.1.2 Enthalpy potential

The enthalpy potential is also mathematically a saddle surface. In this case, Legendre transformation is applied in the extensive parameter V and introduced the intensive parameter P. Further, H keep unaltered with a minimum in relation to the entropy S and N but becomes a maximum on P, and so $H = H(S, P, N)$. Remembering that $dU = TdS - PdV + \mu dN$, then

$$\begin{aligned} d(pV) &= PdV + VdP \\ -PdV &= -d(PV) + VdP, \end{aligned} \tag{33}$$

and substituting Eq. (33) into Eq. (28), we have

$$\begin{aligned} dU &= TdS - d(PV) + VdP + \mu dN \\ d(U + PV) &= TdS + VdP + \mu dN \\ dH &= TdS + VdP + \mu dN, \end{aligned} \tag{34}$$

where

$$H \equiv U + PV \left(\text{being}\, H\, \text{known as enthalpy potential}\right). \tag{35}$$

Due to Legendre transformations, it is possible to conclude that

$$\frac{\partial^2 H}{\partial S^2} > 0, \frac{\partial^2 H}{\partial P^2} < 0, \frac{\partial^2 H}{\partial N^2} > 0, \tag{36}$$

and other inequalities can be obtained the same way as previously presented to energy and entropy functions,i. e., by diagonalization of $H(S, P, N)$.

3.1.3 Gibbs potential

It is possible to write a function obtained by double Legendre transformation in the extensive parameters S and V, namely Gibbs potential. This is a function on introduced intensive variables T and P. To do that, we combine Eqs. (29) and (33) into Eq. (28). Then,

$$\begin{aligned} dU &= TdS - PdV + \mu dN \\ dU &= d(TS) - SdT - d(PV) + VdP + \mu dN \\ dU - d(TS) + d(PV) &= -SdT + VdP + \mu dN \\ d(U - TS + PV) &= -SdT + VdP + \mu dN \\ dG &= -SdT + VdP + \mu dN, \end{aligned} \tag{37}$$

where

$$G \equiv U - TS + PV (\text{being}\, G\, \text{known as Gibbs potential}). \tag{38}$$

Legendre transformations provide the following relations, and $G = G(T, P, N)$ as seen in Eq. (37).

$$\frac{\partial^2 G}{\partial T^2} < 0, \frac{\partial^2 G}{\partial P^2} < 0, \frac{\partial^2 G}{\partial N^2} > 0. \tag{39}$$

Here the second-order derivatives in relation to T and P are negative now as well as the $G = G(T, P, N)$ becomes a surface of maximum in relation of these two parameters. See that energy keeps unaltered in relation to N, and Gibbs potential has a minimum in relation to mole number because Legendre transformations are applied only in S and V, introducing T and P respectively. Besides, by diagonalization of quadratic form obtained by expanding of G, it is possible to compute other inequalities in additon those expressed by Eq. (39), as already discussed to the energy and entropy formalisms.

3.1.4 Grand canonical potential

A function of T, V and μ is known as grand canonical potential J. To obtaining $J = J(T, V, \mu)$ let us introduce the intensive parameter μ of the extensive parameter N as follows. Taking

$$\begin{aligned} d(\mu N) &= Nd\mu + \mu dN \\ \mu dN &= d(\mu N) - Nd\mu, \end{aligned} \tag{40}$$

and combining the above equation with Eq. (29) into (28), we have

$$\begin{aligned} dU &= TdS - PdV + \mu dN \\ dU &= d(TS) - SdT - PdV + d(\mu N) - Nd\mu \\ dU - d(TS) - d(\mu N) &= -SdT - PdV - Nd\mu \\ d(U - TS - \mu N) &= -SdT - PdV - Nd\mu \\ dJ &= -SdT - PdV - Nd\mu, \end{aligned} \tag{41}$$

where

$$J = U - TS - \mu N = F - \mu N (\text{being} J \, \text{known as grand canonical potential}). \tag{42}$$

Thus, by Legendre transformations in S and N, T and μ intensive variables are introduced, respectively, the relations below are naturally obtained.

$$\frac{\partial^2 J}{\partial T^2} < 0, \frac{\partial^2 J}{\partial V^2} > 0, \frac{\partial^2 J}{\partial \mu^2} < 0. \tag{43}$$

These relations indicate that G has now a maximum in relation to intensive parameters T and μ, keeping a minimum on V. Legendre transformations applied in the entropy formalism are also useful to derive other thermodynamic functions that are not treated here. The appropriate choice of the thermodynamic function is relevant in practical problems. Besides, thermodynamic functions are convex functions of their extensive variables (positive signs of the second-order derivatives) and concave functions (negative signs of the second-order derivatives) of their intensive variables [1].

Novel geometric approaches aimed at obtaining thermodynamic relations in a systematic way for a number of thermodynamic potentials and formally derived the classical Gibbs stability condition has been recently investigated [15].

So far, we demonstrate the mathematical conditions that second-order derivatives of the thermodynamic functions must satisfied. In the next section, we use these conditions to directly obtain the mechanical and thermal stability of a general system.

4. The stability conditions of a system

Let us start this section remembering some quantities of physical interest defined below [1–3]:

$$\alpha \equiv \frac{1}{V}\frac{\partial V}{\partial T} \tag{44}$$

$$c_V \equiv \frac{T}{N}\frac{\partial S}{\partial T} \tag{45}$$

$$c_P \equiv \frac{T}{N}\frac{\partial S}{\partial T} \tag{46}$$

$$k_T \equiv -\frac{1}{V}\frac{\partial V}{\partial P} \tag{47}$$

$$k_S \equiv -\frac{1}{V}\frac{\partial V}{\partial P}, \tag{48}$$

where α (at p constant) in Eq. (44) is the coefficient of thermal expansion, c_V and c_P in Eqs. (45) and (46) respectively, are the specific heats at V or P constant, k_T (T constant) in Eq. (47) is the isothermal compressibility and k_S (S constant) in Eq. (48) is the adiabatic compressibility. All these quantities are relevant in physical applications and their exact values as well as their increase or decrease tendencies can say a lot about the stability of the physical system.

The thermal expansion is related to changes in dimensions of physical systems due to temperature variations. We can understand the behavior of materials on the macroscopic or microscopic scale when subjected to temperature changes by the abosolute values of α that can be positive or negative.

Specific heats are useful to understand the thermal properties of physical systems in several length scales (macroscale and microscale). Besides, the specific heats are positive physical quantities associated to the thermal stability of the system, as will be mathematically demonstrated in this section.

The isothermal and adiabatic compressibilities are positive physical quantities, being related to the mechanical stability of the system. A deep comprehension of the physical origin of the mentioned quantities in terms of the signs of the second-order derivatives of thermodynamis functions, it is relevant to theoretical or experimental researchers.

In order to better investigate the physical consequences of the signs of the second-order derivatives of the energy, see the first relation in Eq. (20)

$$\frac{\partial^2 U}{\partial S^2} > 0. \tag{49}$$

Remembering the temperature definition $T = \frac{\partial U}{\partial S}$, we have by derivation of temperature T side by side in relation to the S entropy

$$\frac{\partial T}{\partial S} = \frac{\partial^2 U}{\partial S^2} > 0. \tag{50}$$

Then, if we combine Eq. (50) and the definition of specific heat (at V constant) given by Eq. (45), it is possible to obtain

$$\begin{aligned} \frac{T}{Nc_V} = \frac{\partial^2 U}{\partial S^2} > 0 \\ \Rightarrow c_V > 0. \end{aligned} \tag{51}$$

A positive specific heat ($c_V > 0$) is obtained due to the absolute temperature is positive. Besides, N is a positive amount. The same physical conclusion can be obtained of the first relation in Eq. (32), $\frac{\partial^2 F}{\partial T^2} < 0$. As F is a function of T, V and N ($F = F(T, V, N)$), an infinitesimal of dF is given by

$$dF = \frac{\partial F}{\partial T} dT + \frac{\partial F}{\partial V} dV + \frac{\partial F}{\partial N} dN \tag{52}$$

that compared with Eq. (30) provides

$$-S = \frac{\partial F}{\partial T}, \tag{53}$$

$$-P = \frac{\partial F}{\partial V}, \tag{54}$$

and

$$\mu = \frac{\partial F}{\partial N}. \tag{55}$$

If we take the derivation side by side of Eq. (53) in relation to T considering V and N constant

$$-\frac{\partial S}{\partial T} = \frac{\partial^2 F}{\partial T^2}. \tag{56}$$

It is possible to observe that the left side of Eq. (56) is relationed to the specific heat at V constant and the sign of the second-order derivatives can be checked by comparing with Eq. (32), and so

$$-\frac{\partial S}{\partial T} = \frac{\partial^2 F}{\partial T^2} < 0, \tag{57}$$

and from definition of specific heat in Eq. (45)

$$-\frac{N c_V}{T} < 0 \Rightarrow \frac{N c_v}{T} > 0 \Rightarrow c_v > 0. \tag{58}$$

Note that Eq. (58) represents the same result already obtained in Eq. (51), only taking different formalisms to thermodynamic function, and so analyzing distinct second-order derivatives. The specific heat must be interpreted as the necessary amount of heat to increase or decrease the temperature of the physical system. A negative specific heat would imply in an inexistent physical situation because we would have a system capable of receiving some quantity of heat (postive) and decreasing its temperature (negative dT). There is still another non-physical situation with negative specific heat in the hypothetical situation in which the system loses heat but increases its temperature.

We investigate now the signs of second-order derivatives of Gibbs potential. The relation given by first inequality in Eq. (39) provides an important conclusion to specific heat at P constant, with $c_P > 0$. To demonstrate that, let us take a differential element dG of Gibbs potential $G = G(T, P, N)$

$$dG = \frac{\partial G}{\partial T} dT + \frac{\partial G}{\partial P} dP + \frac{\partial G}{\partial N} dN. \tag{59}$$

The above equation can be compared with Eq. (37), and we obtain

$$-S = \frac{\partial G}{\partial T}, \tag{60}$$

$$V = \frac{\partial G}{\partial P}, \tag{61}$$

and

$$\mu = \frac{\partial G}{\partial N}. \tag{62}$$

Deriving Eq. (60) side by side in relation to T at P constant, we have

$$-\frac{\partial S}{\partial T} = \frac{\partial^2 G}{\partial T^2}, \tag{63}$$

and from definition of specific heat at P constant in Eq. (46) and by comparing with the first inequality in Eq. (39)

$$\begin{aligned} -\frac{Nc_P}{T} &= \frac{\partial^2 G}{\partial T^2} < 0 \\ \frac{Nc_P}{T} &> 0 \Rightarrow c_P > 0. \end{aligned} \tag{64}$$

Notice that specific heat at P constant is also positive. The positivity of the specific heats previous shown is related to the thermal stability of the physical system. Then, it is possible to see that the thermal stability emerge as consequence of the signs of the second-order derivatives previously treated. Thus, appropriately computing the eigenvalues of the matricial energy or other thermodynamic function is essencial to finding the stability conditions.

Resuming Eq. (54) and by derivation of the left and right sides in relation to V keeping T constant

$$\frac{\partial P}{\partial V} = -\frac{\partial^2 F}{\partial V^2}. \tag{65}$$

Comparing Eq. (66) with the definition to isothermal compressibility in Eq. (47), we can obtain

$$\frac{\partial P}{\partial V} = -\frac{1}{Vk_T}. \tag{66}$$

As the sign of the second-order derivative in Eq. (66) is positive, we have

$$\begin{aligned} -\frac{1}{Vk_T} &= -\frac{\partial^2 F}{\partial V^2} \\ \frac{1}{Vk_T} = \frac{\partial^2 F}{\partial V^2} &> 0 \Rightarrow k_T > 0. \end{aligned} \tag{67}$$

Notice that the sign of the second-order derivative of the appropriately chosen potential leads to a relevant relation for the sign of physical quantity of interest. Besides, in the definition given by Eq. (47) that increments of pressure in the system leads to decrease in volume due to the ever positive isothermal compressibility, and this is an intuitive conclusion. From Eq. (67) we mathematically demonstrated that isothermal compressibility is always positive due to specific features of the potentials. In particular, the positive value of k_T appears from curvature of some chosen potential. The same way $k_S > 0$ can be obtained from enthalpy potential through the the sign of the second relation ($\frac{\partial^2 H}{\partial P^2} < 0$) in Eq. (36), and after some algebraic manipulations. A positive value of this physical quantity is associated with the mechanical stability of the physical system, as in k_T.

It is relevant to clarify that α does not to have a positive defined sign that can be obtained from some function. The well-known case of the water shows that volume increases when temperature decreases below at $4^o C$, being negative α in this regime. Yet, thermodynamic books [1–3] show some relations between the physical quantities, as $c_p = c_v + TV\alpha^2/Nk_T$, $c_p/c_v = k_T/k_S$ as well as $c_p \geq c_v$ and $k_T \geq k_s$ obtained by reduction of thermodynamic derivatives and by using Maxwell's relations. But this is not the purpose of this chapter.

It is worthy of emphasis that some stability condition can be deduced by the signs of the second-order derivatives of energy (or any thermodynamic function), as presented in this chapter. In a three-dimensional (or higher) thermodynamic space the complexity in obtain with success the stability conditions for some potential is associated to the matrix order of the second-order derivatives. Besides, to all cases one or several second-order relations must be manipulated to conclude about the thermal and mechanical stability of the system.

5. Conclusions

In this chapter, we show the useful of specific linear algebra topics in addition with many-variable calculus that coupled to minimum energy postulate appear as in important insight to understand the stability of thermodynamic systems. We find the thermal and mechanical stability of physical systems are directly associated with the signs of the second-order derivatves of thermodynamic energy or other taken representation.

We present a general addressing to the energy representation in terms of matrial equations whereby the stability conditions arise of an eigenvalues fundamental problem. Besides, the minimum energy postulate provides the signs of the second-order derivatives. Accordingly, of a physical point of view the stabilility of a system occurs due to minimum energy postulate.

Formal caracteristics of postulational thermodynamic theory and, particularly, about the second-order derivatives of the thermodynamic functions are discussed with relevant consequences on the thermal and mechanical stability. The presented analytical formalism is an important support to conclude how the stability of a system arises, and can be useful in any field of the exact sciences. We hope that this methodology can be extended to higher-order matrices of energy as well as some of the obtained relations can be used in specific problems of applied physics.

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Nonlinear differential equations are ubiquitous in computational science and engineering modeling, fluid dynamics, finance, and quantum mechanics, among other areas. Nowadays, solving challenging problems in an industrial setting requires a continuous interplay between the theory of such systems and the development and use of sophisticated computational methods that can guide and support the theoretical findings via practical computer simulations. Owing to the impressive development in computer technology and the introduction of fast numerical methods with reduced algorithmic and memory complexity, rigorous solutions in many applications have become possible. This book collects research papers from leading world experts in the field, highlighting ongoing trends, progress, and open problems in this critically important area of mathematics.

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